NONASSOCIATIVE CYCLIC EXTENSIONS OF FIELDS AND CENTRAL SIMPLE ALGEBRAS

C. BROWN AND S. PUMPLÜN

ABSTRACT. We define nonassociative cyclic extensions of degree m of both fields and central simple algebras over fields. If a suitable field contains a primitive m th (resp., q th) root of unity, we show that suitable nonassociative generalized cyclic division algebras yield nonassociative cyclic extensions of degree m (resp., qs). Some of Amitsur's classical results on non-commutative associative cyclic extensions of both fields and central simple algebras are obtained as special cases.

INTRODUCTION

Analogously as both for commutative field extensions [6, 2, 3, 26] and for associative cyclic extensions of fields and central simple algebras [5], nonassociative cyclic extensions of degree m of a field or a central division algebra are investigated separately for prime characteristics and for the case that the characteristic is zero or a prime p with $\gcd(p, m) = 1$. Nonassociative cyclic extensions of degree p in characteristic p were already studied in [18].

Let D be a finite-dimensional central division algebra over a field K . An (associative) central division algebra A over a field F is called a non-commutative *cyclic extension of degree m* of D over K , if $\text{Aut}_F(A)$ has a cyclic subgroup of automorphisms of order m which are all extended from id_D , and if A is a free left D -module of rank m [5]. For instance, if F contains a primitive m th root of unity, then generalized cyclic algebras (D, σ, a) are cyclic extensions of D of degree m [5, Theorem 6]. We recall that a *generalized cyclic algebra* (D, σ, a) is a quotient algebra $D[t; \sigma]/(t^m - a)D[t; \sigma]$, where $D[t; \sigma]$ is a twisted polynomial ring, $\sigma \in \text{Aut}(D)$ is an automorphism such that $\sigma|_K$ has finite order m , $F_0 = \text{Fix}(\sigma) \cap K$, and $f(t) = t^m - a \in D[t; \sigma]$ with $d \in F_0^\times$. The special case where $D = F$ and $F_0 = \text{Fix}(\sigma)$ yields the cyclic algebra $(F/F_0, \sigma, a)$ [12, p. 19].

A finite-dimensional central simple algebra A over F is called a *G -crossed product* if it contains a maximal field extension K/F which is Galois with Galois group $G = \text{Gal}(K/F)$. If G is solvable then A is called a *solvable G -crossed product*. In [9] we revisited a result by Albert [1] on solvable crossed products and gave a proof for Albert's result using generalized cyclic algebras following Petit's approach [17], proving that a G -crossed product is solvable if and only if it can be constructed as a chain of generalized cyclic algebras. Hence any solvable G -crossed product division algebra is always a generalized cyclic division algebra. In particular, hence if F contains a primitive m th root of unity, solvable crossed product division algebras over F are non-commutative cyclic extensions.

2010 *Mathematics Subject Classification.* Primary: 17A35; Secondary: 17A60, 16S36.

Key words and phrases. Skew polynomial, Ore polynomial, cyclic algebra, cyclic extension.

A generalization of associative cyclic extensions of simple rings instead of division rings was considered in [13].

In this paper, we define and investigate nonassociative cyclic extensions of degree m of both fields and central simple algebras employing nonassociative generalized cyclic division algebras: Let A be a unital nonassociative division algebra. Then A is called a *nonassociative cyclic extension of D of degree m* , if A is a free left D -module of rank m and $\text{Aut}(A)$ has a cyclic subgroup G of order m , such that for all $H \in G$, $H|_D = \text{id}_D$.

We show that if F contains a primitive m th root of unity (i.e., F has characteristic 0 or characteristic p with $\gcd(m, p) = 1$), then the nonassociative generalized cyclic division algebras $(D, \sigma, a) = D[t; \sigma]/(t^m - a)D[t; \sigma]$ with $a \in D^\times$ are nonassociative cyclic extensions of D of degree m . Additionally, the subgroup of order m in $\text{Aut}_{F_0}(D, \sigma, a)$ that consists of automorphisms extending id_D contains only inner automorphisms (Corollary 10). We also investigate the structure of the automorphism groups of nonassociative generalized cyclic algebras in general.

Note that nonassociative cyclic division algebras $(K/F, \sigma, a)$ are a special case of nonassociative generalized cyclic division algebras. If F contains a primitive m th root of unity the nonassociative cyclic division algebras $(K/F, \sigma, a)$ are nonassociative cyclic extensions of K of degree m . The subgroup of the automorphisms extending id_K has order m , is isomorphic to $\ker(N_{K/F})$, and contains only inner automorphisms. If F has no non-trivial m th root of unity and $a \in K^\times$ is not contained in any proper subfield of K , all automorphisms of $(K/F, \sigma, a)$ are inner and leave id_K fixed (Theorem 3).

We point out that nonassociative generalized cyclic algebras have been recently successfully used both in constructing space-time block codes and linear codes [19, 20, 21, 22].

The paper is organized as follows: After introducing the basic terminology in Section 1, we define nonassociative cyclic extensions and nonassociative generalized cyclic algebras in Section 2 and investigate nonassociative cyclic extensions of a field. In Section 3 we show when generalized cyclic division algebras (D, σ, d) are nonassociative cyclic extensions of D of degree m . We briefly look at the question when a nonassociative overring is a nonassociative cyclic extension of a field or a central simple algebra in Section 4.

The results presented in this paper complements the ones for the nonassociative algebras $(K, \delta, d) = K[t; \delta]/K[t; \delta]f(t)$ for $f(t) = t^p - t - d \in K[t; \delta]$ constructed using a field K of characteristic p together with some derivation δ with minimum polynomial $g(t) = t^p - t \in F[t]$, $F = \text{Const}(\delta)$, and of the nonassociative algebras $(D, \delta, d) = D[t; \delta]/D[t; \delta]f(t)$ for $f(t) = t^p - t - d \in D[t; \delta]$ constructed using a division algebra D , where the derivation δ has minimum polynomial $g(t) = t^p - t \in F[t]$ and the field F characteristic p . [18].

1. PRELIMINARIES

1.1. Nonassociative algebras. Let F be a field and let A be an F -vector space. A is an *algebra* over F if there exists an F -bilinear map $A \times A \rightarrow A$, $(x, y) \mapsto x \cdot y$, denoted simply by juxtaposition xy , the *multiplication* of A . An algebra A is called *unital* if there is an

element in A , denoted by 1, such that $1x = x1 = x$ for all $x \in A$. We will only consider unital algebras without saying so explicitly.

The *associator* of A is given by $[x, y, z] = (xy)z - x(yz)$. The *left nucleus* of A is defined as $\text{Nuc}_l(A) = \{x \in A \mid [x, A, A] = 0\}$, the *middle nucleus* of A is $\text{Nuc}_m(A) = \{x \in A \mid [A, x, A] = 0\}$ and the *right nucleus* of A is $\text{Nuc}_r(A) = \{x \in A \mid [A, A, x] = 0\}$. $\text{Nuc}_l(A)$, $\text{Nuc}_m(A)$, and $\text{Nuc}_r(A)$ are associative subalgebras of A . Their intersection $\text{Nuc}(A) = \{x \in A \mid [x, A, A] = [A, x, A] = [A, A, x] = 0\}$ is the *nucleus* of A . $\text{Nuc}(A)$ is an associative subalgebra of A containing $F1$ and $x(yz) = (xy)z$ whenever one of the elements x, y, z lies in $\text{Nuc}(A)$. The *center* of A is $C(A) = \{x \in A \mid x \in \text{Nuc}(A) \text{ and } xy = yx \text{ for all } y \in A\}$.

An algebra $A \neq 0$ is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a , $L_a(x) = ax$, and the right multiplication with a , $R_a(x) = xa$, are bijective. If A has finite dimension over F , A is a division algebra if and only if A has no zero divisors [24, pp. 15, 16]. An element $0 \neq a \in A$ has a *left inverse* $a_l \in A$, if $R_a(a_l) = a_l a = 1$, and a *right inverse* $a_r \in A$, if $L_a(a_r) = a a_r = 1$. If $m_r = m_l$ then we denote this element by m^{-1} .

An automorphism $G \in \text{Aut}_F(A)$ is an *inner automorphism* if there is an element $m \in A$ with left inverse m_l such that $G(x) = (m_l x) m$ for all $x \in A$. We denote such an automorphism by G_m . The set of inner automorphisms $\{G_m \mid m \in \text{Nuc}(A) \text{ invertible}\}$ is a subgroup of $\text{Aut}_F(A)$. Note that if the nucleus of A is commutative, then for all $0 \neq n \in \text{Nuc}(A)$, $G_n(x) = (n^{-1}x)n = n^{-1}xn$ is an inner automorphism of A such that $G_n|_{\text{Nuc}(A)} = \text{id}_{\text{Nuc}(A)}$.

1.2. Division algebras obtained from twisted polynomial rings. Let D be a unital division ring and σ a ring automorphism of D . The *twisted polynomial ring* $D[t; \sigma]$ is the set of polynomials $a_0 + a_1 t + \cdots + a_n t^n$ with $a_i \in D$, where addition is defined term-wise and multiplication by $ta = \sigma(a)t$ ($a \in D$) [16]. That means, $at^n b t^m = a \sigma^n(b) t^{n+m}$ and $t^n a = \sigma^n(a) t^n$ for all $a, b \in D$ [12, p. 2]. $R = D[t; \sigma]$ is a left principal ideal domain and there is a right division algorithm in R , i.e. for all $g, f \in R$, $f \neq 0$, there exist unique $r, q \in R$ such that $\deg(r) < \deg(f)$ and $g = qf + r$ [12, p. 3]. (Our terminology is the one used by Petit [17] and different from Jacobson's [12], who calls what we call right a left division algorithm and vice versa.)

An element $f \in R$ is *irreducible* in R if it is no unit and it has no proper factors, i.e. there do not exist $g, h \in R$ such that $f = gh$ [12, p. 11].

Let $f \in D[t; \sigma]$ be of degree m and let $\text{mod}_r f$ denote the remainder of right division by f . Then the vector space $R_m = \{g \in D[t; \sigma] \mid \deg(g) < m\}$ together with the multiplication

$$g \circ h = gh \text{ mod}_r f$$

becomes a unital nonassociative algebra $S_f = (R_m, \circ)$ over $F_0 = \{z \in D \mid zh = hz \text{ for all } h \in S_f\}$ (cf. [17, (7)]), and F_0 is a subfield of D . We also denote this algebra R/Rf .

We note that when $\deg(g) + \deg(h) < m$, the multiplication of g and h in S_f is the same as the multiplication of g and h in R [17, (10)].

A twisted polynomial $f \in R$ is *right-invariant* if $fR \subset Rf$. If f is right invariant then Rf is a two-sided ideal and conversely, every two-sided ideal in R arises this way.

S_f is associative if and only if f is right-invariant. In that case, S_f is the usual quotient algebra $D[t; \delta]/(f)$ [17, (9)].

2. NONASSOCIATIVE GENERALIZED CYCLIC ALGEBRAS AND NONASSOCIATIVE CYCLIC EXTENSIONS

In the following, let D be a division algebra which is finite-dimensional over its center $F = C(D)$ and $\sigma \in \text{Aut}(D)$ such that $\sigma|_F$ has finite order m and fixed field $F_0 = \text{Fix}(\sigma) \cap F$. Note that F/F_0 is automatically a cyclic Galois field extension of degree m with $\text{Gal}(F/F_0) = \langle \sigma|_F \rangle$.

2.1. Following Jacobson [12, p. 19], an (associative) *generalized cyclic algebra* is an associative algebra $S_f = D[t; \sigma]/D[t; \sigma]f$ constructed using a right-invariant twisted polynomial

$$f(t) = t^m - d \in D[t; \sigma]$$

with $d \in F_0^\times$. We write (D, σ, d) for this algebra. If D is a central simple algebra over F of degree n , then (D, σ, d) is a central simple algebra over F_0 of degree mn and the centralizer of D in (D, σ, d) is F [12, p. 20]. In particular, if $D = F$, F/F_0 is a cyclic Galois extension of degree m with Galois group generated by σ and $f(t) = t^m - d \in F[t; \sigma]$, we obtain the cyclic algebra $(F/F_0, \sigma, d)$ of degree m .

This definition generalizes to nonassociative algebras as follows:

Definition 1. A *nonassociative generalized cyclic algebra of degree mn* is an algebra $S_f = D[t; \sigma]/D[t; \sigma]f$ over F_0 with $f(t) = t^m - d \in D[t; \sigma]$, $d \in D^\times$. We denote this algebra by (D, σ, d) .

The algebra $A = (D, \sigma, d)$, $d \in D^\times$, has dimension $m^2 n^2$ over F_0 . In particular, if $D = F$ and F/F_0 is a cyclic Galois extension of degree m with Galois group generated by σ , then $(F/F_0, \sigma, d)$ is a nonassociative cyclic algebra [25]. A is associative if and only if $d \in F_0$. If (D, σ, d) is not associative then $\text{Nuc}_l(A) = \text{Nuc}_m(A) = D$ and $\text{Nuc}_r(A) = \{g \in S_f \mid fg \in Rf\}$.

(D, σ, d) is a division algebra over F_0 if and only if $f(t) = t^m - d \in D[t; \sigma]$ is irreducible [17, (7)]. Moreover, we know that $f(t) = t^2 - d \in D[t; \sigma]$ is irreducible if and only if $\sigma(z)z \neq d$ for all $z \in D$, $f(t) = t^3 - d \in D[t; \sigma]$ is irreducible if and only if $d \neq \sigma^2(z)\sigma(z)z$ for all $z \in D$, and $f(t) = t^4 - d \in D[t; \sigma]$ is irreducible if and only if

$$\sigma^2(y)\sigma(y)y + \sigma^2(x)y + \sigma^2(y)\sigma(x) \neq 0 \text{ or } \sigma^2(x)x + \sigma^2(y)\sigma(y)x \neq d$$

for all $x, y \in D$ (cf. [17] or [19], [7, Theorem 3.19], see also [11]). More generally, if F_0 contains a primitive m th root of unity and m is prime then $f(t) = t^m - d \in D[t; \sigma]$ is irreducible if and only if $d \neq \sigma^{m-1}(z) \cdots \sigma(z)z$ for all $z \in D$ ([7, Theorem 3.11], see also [19, Theorem 6]).

Amitsur's definition [5] of cyclic extensions generalizes to the nonassociative setting as follows:

Definition 2. Let $m \geq 2$. Let A be a nonassociative division algebra with center F_0 and D an associative division algebra with center F . Then A is a *nonassociative cyclic extension* of D of degree m , if A is a free left D -module of rank m and $\text{Aut}(A)$ has a cyclic subgroup G of order m , such that for all $H \in G$, $H|_D = \text{id}$.

2.2. Nonassociative cyclic extensions of a field. For a nonassociative cyclic algebra $(K/F, \sigma, d)$ of degree m , and for all $k \in K$ such that $N_{K/F}(k) = 1$, the map

$$H_{id,k}(\sum_{i=0}^{m-1} a_i t^i) = a_0 + \sum_{i=1}^{m-1} a_i (\prod_{l=0}^{i-1} \sigma^l(k)) t^i$$

is an inner F -automorphism of $(K/F, \sigma, d)$ extending id_K . The subgroup generated by the automorphisms $H_{id,k}$ is isomorphic to $\ker(N_{K/F})$ [8, Theorem 19].

The maps $H_{id,k}$ are the only F -automorphisms of $(K/F, \sigma, d)$, unless for some $j \in \{1, \dots, m-1\}$, σ^j can be extended to an F -automorphism of $(K/F, \sigma, d)$ as well. More precisely, the automorphism $\tau = \sigma^j$ with $j \in \{1, \dots, m-1\}$ can be extended to an F -automorphism H of $(K/F, \sigma, d)$, if and only if there is an element $k \in K$ such that

$$(1) \quad \sigma^j(d) = N_{K/F}(k)d.$$

The extension then has the form $H = H_{\tau,k}$ with

$$(2) \quad H_{\tau,k}(\sum_{i=0}^{m-1} a_i t^i) = \tau(a_0) + \sum_{i=1}^{m-1} \tau(a_i) (\prod_{l=0}^{i-1} \sigma^l(k)) t^i$$

[8, Theorem 4]. We then immediately get the following partial generalization of [5, Theorem 6]:

Theorem 1. *Suppose F contains a primitive m th root of unity ω , $A = (K/F, \sigma, d)$ is a nonassociative cyclic division algebra of degree m over F , and $d \in K \setminus F$. Then A is a nonassociative cyclic extension of K of degree m . The generating automorphism of the subgroup of $\text{Aut}_F(A)$ of order m is given by $H_{id,\omega}$.*

Proof. $\langle H_{id,\omega} \rangle$ is a cyclic subgroup of $\text{Aut}_F(A)$ of order m by [8, Theorem 20]. It consists of automorphisms extending id_K , therefore A is a nonassociative cyclic extension of K . $\square$

Corollary 2. *If m is prime, F contains a primitive m th root of unity and K/F is a cyclic Galois extension of degree m , then K has a nonassociative cyclic extension of degree m .*

Proof. Let $d \in K \setminus F$ and suppose $\text{Gal}(K/F) = \langle \sigma \rangle$. Then since m is prime, the nonassociative cyclic algebra $A = (K/F, \sigma, d)$ is a division algebra [25, Corollary 4.5]. Thus A is a nonassociative cyclic extension of K by Theorem 1. $\square$

If F has no non-trivial m th root of unity, we obtain:

Theorem 3. *Suppose F has no non-trivial m th root of unity. Let $A = (K/F, \sigma, d)$ be a nonassociative cyclic algebra of degree m where $d \in K^\times$ is not contained in any proper subfield of K . Then every F -automorphism of A leaves K fixed and*

$$\text{Aut}_F(A) \cong \ker(N_{K/F}).$$

In particular, all automorphisms of A are inner.

Proof. Every automorphism of A has the form $H_{id,k}$: suppose that there exist $j \in \{1, \dots, m-1\}$ and $k \in K^\times$ such that $H_{\sigma^j,k} \in \text{Aut}_F(A)$. This implies $H_{\sigma^j,k}^2 = H_{\sigma^j,k} \circ H_{\sigma^j,k} \in \text{Aut}_F(A)$ and

$$(3) \quad H_{\sigma^j,k}^2 \left(\sum_{i=0}^{m-1} x_i t^i \right) = \sigma^{2j}(x_0) + \sum_{i=1}^{m-1} \sigma^{2j}(x_i) \left(\prod_{q=0}^{i-1} \sigma^{j+q}(k) \sigma^q(k) \right) t^i.$$

Now $H_{\sigma^j,k}^2$ must have the form $H_{\sigma^{2j},l}$ for some $l \in K^\times$, and comparing (2) and (3) yields $l = k\sigma^j(k)$. Similarly, $H_{\sigma^j,k}^3 = H_{\sigma^{3j},s} \in \text{Aut}_F(A)$ where $s = k\sigma^j(k)\sigma^{2j}(k)$. Continuing in this manner we conclude that the automorphisms $H_{\sigma^j,k}, H_{\sigma^{2j},l}, H_{\sigma^{3j},s}, \dots$ all satisfy (1) implying that

$$(4) \quad \begin{aligned} \sigma^j(d) &= N_{K/F}(k)d, \\ \sigma^{2j}(d) &= N_{K/F}(k\sigma^j(k))d = N_{K/F}(k)^2d, \\ &\vdots \\ d &= \sigma^{nj}(d) = N_{K/F}(k)^nd, \end{aligned}$$

where $n = m/\text{gcd}(j, m)$ is the order of σ^j . Note that $\sigma^{ij}(d) \neq d$ for all $i \in \{1, \dots, n-1\}$ since d is not contained in any proper subfield of K . Therefore $N_{K/F}(k)^n = 1$ and $N_{K/F}(k)^i \neq 1$ for all $i \in \{1, \dots, n-1\}$ by (4), i.e. $N_{K/F}(k)$ is a primitive n th root of unity, thus also an m th root of unity, a contradiction. This proves the assertion. $\square$

Note that if $d \in K^\times$ is not contained in any proper subfield of K then $1, d, \dots, d^{m-1}$ are linearly independent over F and thus A is a division algebra [25]. In particular, if m is prime then $1, d, \dots, d^{m-1}$ are linearly independent over F . This yields for a field F of arbitrary characteristic:

Corollary 4. *Suppose that F has no non-trivial m th root of unity. If $d \in K^\times$ is not contained in any proper subfield of K (e.g. if m is prime), and $\ker(N_{K/F})$ has a subgroup of order m , then any cyclic algebra $A = (K/F, \sigma, d)$ is a cyclic extension of K of degree m .*

Example 5. Let $K = \mathbb{F}_{q^m}$ be a finite field, $q = p^r$ for some prime p , σ an automorphism of K of order $m \geq 2$ and $F = \text{Fix}(\sigma) = \mathbb{F}_q$, i.e. K/F is a cyclic Galois extension of degree m with $\text{Gal}(K/F) = \langle \sigma \rangle$. Then $\ker(N_{K/F})$ is a cyclic group of order $s = (q^m - 1)/(q - 1)$ and any division algebra $(K/F, \sigma, d)$ has exactly s inner automorphisms, all of them extending id_K . The subgroup they generate is cyclic and isomorphic to $\ker(N_{K/F})$ [10]. Hence if m divides s , which is the case if F contains a primitive m th root of unity, then there is a subgroup of automorphisms of order m extending id_K and hence $(K/F, \sigma, d)$ is a cyclic extension of K of degree m .

3. NONASSOCIATIVE CYCLIC EXTENSIONS OF A CENTRAL SIMPLE ALGEBRA

3.1. From now until stated otherwise, let $A = (D, \sigma, d)$ be a nonassociative generalized cyclic algebra of degree mn over F_0 , for some $d \in D \setminus F_0$. We first determine the automorphisms of A :

Theorem 6. (i) Suppose $\tau \in \text{Aut}_{F_0}(D)$ commutes with σ . Then τ can be extended to an automorphism $H \in \text{Aut}_{F_0}(A)$, if and only if there is some $k \in F^\times$ such that $\tau(d) = N_{F/F_0}(k)d$. In that case, the extension H of τ has the form $H = H_{\tau,k}$ with

$$H_{\tau,k}\left(\sum_{i=0}^{m-1} a_i t^i\right) = \tau(a_0) + \sum_{i=1}^{m-1} \tau(a_i) \left(\prod_{l=0}^{i-1} \sigma^l(k)\right) t^i.$$

All maps $H_{\tau,k}$ where $\tau \in \text{Aut}_{F_0}(D)$ commutes with σ and where $k \in F^\times$ such that $\tau(d) = N_{F/F_0}(k)d$ (hence $N_{F/F_0}(k)^{mn} = 1$), are automorphisms of A .

In particular, for $\tau \neq \text{id}$ and $d \notin \text{Fix}(\tau)$, $N_{F/F_0}(k) \neq 1$.

(ii) $\text{id} \in \text{Aut}(D)$ can be extended to an automorphism $H \in \text{Aut}_{F_0}(A)$, if and only if there is some $k \in F^\times$ such that $N_{F/F_0}(k) = 1$. In that case, the extension H of id has the form $H = H_{\text{id},k}$ with

$$H_{\text{id},k}\left(\sum_{i=0}^{m-1} a_i t^i\right) = a_0 + \sum_{i=1}^{m-1} a_i \left(\prod_{l=0}^{i-1} \sigma^l(k)\right) t^i.$$

All $H_{\text{id},k}$ where $k \in F^\times$ such that $N_{F/F_0}(k) = 1$ are automorphisms of A .

Proof. (i) Let $H \in \text{Aut}_{F_0}(A)$, then $H|_D \in \text{Aut}_{F_0}(D)$, since H leaves the left nucleus invariant. Thus $H|_D = \tau$ for some $\tau \in \text{Aut}_{F_0}(D)$. Write $H(t) = \sum_{i=0}^{m-1} k_i t^i$ for some $k_i \in D$, then we have

$$H(tz) = H(t)H(z) = \left(\sum_{i=0}^{m-1} k_i t^i\right)\tau(z) = \sum_{i=0}^{m-1} k_i \sigma^i(\tau(z)) t^i,$$

and

$$H(tz) = H(\sigma(z)t) = \tau(\sigma(z)) \sum_{i=0}^{m-1} k_i t^i = \sum_{i=0}^{m-1} \tau(\sigma(z)) k_i t^i$$

for all $z \in D$. Comparing the coefficients of t^i yields

$$k_i \sigma^i(\tau(z)) = k_i \tau(\sigma^i(z)) = \tau(\sigma(z)) k_i \text{ for all } i = \{0, \dots, m-1\}$$

for all $z \in D$ since σ and τ commute. In particular, we obtain

$$k_i(\tau(\sigma^i(z)) - \tau(\sigma(z))) = 0 \text{ for all } i \in \{0, \dots, m-1\}$$

for all $z \in F$, i.e. $k_i = 0$ or $\sigma|_F = \sigma^i|_F$ for all $i \in \{0, \dots, m-1\}$. As $\sigma|_F$ has order m , this means $k_i = 0$ for all $1 \neq i \in \{0, \dots, m-1\}$. For $i = 1$, this yields $k_1 \tau(\sigma(z)) = \tau(\sigma(z)) k_1$ for all $z \in D$, hence $k_1 \in F$. This implies $H(t) = kt$ for some $k \in F^\times$.

Since

$$H(zt^i) = H(z)H(t)^i = \tau(z)(kt)^i = \tau(z) \left(\prod_{l=0}^{i-1} \sigma^l(k)\right) t^i,$$

for all $i \in \{1, \dots, m-1\}$ and all $z \in D$, H has the form

$$H_{\tau,k} : \sum_{i=0}^{m-1} a_i t^i \mapsto \tau(a_0) + \sum_{i=1}^{m-1} \tau(a_i) \left(\prod_{l=0}^{i-1} \sigma^l(k)\right) t^i,$$

for some $k \in F^\times$.

Comparing the constant terms in $H(t)^m = H(t^m) = H(d)$ implies

$$\tau(d) = k\sigma(k) \cdots \sigma^{m-1}(k)d = N_{F/F_0}(k)d.$$

Let $N = N_{F/F_0} \circ N_{D/F}$ be the norm of the F_0 -algebra D . Applying N to both sides of the equation yields $N(d) = N(k)^m N(d)$, so that $N(k)^m = 1$. Now $k \in F^\times$ and D has degree n , thus

$$N(k) = N_{F/F_0}(N_{D/F}(k)) = N_{F/F_0}(k^n) = N_{F/F_0}(k)^n,$$

and so $N(k)^m = N_{F/F_0}(k)^{nm} = 1$.

Finally, the fact that the maps $H_{\tau,k}$ are automorphisms when τ commutes with σ , and $\tau(d) = N_{F/F_0}(k)d$, can be shown similarly to the proof of [8, Theorem 4], see also [7].

(ii) In particular, for $\tau = id$, we get from (i) that H has the form

$$H_{id,k} : \sum_{i=0}^{m-1} a_i t^i \mapsto a_0 + \sum_{i=1}^{m-1} a_i \left(\prod_{l=0}^{i-1} \sigma^l(k) \right) t^i$$

for some $k \in F^\times$ with $k\sigma(k) \cdots \sigma^{m-1}(k) = N_{F/F_0}(k) = 1$. $\square$

The above is proved for a more general set-up in the first author's PhD thesis [7]. Note that the automorphisms $H_{\tau,k}$ are restrictions of automorphisms of the twisted polynomial ring $D[t; \sigma]$.

Corollary 7. (i) *The subgroup of F_0 -automorphisms of A extending $id_D \in \text{Aut}_{F_0}(D)$ is isomorphic to*

$$\{k \in F^\times \mid k\sigma(k) \cdots \sigma^{m-1}(k) = 1\}.$$

(ii) *If F_0 contains a primitive m th root of unity ω , then $\langle H_{id,\omega} \rangle$ is a cyclic subgroup of $\text{Aut}_{F_0}(A)$ of order m .*

3.2. We obtain the following generalization of [5, Theorem 6]:

Corollary 8. *Suppose F_0 contains a primitive m th root of unity. If $f(t) = t^m - d \in D[t; \sigma]$ is irreducible, then A is a nonassociative cyclic extension of D of degree m . In particular, if m is prime and*

$$d \neq \sigma^{m-1}(z) \cdots \sigma(z)z$$

for all $z \in D$, then A is a nonassociative cyclic extension of D of degree m .

Proof. If F_0 contains a primitive m th root of unity ω , then $\langle H_{id,\omega} \rangle$ is a cyclic subgroup of $\text{Aut}_{F_0}(A)$ of order m by Corollary 7 (ii). If m is prime, then $f(t) = t^m - d \in D[t; \sigma]$ is irreducible if and only if

$$d \neq \sigma^{m-1}(z) \cdots \sigma(z)z$$

for all $z \in D$. The rest is trivial. $\square$

Proposition 9. *Every automorphism $H_{id,k}$ of A is an inner automorphism*

$$G_c \left(\sum_{i=0}^{m-1} a_i t^i \right) = (c^{-1} \sum_{i=0}^{m-1} a_i t^i) c$$

for some $c \in F^\times$ satisfying $k = \sigma(c)c^{-1}$.

Proof. For all $k \in F$ such that $N_{F/F_0}(k) = 1$, $H_{id,k}$ is an F -automorphism extending id_D . These are the only F_0 -automorphisms of A , unless $\tau \neq id$ can be also extended. By Hilbert's Satz 90, $N_{F/F_0}(k) = 1$ if and only if there is $c \in F^\times$ such that $k = c^{-1}\sigma(c)$ [14]. So there is $c \in F^\times$ such that $k = c^{-1}\sigma(c)$ and

$$k\sigma(k) \cdots \sigma^{i-1}(k) = c\sigma^i(c), \quad i = 1, \dots, m-1$$

yields that $H_{id,k} = G$ with

$$G\left(\sum_{i=0}^{m-1} a_i t^i\right) = a_0 + a_1 c^{-1}\sigma(c)t + \sum_{i=2}^{m-1} a_i c^{-1}\sigma^i(c)t^i,$$

which is an inner automorphism, since $G = G_c$ with

$$G_c\left(\sum_{i=0}^{m-1} a_i t^i\right) = (c^{-1} \sum_{i=0}^{m-1} a_i t^i)c.$$

Note that here we use that $F = C(D)$. □

Corollary 10. *If F_0 contains a primitive m th root of unity ω , then A is a cyclic extension of D of order m , and all automorphisms extending id_D are inner.*

Example 11. Let F and L be fields and let K be a cyclic Galois extension of both F and L such that $[K : F] = n$, $[K : L] = m$, $\text{Gal}(K/F) = \langle \gamma \rangle$ and $\text{Gal}(K/L) = \langle \sigma \rangle$, and $\sigma \circ \gamma = \gamma \circ \sigma$. Define $F_0 = F \cap L$.

Let $D = (K/F, \gamma, c)$ be a cyclic division algebra of degree n with $c \in F_0$, i.e. $D \cong D_0 \otimes_{F_0} K$ for some cyclic algebra $D_0 = (F/F_0, \gamma, c)$. Let $1, e, \dots, e^{n-1}$ be the canonical basis of D , that is $e^n = c$, $ex = \gamma(x)e$ for every $x \in K$. For $x = x_0 + x_1e + x_2e^2 + \cdots + x_{n-1}e^{n-1} \in D$, define an L -linear map $\sigma \in \text{Aut}_L(D)$ via

$$\sigma(x) = \sigma(x_0) + \sigma(x_1)e + \sigma(x_2)e^2 + \cdots + \sigma(x_{n-1})e^{n-1}$$

(note that $c \in L$ implies $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in D$). Then $\sigma \in \text{Aut}_{F_0}(D)$ has order m . For all $d \in D^\times$,

$$D[t; \sigma]/D[t; \sigma](t^m - d) = (D, \sigma, d)$$

is a generalized nonassociative cyclic algebra of degree mn over F_0 (used for instance in [20]). (D, σ, d) is associative if and only if $d \in F_0$. In the special case that $d \in F^\times$,

$$(D, \sigma, d) = (L/F_0, \gamma, c) \otimes_{F_0} (F/F_0, \sigma, d)$$

is the tensor product of an associative and a nonassociative cyclic algebra.

If F_0 contains a primitive m th root of unity and $d \in D^\times \setminus F_0$ is chosen such that $f(t) = t^m - d \in D[t; \sigma]$ is irreducible, then (D, σ, d) is a cyclic extension of D of order m , and all automorphisms extending id_D are inner (Corollary 10). Recall that if m is prime then $f(t) = t^m - d \in D[t; \sigma]$ is irreducible if and only if $d \neq \sigma^{m-1}(z) \cdots \sigma(z)z$ for all $z \in D$.

For $m = 2$, this algebra is studied in [21], and used in the codes constructed in [15]. For $d \in F^\times$ the algebra is used in [23], see also [20].

3.3. In the following, let D be a division algebra which is finite-dimensional over its center $F = C(D)$, $\sigma \in \text{Aut}(D)$ an automorphism such that $\sigma|_F$ has finite order q and fixed field $F_0 = \text{Fix}(\sigma) \cap F$. If D has degree n then the associative generalized cyclic algebra $A = (D, \sigma, a)$ has degree qn over F_0 . We choose $a \in F_0$ such that A is a division algebra.

Now assume F_0 contains a primitive q th root of unity ω . Then $\tau = H_{id_D, \omega} : A \rightarrow A$ generates a cyclic subgroup of $\text{Aut}_{F_0}(A)$ of order q by [5, Theorem 6] which consists of automorphisms which all extend id_D . We obtain the following generalization of [5, Theorem 7]:

Theorem 12. *Suppose there exists $\rho \in \text{Aut}(A)$, $b \in A$ and $1 \neq k \in F_0$ such that*

- (1) τ commutes with ρ ,
- (2) $\tau(b) = k\rho(k) \cdots \rho^{m-1}(k)b$,
- (3) k^q is a primitive m th root of unity,
- (4) $t^m - b \in A[t; \rho]$ is irreducible, and
- (5) the algebra $B = A[t; \rho]/A[t; \rho](t^m - b)$ is either associative, or finite-dimensional over $F_0 \cap \text{Fix}(\rho)$, or finite-dimensional over $\text{Nuc}_r(B)$.

Then B is a nonassociative cyclic extension of D of degree mq which contains A .

Proof. Since B is a free left A -module of rank m and A is a free left D -module of rank q , B is a free left D -module of rank mq . Furthermore, (4) and (5) yield that B is a division algebra by [17, (7)]. Define the map

$$H_{\tau, k} : B \rightarrow B, \quad \sum_{i=0}^{m-1} x_i t^i \mapsto \tau(x_0) + \sum_{i=1}^{m-1} \tau(x_i) \left(\prod_{l=0}^{i-1} \rho^l(k) \right) t^i \quad (x_i \in A),$$

then (1) and (2) together imply that $H_{\tau, k}$ is an automorphism of B by [8, Theorem 4].

$H_{\tau, k}$ has order mq : We have $\tau(k) = k$ because $k \in F_0 \subset D$. Therefore straightforward calculations yield $H_{\tau, k}^2 = H_{\tau, k} \circ H_{\tau, k} = H_{\tau^2, k\tau(k)} = H_{\tau^2, k^2}$, $H_{\tau, k}^3 = H_{\tau^3, k^3}$ etc., thus $H_{\tau, k}$ will have order at least q . After q steps we obtain $H_{\tau, k}^q = H_{id_A, v}$ with $v = k^q$ and so $H_{\tau, k}$ has order mq by (3).

Finally $H_{\tau, k}|_D = \tau|_D = id_D$, hence we conclude B is a nonassociative cyclic extension of D of degree mq . $\square$

4. WHEN IS A RING A NONASSOCIATIVE CYCLIC EXTENSION?

A nonassociative ring $A \neq 0$ is called a *right division ring*, if for all $a \in A$, $a \neq 0$, the right multiplication with a , $R_a(x) = xa$, is bijective. If D is a division ring and f is irreducible, then $S_f = D[t; \sigma]/D[t; \sigma]f$ is a right division algebra and has no zero divisors ([17, (6)] or [11]).

Theorem 13. (cf. [17, (3), (6)])

(i) *Let S be a nonassociative ring with multiplication $\circ$. Suppose that*

- (1) *S has an associative subring D which is a division algebra and S is a free left D -module of rank m , and there is $t \in S$ such that t^j , $0 \leq i < m$ is a basis of S over D , when defining $t^{j+1} = t \circ t^j$, $t^0 = 1$;*
- (2) *for all $a \in D$, $a \neq 0$, there are $a_1, a_2 \in D$, $a_1 \neq 0$, such that $t \circ a = a_1 \circ t + a_2$;*

(3) for all $a, b, c \in D$, $i + j < m$, $k < m$, we have $[a \circ t^i, b \circ t^j, c \circ t^k] = 0$.

Then $S \cong S_f$ with $f(t) \in D[t; \sigma, \delta]$ and σ, δ defined via $t \circ a = \sigma(a) \circ t + \delta(a)$ and where the polynomial $f(t) = t^m - \sum_{i=0}^{m-1} d_i t^i$ is given by $t^m = \sum_{i=0}^{m-1} d_i t^i$ with $t^0 = 1$, $t^{i+1} = t \circ t^i$, $0 \leq i < m$.

(ii) If S is a right division ring in (i) then f is irreducible.

Theorem 13 yields the nonassociative analogues to the existence conditions for associative cyclic extensions in [5, Theorem 6].

Theorem 14. (i) Let S be a nonassociative ring with multiplication $\circ$, which has a field K as a subring, and is a free left K -vector space of dimension m . Suppose that

(1) there is $t \in S$ such that t^i , $0 \leq i < m$, is a basis of S over K when defining $t^0 = 1$, $t^{i+1} = t \circ t^i$, $0 \leq i < m$;

(2) for all $a \in K$, $a \neq 0$, there is $a' \in K^\times$, such that $t \circ a = a' \circ t$;

(3) for all $a, b, c \in K$, $i + j < m$, $k < m$, we have $[a \circ t^i, b \circ t^j, c \circ t^k] = 0$;

(4) $t^m = d$ for some $d \in K^\times$;

(5) the map $\sigma : K \rightarrow K$, $\sigma(a) = a'$, has order m and fixed field $F = \{a \in K \mid t \circ a = a \circ t\}$ containing a primitive m th root of unity ω , and K/F is a finite cyclic Galois extension.

Then $S \cong S_f = (K/F, \sigma, d)$ with $f(t) = t^m - d \in K[t; \sigma]$.

(ii) If S is a right division ring in (i) then f is irreducible and $S \cong (K/F, \sigma, d)$ is a nonassociative cyclic extension of K of degree m .

Proof. (1), (2) and (3) imply that $S \cong S_f$ with $f \in K[t; \sigma]$ and σ defined via $t \circ a = \sigma(a) \circ t$, i.e. $\sigma(a) = a'$, and where the polynomial $f(t) = t^m - \sum_{i=0}^{m-1} d_i t^i$ is given by $t^m = \sum_{i=0}^{m-1} d_i t^i$ for some suitably chosen d_i (cf. [17, (3)]). (4) implies that indeed $f(t) = t^m - d$. (5) guarantees that $(K/F, \sigma, d)$ where F contains a primitive m th root of unity ω .

(ii) Here we are in the setup of Theorem 1 which yields the assertion: F contains a primitive m th root of unity ω , so $\langle H_{id, \omega} \rangle$ is a cyclic subgroup of order m of the division algebra $(K/F, \sigma, d)$. $\square$

For nonassociative cyclic extensions of a central simple algebra D we obtain from Theorem 13:

Theorem 15. (i) Let S be a nonassociative ring with multiplication $\circ$, which has an associative subring D which is a division algebra and S is a free left D -module of rank m . Suppose that

(1) there is $t \in S$ such that t^i , $0 \leq i < m$, is a basis of S over D when defining $t^0 = 1$, $t^{i+1} = t \circ t^i$, $0 \leq i < m$;

(2) for all $a \in D$, $a \neq 0$, there are $a' \in D$, $a' \neq 0$, such that $t \circ a = a' \circ t$;

(3) for all $a, b, c \in D$, $i + j < m$, $k < m$, we have $[a \circ t^i, b \circ t^j, c \circ t^k] = 0$;

(4) $t^m = d$;

(5) the map $\sigma : D \rightarrow D$, $\sigma(a) = a'$, has order m , fixed field $\{a \in D \mid t \circ a = a \circ t\}$ and D/F is a central simple algebra, where $F_0 = F \cap \text{Fix}(\sigma)$ contains a primitive m th root of unity ω . Then $S \cong S_f = (D, \sigma, d)$ with $f(t) = t^m - d \in D[t; \sigma]$.

(ii) If S is a right division ring and D a central simple algebra in (i), then f is irreducible and S a nonassociative cyclic extension of D of degree m .

Proof. (1), (2) and (3) imply that $S \cong S_f$ with $f \in D[t; \sigma]$ and σ defined via $t \circ a = \sigma(a) \circ t$, i.e. $\sigma(a) = a'$, and where the polynomial $f(t) = t^m - \sum_{i=0}^{m-1} d_i t^i$ is given by $t^m = \sum_{i=0}^{m-1} d_i t^i$ for some suitably chosen d_i (cf. [17, (3)]). (4) implies $f(t) = t^m - d$. (5) guarantees that $S \cong (D, \sigma, d)$ where F contains a primitive m th root of unity ω .

(ii) Here we are in the setup of Theorem 6 which yields the assertion, since F contains a primitive m th root of unity ω , $\langle H_{id, \omega} \rangle$ is a cyclic subgroup of order m of the division algebra (D, σ, d) . $\square$

REFERENCES

- [1] A. A. Albert, *Structure of algebras*. Vol. 24, AMS 1939.
- [2] A. A. Albert, *Cyclic fields of degree p^n over F of characteristic p* . Bull. AMS 40 (1934), 625-631.
- [3] A. A. Albert, *Modern higher algebra*. Chicago 1937, Chapter IX, 192-208.
- [4] A. S. Amitsur, *Differential Polynomials and Division Algebras*. Annals of Mathematics, Vol. 59 (2) (1954) 245-278.
- [5] A. S. Amitsur, *Non-commutative cyclic fields*. Duke Math. J. 21 (1954), 87105.
- [6] E. Artin, O. Schreier, *Über eine Kennzeichnung der reell algebraischen Körper*. Abh. Math. Seminar der Hamburgischen Universitäten, 5 (1927), 225-231.
- [7] C. Brown, *Petit's algebras and their automorphisms*. PhD Thesis, University of Nottingham, 2018.
- [8] C. Brown, S. Pumplün, *The automorphisms of Petit's algebras*. Comm. Algebra 46 (2) (2018), 834-849.
- [9] C. Brown, S. Pumplün, *Solvable crossed product algebras revisited*. Online at arXiv:1702.04605 [math.RA]
- [10] C. Brown, S. Pumplün, A. Steele, *Automorphisms and isomorphisms of Jha-Johnson semifields obtained from skew polynomial rings*. To appear in Comm. Alg. Online at arXiv:1703.02356 [math.RA]
- [11] C. Brown, S. Pumplün, *How a nonassociative algebra reflects the properties of a skew polynomial*. Preprint, 2018.
- [12] N. Jacobson, "Finite-dimensional division algebras over fields." Springer Verlag, Berlin-Heidelberg-New York, 1996.
- [13] K. Kishimoto, *On cyclic extensions of simple rings*. J. Fac. Sci. Hokkaido Univ. Ser. I 19 (1966), 74-85.
- [14] T. Y. Lam, A. Leroy, *Hilbert 90 theorems over division rings*. Trans. Amer. Math. Soc. 345 (2) (1994), 595-622.
- [15] N. Markin, F. Oggier, *Iterated space-time code constructions from cyclic algebras*. IEEE Transactions on Information Theory, 59 (9), September 2013.
- [16] O. Ore, *Theory of noncommutative polynomials*. Annals of Math. 34 (3) (1933), 480-508.
- [17] J.-C. Petit, *Sur certains quasi-corps généralisant un type d'anneau-quotient*. Séminaire Dubriel. Algèbre et théorie des nombres 20 (1966 - 67), 1-18.
- [18] S. Pumplün, *Nonassociative differential extensions of characteristic p* . Results in Mathematics 72 (1-2) (2017), 245-262, DOI 10.1007/s00025-017-0656-x
- [19] S. Pumplün, *Tensor products of nonassociative cyclic algebras*. Journal of Algebra 451 (2016), 145-165.
- [20] S. Pumplün, A. Steele, *Fast-decodable MIDO codes from nonassociative algebras*. Int. J. of Information and Coding Theory (IJICOT) 3 (1) 2015, 15-38.
- [21] S. Pumplün, *How to obtain division algebras used for fast decodable space-time block codes*. Adv. in Math. Comm. 8 (3) (2014), 323 - 342.
- [22] S. Pumplün, A. Steele, *The nonassociative algebras used to build fast-decodable space-time block codes*. Adv. Math. Comm. 9 (4) (2015), 449-469.
- [23] K. P. Srinath, B. S. Rajan, *Fast-decodable MIDO codes with large coding gain*. IEEE Transactions on Information Theory (2) 60 2014, 992-1007.
- [24] R. D. Schafer, "An Introduction to Nonassociative Algebras." Dover Publ., Inc., New York, 1995.
- [25] A. Steele, *Nonassociative cyclic algebras*. Israel Journal of Mathematics 200 (1) (2014), 361-387.
- [26] E. Witt, *Zyklische Körper und Algebren der Charakteristik p vom Grad p^n* . J. Reine Angew. Math. 176 (1936), 126-140.

Email address: christian_jb@hotmail.co.uk; susanne.pumpluen@nottingham.ac.uk

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM
NG7 2RD, UNITED KINGDOM