On The Asphericity of a Family of Positive Relative Group Presentations

Suzana Aldwaik and Martin Edjvet
School of Mathematical Sciences
The University of Nottingham
University Park, Nottingham NG7 2RD, UK

Abstract

Excluding four exceptional cases, the asphericity of the relative presentation $\mathcal{P}= \langle G, x | x^m gxh \rangle$ for $m \geq 2$ is determined. If $H = \langle g, h \rangle \leq G$, then the exceptional cases occur when H is isomorphic to C_5 or C_6.

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1 Introduction

A *relative group presentation* is a presentation of the form $\mathcal{P}= \langle G, \mathbf{x} | \mathbf{r} \rangle$, where G is a group and $\mathbf{x}$ is a set disjoint from G. Denoting the free group on $\mathbf{x}$ by $\langle \mathbf{x} \rangle$, $\mathbf{r}$ is a set of cyclically reduced words in the free product $G * \langle \mathbf{x} \rangle$. The group defined by $\mathcal{P}$ is $\hat{G}= G * \langle \mathbf{x} \rangle / N$, where N is the normal closure in $G * \langle \mathbf{x} \rangle$ of $\mathbf{r}$. A relative presentation is said to be *aspherical* if every spherical picture over it contains a dipole. These notions were defined and studied in [3] where it is shown that if $\mathcal{P}$ is aspherical then group theoretic information about $\hat{G}$ can be deduced.

There has been much interest in determining asphericity of $\mathcal{P}$ particularly when $\mathbf{x} = \{x\}$ and $\mathbf{r} = \{r\}$ both consist of a single element. Indeed, if $r = x^{\varepsilon_1} g_1 \ldots x^{\varepsilon_k} g_k$ where $g_i \in G$, $\varepsilon_i = \pm 1$ and $g_i = 1$ implies $\varepsilon_i + \varepsilon_{i+1} \neq 0$ ($1 \leq i \leq k$, subscripts mod k), then the asphericity of $\mathcal{P}$ has been determined (modulo some exceptional cases) when $k \leq 3$ or $r \in \{ xg_1xg_2xg_3xg_4,\ xg_1xg_2xg_3x^{-1}g_4,\ xg_1xg_2xg_3xg_4xg_5,\ (xg_1)^{l_1}(xg_2)^{l_2}(xg_3)^{l_3} (l_i > 1, 1 \leq i \leq 3)\}$ [1-3] [7-9]. This list includes $x^m gx^{-1}h$ ($g, h \in G \backslash \{1\}$) for $m \leq 3$, and when $m \geq 4$ asphericity (modulo exceptional cases) has been determined in [6].

In this paper we consider $x^m gxh$ ($g, h \in G \backslash \{1\}$). If $m = 2$ then a complete classification of when $\mathcal{P}$ is aspherical has been obtained in [3]. Modulo some exceptions the cases $m = 3$ and $m = 4$ were determined in [2] and [8] respectively. Before stating our main result observe that $x^m gxh = 1$ if and only if $x^{-m}h^{-1}x^{-1}g^{-1} = 1$, and it follows that we can work modulo $g \leftrightarrow h^{-1}$.

We list the following exceptional cases.

(**E1**) $g = h^2$, $|h| = 5$ and $m \geq 5$.

(**E2**) $g \in \{h^2, h^3, h^4\}$, $|h| = 6$ and $m \geq 3$.

Theorem 1.1. *Let $\mathcal{P}$ be the relative presentation $\mathcal{P} = \langle G, x | x^m gxh \rangle$, where $m \geq 2$, $x \notin G$, $g, h \in G \backslash \{1\}$. Suppose that none of the conditions in (**E1**) or (**E2**) holds. Then $\mathcal{P}$ is aspherical if and only if (modulo $g \leftrightarrow h^{-1}$) none of the following holds:*

1. *$g = h^{\pm 1}$ has finite order.*

2. *$g = h^2$ has finite order and $m = 2$.*

3. *$g = h^2$, $|h| = 4$ and $m \geq 3$.*

4. *$g = h^2$, $|h| = 5$ and $3 \leq m \leq 4$.*

5. *$g \in \{h^3, h^4\}$, $|h| = 6$ and $m = 2$.*

6. *$|g| = 2$, $|h| = 3$ and $[g, h] = 1$.*

7. *$\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, where $\frac{1}{\infty} := 0$.*

If $m = 2, 3, 4$ (respectively) then the proof of Theorem 1.1 can be deduced from results in [3], [2], [8] (respectively) apart from two exceptional cases for $m = 3$ (**E4** and **E5** of [2]) which are dealt with here together with the case $m \geq 5$. In Section 2 we discuss the method of the proof where the concept of pictures is needed. In Section 3 some preliminaries results are stated. The proof of Theorem 1.1 is given in Section 4.

2 Method of Proof

2.1 Pictures and Curvature

The definitions of this subsection are taken from [3]. The reader is referred to [3] and [2] for more details.

A *picture* $\mathbb{P}$ is a finite collection of pairwise disjoint discs $\{D_1, \ldots, D_m\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs

$\{\alpha_1, \ldots, \alpha_n\}$ embedded in the closure of $D^2 - \bigcup_{i=1}^{m} D_i$ in such a way that each arc meets $\partial D^2 \cup \bigcup_{i=1}^{m} D_i$ transversely in its end points. The *boundary* of $\mathbb{P}$ is the circle ∂D^2, denoted by $\partial\mathbb{P}$. For $1 \leq i \leq m$, the *corners* of D_i are the closures of the connected components of $\partial D_i - \bigcup_{j=1}^{n} \alpha_j$, where ∂D_i is the boundary of D_i. The *regions* Δ of $\mathbb{P}$ are the closures of the connected components of $D^2 - \left(\bigcup_{i=1}^{m} D_i \cup \bigcup_{j=1}^{n} \alpha_j\right)$. An *inner region* of $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet $\partial\mathbb{P}$. The picture $\mathbb{P}$ is *non-trivial* if $m \geq 1$, is *connected* if $\bigcup_{i=1}^{m} D_i \cup \bigcup_{j=1}^{n} \alpha_j$ is connected, and is *spherical* if it is non-trivial and if none of the arcs meets the boundary of D^2. The number of edges in $\partial\Delta$ is called the *degree* of the region Δ and is denoted by $d(\Delta)$. A region of degree n will be called an *n-region.* If $\mathbb{P}$ is a spherical picture, the number of different discs to which a disc D_i is connected is called the *degree* of D_i, denoted by $d(D_i)$. The discs of a spherical picture $\mathbb{P}$ are also called *vertices* of $\mathbb{P}$.

Suppose that the picture $\mathbb{P}$ is labelled in the following sense: each arc α_j is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$. Each corner of $\mathbb{P}$ is oriented *clockwise* (with respect to D^2) and labelled by an element of G. If κ is a corner of a disc D_i of $\mathbb{P}$, then $W(\kappa)$ will be the word obtained by reading in a clockwise order the labels on the arcs and corners meeting ∂D_i beginning with the label on the first arc we meet as we read the clockwise corner κ. If we cross an arc labelled x in the direction of its normal orientation, we read x, else we read x^{-1}.

A *picture over* $\mathcal{P}$ is a picture $\mathbb{P}$ labelled in such a way the following are satisfied:

1. For each corner κ of $\mathbb{P}$, $W(\kappa) \in \mathbf{r}^*$, the set of all cyclic permutations of $\mathbf{r} \cup \mathbf{r}^{-1}$ which begin with a member of $\mathbf{x}$.

2. If $g_1, ..., g_l$ is the sequence of corner labels encountered in *anticlockwise* traversal of the boundary of an inner region Δ of $\mathbb{P}$, then the product $g_1 g_2 ... g_n$=1 in G. We say that $g_1 g_2 ... g_n$ is the label of Δ, denoted by $l(\Delta) = g_1 g_2 ... g_n$.

A *dipole* in a labelled picture $\mathbb{P}$ over $\mathcal{P}$ consists of corners κ and κ' of $\mathbb{P}$ together with an arc joining the two corners such that κ and κ' belong to the same region and such that if $W(\kappa)$= Sg where $g \in G$ and S begins and ends with a member of $\mathbf{x} \cup \mathbf{x}^{-1}$, then $W(\kappa')$= $S^{-1}g^{-1}$. The picture $\mathbb{P}$ is *reduced* if it does not contain a dipole. A relative presentation $\mathcal{P}$ is called *aspherical* if every connected spherical picture over $\mathcal{P}$ contains a dipole. If $\mathcal{P}$ is not aspherical then there is a reduced spherical picture over $\mathcal{P}$.

The *star graph* $\mathcal{P}^{st}$ of a relative presentation $\mathcal{P}$ is a graph whose vertex set is $\mathbf{x} \cup \mathbf{x}^{-1}$ and edge set is $\mathbf{r}^*$. For $R \in \mathbf{r}^*$, write $R = Sg$ where $g \in G$ and S begins and ends with a member of $\mathbf{x} \cup \mathbf{x}^{-1}$. The initial and terminal functions are given as follows: $\iota(R)$ is the first symbol of S, and $\tau(R)$ is the inverse of the last symbol of S. The labelling function on the edges is defined by $\lambda(R) = g^{-1}$ and is extended to paths in the usual way. A non-empty cyclically reduced cycle (closed path) in $\mathcal{P}^{st}$ will be called *admissible* if it has trivial label in G. Each inner region of a reduced picture over $\mathcal{P}$ supports an admissible cycle in $\mathcal{P}^{st}$.

A *weight function* θ is a real-valued function on the set of edges of $\mathcal{P}^{st}$ which satisfies $\theta(Sg) = \theta(S^{-1}g^{-1})$ where $Sg = R \in \mathbf{r}^*$. The weight of a closed cycle is the sum of the weights of the constituent edges. A weight function is *weakly aspherical* if the following conditions are satisfied:

1. Let $R \in \mathbf{r}^*$, with $R = x_1^{\varepsilon_1} g_1 \dots x_n^{\varepsilon_n} g_n$. Then

$$\sum_{i=1}^{n} (1 - \theta(x_i^{\varepsilon_i} g_i \dots x_n^{\varepsilon_n} g_n x_1^{\varepsilon_1} g_1 \dots x_{i-1}^{\varepsilon_{i-1}} g_{i-1})) \geq 2.$$

2. The weight of each admissible cycle in $\mathcal{P}^{st}$ is at least 2.

If $\mathcal{P}^{st}$ admits a weakly aspherical weight function, then $\mathcal{P}$ is aspherical [3] and this method will be used in the proofs.

Another method is *curvature distribution* (see, for example [7]). Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$. We proceed as follows. An *angle function* on $\mathbb{P}$ is a real-valued function on the set of corners of $\mathbb{P}$. Given this, the *curvature* of a vertex of $\mathbb{P}$ is defined to be 2π less the sum of the angles at that vertex. The *curvature* $c(\Delta)$ of a k-gonal region Δ of $\mathbb{P}$ is the sum of all the angles of the corners of Δ less $(k-2)\pi$. Our method of associating angles ensures that vertices have zero curvature and it follows from this that $\sum c(\Delta) = 4\pi$ where the sum is taken over all the regions Δ of $\mathbb{P}$. Assuming that none of conditions 1-7 holds, our strategy will be to show that the positive curvature that exists in $\mathbb{P}$ can be sufficiently compensated by the negative curvature. To this end we locate each Δ satisfying $c(\Delta) > 0$ and distribute $c(\Delta)$ to near regions $\hat{\Delta}$ of Δ. For such regions $\hat{\Delta}$ define $c^*(\hat{\Delta})$ to equal $c(\hat{\Delta})$ plus all the positive curvature $\hat{\Delta}$ receives during this *distribution procedure*. We prove that $c^*(\hat{\Delta}) \leq 0$ and, since the total curvature of $\mathbb{P}$ is at most $\sum c^*(\hat{\Delta})$, this yields a contradiction which shows that $\mathcal{P}$ is aspherical.

2.2 Construction of pictures and Defined angle functions

For this subsection we assume $g \neq h^{\pm 1}$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}=\langle G, x | x^m gxh\rangle$. Then each vertex (disc) in $\mathbb{P}$ has one of the forms given by Figure 2.2.1(i) and (ii); and the the star graph $\mathcal{P}^{st}$ of $\mathcal{P}$ is given by Figure 2.2.1(iii). Note that when drawing figures the edge arrows shown in Figure 2.2.1 will often be omitted.

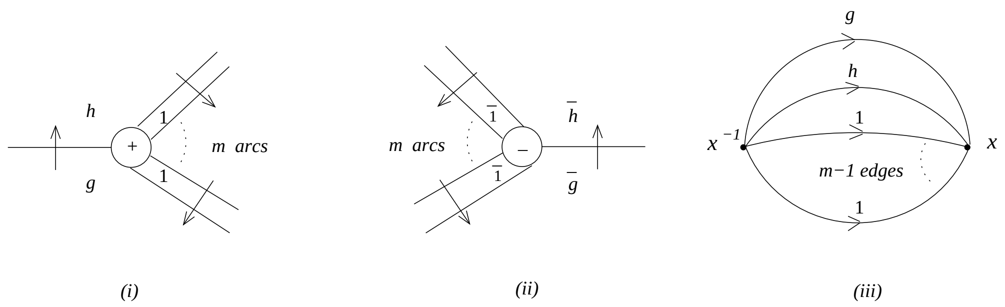

Figure 2.2.1: + disc, - disc and $\mathcal{P}^{st}$.

If there are $k-1$ consecutive regions of degree 2, then the k arcs in the boundary of these regions constitute a *k-bond*. We will refer to a 1-bond as a single bond. Given that $g \neq h^{\pm 1}$ there are (up to inversion) only two types of $(m-1)$-bonds in a reduced picture $\mathbb{P}$ (see Figure 2.2.2). For simplicity, in our figures $(m-1)$-bonds will be drawn as bold 2-bonds (see Figure 2.2.2). Note that there are no m-bonds or $(m+1)$-bonds in $\mathbb{P}$, indeed a vertex of degree 2 can only occur in a reduced picture if $g = h$ or $g = 1$ or $h = 1$. Also, for simplicity, the vertex of degree 3 of the form shown in Figure 2.2.3 (i) will be drawn as shown in Figure 2.2.3 (ii), where $m_1 \geq 2$, $m_2 \geq 2$ and $m_1 + m_2 = m$.

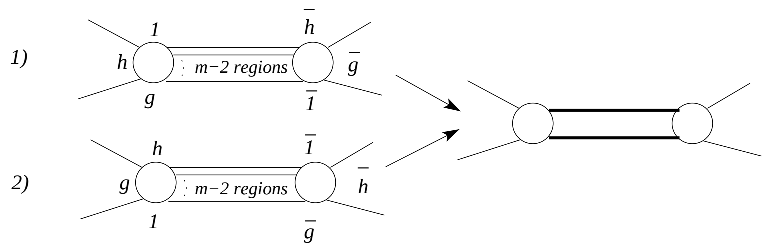

Figure 2.2.2: $(m-1)$-bond.

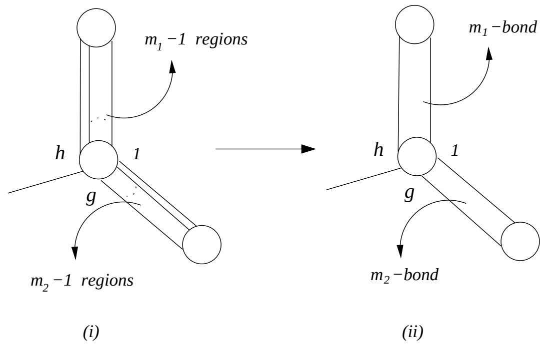

Figure 2.2.3.

Remarks 2.1.

1. *Each arc connects a + disc to a - disc, and so each region has even degree.*

2. *A word w obtained from reading the labels on the edges of a cyclically reduced cycle in $\mathcal{P}^{st}$ does not contain (up to cyclic permutation and inversion) gg^{-1} or hh^{-1} although it can contain 11^{-1} provided different edges in $\mathcal{P}^{st}$ are used. We will call such words w cyclically reduced .*

3. *Each region in a reduced spherical picture $\mathbb{P}$ over $\mathcal{P}$ supports a cyclically reduced word in $\{g, h, 1\}$.*

There are (up to inversion) three types of vertices of degree 3 and these are shown in Figure 2.2.4.

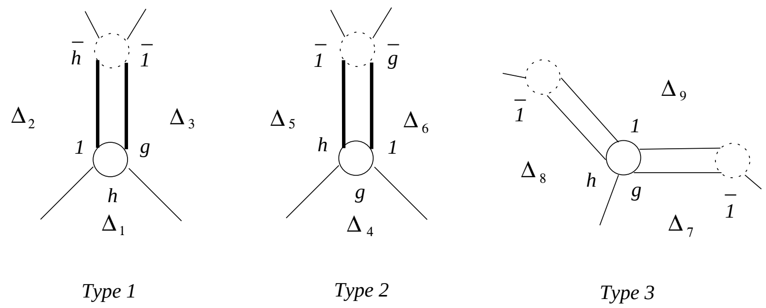

Figure 2.2.4.

For the proofs, we define the following angle functions on the vertices v of $\mathbb{P}$. The *angle function* α is defined as follows. Each corner within a 2-bond has angle zero, while each of the other corners has angle $\frac{2\pi}{d(v)}$. We will refer to α as the *standard angle function.*

The *angle function* α_1 is defined as follows. Again, corners within 2-bonds have angle zero. For vertices of degree 3 of Type 1-3, α_1 is given by Figure 2.2.5. If $d(v) > 3$, then each corner in v has angle $\frac{2\pi}{d(v)}$.

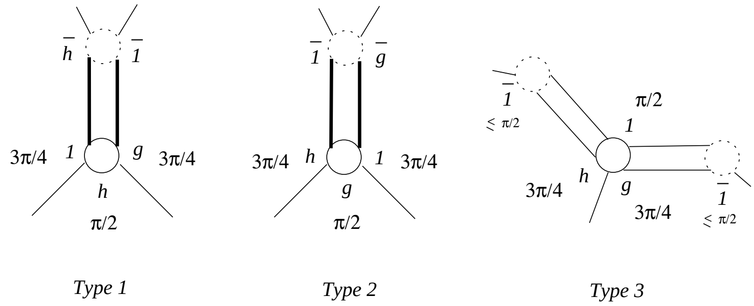

Figure 2.2.5: Angle function α_1 for vertices of degree 3.

Define an *angle function* α_2 on $\mathbb{P}$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners labelled by $h^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 2.2.6). Corners in vertices of degree > 3 have angle $\frac{2\pi}{d(v)}$.

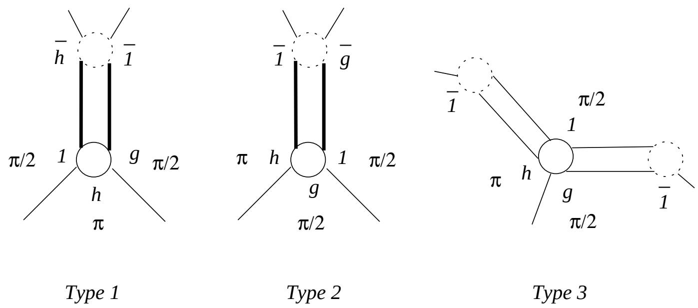

Figure 2.2.6: Angle function α_2 for vertices of degree 3.

Finally, the *angle function* α_3 on $\mathbb{P}$ is given as follows. Corners within 2-bonds have angle zero. For vertices of degree 3, corners labelled by $1^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 2.2.7). Corners in vertices of degree > 3 have angle $\frac{2\pi}{d(v)}$.

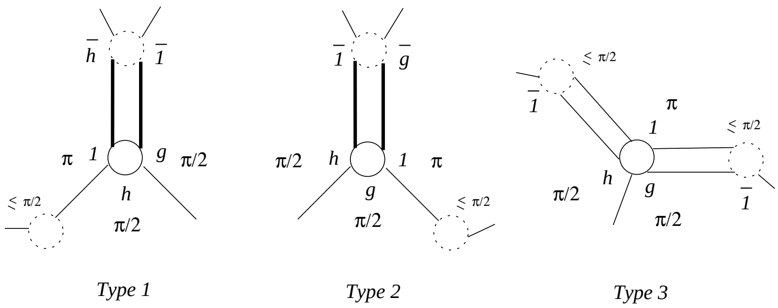

Figure 2.2.7: Angle function α_3 for vertices of degree 3.

Remarks 2.2.

1. *The corners in each 2-bond have angle 0 in each of the above angle functions. It follows that the curvature of regions of degree 2 is 0, and so we can treat each k-bond as a single bond.*

2. *By assigning the angle function α_1 to the corners of $\mathbb{P}$, the following are satisfied:*

   (i) Since $(2-8)\pi + 8.\frac{3\pi}{4} = 0$, positive regions can only have degree 4 or 6.

   (ii) Both corners adjacent to the $(m-1)$-bond in a boundary of a region have angle $\frac{3\pi}{4}$; while the two corners adjacent to the m_1-bond or m_2-bond in a boundary of a region cannot both have angle $\frac{3\pi}{4}$ (see Figure 2.2.5).

3. *By assigning the angle function α_2 to the corners of $\mathbb{P}$, the following are satisfied:*

   (i) In any region Δ of $\mathbb{P}$, there are no consecutive corners with angle π, else $\mathbb{P}$ is not reduced. Hence, $c(\Delta) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} = \pi(\frac{8-n}{4})$ and so positively curved regions can only be 4-regions or 6-regions.

   (ii) If Δ is a positive 4-region, then it has at least one corner labelled by $h^{\pm 1}$ with angle π (otherwise $c(\Delta) \leq -2\pi + 4.\frac{\pi}{2} = 0$).

   (iii) If Δ is a positive 6-region, then it contains at least three $h^{\pm 1}$-corners each with angle π (else $c(\Delta) \leq -4\pi + 2\pi + 4.\frac{\pi}{2} = 0$).

4. *By assigning the angle function α_3 to the corners of $\mathbb{P}$, the following are satisfied:*

   (i) There are no consecutive corners with angle π in the boundary of a region Δ of $\mathbb{P}$ (otherwise $\mathbb{P}$ is not reduced). Thus, $c(\Delta) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} = \pi(\frac{8-n}{4})$ and so positive regions can only be 4-regions or 6-regions.

   (ii) If Δ is a positive 4-region, then it contains at least one corner labelled by $1^{\pm 1}$ with angle π (otherwise $c(\Delta) \leq -2\pi + 4.\frac{\pi}{2} = 0$).

(iii) If Δ is a positive 6-region, then it contains three occurrences of $1^{\pm 1}$-corners each with angle π (else $c(\Delta) \leq -4\pi + 2\pi + 4.\frac{\pi}{2} = 0$).

3 Preliminary Lemmas

Assume that $m \geq 5$. We first state a series of lemmas followed by their proofs. Recall that we assume $g, h \in G \backslash \{1\}$.

3.1 Statement of Lemmas

Lemma 3.1. *If $\mathcal{P}$ is not aspherical, then at least one of the following conditions holds:*

1. $g = h^{\pm 1}$;
2. $g = h^2$ *or* $h = g^2$;
3. $2 \in \{|g|, |h|\}$;
4. $|gh^{-1}| = 2$ *and* $3 \in \{|g|, |h|\}$.

Lemma 3.2. *If $g = h^{\pm 1}$, then $\mathcal{P}$ is aspherical if and only if g has infinite order.*

Lemma 3.3. *Let $g = h^2$. If $|h| = 4$, then $\mathcal{P}$ is not aspherical, while if $|h| > 6$, then $\mathcal{P}$ is aspherical.*

Lemma 3.4. *If $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, then $\mathcal{P}$ is not aspherical.*

Lemma 3.5. *If $|gh^{-1}|$ is infinite, then $\mathcal{P}$ is aspherical.*

Lemma 3.6. *Suppose that $|g| = 2$.*

1. *If $|gh^{-1}| = 2$ and $|h| = \infty$, then $\mathcal{P}$ is aspherical.*
2. *If $|gh^{-1}| = 3$, $|h| \geq 6$ and $\mathcal{P}$ is not aspherical, then $g = h^3$, in particular $|h| = 6$.*
3. *If $|gh^{-1}| \geq 4$, $|h| \geq 4$ and $g \neq h^2$, then $\mathcal{P}$ is aspherical.*
4. *If $|gh^{-1}| \geq 6$ and $|h| = 3$, then $\mathcal{P}$ is not aspherical if and only if $[g, h] = 1$.*

Lemma 3.7. *If $|g| = 3$, $|gh^{-1}| = 2$, $|h| \geq 6$ and $\mathcal{P}$ is not aspherical, then $g = h^4$ and $|h| = 6$.*

3.2 Proof of Lemma 3.1.

Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$. It can be assumed without any loss of generality (**A**) that the number of regions of degree 4 cannot be decreased by bridge moves [5]. Suppose that none of the Conditions 1, 2 or 3 holds.

First assign the standard angle function α to the vertices of $\mathbb{P}$. Since for any n-region Δ in $\mathbb{P}$, $c(\Delta) \leq \pi\left(\frac{6-n}{3}\right)$, $c(\Delta) > 0$ only if $n = 4$. A positively curved 4-region Δ has at least one vertex of degree 3. If $\Delta \in \{\Delta_i : 1 \leq i \leq 8\}$ which are shown in Figure 2.2.4, then at least one corner of Δ is not labelled by $1^{\pm 1}$. By considering all cyclically reduced words of length at most 4 in $\{g^{\pm 1}, h^{\pm 1}\}$ (which are compatible with our hypotheses on g and h), we obtain $l(\Delta) = (gh^{-1})^{\pm 2}$. If $\Delta = \Delta_9$ then $l(\Delta)$ gives a contradiction or Δ is the positive 4-region shown in Figure 3.2.1. Since $m_1 > B$ a sequence of bridge moves transforms Δ into a region of degree > 4 without creating a new region of degree 4. This contradicts assumption (**A**) and so by assigning α we obtain $|gh^{-1}| = 2$.

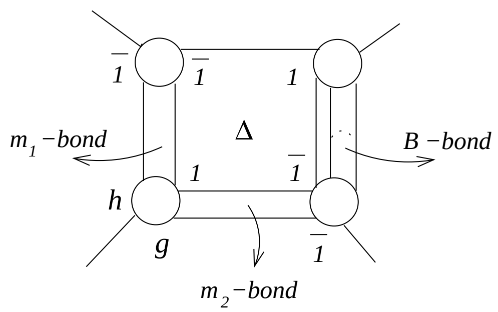

Figure 3.2.1.

Now apply the angle function α_1. By Remark 2.2.(2)(i), positively curved regions can only be 4-regions or 6-regions. A positively curved 4-region Δ has at least one corner with angle $\frac{3\pi}{4}$ in its boundary and so $\Delta = \Delta_i$ for some $i \in \{2, 3, 5, 6, 7, 8\}$. This implies that Δ has at least one corner not labelled by $1^{\pm 1}$. Also, it implies that $l(\Delta) \neq (gh^{-1})^{\pm 2}$. All other choices contradict our assumptions on g and h and so there are no positive 4-regions. It follows that Δ is a 6-region which contains at least five corners with angle $\frac{3\pi}{4}$ in its boundary (else, $c(\Delta) \leq (2-6)\pi + 4.\frac{3\pi}{4} + 2.\frac{\pi}{2} = 0$). By Remark 2.2(2)($ii$) Δ contains at least two $(m-1)$-bonds in its boundary and a third bond which is either an $(m-1)$-bond, an m_1-bond or m_2-bond. If the $(m-1)$-bonds in the boundary of Δ are inwardly oriented, then $l(\Delta) = (g1^{-1})^{\pm 3}$, while if the $(m-1)$-bonds are oriented outward Δ, then $l(\Delta) = (h1^{-1})^{\pm 3}$. It follows that $|gh^{-1}| = 2$ and $3 \in \{|g|, |h|\}$ which is Condition 4, as required.

3.3 Proof of Lemma 3.2.

If $g = h$ then $x^m gxh = 1$ if and only if $x^{m-1}(xg)^2 = 1$. By Lemma 1 in [9], $\mathcal{P}$ is aspherical if and only if $|g| = \infty$.

If $g = h^{-1}$ and g has infinite order, then Lemma 3 in [2] applies to show that $\mathcal{P}$ is aspherical. But $x^m gxg^{-1} = 1$ and $|g| < \infty$ implies $|x| < \infty$ and by Theorem 1 in [2] $\mathcal{P}$ is not aspherical.

3.4 Proof of Lemma 3.3.

Let $g = h^2$. For $|h| = 4$ there is the sphere shown in Figure 3.4.1. On the other hand if $|h| = k > 6$, then the ordinary presentation $\langle x, h | x^m h^2 xh = 1 = h^k \rangle$ is a C(4)-T(4) presentation, hence $\mathcal{P}$ is aspherical (for more details see [2]).

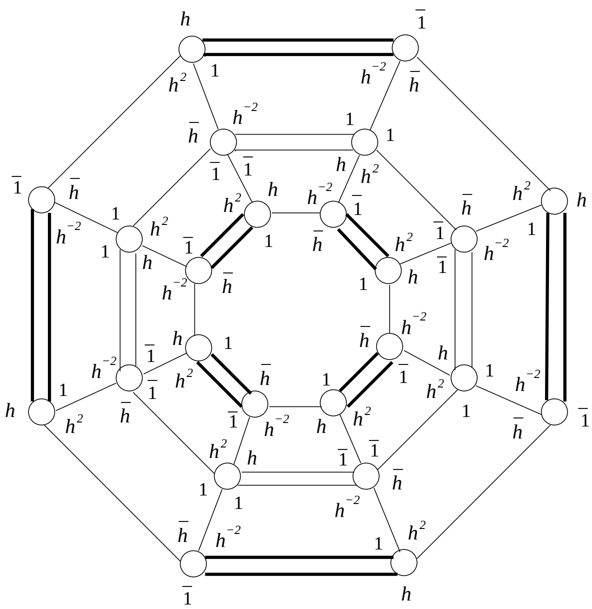

Figure 3.4.1: $g = h^2$ and $|h| = 4$.

3.5 Proof of Lemma 3.4.

If $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$ then there are spherical pictures $\mathbb{P}$ over $\mathcal{P}$. For example if $(|g|, |gh^{-1}|, |h|)$ =(2, 3, 4) then $\mathbb{P}$ is given by Figure 3.5.1. The other spheres are constructed in a similar way, we omit the details.

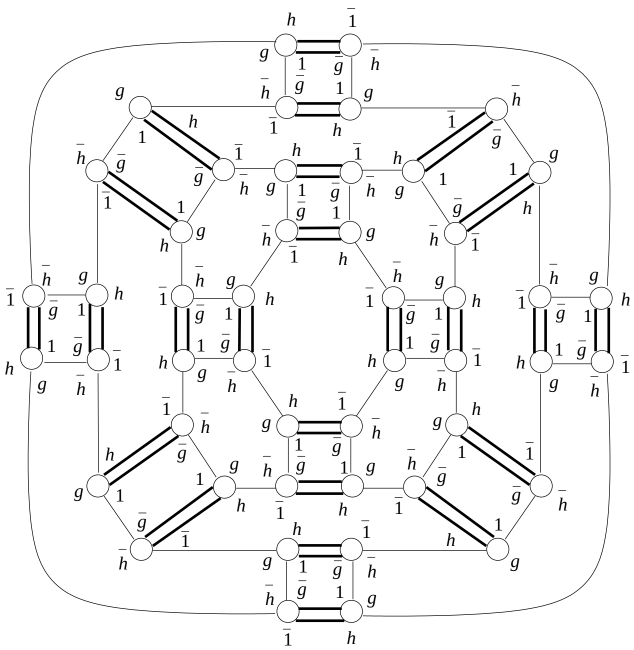

Figure 3.5.1: $\big(|g|, |gh^{-1}|, |h|\big) = (2, 3, 4)$.

3.6 Proof of Lemma 3.5.

Suppose that $|gh^{-1}|$ is infinite. If we have a relation of the form $(gh^{-1})^k g = 1$ or $h^{-1}(gh^{-1})^k = 1$ in G then $H = gp\{g, h\}$ is infinite cyclic generated by gh^{-1}, and so $\mathcal{P}$ is aspherical by Lemma 3 in [2]. So assume otherwise.

Define the following weight function θ on $\mathcal{P}^{st}$ (see Figure 2.2.1(iii)): $\theta(e_g)= 0= \theta(e_h)$ and $\theta(s_i) = 1$ for $(1 \leq i \leq m-1)$, where e_g, e_h, s_i $(1 \leq i \leq m-1)$ are the edges of $\mathcal{P}^{st}$ labelled g, h, 1 (respectively). Clearly Condition 1 of weakly aspherical weight function is satisfied. The assumptions on g and h imply that each admissible cycle in $\mathcal{P}^{st}$ must involve at least 2 edges labelled by the identity, and so has weight at least 2. Therefore θ is an aspherical weight function which proves that $\mathcal{P}$ is aspherical.

3.7 Proof of Lemma 3.6(1): Case$(2, 2, \infty)$

In this case, $|g| = 2$, $|gh^{-1}| = 2$ and $|h| = \infty$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$ and assign the angle function α_2 to $\mathbb{P}$. By Remark 2.2 (3)(i), positive regions can only be 4-regions or 6-regions. By Remark 2.2 (3)(iii), positive 6-regions involve three occurrences of $h^{\pm 1}$-corners and each possible label yields a contradiction. By Remark 2.2 (3)(ii) a positive 4-region must contain $h^{\pm 1}$ forcing the label $(gh^{-1})^{\pm 2}$. Hence, there are (up to inversion) two types of positive regions as shown in Figure 3.7.1. (Note that the maximum possible curvature is always indicated.)

We adopt the notation of [2] and define the following distribution scheme (distributing positive curvature from Δ to $\hat{\Delta}$) which is given in Figure 3.7.1:

$$\Gamma(\Delta,\hat{\Delta}) = \begin{cases} c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond } S \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ such that } S \text{ is adjacent to an} \\ & h^{\pm 1}\text{-corner in } \Delta \text{ with angle } \pi \\ c(\Delta)/2 & \text{if } \frac{\pi}{2} < c(\Delta) \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

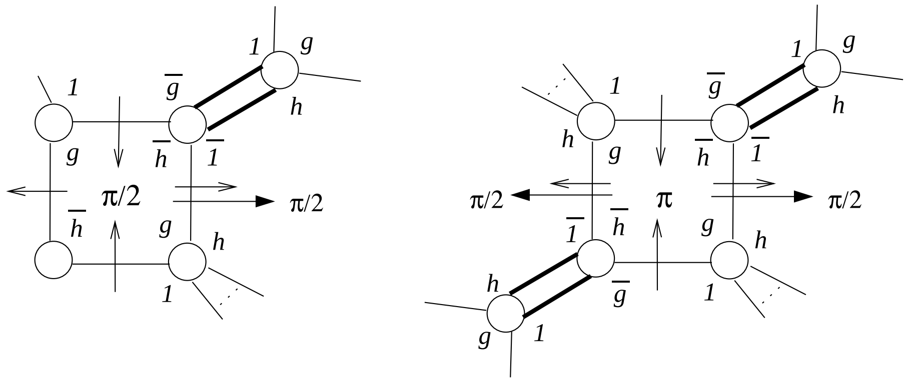

Figure 3.7.1: Positive regions and distribution scheme in Case$(2,2,\infty)$.

Let $\Gamma(\Delta,\hat{\Delta}) > 0$ and let r be the number of corners of angle π in $\hat{\Delta}$. By Remark 2.2(3)(i), $r \leq \frac{n}{2}$ where $n = d(\hat{\Delta})$. Set $\Gamma_2 = \Gamma_2(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta,\hat{\Delta}) = \frac{\pi}{2}\}| \leq \frac{n}{2}$ (since $\hat{\Delta}$ receives $\pi/2$ only across edges that are oriented inwards - see Figure 3.7.1). Then $c^*(\hat{\Delta}) \leq (2-n)\pi + r\pi + (n-r)\frac{\pi}{2} + \Gamma_2.\frac{\pi}{2} = 2\pi - \frac{\pi}{2}(n - r - \Gamma_2) \leq 2\pi$. It follows that if $\Gamma_2 \leq \frac{n}{2} - 4$ then $c^*(\hat{\Delta}) \leq 0$, so assume otherwise.

If $\Gamma_2 = \frac{n}{2}$ or $\frac{n}{2} - 1$, then (see Figure 3.7.1) the labelling of $\hat{\Delta}$ implies that either $h^{\pm\frac{n}{2}} = 1$ or $g = h^{\pm\frac{n}{2}}$, contradicting $|h| = \infty$. This leaves $\Gamma_2 = \frac{n}{2} - 3$ and $r = \frac{n}{2}$; or $\Gamma_2 = \frac{n}{2} - 2$ and $r \geq \frac{n}{2} - 1$ (otherwise $c^*(\hat{\Delta}) \leq 0$). First assume that $r = \frac{n}{2} - 1$. Then $\Gamma_2 = \frac{n}{2} - 2$ and $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$. The fact that $\Gamma_2 = \frac{n}{2} - 2$ means that there are two inwardly oriented edges in $\partial\hat{\Delta}$ across which $\hat{\Delta}$ does not receive $\frac{\pi}{2}$. Figure 3.7.2 (i) shows the first case (consecutive), which forces $l(\hat{\Delta}) = h^{\frac{n}{2}-1}w_1w_2w_3$, where $w_1, w_3 \in \{1^{-1}, g^{-1}\}$ and $w_2 \in \{1, g, h\}$; and it follows that $|h| < \infty$, a contradiction. The second case is given by Figure 3.7.2 (ii) and $l(\hat{\Delta}) = z_1h^{\alpha_1}z_2h^{\alpha_2}$, where $z_1, z_2 \in \{1^{-1}, g^{-1}\}$. If $z_1 = 1^{-1}$ or

$z_2 = 1^{-1}$ then $|h| = \infty$, a contradiction, so assume otherwise. But if $z_1 = g^{-1}$ in Figure 3.7.2 (ii) then either the h-corner in the vertex v_1 has angle $\leq \frac{\pi}{2}$, or Δ_1 contains an m-bond in its boundary and so it cannot be either of the positive regions shown in Figure 3.7.1. (i.e $\hat{\Delta}$ does not receive $\frac{\pi}{2}$ from Δ_1). Either way, $c^*(\hat{\Delta})$ will be decreased by $\frac{\pi}{2}$ and so $c^*(\hat{\Delta}) \leq 0$.

Now let $r = \frac{n}{2}$ in which case $\Gamma_2 = \frac{n}{2} - 2$ or $\frac{n}{2} - 3$ and $c^*(\hat{\Delta}) \leq \pi$. Since $g^2 {=} (gh^{-1})^2 {=} 1$ it follows that any word in g and h can be rewritten in the form $g^{\alpha_1} h^{\alpha_2}$. If $g^{\pm 1}$ appears an odd number of times in $l(\hat{\Delta})$ then $|h| < \infty$. Also, if $g^{\pm 1}$ occurs at least four times in $l(\hat{\Delta})$ then $\Gamma_2 \leq \frac{n}{2} - 4$, a contradiction, and so $g^{\pm 1}$ appears exactly twice in $l(\hat{\Delta})$. Since $r = \frac{n}{2}$, each of these two g^{-1}-corners is adjacent to two h-corners in $\partial\hat{\Delta}$. Thus, arguing as in the case $z_1 = g^{-1}$ above it follows that $c^*(\hat{\Delta}) \leq \pi - 2.\frac{\pi}{2} = 0$.

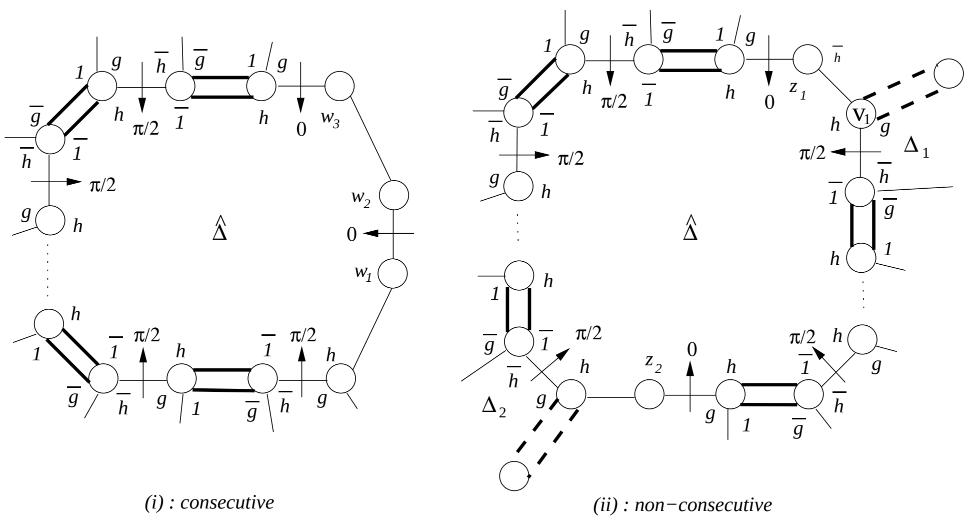

Figure 3.7.2.

3.8 Proof of Lemma 3.6(2): Case$(2, \bar{3}, \bar{6})$

Here we assume that $|g| = 2$, $|gh^{-1}| \geq 3$ and $|h| \geq 6$. Suppose that $\mathcal{P} = \langle G, x | x^m gxh \rangle$ is not aspherical. We show that $H = gp\{g, h\}$ is cyclic of order 6 generated by h and $g = h^3$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$ to which we assign the angle function α_2. All possible labels for a positive 4-region give a contradiction since, by Remark 2.2 (3)(ii), each must involve $h^{\pm 1}$. For positive 6-regions, by Remark 2.2 (3)(iii), there are three occurrences of $h^{\pm 1}$ and the only possible labels not yielding a contradiction imply $(gh^{-1})^{\pm 3} = 1$ or $g = h^3$ (and we are done). Therefore there is (up to inversion) only one positive region which is shown in Figure 3.8.1.

Apply the following distribution scheme:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} c(\Delta)/3 & \text{if } c(\Delta) > 0 \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

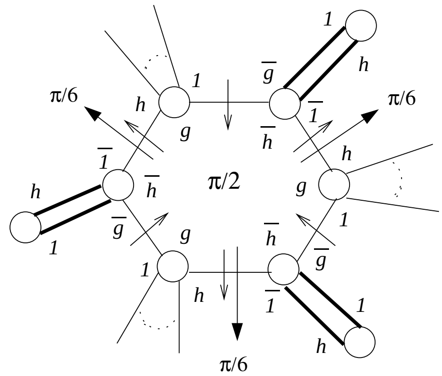

Figure 3.8.1: Positive regions and distribution scheme in Case$(2, \bar{3}, \bar{6})$.

As shown in Figure 3.8.1, if $\Gamma(\Delta, \hat{\Delta}) > 0$, then $(h1^{-1}h)^{\pm 1}$ is a sublabel of $\hat{\Delta}$. For a fixed region $\hat{\Delta}$ set $\Gamma_6(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{6}\}|$.

Remarks 3.8.

1. *Since $\hat{\Delta}$ receives $\pi/6$ only through edges that are oriented towards $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.*
2. *For each $\pi/6$ that $\hat{\Delta}$ receives, there is an $(m-1)$-bond in the boundary of $\hat{\Delta}$ which gives $(h1^{-1})^{\pm 1}$ as a sublabel of $\hat{\Delta}$.*
3. *$l(\hat{\Delta}) = h1^{-1}hw$ and so $d(\hat{\Delta}) > 6$; since if $d(\hat{\Delta})$=6 then $l(\hat{\Delta})$ yields a contradiction or $g = h^3$.*

Observe that by Remarks 2.2 (3)(i) and 3.8.1, $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6}$ and so $c^*(\hat{\Delta}) > 0$ implies $n < 12$.

Let $\hat{\Delta}$= (n, r) denote a region of degree n with $\Gamma_6 = r$. We need to check $c^*(\hat{\Delta})$ for $\hat{\Delta}$= (n, r)= $(10, 5), (10, 4), (10, 3), (10, 2), (10, 1), (8, 4), (8, 3), (8, 2)$ and $(8, 1)$. The region $(n, r) \neq (10, 5)$ or $(8, 4)$ else it gives $h^{\pm 5} = 1$ or $h^{\pm 4} = 1$ (respectively) contradicting $|h| \geq 6$. All possible labels for $\hat{\Delta}$=(n, r)= $(10, 4)$ or $(8, 3)$ yields a contradiction.

For example, $\hat{\Delta}=(8,3)$ gives either $h^{\pm 4}=1$ or $g=h^4$: the first contradicts $|h|\geq 6$ and the second implies $h=1$. For $\hat{\Delta}=(10, r\leq 3)$, $c^*(\hat{\Delta})\leq (2-10)\pi+5\pi+5.\frac{\pi}{2}+3.\frac{\pi}{6}=0$. Finally, since $(2-8)\pi+3\pi+5.\frac{\pi}{2}+2.\frac{\pi}{6}=-\frac{\pi}{6}<0$, $c^*(\hat{\Delta})>0$ for $\hat{\Delta}=(8, r\leq 2)$ only if it contains 4 corners with angle π (up to inversion $\hat{\Delta}$ is shown in Figure 3.8.2), and each possible $l(\hat{\Delta})$ yields a contradiction.

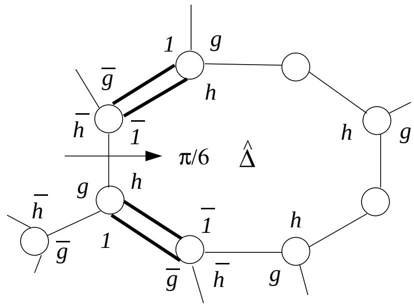

Figure 3.8.2: $\hat{\Delta}=(8, r\leq 2)$ with $c^*(\hat{\Delta})>0$.

3.9 Proof of Lemma 3.6(3): Case$(2,\bar{4},\bar{4})$

Here $|g|=2$, $|gh^{-1}|\geq 4$ and $|h|\geq 4$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$ and assign the angle function α_2 to $\mathbb{P}$. By Remark 2.2(3)(i) a positive region $\hat{\Delta}$ can only have degree 4 or 6. It follows from Remarks 2.2(3)(ii) and (iii) that $l(\hat{\Delta})$ will yield a contradiction. Therefore, in this case $\mathcal{P}$ is aspherical.

3.10 Proof of Lemma 3.6(4): Case$(2,\bar{6},3)$

If $[g,h]=1$ then $|gh^{-1}|=6$, $(gh^{-1})^3=g$ and $(gh^{-1})^2=h$. It follows that $\mathcal{P}=\langle G,x|x^m b^3 x b^2, b^6\rangle$ and this presentation has been shown to be not aspherical by Bogley and Williams [4] (indeed it can be shown that b is conjugate to x^{m+1}). So it can be assumed that $[g,h]\neq 1$. We prove that $\mathcal{P}=\langle G,x|x^m gxh\rangle$ is aspherical. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$ with the assumption (**A**) stated in the proof of Lemma 3.1 and assign the angle function α_3. By Remark 2.2(4)(i) the degree of a positive region Δ can only be 4 or 6. If Δ is a positive 4-region with an $h^{\pm 1}$-corner then $l(\Delta)$ yields a contradiction, so assume otherwise. If now Δ has a $g^{\pm 1}$ corner then $l(\Delta)$ yields the 4-regions shown in Figure 3.10.1. This leaves $l(\Delta)=11^{-1}11^{-1}$ which contradicts (**A**) as in the proof of Lemma 3.1.

If Δ is a 6-region, then either there is a contradiction or $l(\Delta)\in\{1^{-1}11^{-1}11^{-1}1$, $1^{-1}11^{-1}g1^{-1}g$, $1^{-1}h1^{-1}h1^{-1}h\}$. The first two cannot be positive, while the last gives the positive 6-region shown in Figure 3.10.1.

Define the following distribution scheme which is given in Figure 3.10.1:

$$\Gamma(\Delta,\hat{\Delta}) = \begin{cases} c(\Delta)/2 & \text{if } c(\Delta)=\pi \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \\ c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2},\ \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond } S \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ and } S \text{ is adjacent to a 1-corner} \\ & \text{ in } \Delta \text{ with angle } \pi \\ \pi/6 & \text{if } c(\Delta)=\frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \hat{\Delta} \text{ to } \Delta \\ 0 & \text{otherwise} \end{cases}$$

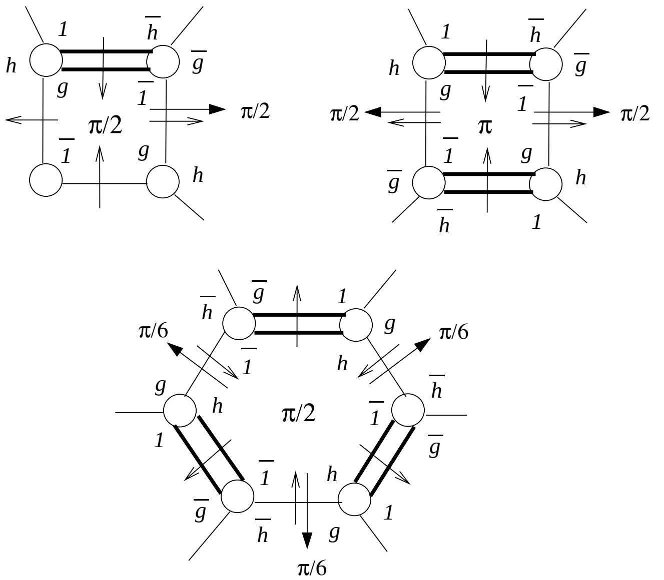

Figure 3.10.1: Positive regions and distribution scheme in Case$(2,\bar{6},3)$.

Let r be the number of corners of angle π in Δ. Then $r \leq \frac{n}{2}$ (by Remark 2.2(4)(i)). Let s denote the number of pairs $(\frac{\pi}{2},\frac{\pi}{6})$ or $(\frac{\pi}{6},\frac{\pi}{2})$ such that $\hat{\Delta}$ receives $\frac{\pi}{2}$ and $\frac{\pi}{6}$ across adjacent edges in $\partial\hat{\Delta}$, with the understanding that each $\frac{\pi}{2}$ and $\frac{\pi}{6}$ that $\hat{\Delta}$ receives appears at most once in these pairs. Denote the remaining number of $\frac{\pi}{2}$ that $\hat{\Delta}$ receives by s_1. Also, let s_2 denote the remaining number of $\frac{\pi}{6}$ that $\hat{\Delta}$ receives. As an example to show how to get the values s, s_1 and s_2 see Figure 3.10.2.

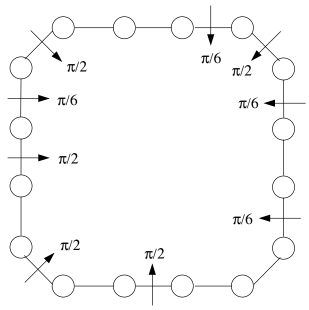

Figure 3.10.2: $n = 16, s = 2, s_1 = 3, s_2 = 2$.

Remarks 3.9.

1. *As shown in Figure 3.10.1,* $l(\hat{\Delta}) \in \{hg^{-1}w, h^{-1}gw\} \Rightarrow d(\hat{\Delta}) > 6$ *for otherwise* $l(\hat{\Delta})$ *yields a contradiction.*

2. $r \leq \frac{n}{2} - (s + s_1 + s_2)$.

3. $s + s_2 \leq \frac{n}{2}$.

Let $\hat{\Delta}$ be a region such that $c^*(\hat{\Delta}) > 0$. Then $c^*(\hat{\Delta}) \leq (2-n)\pi + \left[\frac{n}{2} - (s + s_1 + s_2)\right]\pi + (\frac{n}{2} + s + s_1 + s_2)\frac{\pi}{2} + s(\frac{1}{2} + \frac{1}{6})\pi + s_1.\frac{\pi}{2} + s_2.\frac{\pi}{6} = \frac{\pi}{12}(24 - 3n + 2s - 4s_2)$, and so $c^*(\hat{\Delta}) > 0$ implies $24 - 3n + 2s - 4s_2 > 0 \Rightarrow 3n < 24 - 4s_2 + 2s \leq 24 - 4s_2 + 2(\frac{n}{2} - s_2) = 24 - 6s_2 + n \Rightarrow n < 12$.

Let $n = 10$. Then $c^*(\hat{\Delta}) > 0 \Rightarrow 24 - 3(10) + 2s > 4s_2 \geq 0 \Rightarrow s > 3$. If s=4 or 5, then either $l(\hat{\Delta}) = (g^{-1}h)^4 g^{-1}1$ which contradicts $[g,h] \neq 1$ or $l(\hat{\Delta}) = (g^{-1}h)^5$ which contradicts $|gh^{-1}| \geq 6$. This leaves $n = 8$. But checking the possible labels shows that $l(\hat{\Delta}) = hg^{-1}1g^{-1}h1^{-1}h1^{-1} \Rightarrow c^*(\hat{\Delta}) \leq -\frac{\pi}{3}$ (see Figure 3.10.3).

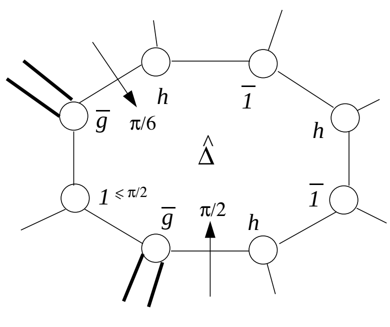

Figure 3.10.3: $n = 8$.

3.11 Proof of Lemma 3.7: Case$(3, 2, \bar{6})$

Here, we assume that $|g| = 3$, $|gh^{-1}| = 2$ and $|h| \geq 6$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$ and assign the angle function α_1 to $\mathbb{P}$. Observe that if $c(\hat{\Delta}) > 0$, then

$l(\hat{\Delta}) \in \{1^{-1}gw, h1^{-1}w\}$ (see Figure 2.2.6). It follows that all positively curved regions are shown in Figure 3.11.1.

Define the following distribution scheme which is given in Figure 3.11.1:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} \pi/6 & \text{if } c(\Delta) = \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by an } (m-1)\text{-bond} \\ c(\Delta)/2 & \text{if } 0 < c(\Delta) \leq \frac{\pi}{4} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by an } (m-1)\text{-bond} \\ 0 & \text{otherwise} \end{cases}$$

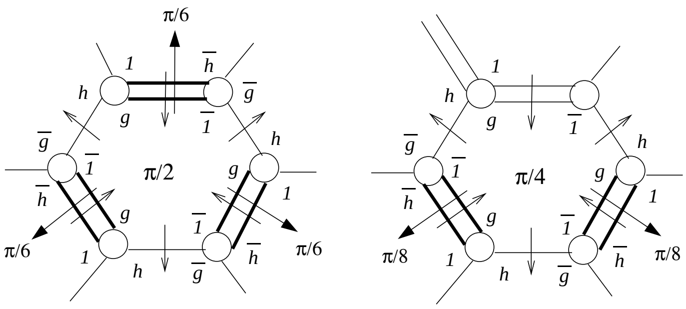

Figure 3.11.1: Positive regions and distribution scheme in Case$(3, 2, \bar{6})$.

For a fixed region $\hat{\Delta}$ again set $\Gamma_6(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{6}\}|$.

Remarks 3.10.

1. *The region $\hat{\Delta}$ receives each $\pi/6$ through an $(m-1)$-bond in its boundary which gives $(1h^{-1})^{\pm 1}$ as a sublabel of $\hat{\Delta}$.*

2. *$\hat{\Delta}$ receives $\pi/6$ only through edges that are oriented outwards $\hat{\Delta}$, and so $\hat{\Delta}$ does not receive $\pi/6$ through consecutive edges in its boundary ($\Gamma_6 \leq \frac{n}{2}$). Also, for each $\pi/6$ that $\hat{\Delta}$ receives, there are two corners in $\hat{\Delta}$ with angle $\frac{3\pi}{4}$. Therefore, $\Gamma_6 \leq \frac{r}{2}$, where r is the number of corners with angle $\frac{3\pi}{4}$ in the boundary of $\hat{\Delta}$.*

3. *As shown in Figure 3.11.1, $l(\hat{\Delta}) = 1h^{-1}w$, which implies that $d(\hat{\Delta}) > 4$ for otherwise $l(\hat{\Delta})$ yields a contradiction.*

By using $\Gamma_6 \leq \frac{r}{2}$, $c^*(\hat{\Delta}) \leq (2-n)\pi + r.\frac{3\pi}{4} + (n-r).\frac{\pi}{2} + \frac{r}{2}.\frac{\pi}{6}$, and so $c^*(\hat{\Delta}) > 0$ $\Rightarrow 2r > 3n - 12$. Since $r \leq n$, this implies that $n < 12$.

Let $\hat{\Delta}=(n,r)$ denote a region of degree n with r corners of angle $\frac{3\pi}{4}$ and assume that $c^*(\hat{\Delta}) > 0$. Since $2r > 3n - 12$ it follows that if $n = 10$ then $r = 10$; if $n = 8$ then $r = 7$ or 8; and if $n = 6$ then $r = 4, 5$ or 6. If $(n,r)= (10,10)$ or $(8,8)$ then $l(\hat{\Delta})$ implies that $h^5 = 1$ or $h^4 = 1$ contradicting $|h| \geq 6$. If $(n,r)= (8,7)$ then $\hat{\Delta}$ is given by Figure 3.11.2 (i) and $l(\hat{\Delta})$ implies either $h^4 = 1$ or $g = h^4$ which is (**E2**). This leaves $d(\hat{\Delta}) = 6$ and checking shows that $l(\hat{\Delta})=1h^{-1}gh^{-1}g1^{-1}$ as in Figure 3.11.2 (ii), otherwise there is a contradiction or condition (**E2**) occurs. But observe that if $r > 3$ in Figure 3.11.2 (ii) then $r = 4$ and since the $\hat{\Delta}$ corners of vertices u and v cannot have angle $\frac{3\pi}{4}$, this forces $x = h^{-1}$, a contradiction which completes the proof.

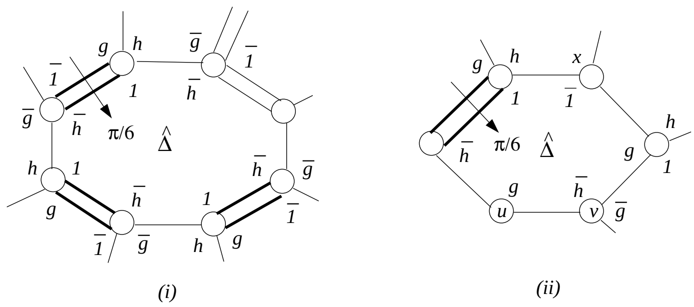

Figure 3.11.2.

4 Proof of Theorem 1.1

As mentioned in the introduction, the proof of Theorem 1.1 has been done previously for m= 2,3 and 4 except for the exceptional cases **E4** and **E5** of [2]. But by following the proof of Lemma 3.6(3), Lemma 8(3) in [2] can be amended as follows: if $|g|$=2, $|gh^{-1}| \geq 4$, $|h| \geq 4$ and $g \neq h^2$, then $\mathcal{P}$ is aspherical even if $[g,h] = 1$; and so in these two cases $\mathcal{P}$ is also aspherical. So it can be assumed that $m \geq 5$. The 'only if' direction of Theorem 1.1 follows from Lemmas 3.2, 3.3, 3.4 and 3.6(4). For the rest of the proof we assume that none of the Conditions (1)-(7) of Theorem 1.1 is satisfied. We show that either $\mathcal{P}$ is aspherical or exceptional.

If none of the conditions of Lemma 3.1 holds, then $\mathcal{P}$ is aspherical. Assume that Condition 1 of Lemma 3.1 holds. Then $|g| = \infty$ (since Condition 1 of Theorem 1.1 does not hold), and so $\mathcal{P}$ is aspherical by Lemma 3.2. So assume from now on that $g \neq h^{\pm 1}$.

If Condition 2 of Lemma 3.1 holds, then it can be assumed without any loss that $g = h^2$. Then $|h| \geq 5$ (by the negation of Condition 3 of Theorem 1.1). If $|h| \in \{5, 6\}$ then $\mathcal{P}$ is exceptional of type (**E1**) or (**E2**, $g = h^2$); and if $|h| \geq 7$, then $\mathcal{P}$ is aspherical by Lemma 3.3. So assume from now on that $g \neq h^2$.

If Condition 3 of Lemma 3.1 holds, then it can be assumed without any loss that $|g| = 2$. Since $g \neq h$, $|gh^{-1}| \geq 2$. If $|gh^{-1}| = 2$ then $|h| = \infty$ (Condition 7 of Theorem 1.1) and it follows that $\mathcal{P}$ is aspherical by Lemma 3.6(1). If $|gh^{-1}| = 3$, then $|h| \geq 6$ (Condition 7 of Theorem 1.1). By Lemma 3.6(2), $\mathcal{P}$ is aspherical if $g \neq h^3$, while if $g = h^3$ then $\mathcal{P}$ is exceptional of type (**E2**, $g = h^3$). If $|gh^{-1}| = 4$ or 5 then $|h| \geq 4$ (Condition 7 of Theorem 1.1), and so $\mathcal{P}$ is aspherical by Lemma 3.6(3). Now suppose that $|gh^{-1}| \geq 6$. By Lemma 3.5, if $|gh^{-1}| = \infty$ then $\mathcal{P}$ is aspherical, so assume otherwise. Then $|h| \geq 3$ (Condition 7 of Theorem 1.1). If $|h| = 3$ then $[g, h] \neq 1$, otherwise Condition 6 of Theorem 1.1 holds, and so $\mathcal{P}$ is aspherical by Lemma 3.6(4). If $|h| \geq 4$, then $\mathcal{P}$ is aspherical by Lemma 3.6(3).

Finally, if Condition 4 of Lemma 3.1 is satisfied then it can be assumed without loss that $|g| = 3$ and $|gh^{-1}| = 2$. Hence $|h| \geq 6$ (else, Condition 7 of Theorem 1.1 applies). If $g = h^4$ then $\mathcal{P}$ is exceptional of type (**E2**, $g = h^4$); otherwise $\mathcal{P}$ is aspherical by Lemma 3.7.

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