Derivatives of meromorphic functions of finite order

J.K. Langley

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Abstract

A result is proved concerning meromorphic functions f of finite order in the plane such that all but finitely many zeros of f'' are zeros of f'. A.M.S. MSC 2000: 30D35.

1 Introduction

The starting point of this paper is the following theorem from [8].

Theorem 1.1 ([8]) *Assume that the function f is meromorphic of finite lower order in the plane and that $f^{(k)}$ has finitely many zeros, for some $k \geq 2$. Assume further that there exists a positive real number M such that if ζ is a pole of f of multiplicity m_ζ then*

$$m_\zeta \leq M + |\zeta|^M. \tag{1}$$

Then f has finitely many poles.

Condition (1) is evidently satisfied if f has finite order. Theorem 1.1 fails for $k = 1$, as shown by simple examples, and for $k \geq 2$ and infinite lower order, in which case an example is constructed in [7] with infinitely many poles, all simple, such that $f^{(k)}$ has no zeros at all. The result was inspired by the conjecture made by A.A. Gol'dberg, to the effect that for $k \geq 2$ and a meromorphic function f in the plane, regardless of growth, the frequency of distinct poles of f is controlled by the frequency of zeros of $f^{(k)}$, up to an error term which is small compared to the Nevanlinna characteristic. Yamanoi has now proved this conjecture in a landmark paper [12]; however, because of the error terms involved, his result does not imply Theorem 1.1 directly.

This paper is concerned with a generalisation of Theorem 1.1 in a different direction. The assumption there that $f^{(k)}$ has finitely many zeros is a strong one, so that it is natural to ask whether it may be replaced by something less restrictive. A reasonable candidate is the condition that all but finitely many zeros of $f^{(k)}$ have the same image under $f^{(k-1)}$, which may then be assumed to be 0, but the following example shows that this does not by itself imply that f has finitely many poles. Set

$$f(z) = z - \tan z, \quad f'(z) = 1 - \sec^2 z = -\tan^2 z, \quad f''(z) = -2\tan z \sec^2 z. \tag{2}$$

Here all zeros of f'' are zeros of f' and fixpoints of f, all zeros and poles of f' have the same multiplicity, and 1 is an asymptotic value of f'. More generally it may be observed that, for any even positive integer n, the antiderivative of $\tan^n z$ is meromorphic in $\mathbb{C}$. The example (2) shows that the following theorem, which evidently implies Theorem 1.1, is essentially sharp.

Theorem 1.2 *Let $k \geq 2$ be an integer and let f be a meromorphic function of finite lower order in the plane with the following properties:*
(i) the zeros of $f^{(k-1)}$ have bounded multiplicities;
(ii) all but finitely many zeros of $f^{(k)}$ are zeros of $f^{(k-1)}$;
(iii) there exists $M \in (0, +\infty)$ such that if ζ is a pole of f of multiplicity m_ζ then (1) holds;
(iv) for each $\varepsilon > 0$, all but finitely many zeros z of $f^{(k)}$ satisfy either $|f^{(k-2)}(z)| \leq \varepsilon|z|$ or $\varepsilon|f^{(k-2)}(z)| \geq |z|$.
Then $f^{(k)}$ has a representation $f^{(k)} = Re^P$ with R a rational function and P a polynomial. In particular, f has finite order and finitely many poles, and $f^{(k)}$ has finitely many zeros.

It suffices to prove Theorem 1.2 for $k = 2$ and, as already noted, condition (iii) holds when f has finite order. If f is a meromorphic function of finite lower order in the plane satisfying condition (ii) of Theorem 1.2, with $k = 2$, then f' has finitely many critical values and so finitely many asymptotic values, by a result of Bergweiler and Eremenko [2] and its extension by Hinchliffe [6] to functions of finite lower order (see Section 3). Therefore Theorem 1.2 follows from the next result, which fails for infinite lower order, because of the same example from [7] mentioned after Theorem 1.1.

Theorem 1.3 *Let f be a meromorphic function of finite lower order in the plane satisfying conditions (i), (ii) and (iii) of Theorem 1.2, with $k = 2$. Assume that there exist positive real numbers κ and R_0 such that if z is a zero of f'' with $|z| \geq R_0$ then $|f(z) - \alpha z| \geq \kappa |z|$ for all finite non-zero asymptotic values α of f'. Then $f'' = Re^P$ with R a rational function and P a polynomial.*

2 Lemmas needed for Theorem 1.3

Throughout this paper $B(z_0, r)$ will denote the disc $\{z \in \mathbb{C} : |z - z_0| < r\}$ and $S(z_0, r)$ will be the circle $\{z \in \mathbb{C} : |z - z_0| = r\}$. The following results are both well known.

Lemma 2.1 ([11], p.116) *Let D be a simply connected domain not containing the origin, and let z_0 lie in D. Let r satisfy $0 < 4r < |z_0|$ or $4|z_0| < r < \infty$. Let $\theta(t)$ denote the angular measure of $D \cap S(0, t)$, and let D_r be the component of $D \setminus S(0, r)$ which contains z_0. Then the harmonic measure of $S(0, r)$ with respect to the domain D_r, evaluated at z_0, satisfies*

$$\omega(z_0, S(0, r), D_r) \leq C \exp\left(-\pi \int_I \frac{dt}{t\theta(t)}\right), \tag{3}$$

with C an absolute constant, $I = [2|z_0|, r/2]$ if $4|z_0| < r$, and $I = [2r, |z_0|/2]$ if $4r < |z_0|$.

Lemma 2.2 ([5], p.366) *Let Q be a positive integer and let $w_1, \ldots, w_Q$ be complex numbers. For each $\Lambda > 0$ the estimate*

$$\prod_{j=1}^{Q} |z - w_j| \geq \Lambda^Q \tag{4}$$

holds for all z outside a union of discs having sum of radii at most 6Λ.

3 Critical points and asymptotic values

Suppose that the function h is transcendental and meromorphic in the plane, and that $h(z)$ tends to $a \in \mathbb{C}$ as z tends to infinity along a path γ. Then a is an asymptotic value of h, and the inverse function h^{-1} has a transcendental singularity over a [2, 10]. For each $t > 0$, let $C(t)$ be that component of $C'(t) = \{z \in \mathbb{C} : |h(z) - a| < t\}$ which contains an unbounded subpath of γ. The singularity of h^{-1} over a corresponding to γ is called direct [2] if $C(t)$, for some $t > 0$, contains no zeros of $h(z) - a$. Singularities over ∞ are classified analogously.

Recall next some standard facts from [10, p.287]. Suppose that G is a transcendental meromorphic function with no asymptotic or critical values in $1 < |w| < \infty$. Then every component C_0 of the set $\{z \in \mathbb{C} : |G(z)| > 1\}$ is simply connected, and there are two possibilities. Either (i) C_0 contains one pole z_0 of G of multiplicity k, in which case $G^{-1/k}$ maps C_0 univalently onto $B(0,1)$, or (ii) C_0 contains no pole of G, but instead a path tending to infinity on which G tends to infinity. In case (ii) the function $w = \log G(z)$ maps C_0 univalently onto the right half plane.

Lemma 3.1 ([8]) *There exists a positive absolute constant C with the following property. Suppose that G is a transcendental meromorphic function in the plane and that G' has no asymptotic or critical values w with $0 < |w| < d_1 < \infty$. Let D be a component of the set $\{z \in \mathbb{C} : |G'(z)| < d_1\}$ on which G' has no zeros, but such that D contains a path tending to infinity on which $G'(z)$ tends to 0. If z_1 is in D and $\log |d_1/G'(z_1)| \geq 1$ then*

$$|G(z_1)| \leq S + \frac{C|z_1 G'(z_1)|}{\log |d_1/G'(z_1)|},$$

in which the positive constant S depends on G and D but not on z_1.

□

Suppose next that the function F is meromorphic of finite lower order in the plane, and that all but finitely many zeros of F' are zeros of F. Then F has finitely many critical values. By Hinchliffe's extension [6] to the finite lower order case of a theorem of Bergweiler and Eremenko [2], the function F has finitely many asymptotic values. Furthermore, all asymptotic values of F give rise to direct transcendental singularities of the inverse function F^{-1} and, by the Denjoy-Carleman-Ahlfors theorem [2, 5, 10], there are finitely many such singularities. The following facts are related to the argument from [7, Section 4]. Let J be a polygonal Jordan curve in $\mathbb{C} \setminus \{0\}$ such that every finite non-zero critical or asymptotic value of F lies on J, but is not a vertex of J, and such that the complement of J in $\mathbb{C} \cup \{\infty\}$ consists of two simply connected domains B_1 and B_2, with $0 \in B_1$ and $\infty \in B_2$. Fix conformal mappings

$$h_m : B_m \to B(0,1), \quad m = 1, 2, \quad h_1(0) = 0, \quad h_2(\infty) = 0. \tag{5}$$

The mapping h_1 may then be extended to be quasiconformal on the plane, fixing infinity, and there exist a meromorphic function G and a quasiconformal mapping ψ such that $h_1 \circ F = G \circ \psi$ on $\mathbb{C}$. It follows that for $j = 1, 2$ all components of $F^{-1}(B_j)$ are simply connected and all but finitely many are unbounded, since all but finitely many zeros z of G' have $G(z) = 0$.

4 Proof of Theorem 1.3: first part

Let the function f be as in the hypotheses. If f''/f' is a rational function then f' is a rational function multiplied by the exponential of a polynomial, and so is f''. Assume henceforth that f''/f' is transcendental: then obviously so is f. Apply the reasoning and notation of Section 3, with $F = f'$. The following is an immediate consequence of Lemma 3.1.

Lemma 4.1 *Arbitrarily small positive real numbers ε_1 and ε_2 may be chosen with the following properties. There exist finitely many unbounded simply connected domains U_n, each of which is a component of the set $\{z \in \mathbb{C} : |f'(z) - b_n| < \varepsilon_1\}$, such that U_n contains a path tending to infinity on which $f'(z)$ tends to the finite asymptotic value b_n. Here $f'(z) \neq b_n$ on U_n and $|f(z) - b_n z| < \varepsilon_2 |z|$ for all large z in U_n. If Γ is a path tending to infinity on which f' tends to a finite asymptotic value α, then there exists n such that $\alpha = b_n$ and $\Gamma \setminus U_n$ is bounded. The b_n need not be distinct, and some of them may be 0.*

$\square$

Lemma 4.2 *There exists a positive real number $s_1 < \varepsilon_1$ with the following property. Let b_p be a finite non-zero asymptotic value of f'. Then the conformal map $h_1 : B_1 \to B(0, 1)$ extends to be analytic and univalent on $B_1 \cup B(b_p, s_1)$.*

Proof. This follows from the Schwarz reflection principle and the fact that each non-zero b_p lies on the polygonal Jordan curve $J = \partial B_1$ but is not a vertex of J. $\square$

Definitions 4.1 *Fix positive real numbers ρ, σ and τ with $\tau < s_1 < \varepsilon_1$ and σ/τ and ρ/σ small. Fix $W_0 \in \mathbb{C}$ such that $f'(W_0)$ is large.*

Lemma 4.3 *With the notation of Definitions 4.1, there exist positive real numbers M_1, M_2, M_3 having the following properties. Let z_0 be large with $|f'(z_0)| < \tau$ and assume that z_0 lies in a component C of $(f')^{-1}(B_1)$ satisfying one of the following two conditions:*
(A) there is at least one zero of f'' in C;
(B) the function f' is univalent on C, and $C \cap U_p$ and $C \cap U_q$ are both non-empty, where U_p and U_q are as in Lemma 4.1 with $0 \neq b_p \neq b_q \neq 0$.
Then $|z_0 f''(z_0)| \leq M_1$ and there exists a disc $B(z_0^*, M_2|z_0^*|) \subseteq B\left(z_0, \frac{1}{2}|z_0|\right) \cap C$ on which

$$\left| \frac{f'''(\zeta)}{f''(\zeta)} \right| \leq \frac{M_3}{|\zeta|}. \tag{6}$$

Proof. Observe that conditions (A) and (B) are mutually exclusive. Denote positive constants by c_j and small positive constants by δ_j; these will be independent of z_0 and C. In case (A) there is exactly one point in C at which f'' vanishes, and it must be a zero of f'. In both cases $f'(C) = B_1$ (see Section 3), and C contains precisely one zero z_1 of f', of multiplicity $m \leq c_1$, by hypothesis (i) of the theorem, with $m = 1$ in Case (B). There exist only finitely many components C_1 of $(f')^{-1}(B_1)$ which are bounded or have a zero of f'' on their boundary, and if one of these contains a zero of f' then the set $\{z \in C_1 : |f'(z)| \leq \tau\}$ is compact. Therefore since z_0 is large the component C is unbounded and simply connected and its boundary ∂C contains no zeros of f''. Now set $v_0 = (h_1 \circ f')^{1/m}$, with h_1 as in (5). Then v_0 maps C univalently onto $B(0,1)$, and $u_0 = v_0(z_0)$ satisfies $|u_0| \leq \delta_1$, since $m \leq c_1$ and τ is small.

Let Γ be a component of ∂C. Then Γ is a simple curve tending to infinity in both directions and, as z tends to infinity in either direction along Γ, the image $f'(z)$ must tend to a finite non-zero asymptotic value of f'; this is because v_0 is univalent on C. Hence there exists z_1 lying close to Γ, such that $z_1 \in C \cap U_n$, for some U_n as in Lemma 4.1, with $b_n \neq 0$ and $|f'(z_1) - b_n| < \varepsilon_1$. By construction, b_n lies on the polygonal Jordan curve J but is not a vertex of J. Thus analytic continuation of $(f')^{-1}$ along a path in the semi-disc $B(b_n, \varepsilon_1) \cap B_1$ then gives a point $z_2 \in C \cap U_n$ with $|f'(z_2) - b_n| < \varepsilon_1$, as well as $|h_1(f'(z_2))| \leq 1 - \delta_2$, which implies in turn that $|v_0(z_2)| \leq 1 - \delta_3$.

Let $G_0 : B(0,1) \to C$ be the inverse function of v_0, and suppose that $G_0'(u_0) = o(|z_0|)$. Then Koebe's distortion theorem implies that $G_0'(u) = o(|z_0|)$ for $|u| \leq 1 - \delta_3$. In Case (A) this gives a path γ in C, of length $o(|z_0|)$, joining $z_3 = G_0(0)$ to z_2 via z_0, and with $|f'(z)| \leq c_2$

on γ. Since z_0 is large so are z_2 and z_3. Thus Lemma 4.1 and integration of f' yield

$$f(z_3) = f(z_2) + o(|z_0|), \quad |f(z_3) - b_n z_3| \leq \varepsilon_2 |z_2| + o(|z_0|) \leq (\varepsilon_2 + o(1))|z_3|. \tag{7}$$

But by the assumption of Case (A), f'' has a zero in C, which must be at z_3, so that, by the hypotheses of the theorem, $|f(z_3) - b_n z_3| \geq \kappa |z_3|$. This contradicts (7), since ε_2 is small. Next, in Case (B) the above analysis may be applied twice, to give a path γ in C of length $o(|z_0|)$, on which $|f'(z)| \leq c_2$, such that γ joins points $w_p \in C \cap U_p$ and $w_q \in C \cap U_q$ via z_0, where b_p and b_q are distinct and non-zero, and $|f'(w_j) - b_j| < \varepsilon_1$ for $j = p, q$. Therefore the w_j satisfy $w_j \sim z_0$ and $|f(w_j) - b_j w_j| \leq \varepsilon_2 |w_j| \leq 2\varepsilon_2 |z_0|$ for $j = p, q$. Since ε_2 is small and integration of f' along γ leads to $f(w_p) - f(w_q) = o(|z_0|)$, this case also delivers a contradiction.

It follows in both cases that $|G_0'(u_0)| \geq c_3 |z_0|$, which implies at once that $|z_0 v_0'(z_0)| \leq c_4$. Writing $f'(z) = h_1^{-1}(v_0(z)^m)$ and using the fact that $|f'(z_0)| < \tau$ and $m \leq c_1$ gives $|z_0 f''(z_0)| \leq c_5$. To prove the last assertion requires a disc on which f' is univalent. To this end, observe that $|G_0'(u_0)| \leq c_6 |z_0|$, since z_0 is large but C does not contain the point W_0 chosen in Definitions 4.1. Now choose u_0^* with $|u_0^* - u_0| \leq \delta_4$ and $|u_0^*| \geq \delta_4$, and choose δ_5 so small that the function u^m is univalent on $B(u_0^*, \delta_5)$. Then Koebe's distortion theorem implies that the image X_0 of $B(u_0^*, \delta_5)$ under G_0 lies in $B\left(z_0, \frac{1}{2}|z_0|\right) \cap C$ and contains a disc $B(z_0^*, 2M_2|z_0^*|)$, where $z_0^* = G_0(u_0^*)$ and $M_2 = \delta_6$: this requires only that δ_4 and δ_5/δ_4 be small enough, independent of z_0. The function $v_0(z)^m$ is univalent on X_0 and therefore so is f'. Now take ζ in $B(z_0^*, M_2|z_0^*|)$ and set

$$g(z) = \frac{f'(\zeta + M_2|z_0^*|z) - f'(\zeta)}{M_2|z_0^*|f''(\zeta)} = z + \sum_{\mu=2}^{\infty} A_\mu z^\mu$$

for $|z| < 1$, so that the estimate (6) follows from Bieberbach's bound $|A_2| \leq 2$. □

It will be seen that hypothesis (i) of Theorem 1.3 plays a key role in the above proof of Lemma 4.3, principally by preventing z_0 from lying too close to the boundary of C.

Lemma 4.4 *With the notation of Lemma 4.1 and Definitions 4.1, let z_1 be large and satisfy*

$$z_1 \in U_p, \quad b_p \neq 0, \quad \sigma < |f'(z_1) - b_p| < \tau < s_1, \quad f'(z_1) \in J = \partial B_1, \tag{8}$$

and let C be the component of $(f')^{-1}(B_1)$ with $z_1 \in \partial C$. Assume that one of the following two mutually exclusive conditions holds:

(a) the function f' is not univalent on C;

(b) the function f' is univalent on C, and $C \cap U_q$ is non-empty, for some q with $0 \neq b_q \neq b_p$.

Then there exists an open set H_1, with

$$H_1 \subseteq B\left(z_1, \frac{1}{2}|z_1|\right) \cap C \quad \textit{and} \quad \partial H_1 \cap \partial C = \{z_1\}, \tag{9}$$

such that f' maps H_1 onto an open disc $K_1 \subseteq B_1$, of diameter less than ρ, which is tangent to $J = \partial B_1$ at $f'(z_1)$. Furthermore, H_1 contains an open disc L_1 of radius $M_4|z_1|$ on which (6) holds; here both M_3 and M_4 are independent of z_1 and C.

Proof. The component C is unique because z_1 is large and f'' has finitely many zeros which are not zeros of f'. As in Lemma 4.3 denote small positive constants by δ_j, and positive constants by c_j; these will again be independent of z_1 and C. Let γ_0 be the straight line segment

$$u = tu_1, \quad \delta_1 \leq t \leq 1, \quad u_1 = h_1(f'(z_1)) \in S(0,1),$$

where δ_1 is chosen sufficiently small that $|h_1(w)| \leq \delta_1$ implies that $|w| \leq \delta_2 < \tau < \varepsilon_1$. Using (8) and the conformal extension of h_1 to $B_1 \cup B(b_p, s_1)$ given by Lemma 4.2, define domains $F_1 \subseteq B_1 \cup \{\zeta \in \mathbb{C} : \rho < |\zeta - b_p| < s_1\}$ and E_1 by

$$E_1 = \{u \in \mathbb{C} : \mathrm{dist}\,\{u, \gamma_0\} < \delta_3\} = h_1(F_1),$$

in which δ_3 is small compared to δ_1, which ensures that $0 \notin E_1$. Then F_1 contains no singular values of the inverse function $(f')^{-1}$, and z_1 lies in a component D of $(f')^{-1}(F_1)$ such that $h_1 \circ f'$ maps D conformally onto E_1. Let $G_1 : E_1 \to D$ be the inverse function of $h_1 \circ f'$, and choose $z_2 \in D$ with $u_2 = h_1(f'(z_2)) = \delta_1 u_1$ and hence $|f'(z_2)| \leq \delta_2 < \tau$. Observe that z_2 lies in C. Repeated application of the Koebe distortion theorem yields $c_1|G_1'(u_1)| \leq |G_1'(u)| \leq c_2|G_1'(u_1)|$ on the line segment γ_0, and the image $\sigma_1 = G_1(\gamma_0)$ is a path of length at most $c_3|G_1'(u_1)|$ from z_1 to z_2 in D.

Suppose first that $G_1'(u_1) = o(|z_1|)$. Then $z_2 \sim z_1$ and $G_1'(u_2) = o(|z_1|)$, from which it follows that $z_2 f''(z_2)$ is large. Hence C satisfies neither condition (A) nor condition (B) of Lemma 4.3, and so cannot satisfy (b), because (b) implies (B) since $C \cap U_p \neq \emptyset$ and $b_p \neq 0$. Hence f' is not univalent on C but C contains no zero of f''. Thus C must contain a path Γ tending to infinity on which $f'(z)$ tends to 0, and C meets one of the components U_n with

$b_n = 0$. Moreover, $\log(h_1 \circ f')$ maps C univalently onto the left half plane (see Section 3). Therefore, since $|h_1(f'(z_2))| \leq \delta_1$, there exists a path Γ' in C joining z_2 to some $z_3 \in \Gamma$ on which $|h_1(f'(z))| \leq \delta_1$ and $|f'(z)| < \varepsilon_1$, and hence $z_2 \in U_n$. Since z_1 is large, and $z_2 \sim z_1$, Lemma 4.1 gives $|f(z_2)| \leq \varepsilon_2|z_2|$ and $|f(z_1) - b_p z_1| \leq \varepsilon_2|z_1|$, in which $b_p \neq 0$. On the other hand $|f'(z)| \leq c_4$ on σ_1, and so integration yields $f(z_1) = f(z_2) + o(|z_1|)$ and a contradiction.

It must therefore be the case that $|G_1'(u_1)| \geq c_5|z_1|$. However, the point W_0 chosen in Definitions 4.1 is not in D and so $|G_1'(u_1)| \leq c_6|z_1|$. Now let $G_2 = G_1 \circ h_1 : F_1 \to D$ be the inverse function of f', and set $v_1 = f'(z_1) = h_1^{-1}(u_1) \in J$. Then (8) yields $c_7|z_1| \leq |G_2'(v_1)| \leq c_8|z_1|$, as well as $B(v_1, 2\delta_4) \subseteq F_1$ for some $\delta_4 < \rho$, and Koebe's distortion theorem gives $c_9|z_1| \leq |G_2'(v)| \leq c_{10}|z_1|$ on $B(v_1, \delta_4)$. Hence $G_2(B(v_1, \delta_5)) \subseteq B\left(z_1, \frac{1}{2}|z_1|\right)$, provided $\delta_5 \leq \delta_4$ is chosen small enough. Let $K_1 \subseteq B(v_1, \delta_5) \cap B_1$ be an open disc of radius $\delta_6 \leq \frac{1}{4}\delta_5$, which is tangent to J at v_1. Then $H_1 = G_2(K_1)$ satisfies (9), and H_1 contains a disc $B(z_1^*, 2M_4|z_1|)$, with $M_4 = \delta_7$. It may now be assumed that M_3 is large enough that (6) holds on $L_1 = B(z_1^*, M_4|z_1|)$, since Bieberbach's theorem may be applied as in the proof of Lemma 4.3. □

5 The frequency of poles of f and zeros of f''

Lemma 5.1 *Let $w_1, \ldots, w_Q$ be pairwise distinct poles of f with $|w_j|$ large. For $1 \leq j \leq Q$ let D_j be the component of $(f')^{-1}(B_2)$ in which w_j lies. Then for each j there exists $p_j \in \mathbb{Z}$ such that ∂D_j contains a Jordan arc λ_j which is mapped univalently by f' onto a line segment μ_j of length at least σ, and these may be chosen so that*

$$\lambda_j \subseteq U_{p_j}, \quad \mu_j \subseteq \{\zeta \in J = \partial B_2 : \sigma < |\zeta - b_{p_j}| < \tau\}, \quad b_{p_j} \neq 0, \tag{10}$$

where U_{p_j} and b_{p_j} are as in Lemma 4.1, while σ and τ are as in Definitions 4.1.

Moreover, if points z_j are chosen such that $z_j \in \lambda_j$ for $1 \leq j \leq Q$, then each $|z_j|$ is large and for each j there exists an open disc $L_j \subseteq B\left(z_j, \frac{1}{2}|z_j|\right)$ of radius $M_4|z_j|$, on which (6) holds, where M_4 is as in Lemma 4.4. The L_j are pairwise disjoint.

Proof. By the discussion in Section 3, each D_j is unbounded and simply connected and the boundary ∂D_j contains no zeros of f''. Each component of ∂D_j is a simple path tending to infinity in both directions, and there exists a component Γ_j of ∂D_j which separates w_j from the point W_0 chosen in Definitions 4.1. Since D_j contains a pole of f it follows that f' is finite-valent on D_j. Thus as z tends to infinity in either direction along Γ_j the image $f'(z)$ must tend to a non-zero finite asymptotic value of f'. In particular, Γ_j meets some U_p as in Lemma 4.1 with $b_p \neq 0$, and following Γ_j while staying in U_p gives λ_j and μ_j as in (10). Furthermore, each w_j is large and, for any $M_5 > 0$, the disc $B(0, M_5)$ meets only finitely many components of $(f')^{-1}(B_2)$, each of which contains at most one pole of f. Hence if $z_j \in \lambda_j$ then z_j is large.

To prove the existence of the L_j, choose for each j a component E_j of $(f')^{-1}(B_1)$ with $\Gamma_j \subseteq \partial E_j$. Since Γ_j separates the pole w_j of f from W_0 it follows that Γ_j is not the whole boundary ∂E_j. In particular, if f' is univalent on E_j then Γ_j must meet components U_p and U_q with b_p and b_q distinct and non-zero. Thus each of these components E_j of $(f')^{-1}(B_1)$ satisfies one of the conditions (a), (b) of Lemma 4.4, which may now be applied with z_1 replaced by each z_j. This gives open sets $H_j \subseteq B\left(z_j, \frac{1}{2}|z_j|\right) \cap E_j$, each containing an open disc L_j of radius $M_4|z_j|$ on which (6) holds. Moreover, f' maps H_j onto a disc $K_j \subseteq B_1$ which is tangent to J at $f'(z_j)$ and has diameter less than ρ.

To show that the L_j are disjoint, suppose that $1 \leq j < j' \leq Q$ and that $H_j \cap H_{j'} \neq \emptyset$, from which it follows of course that $K_j \cap K_{j'} \neq \emptyset$. Since ρ is small compared to σ and $z_j \in \lambda_j$, the open disc $U = B(f'(z_j), 3\rho)$ contains no singular value of $(f')^{-1}$, by (10). But K_j and $K_{j'}$ have diameter less than ρ, and so their closures lie in U. Thus H_j and $H_{j'}$ both lie in the same component of $(f')^{-1}(U)$, as do z_j and $z_{j'}$, which forces $\Gamma_j = \Gamma_{j'}$ and gives a contradiction. $\square$

Lemma 5.2 *Let $L(r) \to \infty$ with $L(r) \leq \frac{1}{8}\log r$ as $r \to \infty$, and for $k > 0$ and large r define the annulus $A(k)$ by $A(k) = \{z \in \mathbb{C} : re^{-kL(r)} \leq |z| \leq re^{kL(r)}\}$. Then the number N_1 of distinct poles of f and zeros of f'' in $A(1)$ satisfies*

$$N_1 = O(\phi(r)) \quad \text{as } r \to \infty, \text{ where } \quad \phi(r) = L(r) + \frac{\log r}{L(r)}. \tag{11}$$

Proof. Assume that r is large and that $A(1)$ contains $Q = 2N$ distinct poles $w_1, \ldots, w_{2N}$ of f, with $\phi(r) = o(N)$. For $j = 1, \ldots, Q$ let D_j be the component of $(f')^{-1}(B_2)$ in which w_j lies, let q_j be the multiplicity of the pole of f' at w_j. Each D_j is unbounded and simply connected and may be assumed not to contain the origin. Let $v_j = (h_2 \circ f')^{1/q_j}$, so that v_j maps D_j conformally onto $B(0, 1)$, with $v_j(w_j) = 0$.

For $0 < t < \infty$ let $\theta_j(t)$ be the angular measure of $D_j \cap S(0,t)$. Let c denote positive constants, not necessarily the same at each occurrence, but not depending on $r, L(r)$ or N. For $m \in \mathbb{N}$ the Cauchy-Schwarz inequality gives $m^2 \leq 2\pi \sum_{j=1}^m 1/\theta_j(t)$ so that, as in [7], at least N of the D_j have

$$\int_{2re^{L(r)}}^{(1/2)re^{2L(r)}} \frac{dt}{t\theta_j(t)} > cNL(r), \quad \int_{2re^{-2L(r)}}^{(1/2)re^{-L(r)}} \frac{dt}{t\theta_j(t)} > cNL(r). \tag{12}$$

It may be assumed after re-labelling if necessary that (12) holds for $D_1, \ldots, D_N$. Since w_j lies in $A(1)$, it follows from Lemma 2.1 that

$$\omega(w_j, \sigma_j, D_j) \leq c \exp\left(-\pi \int_{2re^{L(r)}}^{(1/2)re^{2L(r)}} \frac{dt}{t\theta_j(t)}\right) + c \exp\left(-\pi \int_{2re^{-2L(r)}}^{(1/2)re^{-L(r)}} \frac{dt}{t\theta_j(t)}\right).$$

Combining this with (11), (12) and condition (iii) of the theorem shows that $\omega(w_j, \sigma_j, D_j) = o(1/q_j)$ for $j = 1, \ldots, N$, where $\sigma_j = \partial D_j \setminus A(2)$. But Lemma 5.1 gives an arc $\lambda_j \subseteq \partial D_j$, mapped by f' onto a line segment $\mu_j \subseteq J$ as in (10), of length at least σ. Since b_{p_j} in (10) is not a vertex of J, while τ is small, an application of the Schwarz reflection principle to h_2 shows that $h_2 \circ f'$ maps λ_j to an arc of $S(0,1)$ of length at least c, and $v_j(\lambda_j)$ has angular measure at least c/q_j. The conformal invariance of harmonic measure under v_j implies that λ_j cannot be contained in σ_j, and so there exists $z_j \in \lambda_j \cap A(2)$. The corresponding N pairwise disjoint discs L_j given by Lemma 5.1 lie in the annulus $A(3)$, and hence

$$cN \leq \sum_{j=1}^N \int_{L_j} |z|^{-2} dxdy \leq \int_{A(3)} |z|^{-2} dxdy \leq cL(r) \leq c\phi(r) = o(N).$$

This is a contradiction and the asserted upper bound for the number of distinct poles in $A(1)$ is proved. The same upper bound for the number of distinct zeros ζ_j of f'' in $A(1)$ follows at once from Lemma 4.3, because such zeros give rise to pairwise disjoint discs $B(\zeta_j^*, M_2|\zeta_j^*|) \subseteq A(2)$. $\square$

Since all but finitely many zeros of f'' are zeros of f', which have bounded multiplicities by assumption, choosing $L(r) = \frac{1}{8}\log r$ in Lemma 5.2 gives

$$\overline{n}(r^{9/8}, f) - \overline{n}(r^{7/8}, f) + n(r^{9/8}, 1/f'') - n(r^{7/8}, 1/f'') = O(\log r),$$

and so

$$\overline{N}(r, f) + N(r, 1/f'') = O(\log r)^2 \quad \text{as} \quad r \to \infty. \tag{13}$$

Lemma 5.3 *The lower order of f''/f' is at least $\frac{1}{2}$.*

Proof. If this is not the case then the function f'/f'' has finitely many poles and is transcendental of lower order less than $\frac{1}{2}$. The $\cos \pi\lambda$ theorem [1] now gives $r_j \to +\infty$ such that $f''(z)/f'(z) = O(r_j^{-2})$ on $S(0, r_j)$. Moreover, the main result of [9] gives a path γ tending to infinity with

$$\int_\gamma \left| \frac{f''(z)}{f'(z)} \right| |dz| < \infty.$$

This implies that, as z tends to infinity in the union of γ and the $S(0, r_j)$, the image $f'(z)$ tends to some b_n as in Lemma 4.1, contradicting the fact that the U_n are simply connected. □

Lemma 5.4 *The function f'' has the form $f'' = \Pi_1/\Pi_2$, where Π_1 and Π_2 are entire such that Π_2 has finite order and $\Pi_1 \not\equiv 0$ has order 0. Moreover, the lower order of Π_2 is at least $1/2$.*

Proof. Using (1) and (13) shows that $N(r, f'')$ has finite order and $N(r, 1/f'')$ has order 0. Since f'' has finite lower order, this gives the asserted representation for f''. On the other hand, Lemma 5.3 implies that f' has lower order at least $1/2$ and so has f'', and hence so has Π_2. □

Lemma 5.5 *Let $h(z) = zf'''(z)/f''(z)$. For all $s \geq 1$ lying outside a set E_0 of finite logarithmic measure, there exists ζ_s with $|\zeta_s| = s$ and $|h(\zeta_s)| > s^{1/3}$.*

Proof. Take Π_1 and Π_2 as in Lemma 5.4. Applying the Wiman-Valiron theory [4, Theorem 12] and standard estimates for logarithmic derivatives [3] makes it possible to write, for $|\zeta_s| = s$ with $|\Pi_2(\zeta_s)| = M(s, \Pi_2)$ and s outside a set of finite logarithmic measure,

$$\frac{f'''}{f''} = \frac{\Pi_1'}{\Pi_1} - \frac{\Pi_2'}{\Pi_2}, \quad \left| \frac{\Pi_2'(\zeta_s)}{\Pi_2(\zeta_s)} \right| \sim \frac{\nu(s)}{s}, \quad \left| \frac{\Pi_1'(\zeta_s)}{\Pi_1(\zeta_s)} \right| \leq s^{-3/4}.$$

Here $\nu(s)$ is the central index of Π_2 and has lower order at least $1/2$. □

6 Completion of the proof of Theorem 1.3

Lemma 5.4 shows that f has finite order $\rho(f)$. Thus it remains only to prove that f has finitely many poles and f'' has finitely many zeros, so assume that this is not the case. Lemmas 4.3 and

5.1 give a positive real number d_1 and $w \in \mathbb{C}$ with $|w| = r$ arbitrarily large, such that (6) holds on the disc $B(w, d_1 r)$. Let ε and K be positive, with ε small, and let

$$U_K = \left\{ z \in \mathbb{C} : \frac{1}{K} < |z| < K, \quad |z - 1| > d_1 \right\}.$$

Here K is chosen so large that the harmonic measure with respect to U_K satisfies

$$\omega(z, S(0,1/K) \cup S(0,K), U_K) < \varepsilon \quad \text{for} \quad z \in U_K, \quad \frac{1}{2} < |z| < 2. \tag{14}$$

Denote by d_j positive constants which are independent of r, ε, K and S. Standard estimates from [3] give a real number $S = S_r$ such that

$$K < S < 2K \quad \text{and} \quad |h(z)| \leq |z|^{d_2} \quad \text{for} \quad |z| = \frac{r}{S} \quad \text{and} \quad |z| = rS, \tag{15}$$

in which $h(z) = zf'''(z)/f''(z)$ as in Lemma 5.5 and $d_2 = \rho(f) + 1$. Let $w_1, \ldots, w_Q$ be the poles of h in $r/S \leq |z| \leq rS$. Applying Lemma 5.2 with $L(r) = (\log r)^{1/2}$ shows that $Q \leq d_3 (\log r)^{1/2}$.

On the annulus A given by $r/S \leq |z| \leq rS$ set

$$u(z) = \log|h(z)| - \log M_3 + \sum_{1 \leq j \leq Q} \log \frac{|z - w_j|}{4Kr} \leq \log|h(z)| - \log M_3, \tag{16}$$

where M_3 is as in (6) and may be assumed to be at least 1, and the sum is empty if there are no poles w_j. Then u is subharmonic on A, with $u(z) \leq 0$ on the closure of $B(w, d_1 r)$ by (6), and

$$u(z) \leq \log|h(z)| \leq d_2 \log|z| \leq d_2 \log(2Kr) \quad \text{for} \quad z \in S(0, r/S) \cup S(0, rS), \tag{17}$$

by (15). Hence (14) and the monotonicity of harmonic measure yield

$$u(z) \leq \varepsilon d_2 \log(2Kr) \quad \text{for} \quad \frac{r}{2} < |z| < 2r. \tag{18}$$

Now Lemma 2.2 shows that (4) holds, with $\Lambda = r/24$, for all z outside a union P_r of discs having sum of radii at most $r/4$. Choose $s \in (r/2, 2r) \setminus E_0$, with E_0 as in Lemma 5.5, such that the circle $S(0, s)$ does not meet P_r. Thus Lemma 5.5 and (18) give rise to $\zeta_s \in S(0, s)$ such that

$$\begin{aligned} \frac{1}{3} \log s \leq \log|h(\zeta_s)| &\leq \varepsilon d_2 \log(2Kr) + \log M_3 + \sum_{1 \leq j \leq Q} \log \frac{4Kr}{|\zeta_s - w_j|} \\ &\leq \varepsilon d_2 \log(2Kr) + \log M_3 + Q \log(96K) \\ &\leq \varepsilon(\rho(f) + 1) \log(4Ks) + \log M_3 + d_3 (\log 2s)^{1/2} \log(96K). \end{aligned}$$

Since ε may be chosen arbitrarily small, while s is large, this gives a contradiction and the proof of Theorem 1.3 is complete. . $\Box$

Remark. Hypothesis (iii) on the multiplicities of poles may not be really essential for Theorem 1.3 but it does play a key role in the above proof. If it is assumed merely that f has finite lower order, then techniques such as Pólya peaks should give annuli on which the analysis of Lemma 5.2 can be applied, but it seems difficult to ensure that these contain enough distinct poles of f that the discs on which (6) holds are not so remote that the method of Section 6 fails.

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School of Mathematical Sciences, University of Nottingham, NG7 2RD.

jkl@maths.nott.ac.uk