What Do Fund Flows Reveal about Asset Pricing Models and Investor Sophistication?

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Recent evidence indicates that market model alphas are stronger predictors of mutual fund flows than alphas with other models. Some recent papers have interpreted this evidence to mean that CAPM is the best asset pricing model, but some others have interpreted it as evidence against investor sophistication. We evaluate the merits of these mutually exclusive interpretations. We show that no tenable inference about the validity of any asset pricing model can be drawn from this evidence. Rejecting the investor sophistication hypothesis is tenable, but the appropriate benchmark to judge sophistication is different from that used in this literature. (JEL G4, G11, G12, D81)

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An extensive literature documents that net fund flows into mutual funds are driven by funds’ past performance. For example, Patel, Zeckhauser, and Hendricks (1994) document that equity mutual funds with bigger returns attract more cash inflows and they offer various explanations for this phenomenon. Ippolito (1992), Chevalier and Ellison (1997), and Sirri and Tufano (1998) document a positive relation between fund flows and past performance.

Some papers in the early literature also examine whether abnormal performance (or alphas) measured with some benchmarks better predict fund flows than others. For example, Gruber (1996) compares the mutual fund flow-performance relation for alphas measured with 1- and 4-factor models, whereas Del Guercio and Tkac (2002) compare sensitivity of flows to raw returns with that to market model alphas for mutual funds and pension funds. Fung et al. (2008) make similar comparisons with a different set of factor models for hedge funds.

While comparison of flow-alpha relations across models was not the primary focus of earlier papers, recent papers in this area have shown a renewed interest in such comparisons using a broader range of asset pricing and factor models. Driving this resurgence is the idea that these comparisons can potentially help us answer important economic questions that extend beyond a descriptive analysis of mutual fund flows. For example, Barber, Huang, and Odean (2016) (hereafter “BHO”) compare the relation between fund flows and alphas measured with various models to evaluate whether mutual fund investors are sophisticated, or equivalently whether they rationally use all available information. They hypothesize that sophisticated investors should use alphas computed with a model with all common factors to evaluate fund performance regardless of the underlying true asset pricing model, but they find that market model alphas are the strongest predictors of mutual fund flows. BHO conclude that investors in aggregate are not sophisticated in assessing fund performance using past returns.
Berk and van Binsbergen (2016) (hereafter “BvB”), however, claim that such flow-alpha comparisons serve as a new and fundamentally different test of asset pricing models and the results can reveal the true asset pricing model. Because of potential asset pricing model implications, BvB’s comparisons include multifactor models and several versions of consumption-capital asset pricing model. Agarwal, Green, and Ren (2018) and Blocher and Molyboga (2017) carry out similar tests with hedge funds.

BvB also find that fund flows are most highly correlated with alphas computed with the market model in their tests. They therefore conclude that the CAPM is “the best method to use to compute the cost of capital of an investment opportunity” (Berk and van Binsbergen 2016, p. 17). The true asset pricing model has been a holy grail of the finance literature and hence BvB’s findings potentially have broad implications that go well beyond the mutual fund literature. For instance, Berk and van Binsbergen (2017) prescribe the use of the CAPM to practitioners who make capital budgeting decisions based on BvB’s evidence.

Although BHO’s and BvB’s flow-alpha horse races yield similar results, their inferences are mutually exclusive. Specifically, BvB’s asset pricing model interpretation assumes rational expectations but BHO’s interpretation implies that investors’ actions violate the rational expectations hypothesis. Because the inferences in BHO, BvB, and related papers have far-reaching implications, we examine whether such inferences are conceptually and empirically tenable.

We address the conceptual issues using a rational expectations model where investors extract information about mutual fund manager skills from funds’ past performance and optimally decide on investments into and withdraws from mutual funds. Our model augments Berk and
Green’s (2004) model with a multifactor return-generating process and grants investors the flexibility to compute alphas with respect to any factor model to update their priors about fund manager skills. Investors in the model know the true asset pricing model, and therefore which factors are priced. Although investors can compute alphas with only the priced factors, we show that investors optimally use alphas computed with the model with all common factors, both priced and unpriced, to decide on fund flows.

We then consider the flow-alpha horse race that empiricists run when they do not have all the information that agents in the model economy possess. Specifically, unlike the agents in the model, empiricists do not know the true asset pricing model. Also, empiricists do not know true factor betas and they can only estimate them with error. We show that empiricists’ alphas that most precisely estimate funds’ skills will win the horse race under the hypothesis that fund flows are determined in a rational expectations economy.

We use the results from our model to assess empirically whether we can identify the true asset pricing model based on the flow-alpha horse race with a sample of actively managed mutual funds. We use a 7-factor model as in BHO in our empirical analysis, where the seven factors are the Fama-French factors (market, SMB, HML), the momentum factor (UMD) proposed by Carhart (1997) and based on Jegadeesh (1990) and Jegadeesh and Titman (1993), and the three industry factors. We compute the precision of alphas with models that include all seven factors and with subsets of these factors under the hypothesis that each of the following asset pricing models is true: (a) none of the risk factors are priced (or true expected returns are unrelated to factors betas), (b) CAPM, (c) Fama-French 3-factor model, and (d) Fama-French-Carhart 4-factor model.
We find that 4-factor alphas are the most precise when true betas are unknown. Therefore, if fund flows are determined under the rational expectations hypothesis, then the 4-factor alpha should win the empiricists’ horse race regardless of which asset pricing model is true. For instance, 4-factor alphas would win whether the CAPM or the Fama-French 3-factor model is the true asset pricing model.

We also conduct simulation experiments with parameters that match the data. We generate simulated fund flows according to our model and test model predictions. We also conduct a number of robustness tests. The simulation results are similar to our empirical results. Specifically, the 4-factor model alphas are the most precise estimates when we estimate betas using traditional time-series regressions.¹ When flows are generated according to the rational expectations model, the most precise estimator always wins the horse race. In addition, the precision of the alpha estimator and the winner of the horse race does not depend on the true asset pricing model.

The results of our model and our empirical tests indicate that the horse race cannot uniquely identify the true underlying asset pricing model. Therefore, flow-alpha horse is not a tenable test of asset pricing models. Because our conclusions are contrary to BvB’s, we take a closer look at their model to resolve the contradiction. We show that a faulty foundational assumption in BvB’s model is the source of their mistaken inference.

Our empirical findings also contradict BHO’s hypothesis that alphas with a model that includes all common factors will win the horse race if investors were sophisticated. The main

¹ Our empirical tests and the main tests in the simulation estimate betas using OLS regressions as in BHO. However, when we use the more precise Vasicek (1973) shrinkage estimator to estimate betas in the simulation, we find that the alphas computed with all factors are the most precise estimates. We also find similar results when we replace BHO’s three industry factors with the first three principal components that we compute from the 4-factor model residuals for the mutual fund sample.
reason 4-factor alpha is more precise than the 7-factor alpha, and therefore wins the horse race under the investor sophistication hypothesis, is that estimation errors in the three industry factor betas are relatively large. However, even if we use the 4-factor model as the benchmark BHO’s finding that the single-factor alpha wins their horse race rejects the investor sophistication hypothesis.

1. Fund Flows and Alphas: Foundation for Empirical Tests and Inferences

This section presents a model that forms the basis for our analysis of the implications of flow-alpha relations as tests of asset pricing models and investor sophistication. Broadly speaking, we use the model to answer the following questions:

(a) How do investors optimally update their priors about unobservable skills of fund managers when they observe fund returns?

(b) How are equilibrium fund flows related to the information investors use to update their priors?

(c) What are the implications of the answers to the above questions for interpreting the results of a flow-alpha horse race with alphas computed using different multifactor models?

We answer these questions using a rational expectations model as in Berk and Green (2004) augmented with a multifactor return-generating process and an equilibrium asset pricing model.

1.1 Asset pricing model and the return-generating process

The following $K$-factor model is the true asset pricing model:

$$E[r_t] = \sum_{k=1}^{K} \beta_{k,t} Y_k,$$  \hspace{1cm} (1)
where \( \tau_i \) is the return in excess of the risk-free rate or excess returns, \( E[\tau_i] \) is the expected excess return on asset \( i \), \( \beta_{k,i} \) is the beta of asset \( i \) with respect to factor \( k \), and \( \gamma_k \) is the premium for a unit of factor risk. For the CAPM, \( K = 1 \), and for Fama-French 3-factor model, which we refer to as FF3, \( K = 3 \). We also define a model with \( K = 0 \), where the expected returns are equal for all assets regardless of any differences in their factor betas, that is, \( \gamma_k = 0 \ \forall \ k \) in Equation (1). Because there is no beta risk premium under this model, we abbreviate it as “NBRP.”

Asset returns follow the \( J \)-factor model below:\(^2\)

\[
\tau_{i,t} = E[\tau_i] + \sum_{k=1}^{J} \beta_{k,i} f_{k,t} + \xi_{i,t},
\]

where \( f_{k,t} \) is the realization of the common factor \( k \), and \( \xi_{i,t} \) is asset specific return at time \( t \). Factor realization \( f_{k,t} \) is the innovation or the unexpected component of factor \( k \). For instance, let \( F_{k,t} \) be the total factor realization of the \( k \)th factor, and then \( f_{k,t} = F_{k,t} - E[F_{k,t}] \) and \( E[f_{k,t}] = 0 \). Because we consider only traded factors, \( E[F_{k,t}] = \gamma_k \ \forall \ k \leq K \), and for the unpriced factors \( E[F_{k,t}] = 0 \ \forall \ k > K \).

In general, the \( J \) factors in the multifactor model (2) include the \( K \) priced factors from the asset pricing model as well as additional unpriced factors that describe realized returns. For example, the \( J \) factors could include industry factors that are unpriced, because they are uncorrelated with changes in investment opportunity set or with consumption. Therefore, in general \( J \geq K \). Factor returns and asset specific returns are all normally distributed.

1.2 The model

\(^2\) Equation (2) imposes the condition that the intercept of the return-generating process for each asset equals its expected return from the asset pricing model.
This subsection presents a rational expectations model that identifies alphas that investors use to make their mutual fund investment decisions. We make the following assumptions:

(a) **Rational economy:** All agents in the rational expectations economy are symmetrically informed.

(b) **Mutual funds and skill:** There are \( N \) mutual funds in the economy and \( N \to \infty \). Fund \( p \) is endowed with stock selection skills that allow it to generate a gross return of \( \Phi_p \) in excess of the \( K \)-factor asset pricing benchmark. Investors know the true asset pricing model. Fund manager skill \( \Phi_p \sim N(\phi_0, 1/\nu) \), where \( \phi_0 \) is average skill and \( \nu \) is the precision of the distribution of skill at the time of a fund’s inception. \( \phi_0 \) and \( \nu \) are common knowledge, and \( \Phi_p \) is constant over time.

(c) **Costs of active management:** Funds incur certain costs for active management which is a function of total assets under management (AUM), denoted as \( q \), and \( c_t(q) \) is the total cost per unit of AUM at time \( t \). The cost \( c_t(q) \) includes fund fees and administrative costs, brokerage costs and price impact of trades. Funds experience decreasing returns to scale and hence \( c_t(q) \) is an increasing function of \( q \).

(d) **Gross and net returns:** Let \( R_{p,t} \) and \( r_{p,t} \) be fund \( p \)'s gross and net excess returns at time \( t \), respectively. \( R_{p,t} = r_{p,t} + c_{t-1}(q_{p,t-1}) \). Funds’ net returns are observable by both investors in the model economy and econometricians. Investors can also compute \( R_{p,t} \), because they know \( q_{p,t-1} \) and \( c_{t-1}(q_{p,t-1}) \). However, econometricians observe only \( q_{p,t-1} \).

(e) **Competitive market:** The mutual fund market is perfectly competitive. Therefore, expected alpha net of fees and costs for an investment in any mutual fund equals zero in equilibrium:
\[ \phi_{p,t} - c_t(q_{p,t}) = 0, \tag{3} \]

where \( \phi_{p,t} \) is the mean of investors’ posterior about fund manager skill at time \( t \).

(f) **Expected return and return-generating process**: Equations (1) and (2) specify expected returns and the return-generating process in this economy, both of which are common knowledge. Fund betas are constant and common knowledge as well. The net return at time \( t \) is:

\[
r_{p,t} = \Phi_p + \sum_{k=1}^{K} \beta_{k,p} E[F_{k,t}] + \sum_{k=1}^{J} \beta_{k,p} f_{k,t} + \xi_{p,t} - c_{t-1}(q_{p,t-1}). \tag{4}
\]

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<tr>
<th>Expected return, Equation (1)</th>
<th>Unexpected return, Equation (2)</th>
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Assumptions (a) through (e) are the same as those in Berk and Green (2004). We add assumption (f) about expected asset returns and the return-generating factor model.

Investors make their mutual fund investment decisions based on their assessment of fund manager skills. Investors assign a skill of \( \phi_0 \) to all funds at their origin. Subsequently, investors observe net fund returns and factor realizations each period and optimally update their priors. To update their priors, investors could compute alphas relative to any \( \eta \)-factor model, which we denote as \( \hat{\alpha}_{p,\eta,t} \), as follows:

\[
\hat{\alpha}_{p,\eta,t} = r_{p,t} - r_{market,t} \text{ if } \eta = 0 \text{ and } \tag{5}
\]

\[
\hat{\alpha}_{p,\eta,t} = r_{p,t} - \sum_{k=1}^{\eta} \beta_{k,p} F_{k,t}, \text{ if } \eta > 0, \tag{6}
\]

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3 Fund \( p \)'s gross returns follow the return-generating process (2) plus \( \Phi_p \). Investors earn net returns in (4) after all costs.

4 \( \Phi_p \) in Equation (4) denotes managerial skill in our model, which is denoted as \( \alpha \) in Berk and Green. We use \( \alpha \) to denote ex post abnormal returns following a common practice in the empirical mutual fund literature.
where \( r_{\text{market}, t} \) is the market return \( F_{k,t} \) is realized factor returns.

The proposition below describes the Bayesian rule that investors use to update their priors recursively, conditional on using a particular \( \eta \)-factor model to compute alphas.

**Proposition 1:** Let \( \phi_{p,\eta,t-1} \) be investors’ assessment of fund \( p \)’s skill prior to the realization of \( r_{p,t} \) and let \( \phi_{p,\eta,t} \) be the mean of investors’ posterior after observing \( r_{p,t} \). Suppose the competitive market condition in Equation (3) holds and suppose investors compute \( \hat{\alpha}_{p,\eta,t} \) with an \( \eta \)-factor model in Equation (5) or Equation (6) to recursively update their priors about fund manager skills. Then

\[
\phi_{p,\eta,t} = \phi_{p,\eta,t-1} + \frac{\vartheta_{\hat{\alpha},\eta}}{\nu + \text{Age}_{p,t}} \times \hat{\alpha}_{p,\eta,t},
\]

where \( \text{Age}_{p,t} \) is the fund’s age at time \( t \). The precision of investors’ posterior is \( \nu + \text{Age}_{p,t} \times \vartheta_{\hat{\alpha},\eta} \), where \( \vartheta_{\hat{\alpha},\eta} = \frac{1}{\sigma_{\hat{\alpha},\eta}} \).

**Proof:** See Appendix A.

Which particular \( \eta \)-factor model would investors use to compute alphas? Next, we examine the properties of the posteriors with different alphas to determine the answer to this question.

**Proposition 2:** (a) Investors obtain an unbiased estimate of true skill conditional on factor realizations if and only if they use \( \hat{\alpha}_{p,J,t} \) to update their priors; (b) \( \text{Var}(\hat{\alpha}_{p,J,t}) < \text{Var}(\hat{\alpha}_{p,\eta,t}) \) \( \forall \eta < J \).

**Proof:** See Appendix A.

**Proposition 3:** Investors optimally use \( \hat{\alpha}_{p,J,t} \) to minimize mean squared error risk.

**Proof:** From Proposition 1, investors’ posterior of fund \( p \)’s skill is distributed \( N(\phi_{p,\eta,t}, 1/\nu_{p,\eta,t}) \). From Proposition 2, \( \phi_{p,J,t} \) is unbiased and has the smallest variance and hence the smallest MSE.

**Proposition 4:** In a competitive equilibrium, investors update their priors using \( \hat{\alpha}_{p,J,t} \).

**Proof:** Suppose the contrapositive that a competitive equilibrium obtains when investors use \( \hat{\alpha}_{p,\eta,t} \) with \( \eta < J \) to update their priors and determine fund flows. The competitive market condition under the contrapositive implies \( \phi_{p,\eta,t} - c(q_{p,t}) = 0 \). But, suppose \( f_{k^*,t} \neq 0 \) for some \( k^* > \eta \). For any fund with \( \beta_{p,k^*} \neq 0 \), \( E_t(\Phi_{p,}/f_{k^*}) = \phi_{p,\eta,t} - \beta_{p,k^*} f_{k^*,t} \) and \( E_t(\Phi_{p,}/f_{k^*}) - c(q_{p,t}) \neq 0 \). Hence under the contrapositive, nonzero NPV investments exist which violate the competitive market condition. Therefore, the contrapositive is not consistent with a competitive market equilibrium.
Both Propositions 3 and 4 indicate that in equilibrium investors use $\hat{\alpha}_{p,J,t}$, the J-factor alpha, to update their priors about fund manager skills. Intuitively, investors know the true asset pricing model, betas and factor realizations and what they do not know is what portion of a fund’s benchmark-adjusted return is due to the difference between its true skill and investors’ priors $(\Phi_p - \Phi_{p,t-1})$ and what portion is due to $\xi_{p,t}$. Investors optimally use $\hat{\alpha}_{p,J,t}$ because it is orthogonal to the information that they already know. Because $\hat{\alpha}_{p,J,t}$ is orthogonalized to both priced and unpriced factors it does not contain any information to differentiate between them.

1.3 Alphas and fund flows

Investors update their priors each period using $\hat{\alpha}_{p,J,t}$ and make their investment decisions each period. In a competitive equilibrium $\phi_{p,J,t} = c_t(q_{p,t})$, where $q_{p,t}$ is fund $p$’s AUM after time $t$ net fund flows. Therefore, $c_t(q_t)$ also follows a recursive equation analogous to Equation (7). Specifically,

$$c_t(q_{p,t}) = c_{t-1}(q_{p,t-1}) + \frac{\partial \hat{\alpha}_J}{\partial q_{p,t}} \times A e_{p,t} \times \hat{\alpha}_{p,J,t}.$$  

(8)

The net flow $\Gamma_{p,t}$ into mutual fund $p$ in each period is given by

$$\Gamma_{p,t} = \frac{q_{p,t} - q_{p,t-1}(1 + r_{p,t})}{q_{p,t-1}} = \frac{q_{p,t} - q_{p,t-1}}{q_{p,t-1}} - r_{p,t}.$$  

(9)

To determine a functional relation between $\hat{\alpha}_{p,J,t}$ and fund flows, we assume that the cost function is given by

$$c_{p,t}(q_{p,t}) = \delta_{p,t} \times q_{p,t},$$  

(10)

where $\delta_t$ is a time-varying cost per unit of AUM.
We specify the time-varying cost function as
\[
\delta_{p,t} = \frac{\delta_{p,t-1}}{(1 + r_{p,t})}.
\] (11)

This cost function assumes that the total cost of active management does not change with changes in fund size due to funds’ own returns and any change in total cost is only due to net fund flows.\(^5\)

With this cost function and Equations (8) and (9), equilibrium fund flows are
\[
\Gamma_{p,t} = \frac{q_{p,t} - q_{p,t-1}(1 + r_{p,t})}{q_{p,t-1}} = \frac{\vartheta_{\hat{\alpha},J}}{\nu + \text{Age}_{p,t} \times \vartheta_{\hat{\alpha},J}} \times \frac{(1 + r_{p,t})}{\delta_{t-1} q_{p,t-1}} \times \hat{\alpha}_{p,J,t}.
\] (12)

Equation (12) indicates that in addition to \(\hat{\alpha}_{p,J,t}\), fund flow is a function of fund’s marginal cost and the precision of investors’ posterior distribution. As we noted earlier, \(\hat{\alpha}_{p,J,t}\) does not differentiate between priced and unpriced factors. Therefore, \(\Gamma_{p,t}\) also does not differentiate between priced and unpriced factors and it contains no information to identify which factors are priced or unpriced in the true asset pricing model.

1.4 Econometricians’ information set and an alpha-fund flows horse race

The literature runs a horse race based on the relation between fund flows and alphas computed under various \(\eta\)-factor models. Because the horse race is run by empiricists, alphas should only use information available to them. Empiricists have the same information as investors in the model except that empiricists do not know (a) the true asset pricing model and (b) true betas. Therefore, the \(\eta\)-factor model alpha computed by empiricists is

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\(^5\) Because \(q_t = q_{t-1}(1 + r_t) + q_{t-1} \Gamma_t\), the total cost of active management at \(t\) under this specification is \(c_t(q_t) = \delta_{t-1} q_{t-1} \Gamma_t = c_{t-1}(q_{t-1}) + \delta_{t-1} q_{t-1} \Gamma_t\). Therefore, the change in the total cost from \(t - 1\) to \(t\) results from new money only. In contrast, a time-invariant cost function (e.g., \(\delta_t \equiv \text{constant}\)) would imply that aggregate fund flows would be negatively correlated with market returns, because the average alpha is zero and costs vary with the aggregate AUM regardless of whether the change in AUM is due to fund returns or due to flow of new funds. Also, because expected fund returns are positive, a time-invariant cost function would result in an average net outflow of funds to offset funds’ capital gains.
\[ \hat{\alpha}_{p,\eta,t}^E = r_{p,t} - \sum_{k=1}^{n} \hat{\beta}_{k,p} F_{k,t}, \]  

(13)

where \( \hat{\beta}_{k,p} \)'s are empiricists' unbiased beta estimates. The superscript \( E \) on alpha denotes that it is computed with the econometrician’s information set. As before, \( \hat{\alpha}_{p,0,t}^E = r_{p,t} - r_{market,t} \).

The literature typically runs the following horse race regression between flow and \( \hat{\alpha}_{p,\eta,t}^E \) to draw inferences about the true asset pricing model and investor sophistication:

\[ \Gamma_p = a_{\eta} + b_{\eta} \times \hat{\alpha}_{p,\eta,t}^E + \omega_{p,\eta,t}. \]  

(14)

The probability limit of the ordinary least squares (OLS) estimate of the slope coefficient is:

\[ \text{plim } b_{\eta} = \frac{\text{COV}\left(\Gamma_{p,t}, \hat{\alpha}_{p,\eta,t}^E\right)}{\sigma^2_{\hat{\alpha}_{p,\eta,t}}}, \]  

(15)

where \( \sigma^2_{\hat{\alpha}_{p,\eta,t}} \) is cross-sectional variance of \( \hat{\alpha}_{p,\eta,t}^E \). Our empirical tests follow the Fama-MacBeth approach and fit regression (14) for each \( t \) and the time-series average of monthly slope coefficients is the sample estimate of \( b_{\eta} \). The winner of the horse race regression (14) is the \( \eta \)-factor model with the biggest \( b_{\eta} \).

Which \( \eta \)-factor model would have the biggest \( b_{\eta} \)? Because flow in the model is determined by \( \hat{\alpha}_{p,j,t} \), heuristically the winner of the horse race would depend on how accurately \( \hat{\alpha}_{p,\eta,t}^E \) measures \( \hat{\alpha}_{p,j,t} \). From Equations (4) and (13), \( \hat{\alpha}_{p,\eta,t}^E \) is
\[
\hat{\alpha}_{p,j,t}^E = \begin{cases}
\alpha_{p,j,t} + \sum_{k=\eta+1}^{j} \beta_{k,p} \bar{f}_{k,t} - \left( \sum_{k=K+1}^{\eta} \beta_{k,p} \bar{F}_k + \sum_{k=1}^{\eta} (\hat{\beta}_{k,p,t} - \beta_{k,p}) F_{k,t} \right) & \text{for } \eta \geq K, \\
\alpha_{p,j,t} + \sum_{k=\eta+1}^{K} \beta_{k,p} \bar{F}_k + \sum_{k=\eta+1}^{j} \beta_{k,p} f_{k,t} - \sum_{k=1}^{\eta} (\hat{\beta}_{k,p,t} - \beta_{k,p}) F_{k,t} & \text{for } \eta < K,
\end{cases}
\]  \tag{16}

where \( \bar{F}_k \) is the sample mean of factor \( k \). The unconditional factor mean equals the corresponding factor risk premium for all priced factors.

In a frictionless economy, unconditional mean for unpriced factors (i.e., for \( k > K \)) should equal zero to preclude arbitrage. Empirically, however, the sample mean of unpriced factors could differ from zero because arbitrage is costly. For example, if CAPM were the true asset pricing model then the fact that the mean of HML is positive is an anomaly and \( \beta_{HML,p} \bar{F}_{HML} \) is not a component of expected returns. Because empiricists do not know the true asset pricing model and whether the \( k \)th factor is priced, the measurement error in \( \hat{\alpha}_{p,\eta,t}^E \) due to model misspecification is \( \sum_{k=\eta+1}^{K} \beta_{k,p} \bar{f}_k \) if \( \eta < K \) and \(-\sum_{k=K+1}^{\eta} \beta_{k,p} \bar{f}_k \) if \( \eta > K \).

From Equation (16), \( \sigma_{\hat{\alpha}_{p,\eta,t}^E}^2 \), the denominator of Equation (15), is

\[
\sigma_{\hat{\alpha}_{\eta}}^2 = \sigma_{\alpha_j}^2 + \sum_{k=K+1}^{\eta} \sigma_{p_k}^2 \bar{F}_k^2 + \sum_{k=\eta+1}^{j} \sigma_{\beta_{k,p}}^2 \bar{f}_k^2 E(F_{k,t}^2) + \sum_{k=\eta+1}^{j} \sigma_{\beta_k}^2 \sigma_{\hat{\beta}_k}^2 f_k^2 + \sum_{k=\eta+1}^{j} \sigma_{\beta_k}^2 \sigma_{\hat{\beta}_k}^2 [E(\beta_k)]^2, \tag{17}
\]

where \( \sigma_{p_k}^2 \) is the cross-sectional variance of factor beta, \( \sigma_{\hat{\beta}_k}^2 \) is factor variance, \( \sigma_{\beta_k}^2 \sigma_{\hat{\beta}_k}^2 \) is the variance of beta measurement error, \( \bar{F}_k \) is the expected value of factor \( k \) and \( \bar{\beta}_k \) is the cross-sectional average of corresponding factor beta.\(^6\) Equation (17) assumes that betas on various factors are uncorrelated in the cross-section for expositional convenience. For example, this assumption implies that the

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\(^6\) When \( \eta = 0 \), Equation (5) subtracts market return from fund returns and hence the last term in Equation (17) should be modified as \( \sigma_{\text{market}}^2 [E(\beta_1 - 1)]^2 + \sum_{k=2}^{j} \sigma_{\hat{\beta}_k}^2 [E(\beta_k)]^2 \). If average factor betas of the funds in the sample equal corresponding factor betas for the market portfolio, then the last term equals zero.
market beta of a fund relative to other funds has no information for the relative HML beta of that fund. However, when we later empirically estimate the components of $\sigma_{\tilde{a}}^2$, we estimate all cross-sectional covariances of betas from the data.

The first term on the right-hand side (RHS) is the variance of alphas across the cross-section of funds if one could estimate alphas with investors’ information set. The remaining terms are sources of incremental error, because empiricists do not have all of investors’ information. Consider each of these three terms:

- **Asset pricing model (APM) misspecification error**: Because factors $k > K$ are unpriced according to the true asset pricing model, realized fund returns are driven only by the unexpected component of these factors and not by $\bar{F}_k$. Suppose $K$ factors are priced under the true asset pricing model but we use $\eta$-factor model to compute alphas, variance due to APM misspecification equals $\sum_{k=K+1}^{\eta} \sigma_{\tilde{\beta}_k}^2 \bar{F}_k^2$.\(^7\) Empiricists’ compute alphas with all priced factors but without any unpriced factor when $\eta = K$ and in this case this term equals zero.

- **Beta measurement error**: Because empiricists estimate betas from the data, the factors used to compute alphas in Equation (13) contribute an incremental error that equals $\sum_{k=1}^{\eta} \sigma_{\tilde{\beta}_k}^2 \sigma_{\hat{\beta}_k}^2 + \sum_{k=\eta+1}^{\eta+1} \sigma_{\hat{\beta}_k}^2 \bar{E}(\beta_k)^2$ to alpha estimation errors. The first part of this sum is due to the cross-sectional variance of betas. This part would be zero if the factor betas of all funds are the same, because in this case adjusting for betas would not affect the cross-sectional rank of a fund’s performance. The second part is a function of the cross-

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\(^7\) When $\eta < K$, the limits of the summation for the second term in Equation (17) is from $\eta + 1$ to $K$. For brevity, we present formulas for $\eta \geq K$ and analogous changes yield the corresponding formulas for $\eta < K$. 

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sectional mean of factor betas. For \( \eta = J \), all factors are used to compute alphas and hence this component is zero.

Because \( \sigma^2_{\widehat{\alpha}_{p,\eta,t}} \) is the denominator of Equation (15), the \( \eta \)-factor model that yields the most precise alpha estimator would win the horse race, ceteris paribus.\(^8\) From Equation (17), estimation error in \( \widehat{\alpha}_{p,\eta,t}^E \) due to omitted factors increases with the exclusion of any factor. However, inclusion of unpriced factors to compute alphas makes the estimate less precise because each unpriced factor adds to APM misspecification error and to beta measurement error. The trade-offs between these two opposite effects will determine whether inclusion of a particular factor leads to a less or more precise alpha estimator.

BHO hypothesize that sophisticated investors would optimally use all \( J \) factors to compute alphas. Investors in our model do use all \( J \) factors because they know the true asset pricing model and true betas. However, because empiricists do not have the same information, their most precise estimator would exclude some of the \( J \) factors if their beta measurement errors and if APM misspecification errors are sufficiently large.

The winner of the horse race also depends on the numerator in Equation (15). However, we prove in Appendix B that the numerator does not vary with \( \eta \). Specifically,

\[
\text{Cov}(\Gamma_{p,t}, \widehat{\alpha}_{p,\eta,t}^E) = \text{Cov}(\Gamma_{p,t}, \widehat{\alpha}_{p,J,t}) \forall \eta.
\]

This result may seem somewhat counterintuitive, because Equation (12) shows that \( \Gamma_{p,t} \) is a function of \( r_{p,t} \), and therefore it shares some of the common factors with \( \widehat{\alpha}_{p,\eta,t}^E \) for \( \eta < J \), but \( \widehat{\alpha}_{p,J,t} \) is orthogonal to all common factors. Therefore, how is \( \text{Cov}(\Gamma_{p,t}, \widehat{\alpha}_{p,\eta,t}^E) \) the same for all \( \eta \) when

\(^8\) One component of \( \sigma^2_{\widehat{\alpha}_{p,\eta,t}} \) is the cross-sectional variance of the true skill of fund managers. Because this component is common across all \( \eta \)'s, the most precise alpha estimator also has the smallest \( \sigma^2_{\widehat{\alpha}_{p,\eta,t}} \).
the common factors included in $\tilde{\alpha}_{p,\eta,t}^E$ vary with $\eta$? Equation (18) obtains because of some key features of our results. One important feature is that $r_{p,t}$ enters $\Gamma_{p,t}$ only in the product form $(1 + r_{p,t}) \times \tilde{\alpha}_{p,J,t}$ in Equation (12), and another is that $\tilde{\alpha}_{p,J,t}$ is uncorrelated with any of the other components of $\tilde{\alpha}_{p,\eta,t}^E$ in Equation (16). These features and the fact that $E[\tilde{\alpha}_{p,J,t}] = 0$ yield Equation (18) and Appendix B contains the technical details. Because the numerator of Equation (15) does not depend on $\eta$, the estimator with the smallest $\sigma_{\tilde{\alpha}_{\eta,t}^E}^2$ based on empiricists’ information will win the empiricists’ horse race under the rational expectations hypothesis.

2. Empirical Tests

Our model shows that the most precise alpha estimator will win the flow-alpha horse race under the hypothesis that a rational expectations equilibrium determines fund flows. Our first set of tests examine the precision of alphas estimated using Equation (13) with various $\eta$-factor models with a sample of mutual funds. One component of the precision of alphas in Equation (16) depends on the true asset pricing model, but we do not know the true model. Therefore, we compute the precision under each of the following candidate asset pricing models: NBRP, CAPM, FF3, and FFC4, that is, $K = 0, 1, 3, \text{and } 4$.

We use the 7-factor model from BHO as the $J$-factor model that generates returns. The seven factors are the three Fama-French factors (market $\equiv mkt - r_f$, $SMB$ and $HML$), Carhart (1997) momentum factor (UMD), and three industry factors ($IND_1, IND_2, \text{and } IND_3$). We construct the three industry factors as the first three principal components of residuals from regressing Fama-French 17 equally weighted industry portfolios on FFC4 factors, as in BHO.

We obtain our sample of mutual funds from the CRSP survivor bias-free mutual fund database. Our sample is comprised of all actively managed domestic equity funds excluding sector funds.
Specifically, we consider funds that CRSP refers to as style-based or cap-based and assigns objective codes “EDC,” “EDYG,” “EDYB,” or “EDYI.” When a fund has multiple share classes, we add assets in all share classes to compute its total net assets (TNAs), and we compute fund level return as the weighted average of returns of individual share classes with lagged TNAs of each class as weights.

Our sample period is from January 1990 to June 2017. Our sample includes all funds with at least $10 million assets under management as of the end of the previous month. Also, the sample for month $t$ includes only funds that have returns data for all months from $t-60$ to $t-1$.\footnote{This criterion excludes funds during the first 60 months of their existence. Therefore, our sample is not exposed to the potential incubation bias that Evans (2010) and Elton, Gruber, and Blake (2001) document.} We follow BHO and exclude funds that had flows smaller than -90% or greater than 1,000% in any month from the sample to avoid the effect of outliers.

Table 1 presents sample summary statistics. The sample comprises 2,969 funds with 1,224 funds per month on average. The average monthly fund flow into a fund is 0.25% of its TNA the previous month.

2.1 Precision of alphas

The decomposition in Equation (17) indicates that one important determinant of the precision of alpha is $\sigma^2_{\beta_k} \sigma^2_{f_k}$, which when normalized by the variance of fund returns roughly equals the incremental $R^2_{adj}$ attributable to common factor $k$. The other determinant is beta measurement
error. To evaluate the individual contribution of each factor to the precision of alpha estimates we first examine these two components separately. We then empirically estimate the variance of alphas from each $\eta$-factor model and the contribution of various components.

2.1.1 $R_{adj}^2$ and beta estimation error.

We fit the following time series regression with $\eta$ factors for month $t$ using data for each fund from months $t - 60$ to $t - 1$ and compute average $R_{adj}^2$ for each model:

$$r_{p,\tau} = a_{p,\eta,\tau} + \sum_{k=1}^{\eta} \beta_{k,p,\tau} F_{k,\tau} + e_{p,\eta,\tau}, \quad \tau = t - 60 \text{ to } t - 1.$$ (19)

Table 2 reports average OLS $R_{adj}^2$ of Equation (19). We compute average $R_{adj}^2$ across all funds each month and the table reports the time-series average. $R_{adj}^2$ for the single-factor market model is 0.820 and it increases to 0.892 for the 3-factor model, but the increase is fairly gradual as we go from the 3-factor model to the 7-factor model. Table 2 also reports $R_{adj}^2$ that we compute based on the explanatory power of $\hat{\alpha}^E_{p,\eta}$, which we define as $R_{adj}^2 = 1 - \frac{\text{var}(\hat{\alpha}^E_{p,\eta}) \times (T_p - 1)}{\text{var}(r_p) \times (T_p - \eta - 1)}$, where $T_p$ is number of months the fund is in the sample. Market-adjusted returns have the smallest $R_{adj}^2$ of 0.761 and $R_{adj}^2$ for the single-factor market model is bigger at 0.829. $R_{adj}^2$ increases to 0.883 for the 3-factor model and then marginally to 0.884 for the 7-factor model.

Another important component of the precision of alpha is the variance of beta measurement error across funds ($\sigma^2_{\beta - \hat{\beta}}$). The term $\sigma^2_{\beta - \hat{\beta}}$ includes OLS estimation error. In addition, it includes
any difference between average betas during the estimation period and month \( t+1 \) beta because of any time variation in betas due to turnover of funds’ holdings.

To estimate the magnitude of this error we first estimate the following regressions for each fund for each month:

\[
(r_{p,\tau} - r_{f,\tau}) = \alpha_{p,k,t}^{\text{past}} + \sum_{k=1}^{7} \beta_{p,k,t}^{\text{past}} F_{k,\tau} + e_{p,k,\tau}, \quad \tau = t - 60 \text{ to } t - 1, \tag{20}
\]

\[
(r_{p,\tau} - r_{f,\tau}) = \alpha_{p,k,t}^{\text{future}} + \sum_{k=1}^{7} \beta_{p,k,t}^{\text{future}} F_{k,\tau} + e_{p,k,\tau}, \quad \tau = t \text{ to } t + 11,
\]

where \( F_{k,\tau} \) is the factor with respect to which betas are estimated. Suppose betas for a particular fund are constant over time.

\[
\hat{\beta}_{p,k,t}^{\text{past}} = \beta_{p,k} + u_{p,k,t}^{\text{past}}, \quad \text{and}
\]

\[
\hat{\beta}_{p,k,t}^{\text{future}} = \beta_{p,k} + u_{p,k,t}^{\text{future}}, \tag{21}
\]

where \( \beta_{p,k} \) is fund \( p \)'s true beta with respect to factor \( k \).

Consider the following cross-sectional regression for month \( t \):

\[
\hat{\beta}_{p,k,t}^{\text{future}} = a_t + b_t \times \hat{\beta}_{p,k,t}^{\text{past}} + e_{p,t}. \tag{22}
\]

Because we use nonoverlapping sample periods to estimate \( \beta_{p,k,t}^{\text{past}} \) and \( \beta_{p,k,t}^{\text{future}} \), \( u_{p,k,t}^{\text{past}} \) and \( u_{p,k,t}^{\text{future}} \) are uncorrelated. The probability limit of the slope coefficient is

\[
\text{plim } b_t = \frac{\text{var}(\beta_{p,k})}{\text{var}(\beta_{p,k}) + \text{var}(u_{p,k,t}^{\text{past}})}. \tag{23}
\]

Therefore, the slope coefficient of regression (22) is the ratio of the cross-sectional variance of the factor betas divided by the sum of this variance plus the variance of the measurement error.
We fit regression (22) each month for each of the betas estimated using multiple regressions of fund returns on the seven factors. Table 3 reports the time-series averages of the slope coefficients for each beta. The slope coefficients are bigger with respect to the three Fama-French factors and UMD compared with industry factor betas. This result combined with the evidence that the incremental $\hat{R}^2_{adj}$ from adding the three industry factors is small suggests that the incremental benefit of adding industry factors is likely small as well.

### 2.1.2 Precision of alpha estimates and implications.

This subsection compares the precision of various $\eta$-factor model alphas ($\sigma^2_{\hat{\alpha}_E}$). We use OLS estimates of regression (19) and compute $\hat{\alpha}^{E}_{p,\eta,t}$ using Equation (13). We compute the cross-sectional variance of $\hat{\alpha}^{E}_{p,\eta,t}$ each month and the time-series average of monthly variance is our estimate of $\sigma^2_{\hat{\alpha}_E}$.

Table 4 presents $\sigma^2_{\hat{\alpha}_E}$ for each $\eta$-factor model. The variance monotonically declines from 650.5 to 357.7 as we go from the single-factor model to the 4-factor model but increases to 363.2 for the 7-factor model.\(^{10}\) Therefore, the 4-factor alpha is the most precise estimate. What does this result imply for interpretations about the true asset pricing model? For instance, can we conclude that the 4-factor model is the true asset pricing model based on this result? Also, why is the 7-factor model alpha not the most precise estimator as hypothesized by BHO?

\(^{10}\) The table reports variances multiplied by $10^6$. 
To answer these questions, we need to know the components of $\sigma^2_{\tilde{\alpha}_{\tilde{\eta}}}$ that we discussed earlier. For example, one component of $\sigma^2_{\tilde{\alpha}_{\tilde{\eta}}}$ is APM misspecification error and the alpha-fund flow horse race can be used as a test of asset pricing models as proposed by BvB only if this component is sufficiently large to make the other models less precise. A sufficiently large misspecification component could also explain why the 7-factor model alpha is not the most precise.

We empirically estimate each component of $\sigma^2_{\tilde{\alpha}_{\tilde{\eta}}}$ to examine these issues. Equation (17) presents the components of $\sigma^2_{\tilde{\alpha}_{\tilde{\eta}}}$, but for expositional convenience that equation assumes funds’ factor betas are not cross-sectionally correlated. Empirically, however, funds’ factor betas are cross-sectionally correlated. For example, funds with bigger market betas on average have smaller HML betas in the data. Allowing for beta correlations, $\sigma^2_{\tilde{\alpha}_{\tilde{\eta}}}$, conditional on a $K$-factor model being the true asset pricing model is

$$
\sigma^2_{\tilde{\alpha}_{\tilde{\eta}}}|K = \sigma^2_{\tilde{\alpha}_{\tilde{\eta}}.t} + F'_{(K+1,\eta)} \left( \text{Cov}(\beta'_{(K+1,\eta)}, \beta'_{(K+1,\eta)}) \right) F_{(K+1,\eta)} + \\
f_{(\eta+1,j).t} \left( \text{Cov}(\beta'_{(\eta+1,j)}, \beta'_{(\eta+1,j)}) \right) f_{(\eta+1,j).t} + E[\beta_{(\eta+1,j)}]' (f_{(\eta+1,j).t} f'_{(\eta+1,j).t}) E[\beta_{(\eta+1,j)}] - \\
\text{Omitted factors} \\
\frac{f'_{(\eta+1,j).t} \left( \text{Cov}(\beta'_{(\eta+1,j)}, \beta'_{(K+1,\eta)}) \right) F_{(K+1,\eta)}.t}{\text{Covariance}} + \\
\frac{f'_{(1,\eta).t} \left( \text{Cov}(\beta'_{(1,\eta)}, \beta'_{(1,\eta)}) \right) F_{(1,\eta)}.t}{\text{Beta measurement error}}.
$$

Equation (24) expresses factors and factor betas as vectors. We use the same notations for vectors as the corresponding scalars but boldface denotes vectors. Also, the subscripts for vectors within parentheses indicate their first and last items. For example, $\beta'_{(\eta+1,j)} \equiv [\beta_{\eta+1}, \beta_{\eta+2}, \ldots, \beta_j]$. 

21
Equation (24) obtains for \( \eta \geq K \) and an analogous expression with dimensions of vectors with subscripts \((K + 1, \eta)\) replaced by \((\eta + 1, K)\) obtains for \( \eta < K \).

Equation (24) is a straightforward generalization of Equation (17) with the addition of terms that include cross-sectional covariance of factor betas. The term labeled “covariance” captures the potential effect of any cross-sectional covariance between betas of unpriced factors included in \( \eta \) and omitted factors that are part of fund returns. This term is nonzero when betas on factors are correlated in the cross-section.

We compute the components of \( \sigma_{\hat{\alpha}_{K,t}}^2 \) in Equation (24) as follows: because we use OLS estimates, \( Cov \left( [\hat{\beta}_{(1,\eta)}p,t - \beta_{(1,\eta)}p,t]’, [\hat{\beta}_{(1,\eta)}p,t - \beta_{(1,\eta)}p,t] \right) \) is \( \sigma_{\epsilon p,t}^2 (X_{(1,\eta),t}X_{(1,\eta),t})^{-1} \), where \( X_{(1,\eta),t} \) is the matrix of factors used in regression (19). The quadratic product of this estimate with \( F_{(1,\eta),t} \) is the \( \hat{\beta} \) measurement error component for that month. We need the covariance matrix of true betas to compute the APM misspecification error, which is

\[
Cov(\beta’_{(1,\eta)}, \beta_{(1,\eta)}) = \frac{1}{T} \sum_t \left( Cov(\hat{\beta}_{(1,\eta),t}’, \hat{\beta}_{(1,\eta),t}) - \left( \frac{1}{P_t} \sum_p \sigma_{\epsilon p,t}^2 (X_{(1,\eta),t}X_{(1,\eta),t})^{-1} \right) \right),
\]

(25)

where \( P_t \) is the number of funds in the sample in month \( t \), and \( T \) is the number of months in the sample period. We compute \( Cov(\hat{\beta}_{(1,\eta)}, \hat{\beta}_{(1,\eta)}) \) by taking the average of the cross-sectional covariance of \( \hat{\beta}_{(1,\eta),p,t} \) each month. The APM misspecification component of variance is the quadratic product of \( F_{(K+1,\eta)} \) with the corresponding submatrix of \( v(\beta’_{(1,\eta)}, \beta_{(1,\eta)}) \). Appendix C describes how we compute the other components.
Table 5 presents sample means and standard deviations of market, SMB, HML, and UMD, which are all significantly positive. Therefore, if CAPM were the true model the nonzero means of the other factors contribute to APM misspecification error in alphas computed using a 4-factor model. Table 5 also presents the covariance matrix of true betas estimated using the procedure described above for the case $\eta = 7$.

Table 6 presents the estimates of various components in Equation (24). To evaluate the net effect of using unpriced factors to compute alphas on $\sigma^2_{\hat{\alpha}}$, consider the NBRP model where all $J$-factors are unpriced. When $\eta = 0$, the benefit of excluding all unpriced factors is that APM misspecification error is zero, but the cost is added variance due to omitted factors, which equals 340.9. When $\eta = J$, variance due to omitted factors is zero but now the APM misspecification error variance is 1.94. Although the contributions from omitted factors and APM misspecification typically go in opposite directions, the contribution of the former is orders of magnitude bigger than that from the latter.

Column 1 of the Table 6 presents $\sigma^2_{\alpha_J}$ for each asset pricing model. $\sigma^2_{\alpha_J}$ varies across $K$ because $\sigma^2_{\hat{\alpha}}$ is the empirical cross-sectional variance (therefore independent of hypothesized $K$) but its components vary across $K$. Column 7 presents the sum of the four components excluding

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11 Industry factors are arbitrarily scaled, and, hence, their means and variances have no particular economic meaning. Therefore, we do not report them in the table.

12 When we assume NBRP is the true model, the contribution of APM misspecification component in Table 6 is smaller for $\eta = 1$ than for $\eta = 3$, although more unpriced factors are used to compute alphas when $\eta = 3$. This seemingly counterintuitive result comes from the negative cross-sectional covariance between market and HML betas.
\( \sigma^2_{\hat{a}_j} \), which ranges from 48.0 to 340.9 as we vary \( \eta \) from 0 to 7 for \( K = 0 \). For any given \( \eta \), this sum varies little with changes in the hypothesized “true” asset pricing model.

The results in Table 4 indicate that the 4-factor alpha estimator is empirically the most precise when we estimate beta from the data using the time-series regression (19) regardless of the true asset pricing model. What would be the most precise alpha estimator if we know true betas, but not the true asset pricing model? To answer this question, we compare the sum of the components of variance excluding the beta measurement error component and \( \sigma^2_{\hat{a}_j} \) under each of the hypothesized asset pricing models.

Column 6 reports this sum, which ranges from 1.9 to 340.9 as we vary \( \eta \) from 0 to 7 for \( K = 0 \). We get the most precise estimator with \( \eta = 7 \) regardless of the true asset pricing model. Therefore, if fund flows are determined under the rational expectations hypothesis the 4-factor alpha will win the horse when betas are estimated from the data using OLS regression (19), but the 7-factor model will win if true betas are known.

The results in Table 6 quantify the trade-offs among the components of measurement error as we add more factors to compute alphas. For example, when NBRP is the true asset pricing model, each common factor used to compute alphas increases the APM misspecification error but decreases the omitted factor component. The biggest value of the APM misspecification component is 1.94 when \( \eta = 7 \), but the omitted factor component decreases from 340.9 to 0 as \( \eta \) varies from zero to seven. The APM misspecification component is at least an order of magnitude smaller than the omitted factor component if any factor is excluded from the alpha estimator. Therefore, APM misspecification makes a trivial contribution to the overall precision of alpha estimators.
The beta measurement error component also increases with $\eta$ because the number of factor betas that are estimated increases with $\eta$. The marginal change in this component is bigger than that for the omitted factor component only when $\eta$ increases from four to seven. Therefore, alpha estimation error is smaller with the 4-factor model than with the 7-factor model.

3. Simulation Experiment

Our next set of tests simulate the rational expectations economy we model with parameters that match the data. We generate fund flows in the simulation according to our model and test the model predictions. Specifically, we compute the precision of alphas with various $\eta$-factor models and test the model prediction that the most precise alpha will win the flow-alpha horse race. We also test BvB’s and BHO’s hypotheses that either the $K$- or $J$-factor model would win the horse race when fund flows are determined in a rational expectations economy. Additionally, we conduct a number of robustness checks by changing various parameters of the model economy and estimation methodology.

3.1 Experimental design

The simulation generates fund returns with a 7-factor model with parameters determined from the data. The number of funds in the simulated sample exactly matches the data each month. Mutual fund skill is unobservable and investors start with priors about the unconditional distribution of fund manager skills and recursively update their priors after observing fund returns and make their investment decisions as the model describes.

The following are the simulation details:

a. **Fund origin:** We start the simulation with the number of funds equal to that in the sample on January 1985.
b. **Skill** ($\Phi_p$): When a fund enters the sample, we randomly draw its skill from a normal distribution with mean ($\phi_0$) equal to 0.15% and standard deviation of 0.2% per month. The average 4-factor alpha in our sample of domestic equity funds, gross of fund fees and expenses is around 5 bps per month, and we add 10 bps per month to this estimate to account for average trading costs incurred by actively managed funds.\(^{13}\) The standard deviation of fund skill matches the estimate we obtain from the data.\(^{14}\)

c. **Betas**: We generate the 7-factor betas $[\beta_{mkt}, \beta_{SMB}, \beta_{HML}, \beta_{UMD}, \beta_{IND1}, \beta_{IND2}, \beta_{IND3}]'$ jointly for each fund from a multivariate normal distribution with the mean vector $[1, 0, 0, 0, 0, 0, 0]'$ and covariance matrix of true betas reported in Table 5.\(^{15}\)

d. **Fund specific return**: We draw $\epsilon_{p,t}$ for each fund from a normal distribution with mean equal to zero and standard deviation equal to 1.75%, which matches our estimates from the data.

e. **Asset pricing model and expected returns**: We conduct simulations under four asset pricing models and expected excess returns under each model are computed as follows:

- **NBRP model**: $E^{NR}(r_p - r_f) = 0.699\%$
- **CAPM**: $E^{CAPM}(r_p - r_f) = \beta_{p,m} \times (\bar{mkt} - \bar{r_f})$, \hspace{1cm} (26)
- **Fama-French 3-factor model** (FF3): $E^{FF3}(r_p - r_f) = \beta_{p,m} \times (mkt - r_f) + \beta_{p,smb} \times (\bar{SMB}) + \beta_{p,html} \times (HML)$,

\(^{13}\) Edelen, Evans, and Kadlec (2013) report that the transaction costs are of the same order of magnitude as expense ratios, which average to around 10 bps per month.

\(^{14}\) The monthly cross-sectional variance of $\hat{\alpha}$s in the real data is the variance of true alphas plus the measurement error of alphas. The measurement error variance in $\hat{\alpha}$s is the average squared OLS standard errors from the time-series regressions used to estimate alphas. The average difference of cross-sectional variance and measurement error variance in $\hat{\alpha}$s across models results in the standard deviation of true alphas to be around 0.2% per month.

\(^{15}\) The average fund betas in the sample are [1, 0.245, 0.012, 0.015, -0.003, 0.016, -0.001], which we approximate with the corresponding factor betas for the market portfolio. None of our results are sensitive to changes in average betas.
- Fama-French-Carhart 4-factor model (FFC4): \( E^{FFC4}(r_p - r_f) = \beta_{p,m} \times (mkt - r_f) + \beta_{p,smb} \times (SMB) + \beta_{p,hml} \times (HML) + \beta_{p,umd} \times (UMD) \).

The bars above the common factor returns represent sample means. The average fund excess returns under all asset pricing models equal average of market excess returns.

f. **Net fund returns**: Net fund return each period is given by the following 7-factor model:

\[
r_{p,t} = \Phi_p - c_{t-1}(q_{t-1}) + E^{model}(r_p) + \beta_{p,m} \times (mkt - r_f)_t + \beta_{p,smb} \times \overline{SMB}_t + \beta_{p,hml} \times \overline{HML}_t + \beta_{p,umd} \times \overline{UMD}_t + \beta_{p,ind1} \times \overline{IND1}_t \\
+ \beta_{p,ind2} \times \overline{IND2}_t + \beta_{p,ind3} \times \overline{IND3}_t + \epsilon_{p,t},
\]

(27)

where \( \Phi_p \) is the fund manager skill, \( c_{t-1}(q_{t-1}) \) is the cost per unit size, the variables under \( \text{tilde} \) are demeaned realizations of the seven common factors. We do not observe \( c_{t-1}(q_{t-1}) \), but the competitive equilibrium condition implies \( c_{t-1}(q_{t-1}) = \phi_{p,t-1} \).

The simulations start with \( \phi_{p,t,0} = \phi_0 \) at \( t=0 \) for all funds. Total unexpected return for \( t = 1 \) is the sum of beta times unexpected factor realizations for that month and \( \epsilon_{p,1} \). We add \( \Phi_p - \phi_0 + E^{model}(r_p) \) to compute \( r_{p,1} \). We then compute alpha \( \hat{\alpha}_{p,t,1} \) using Equation (5) and \( \phi_{p,t,1} \) using Equation (7). We recursively follow these steps for each \( t \).

g. **Fund flow**: Investors observe \( r_{p,t} \) and update their priors using Equation (7). Fund flow is given by Equation (12).
h. **Fund exit and entry:** If $\phi_{p,t}$, the posterior of fund skill, drops below a critical value the fees fund earns will not be sufficient to cover its fixed costs and therefore the fund shuts down. We set this critical value to $\phi_0/100$.\(^{16}\)

To match the number of funds in the simulation to the number of funds in the data, we add new funds when the number funds in simulated sample in any month is smaller than that in the data. If it is greater, the appropriate number of funds with the smallest values of $\phi_{p,t}$ exit the simulated sample for month $t$.

### 3.2 Tests and results

Table 7 presents the simulation parameters. We first examine the relation between fund flows and alphas under various asset pricing models. Table 8 presents $\sigma_{a\eta}^2$ in the simulations and its components from Equation (24). Because we know the true asset pricing model, fund skill, and factor betas in the simulation, we use them when necessary to compute the components of $\sigma_{a\eta}^2$ in the simulation experiment.

Variance due to omitted factors in Table 8 decreases monotonically from 324.1 to 0 as $\eta$ increases from 0 to 7. In comparison, APM misspecification variance ranges from .93 to 1.95 when

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\(^{16}\) The critical value is nonzero for any positive fixed costs. Here, we choose a small positive value, and we will examine the robustness of our results with respect to changes in critical value later.
we set $K = 0$. These results are similar to that in Table 6, and they confirm that the contribution of APM misspecification component is orders of magnitude smaller than that due to omitted factors. Total variance in addition to $\sigma_{\tilde{\alpha}_f}^2$ excluding the beta measurement error component ranges from 2 to 324.1 when $K = 0$.

We find similar results for simulations under the other asset pricing models. When we ignore the beta measurement error component, the most precise alpha estimator is with $\eta = 7$ for all asset pricing models. When we estimate factor betas from simulated returns, the variance of beta measurement error increases monotonically from 0 to 56.7 as we increase $\eta$ and now 4-factor alpha is the most precise estimator. These results indicate that $\sigma_{\tilde{\alpha}_f}^2$ and its components in the simulation are similar in magnitude and pattern to what we find in Tables 4 and 6. Therefore, our decomposition of alpha measurement errors based on asymptotic analytics holds in finite samples.

Insert Table 9 about here

Next, we generate fund flows using Equation (12) each period, and we fit the horse race regression using Fama-MacBeth approach. Table 9 reports the slope coefficients. When we compute alphas using true betas, the slope coefficient increases monotonically as we add factors. For example, under the CAPM, the slope increases from 2.19 for $\eta = 0$ to 3.50 for $\eta = 7$. The 7-factor alpha is the winner under all asset pricing models.

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17 The total variance for a given $\eta$ varies slightly as we change $K$, because $E_{model}(\tau_p)$ in Equation (27) varies with $K$.

18 Overall, the distribution of fund returns in the simulation also matches the data. For example, the average $R_{adj}^2$ in the simulation varies from 76.9 to 86.3 as $\eta$ varies from 0 to 7, which is close to the corresponding statistics in Table 2.
Panel B reports the slope coefficients when we use betas estimated from simulated returns to compute alphas. Now we get the biggest slope coefficient for $\eta = 4$ and not for $\eta = 7$. For example, for the CAPM, the slope coefficients are 3.05 and 3.02 for $\eta = 4$ and $\eta = 7$, respectively. The 4-factor alpha wins the horse race under all asset pricing models.

Overall, the simulation results indicate that 4- and 7-factor alphas are the most precise estimators depending on whether factors betas are estimated with error or whether true betas are known. The horse race results indicate that when flows are generated under the rational expectations hypothesis the most precise alpha wins the horse race. The simulation results also indicate that the outcome of the horse race does not depend on the true asset pricing model. For example, FFC4 wins the horse race in Table 9 when the true asset pricing model is FFC4 and when the true asset pricing model is NBRP, CAPM or FF3. Therefore, the winner of empiricists’ horse race does not contain any information about the true asset pricing model. All these results confirm our model predictions.

3.3 **Robustness tests**

3.3.1 **Parameters.**

We conduct a number of robustness tests to evaluate the sensitivity of the simulation results including the following: (a) vary the mean and variance of $\Phi_p$, (b) set the critical value of posterior about fund skill for exit to $\phi_0 / 50$ or $\phi_0 / 10$, and (c) vary the variance of true factor betas from 1/4th of the variance in the main simulation to twice the variance. Our conclusion that the true asset pricing model has no effect on the precisions of alpha estimates or on the winner of the horse race is robust to all these changes.
Our result that the 7-factor model wins the horse race if betas are measured without error is also robust. However, when we set the variance of true factor betas to 150% of the variance in Table 5 or larger, the 7-factor model is always the winner even when betas are measured with error. Intuitively, at this level the benefit of including an unpriced factor because of the omitted factor effect, that is, the effect of $\sigma^2_{\beta_k} \sigma^2_{f_k}$ in Equation (17), outweighs the cost due to beta measurement error. These findings are consistent with our analytic results that the benchmark for investor sophistication depends on the properties of betas of the assets in the sample and therefore must be determined from the sample under consideration.

3.3.2 Time-varying betas.

Factor betas of funds could vary over time as they turnover their holdings. To capture such time variation, we assume that true factor betas follow an AR(1) process as specified below:

$$\beta_{(1,j)p,t} = (1 - \rho)\bar{\beta}_{(1,j)p} + \rho \beta_{(1,j)p,t-1} + \zeta_{(1,j)p,t},$$

where $\text{Cov}(\zeta_{(1,j)}) = (1 - \rho^2)\text{Cov}(\beta_{(1,j)})$ and $\bar{\beta}_{(1,j)p}$ is unconditional mean of factor betas. We consider values of $\rho$ ranging from 0.2 to 0.9 and set the covariance of $\zeta$ to match the average covariance of fund level betas. We draw the first value of $\bar{\beta}_{(1,j)p}$ for a fund 60 months before its entry date from a normal distribution with the mean vector $[1,0,0,0,0,0,0]'$ and covariance equal to the covariance reported in panel B of Table 5 minus covariance of $\zeta$.

In untabulated results, we find that the relative precision of alphas and the winner of the horse race were identical to what we find with constant betas. Specifically, the winner is always the 7-factor model when we use true betas to run the horse race and the 4-factor model when we estimate betas from the data with regression (19). The true asset pricing model has a trivial effect on the precision of alphas or on the slope coefficients of the horse race regression.

3.3.3 Precision of alpha estimators: Beta shrinkage and alternative factors.
We find that the 4-factor model wins the horse race over the 7-factor model when betas are measured with error. Would this result change if we estimate betas more precisely? Vasicek (1973) shows that market betas shrunk towards one is a more precise estimate of future betas than OLS betas. To examine whether such shrinkage increases precision of alphas, we shrink the betas towards their population means with weights equal to the corresponding slope coefficients in Table 3 and compute alphas in the data for various factor models.\(^{19}\) With the shrunk beta, \(\sigma_{\alpha_{ij}}^2\) for the 4- and 7-factor models are 350.2 and 345.6 compared with 357.7 and 363.2 in Table 4. Therefore, alphas are more precisely estimated in the data with shrunk betas and also 7-factor alphas are more precise than 4-factor alphas.

We could also potentially improve the precision of alpha estimates by suitably modifying the return-generating process that we assume. The first four factors, that is, market, SMB, HML, and UMD, are specified by theoretical or empirical asset pricing models. However, the industry factors are statistically defined. Statistical factors computed with sample fund returns could possibly better identify sample-specific common factors.

To identify these common factors from the sample, we first fit a time-series regression analogous to regression (19) with \(\eta = 4\) for each fund over its entire life. We extract the principal components of covariance matrix of the residuals using the approach in Connor and Korajczyk (1987). We use the first three principal components in place of the industry factors in the return-generating process.

\(^{19}\) For instance, for market beta the shrinkage estimator equals \((1 - .656) \times 1 + .656 \times \beta_{\text{past}}^{\text{market}}\), where \(\beta_{\text{past}}^{\text{market}}\) is the estimate from the time-series regression with past returns.
With this model, $\sigma_{\hat{\alpha}_E}^2$ for the 7-factor model is 321.4 which is smaller than 363.2 for the corresponding model with industry factors in Table 4. Alphas estimated with this model are also more precise than that with the 4-factor model in Table 4. Therefore, sample specific common factors could potentially increase the precision of alpha estimates.\(^{20}\)

We also run our simulation experiments using these modifications. First, we use shrunk betas to compute alpha in the simulation and fit horse race regression (14). In untabulated results we find that the 7-factor model wins the horse race with shrunk betas. Similarly, we simulate returns with factors from fund principal components in place of industry factors and fit the horse race regression. Here, again, the 7-factor alpha wins the horse race.\(^{21}\)

Overall, the robustness test results are consistent with our analytic results that the winner of the horse race under the rational expectations hypothesis depends on the characteristics of the sample, such as dispersion of factor betas and measurement error in betas. Specifically, when betas are estimated more precisely with Vasicek (1973) shrunk betas and when cross-sectional dispersion of betas is bigger than in our sample, the optimal alpha estimator includes more factors and the 7-factor model wins. But large beta measurement errors, for instance, because of time-varying betas, and smaller dispersion in factor betas favor a model with fewer factors and the 4-factor model wins. The true asset pricing model has no effect on the outcome of the horse race.

3.3.4 Fund flows.

When we generate fund flows according to our model in the simulation, the slope coefficient on the 7-factor alpha in horse race regression in panel B of Table 9 is about 3.02. We fit the

\(^{20}\) However, an advantage with industry factors is that they are not specific to a particular sample.
\(^{21}\) One could also improve the precision of beta estimates using the daily returns data available on CRSP starting from September 1998.
following regression with actual data to compare the empirical flow and alpha with model flow alpha relation:

$$\Gamma_{p,t} = a + b \times \hat{\alpha}_{p,7,t} + \psi_{p,t}. \quad (29)$$

The slope coefficient of this regression is 0.198. The smaller empirical correlation indicates that fund flows in practice depend on other factors besides funds’ past performance. Ibert et al. (2017) document that past performance does not fully explain fund flows and suggest that factors such as managerial fundraising skill, advertising, and broker-intermediated flows could also affect fund flows. Additional factors such as investors’ personal liquidity demands and recommendations by advisory services, such as Morningstar, also potentially drive a wedge between empirical and model flows.\(^{22}\)

We examine robustness of our results if fund flows are generated according to Equation (29) rather than Equation (12). We assume that the difference between model flows and empirical flows are due to factors that are exogenous to the model. Heuristically, we also assume that rational investors in the model who optimally extract information about fund skills from returns are aware of these exogenous flows and account for them in their investment decisions so that we get to a competitive equilibrium. Propositions 1 through 3 apply in a competitive equilibrium regardless of the underlying factors that drive flows and therefore \(\hat{\alpha}_J\) determines flows from rational investors.

We run the horse race with simulated flows that match empirical fund flows. Regression (29) uses \(\hat{\alpha}_{p,7,t}^E\) as the explanatory variable, but, for the simulation, we need the relation between

\(^{22}\) For example, Jain and Wu (2000), Gallaher, Kaniel, and Starks (2006), and Kaniel and Parham (2017) find that fund flows are positively correlated with advertising activities; Del Guercio and Tkac (2008) and Ben-David et al. (2018) find that fund flows are correlated with Morningstar ratings; and Christoffersen, Evans, and Musto (2013) find that flows in broker-sold funds are affected by the incentives of the brokers.
\( \hat{\alpha}_{p,7,t} \) that investors in the model use and fund flows. Because \( \hat{\alpha}_{p,7,t} \) equals the sum of \( \hat{\alpha}_{p,7,t} \) and measurement error, we can get the slope coefficient with respect to the latter by scaling up the slope coefficient estimate from regression (29) by a factor equal to \( \frac{\sigma^2_{\hat{\alpha}_E}}{\sigma^2_{\hat{\alpha}_7}} \) from Tables 4 and 8.

With this scaling, we generate fund flows in the simulation using the following equation:

\[
\Gamma_{p,t} = -0.00225 + 0.233 \times \hat{\alpha}_{p,7,t} + \psi_{p,t}. \quad (30)
\]

We randomly draw \( \psi_{p,t} \) from a mean zero normal distribution with variance equal to 9.33%, to match the empirical variance.

Because fund returns are generated using the same parameters as before, the precision of alpha estimates is same as that in Table 8. The untabulated results for the horse race regression are similar to that in Table 9. Specifically, the true asset pricing model has a negligible effect on the slope coefficient estimates, the 7-factor alpha wins without beta estimation error and the 4-factor model wins when betas are measured with error.

4. Binary Variable Regression

Our analyses so far use a linear regression for the alpha-fund flow horse race but the true relation need not be linear. For example, the coefficient on alpha in Equation (12) varies cross-sectionally with fund returns and fund age. Also, in Berk and Green (2004), the equilibrium relation between alpha and fund flow is nonlinear. Because of potential nonlinearity, BvB transform flows and alpha estimates to binary variables and run the horse race with these transformed variables. Specifically, the transformed binary variables are defined as follows:

\[
Q_x = \begin{cases} 
1 & \text{if } x \geq 0 \\
-1 & \text{if } x < 0 
\end{cases} \quad (31)
\]
where $x$ is any random variable. BvB run the following OLS regression:

$$
Q_{t_p} = A_\eta + B_\eta \times Q_{\hat{p}_\eta} + o_{p,\eta},
$$

(32)

and compare $\hat{B}_\eta$. To relate the analyses based on regressions (14) and (32), we first establish the following proposition:

**Proposition 5:** Let $\hat{\alpha}^E_{p,\eta_1}$ and $\hat{\alpha}^E_{p,\eta_2}$ be the alphas computed by the empiricist with respect to $\eta_1$- and $\eta_2$-factor models using Equation (13) and suppose the model misspecification term is sufficiently small. $\hat{B}_{\eta_1}$ and $\hat{B}_{\eta_2}$ are the corresponding regression (14) slope coefficients and $\hat{B}_{\eta_1}$ and $\hat{B}_{\eta_2}$ are the corresponding regression (32) slope coefficients. Under the assumptions of our model, if $\hat{b}_{\eta_1} > \hat{b}_{\eta_2}$ then $\hat{B}_{\eta_1} > \hat{B}_{\eta_2}$, when the number of funds in the sample is sufficiently large.

**Proof:** See Appendix D.

**Corollary:** The ordering of the slope coefficients of regressions (14) and (32) are identical.

Proposition 5 and its corollary show that our analysis of the horse race based on regression (14) applies exactly to the horse race based on regression (32) if we ignore the model misspecification term, and we find that this term is indeed empirically small. Nevertheless, we directly run a horse race with Equation (32) to examine whether Proposition 3 holds if we do not ignore the model misspecification term.

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23 We can show that this proposition also obtains if we replace the supposition in Proposition 5 that “the model misspecification term is sufficiently small” with an assumption that average factor betas of funds equal corresponding betas for the market portfolio.
Table 10 reports the slope coefficients of regression (32). As Proposition 5 predicts, the ordering of the slope coefficients in Table 10 is identical to that for regression (14) in Table 9. Therefore, our results are not sensitive to regression specifications.\textsuperscript{24}

5. Model Robustness

Our analytic results have two broad parts. The first shows that rational investors use the most precise alpha to update their priors and inform their investment decisions. We derive this result under the assumptions that fund skills are unobservable but constant and investors know fund betas. This section considers generalizations of these assumptions along various dimensions proposed in the literature.

Roussanov, Ruan, and Wei (2019) assume that fund manager skill follows an AR(1) process while skill is constant in BG. Investors’ posterior in Roussanov, Ruan, and Wei (2019) is also a linear function of their priors and alphas as in Proposition 1, but their weights are different. Proposition 3 does not depend on the weights assigned to alphas and therefore our result that investors use $\hat{\alpha}_{p, t}$ to update priors in a competitive equilibrium applies in this case as well.

Franzoni and Schmalz (2017) consider a model where investors do not know true factor betas but learn about them through funds’ past performance. Investors’ posterior in this model is also a linear function of their priors and alphas as in Equation (7), but the coefficient of alpha includes a term related to the uncertainty about betas. We need to specify investors’ uncertainty about factor betas to determine the exact factor model that investors would use to update their priors. However, whether or not investors use a particular factor to compute alphas depends on the uncertainty about its betas and not the true asset pricing model.

\textsuperscript{24} Table 10 reports slope coefficients of regression (32) when we compute alphas and flows at monthly horizon. The regression slope coefficient does not vary with the horizon over which we compute alphas and model flows, because flows are uncorrelated with lagged- or lead-month alphas.
To examine the effect of investors’ uncertainty about true betas, we run a modified simulation experiment. We let betas follow the AR(1) process as in Equation (28), but we assume that at time $t$ investors observe $\beta_{(1,j),p,t-1}$, but not $\beta_{(1,j),p,t}$. Investors’ optimal estimate of time $t$ factor betas is $(1 - \rho)\overline{\beta}_{(1,j),p} + \rho \beta_{(1,j),p,t-1}$, which they use to compute alphas. We find in untabulated results that none of our conclusions from earlier simulations change.

Koijen (2014) presents a structural model where funds actively manage a time-varying fraction of their AUM and passively index the rest. Fund betas in this model could vary through time if factor betas of the actively managed portion of the fund are different from the passively indexed portion. In our robustness tests we find virtually the same results with time-varying betas and constant betas. Although our robustness tests model beta time variation as an AR(1) process, our results with a wide range of AR(1) coefficients suggest that beta time variation per se is unlikely to qualitatively change our main results.

Fund managers in BG also actively manage a time-varying fraction of their AUM and passively index the rest and funds also set their fees to maximize their revenues. Propositions 1, 2 and 3 depend only on competitive market equilibrium and they do not depend on how funds manage their AUM. Therefore, our result that investors use $\tilde{\alpha}_{p,j,t}$ to inform their investment decisions is not sensitive to BG’s model of funds’ investment decisions.

6. Results in Perspective

BvB, BHO, Agarwal, Green, and Ren (2018), and Blocher and Molyboga (2017) report that single-factor alpha wins their horse race with samples of mutual funds and hedge funds. BvB and some other papers conclude that these results indicate that the CAPM is the true asset pricing model. However, BHO conclude that these results indicate that investors lack sophistication
because they do not use a model with all common factors to estimate alphas to inform their investment decision. Are such inferences tenable?

A fundamental concept in finance is that investors make investment decisions based on their assessment of future risk-adjusted returns. Therefore, it may appear on the surface that one could identify the particular asset pricing model that investors use for risk-adjustment from their investments into and out of mutual funds. While rational investors indeed make decisions based on expected future risk-adjusted performance, we show that they optimally extract information from past returns with alphas orthogonalized to both priced and unpriced factors. Therefore, alphas that investors use do not contain any information to differentiate between priced and unpriced factors.

BvB justify their inferences about asset pricing model based on a proposition built on their assumption that “if a true risk model exists, any false risk model cannot have additional explanatory power” (p. 6) for fund flows. BvB’s Equation (7) presents a mathematical representation of this assumption, which in our notations is

\[
\text{Probability}\left[\Gamma_{p,t} > 0 \mid \hat{\alpha}_{p,K,t} > 0, \hat{\alpha}_{p,K^*,t} > 0\right] = \text{Probability}\left[\Gamma_{p,t} > 0 \mid \hat{\alpha}_{p,K,t} > 0\right],
\]

(33)

where \(\hat{\alpha}_{p,K,t}\) is alpha computed with only the \(K\) priced factors in the true asset pricing model and \(\hat{\alpha}_{p,K^*,t}\) is alpha computed with respect to any other model. Is this assumption tenable? We show that investors optimally use \(\hat{\alpha}_{p,J,t}\) to update their priors about manager skills and hence flows are determined by \(\hat{\alpha}_{p,J,t}\) and not by alpha with respect to the \(K\)-factor asset pricing model. Therefore,

\[\text{BvB’s Equation (7) defines the conditioning variables in Equation (33) as fund net return minus benchmark returns, and BvB’s equation (9) defines benchmark returns for various factor models. The monthly fund return minus the benchmark return in their equation (9) produces the same alphas as our Equation (13).}\]
\(\hat{\alpha}_{p,t} \) and \(\Gamma_{p,t} \) have the same sign and \(\text{Probability}[\Gamma_{p,t} > 0 | \hat{\alpha}_{p,t} > 0] = 1. \) If \( K \neq J, \) that is, if there is at least one unpriced factor, then BvB’s assumption represented by Equation (33) is false. BvB’s assumption holds in a rational expectations economy if and only if all common factors are priced. But such an assumption would predetermine the true asset pricing model and render any asset pricing model test moot.

Our empirical results also indicate that the true asset pricing model has a negligible effect on the outcome of the horse race. For example, the 4-factor model alpha is empirically the most precise estimate when betas are estimated from the data and our empirical decomposition of the components of alpha measurement error indicates that this model would win the horse race under the rational expectations hypothesis even if the true asset pricing model were CAPM or FF3. Our results in a simulated rational expectations economy confirm this result. So the winner does not reveal the true asset pricing model, and there is neither a theoretical nor an empirical justification to use the horse race as a test of asset pricing models.

BHO, citing Grinblatt and Titman (1989) and Pastor and Stambaugh (2002), hypothesize that if investors are sophisticated, then the \( J \)-factor model alpha should win the horse race. BHO use this model alpha as the benchmark for investor sophistication. However, we show that the \( J \)-factor model alpha need not win empiricists’ horse race under the rational expectations hypothesis because empiricists do not know the true asset pricing model and true factor betas. Our empirical results indicate that when empiricists follow the common practice of estimating betas using time-series regressions with 60 months of data, 4-factor model alphas win the horse race under the rational expectations hypothesis, and not 7-factor model alphas. Even with the 4-factor model as
the benchmark, however, BHO’s result that the single-factor model alpha wins the horse race suggests rejection of the investor sophistication hypothesis.\textsuperscript{26, 27}

7. Conclusion

Investors reveal their preferences for mutual funds through investments in or withdrawals from them. In a rational expectations economy, investors update their priors about fund manager skills based on funds’ past performance and make their investment decisions. Because flows reveal the model that investors use to update their priors, recent literature proposes that a comparison of relations between fund flows and alphas computed with different models can be used to test asset pricing models and also to assess investor sophistication. We examine whether these proposals are conceptually and empirically tenable.

To examine the conceptual issues, we build a rational expectations model where investors extract information about mutual fund manager skills from funds’ past performance and optimally decide on fund flows. We show that investors use alphas computed with a multifactor model that includes all priced and unpriced factors to update their priors. Because alphas that determine fund flows are orthogonal to both priced and unpriced factors, flows do not contain any information to differentiate between factors that are priced under the true asset pricing model and unpriced factors.

\textsuperscript{26} Mathematically, BHO’s findings indicate that $\text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,1,t})$, the numerator in Equation (15), is bigger than the covariance between flow and alphas computed with more than one factor, because empirically $\sigma_{\hat{\alpha}_{p,1,t}}^2 < \sigma_{\hat{\alpha}_{p,1,t}}^2$ for $\eta > 1$. If investors use $\hat{\alpha}_{p,1,t}$ to inform their investment decisions, they conflate abnormal performance due to skill with that due to omitted factor realization. Therefore, $\text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,1,t}) = \text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,1,t} + \sum_{k=2}^{n} \beta_{p,k} f_k) > \text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,1,t})$.

\textsuperscript{27} Ben-David et al. (2019) report that investors rely on recent fund returns and Morningstar ratings, and not on market model alphas. Ben-David et al. (2019) suggest that investors outsource risk assessment to Morningstar.
We then analyze the flow-alpha horse race that the literature runs under the hypothesis that flows are generated in a rational expectations economy. Unlike investors in the model economy, empiricists who run the horse race do not know the true asset pricing model and true betas. We show that the most precise alphas based on empiricists’ information set will win the flow-alpha horse race under the rational expectations hypothesis.

We empirically examine the precision of alphas computed with various models with a sample of actively managed mutual funds. Our empirical tests use BHO’s 7-factor model with Fama-French factors (market, SMB, HML), momentum factor (UMD), and three industry factors. We compute the precision of alphas under the hypothesis that each of the following asset pricing models is true: (a) none of the risk factors are priced (or true expected returns are unrelated to factors betas), (b) CAPM, (c) Fama-French 3-factor model, or (d) Fama-French-Carhart 4-factor model.

We find that alphas with a 4-factor model that excludes the three industry factors are the most precise regardless of the true asset pricing model. Therefore, our model implies that a 4-factor alpha will always win the horse race if flows are determined in a rational expectations economy. We also conduct a simulation experiment with parameters that match the data. We generate fund flows in the simulation according to our model under each candidate asset pricing model and test our predictions. We find that 4-factor alphas are the most precise in our simulations as well, and they always win the flow-alpha horse race regardless of the true asset pricing model.

Our findings show that the winner of the flow-alpha horse race cannot be used to identify the true asset pricing model. In contrast, BvB present a model to justify their inference that CAPM is the best asset pricing model based on their evidence that the market model alpha wins the horse
race. We show that a faulty foundational assumption in BvB’s model is the source of their mistaken inference.

Our findings also contradict BHO’s hypothesis that alphas computed with all seven factors will win the horse race under the investor sophistication hypothesis, or equivalently, the rational expectations hypothesis. We show that 4-factor alphas are more precise than the 7-factor alphas because of estimation errors in industry betas. Even with the 4-factor model as the benchmark, however, BHO’s finding that the single-factor model alpha wins the horse race suggests rejection of the investor sophistication hypothesis.
References


Table 1
Summary statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>SD</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of funds each month</td>
<td>1,224</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Flow (%)</td>
<td>0.25</td>
<td>10.8</td>
<td>-0.42</td>
</tr>
<tr>
<td>TNAs ($ mn)</td>
<td>1,120</td>
<td>4,507.4</td>
<td>223.6</td>
</tr>
<tr>
<td>Age (months)</td>
<td>376.8</td>
<td>306.6</td>
<td>299.2</td>
</tr>
<tr>
<td>Expense ratio (%)</td>
<td>1.22</td>
<td>0.45</td>
<td>1.19</td>
</tr>
<tr>
<td>Load dummy</td>
<td>0.49</td>
<td>0.50</td>
<td>0</td>
</tr>
<tr>
<td>Return volatility ($t-1, t-12)$</td>
<td>4.7</td>
<td>2.3</td>
<td>4.2</td>
</tr>
</tbody>
</table>

This table presents the summary statistics for the sample of actively managed domestic equity funds used in this study. The number of fund-month observations is 404,042. We compute the respective statistics across funds each month and report the averages over the entire sample period. The sample period is from January 1990 to June 2017.
Table 2  
Factor model $R^2$

<table>
<thead>
<tr>
<th>Estimated factor model (η):</th>
<th>From OLS</th>
<th>Adj. $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market adj. return</td>
<td>.820</td>
<td>.761</td>
</tr>
<tr>
<td>Market model</td>
<td>.829</td>
<td></td>
</tr>
<tr>
<td>FF3</td>
<td>.892</td>
<td>.829</td>
</tr>
<tr>
<td>FFC4</td>
<td>.901</td>
<td>.883</td>
</tr>
<tr>
<td>FFC4 + 3 IND</td>
<td>.910</td>
<td>.883</td>
</tr>
</tbody>
</table>

This table fits the following regression:

$$(r_{p,t} - r_{f,t}) = \alpha_{p,\eta,t} + \sum_{k=1}^{\eta} \beta_{k,p} F_{k,t} + e_{p,\eta,t},$$

where $r_{p,t}$, $r_{f,t}$, and $F_{k,t}$ are fund return, the risk-free rate, and the realization of factor $k$ in month $\tau$, respectively. For each fund $p$ and month $t$, the regression is fitted for various $\eta$-factor models from $\tau = t - 60$ to $t - 1$ using an OLS regression. With these estimates, we compute the abnormal return for fund $p$ in month $t$ under each $\eta$-factor model as $\hat{\alpha}_{p,\eta,t} = r_{p,t} - \sum_{k=1}^{\eta} \hat{\beta}_{k,p} F_{k,t}$. Column 1 reports the cross-sectional averages of time-series means of adjusted $R^2$ from the OLS regressions. Column 2 reports the cross-sectional averages of time-series means of monthly adjusted $R^2$ computed using the formula $1 - \left[ \text{Var}(\hat{\alpha}_{p,\eta}) \times (T_p - 1) / \text{Var}(r_p) \times (T_p - \eta - 1) \right]$, where $T_p$ is the number of months the fund is in the sample. The sample period is from January 1990 to June 2017.
### Table 3
### Measurement errors in betas

<table>
<thead>
<tr>
<th>Betas</th>
<th>Average $b_t$</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market</td>
<td>0.656***</td>
<td>0.06</td>
</tr>
<tr>
<td>SMB</td>
<td>0.894***</td>
<td>0.02</td>
</tr>
<tr>
<td>HML</td>
<td>0.720***</td>
<td>0.05</td>
</tr>
<tr>
<td>UMD</td>
<td>0.523***</td>
<td>0.06</td>
</tr>
<tr>
<td>IND1</td>
<td>0.403***</td>
<td>0.07</td>
</tr>
<tr>
<td>IND2</td>
<td>0.315***</td>
<td>0.05</td>
</tr>
<tr>
<td>IND3</td>
<td>0.270***</td>
<td>0.04</td>
</tr>
</tbody>
</table>

This table reports the slope coefficients from the following cross-sectional regressions:

$$\hat{\beta}_{p,k,t}^{\text{future}} = a_t + b_t \times \hat{\beta}_{p,k,t}^{\text{past}} + e_{p,t},$$

where for each fund $p$, $\hat{\beta}_{p,k,t}^{\text{future}}$, and $\hat{\beta}_{p,k,t}^{\text{past}}$ are estimated using time-series regressions with data from $t$ to $t + 11$ and $t - 1$ to $t - 60$, respectively. Backward-looking and forward-looking betas are estimated monthly using multiple regressions of fund returns on the seven factors. The above regression is then fitted each month for betas with respect to each factor and the table reports time-series averages of the slope coefficients. Standard errors from the second stage of Fama-MacBeth regressions are adjusted for serial correlation using Newey-West correction with lag length of 11 months. The sample period for these regressions is from January 1990 to July 2016. *$p < .1$; **$p < .05$; ***$p < .01$. 

Table 4
Variance of empiricist alpha estimates

<table>
<thead>
<tr>
<th>Alpha estimated using (η):</th>
<th>$\sigma^2_{\hat{a}_\eta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mkt adj. ret.</td>
<td>650.5</td>
</tr>
<tr>
<td>Market model</td>
<td>544.4</td>
</tr>
<tr>
<td>FF3</td>
<td>370.9</td>
</tr>
<tr>
<td>FFC4</td>
<td>357.7</td>
</tr>
<tr>
<td>FFC4+3 IND</td>
<td>363.2</td>
</tr>
</tbody>
</table>

This table reports variance of empiricist’s alpha estimates from different $\eta$-factor models. We use $\hat{\beta}$ estimates from the time series regression (19) to compute $\hat{a}_\eta^E$ each month and its cross-sectional variance. We average the monthly estimates over time and report the value multiplied by $10^6$. The sample period is from January 1990 to June 2017.
### Table 5
Summary statistics on factors and factor betas

<table>
<thead>
<tr>
<th>Factor</th>
<th>Mean (%)</th>
<th>SD (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mktrf</td>
<td>.64</td>
<td>4.26</td>
</tr>
<tr>
<td>SMB</td>
<td>.15</td>
<td>3.22</td>
</tr>
<tr>
<td>HML</td>
<td>.21</td>
<td>3.01</td>
</tr>
<tr>
<td>UMD</td>
<td>.51</td>
<td>4.82</td>
</tr>
</tbody>
</table>

### A. Factor statistics

### B. Covariance matrix of true betas

<table>
<thead>
<tr>
<th></th>
<th>$\beta_{mktrf}$</th>
<th>$\beta_{smb}$</th>
<th>$\beta_{hml}$</th>
<th>$\beta_{umd}$</th>
<th>$\beta_{ind1}$</th>
<th>$\beta_{ind2}$</th>
<th>$\beta_{ind3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{mktrf}$</td>
<td>0.023</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{smb}$</td>
<td>0.016</td>
<td>0.114</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{hml}$</td>
<td>-0.018</td>
<td>-0.003</td>
<td>0.080</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{umd}$</td>
<td>0.006</td>
<td>0.008</td>
<td>-0.020</td>
<td>0.017</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{ind1}$</td>
<td>0.001</td>
<td>0.001</td>
<td>-0.002</td>
<td>0.001</td>
<td>0.002</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{ind2}$</td>
<td>0.000</td>
<td>0.001</td>
<td>0.003</td>
<td>0.000</td>
<td>0.000</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>$\beta_{ind3}$</td>
<td>-0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Panel A of this table reports the sample mean and standard deviations of Market, SMB, HML, and UMD factors. Panel B presents the covariance matrix of “true” factor betas. We estimate the true beta covariance matrix from OLS beta covariance matrix using Equation (25). IND1, IND2, and IND3 are industry factors that are the first three principal components of regression residuals of Fama-French 17 equally weighted industry portfolios regressed on FFC4 factors. The table reports the average of monthly estimates. The sample period is from January 1990 to June 2017.
### Table 6
Components of alpha estimation error in the mutual fund sample

<table>
<thead>
<tr>
<th>True asset pricing model (K):</th>
<th>Alpha estimated using (η):</th>
<th>$\sigma^2_{\hat{\alpha}_1}$</th>
<th>APM misspecification error variance</th>
<th>Omitted factors variance</th>
<th>Covariance</th>
<th>Beta measurement error variance</th>
<th>Variance in addition to $\sigma^2_{\hat{\alpha}_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NBRP</td>
<td>Mkt adj. ret.</td>
<td>309.7</td>
<td>0</td>
<td>340.9</td>
<td>0</td>
<td>0</td>
<td>340.9</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>309.7</td>
<td>1.94</td>
<td>222.5</td>
<td>-0.76</td>
<td>11.0</td>
<td>223.7</td>
</tr>
<tr>
<td></td>
<td>FF3</td>
<td>309.7</td>
<td>1.44</td>
<td>38.4</td>
<td>0.00</td>
<td>21.4</td>
<td>39.9</td>
</tr>
<tr>
<td></td>
<td>FFC4</td>
<td>309.7</td>
<td>1.85</td>
<td>15.7</td>
<td>0.07</td>
<td>30.4</td>
<td>17.6</td>
</tr>
<tr>
<td></td>
<td>FFC4+3 IND</td>
<td>309.7</td>
<td>1.94</td>
<td>0</td>
<td>0</td>
<td>51.6</td>
<td>1.9</td>
</tr>
<tr>
<td></td>
<td>Mkt adj. ret.</td>
<td>310.8</td>
<td>1.94</td>
<td>340.9</td>
<td>-3.12</td>
<td>0</td>
<td>339.7</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>310.8</td>
<td>0</td>
<td>222.5</td>
<td>0</td>
<td>11.0</td>
<td>222.5</td>
</tr>
<tr>
<td></td>
<td>FF3</td>
<td>310.8</td>
<td>0.64</td>
<td>38.4</td>
<td>-0.38</td>
<td>21.4</td>
<td>38.7</td>
</tr>
<tr>
<td></td>
<td>FFC4</td>
<td>310.8</td>
<td>0.77</td>
<td>15.7</td>
<td>-0.03</td>
<td>30.4</td>
<td>16.4</td>
</tr>
<tr>
<td></td>
<td>FFC4+3 IND</td>
<td>310.8</td>
<td>0.76</td>
<td>0</td>
<td>0</td>
<td>51.6</td>
<td>0.8</td>
</tr>
<tr>
<td>FF3</td>
<td>Mkt adj. ret.</td>
<td>311.1</td>
<td>1.44</td>
<td>340.9</td>
<td>-2.87</td>
<td>0</td>
<td>339.4</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>311.1</td>
<td>0.64</td>
<td>222.5</td>
<td>-0.90</td>
<td>11.0</td>
<td>222.3</td>
</tr>
<tr>
<td></td>
<td>FF3</td>
<td>311.1</td>
<td>0</td>
<td>38.4</td>
<td>0</td>
<td>21.4</td>
<td>38.4</td>
</tr>
<tr>
<td></td>
<td>FFC4</td>
<td>311.1</td>
<td>0.44</td>
<td>15.7</td>
<td>0.04</td>
<td>30.4</td>
<td>16.2</td>
</tr>
<tr>
<td></td>
<td>FFC4+3 IND</td>
<td>311.1</td>
<td>0.50</td>
<td>0</td>
<td>0</td>
<td>51.6</td>
<td>0.5</td>
</tr>
<tr>
<td>FFC4</td>
<td>Mkt adj. ret.</td>
<td>311.6</td>
<td>1.85</td>
<td>340.9</td>
<td>-3.77</td>
<td>0</td>
<td>338.9</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>311.6</td>
<td>0.77</td>
<td>222.5</td>
<td>-1.50</td>
<td>11.0</td>
<td>221.8</td>
</tr>
<tr>
<td></td>
<td>FF3</td>
<td>311.6</td>
<td>0.44</td>
<td>38.4</td>
<td>-0.92</td>
<td>21.4</td>
<td>38.0</td>
</tr>
<tr>
<td></td>
<td>FFC4</td>
<td>311.6</td>
<td>0</td>
<td>15.7</td>
<td>0</td>
<td>30.4</td>
<td>15.7</td>
</tr>
</tbody>
</table>

52
The table presents the components of cross-sectional variance of multifactor model alpha estimates, decomposed as in Equation (24) for various combinations of hypothesized true asset pricing models ($K = 0, 1, 3, 4$) and multifactor models. Alphas are estimated as the Mkt adj. return, which is fund return minus market returns, and the other alphas are estimated using the indicated models. The hypothesized true asset pricing models are No-beta risk premium model (NBRP) where none of the common factors are priced factors, CAPM, FF3, and FFC4. The column $\sigma_{\hat{\alpha}}^2$ presents the cross-sectional variance of alphas estimated under the assumption that the true asset pricing model and true betas are known. The other columns present the variance as follows: (a) APM misspecification: unpriced factors used to compute alphas; (b) “omitted factors”: common factors excluded from the computation of alphas; (c) “covariance”: covariance of betas on excluded unpriced factors and betas of included priced factors; and (d) “beta measurement error”: factor beta estimation errors. The sample comprises actively managed equity funds from January 1990 to June 2017.

<table>
<thead>
<tr>
<th>FFC4+3 IND</th>
<th>311.6</th>
<th>0.02</th>
<th>0</th>
<th>0</th>
<th>51.6</th>
<th>0.0</th>
<th>51.7</th>
</tr>
</thead>
</table>

This table presents the components of cross-sectional variance of multifactor model alpha estimates, decomposed as in Equation (24) for various combinations of hypothesized true asset pricing models ($K = 0, 1, 3, 4$) and multifactor models. Alphas are estimated as the Mkt adj. return, which is fund return minus market returns, and the other alphas are estimated using the indicated models. The hypothesized true asset pricing models are No-beta risk premium model (NBRP) where none of the common factors are priced factors, CAPM, FF3, and FFC4. The column $\sigma_{\hat{\alpha}}^2$ presents the cross-sectional variance of alphas estimated under the assumption that the true asset pricing model and true betas are known. The other columns present the variance as follows: (a) APM misspecification: unpriced factors used to compute alphas; (b) “omitted factors”: common factors excluded from the computation of alphas; (c) “covariance”: covariance of betas on excluded unpriced factors and betas of included priced factors; and (d) “beta measurement error”: factor beta estimation errors. The sample comprises actively managed equity funds from January 1990 to June 2017.
We generate net returns each month using the following 7-factor model:

\[
r_{p,t} = \Phi_p - c_{t-1}(q_{t-1}) + E^{model}(r_{p,t}) + \beta_{p,m}(\tilde{mkt} - \tilde{rf})_t + \\
\beta_{p,smb}\tilde{SMB}_t + \beta_{p,hml}\tilde{HML}_t + \beta_{p,umd}\tilde{UMD}_t + \beta_{p,ind1}\tilde{IND1}_t + \\
\beta_{p,ind2}\tilde{IND2}_t + \beta_{p,ind3}\tilde{IND3}_t + \epsilon_{p,t},
\]

where \(\Phi_p\) is the fund manager skill and \(c_{t-1}(q_{t-1})\) is the cost per unit size. The variables under \(\tilde{\cdot}\) are demeaned realizations of the seven common factors, and \(\beta_s\) are the corresponding factor sensitivities. We use the factor realizations in the data over January 1990 to June 2017 sample period in our simulations. We draw 7-factor betas for each fund from a multivariate Normal distribution as \(\beta_{7 \times 1} \sim MVN([1,0,0,0,0,0,0]', \Omega)\) with covariance matrix \(\Omega\) reported in panel B of Table 5. We generate monthly flow according to Equation (12). We draw all random variables from normal distributions with means and variances shown above.
Table 8
Components of alpha estimation error in the simulated sample

<table>
<thead>
<tr>
<th>True asset pricing model (K):</th>
<th>Alpha estimated using ( \eta ):</th>
<th>( \sigma^2_{\hat{\alpha}} )</th>
<th>APM misspecification error variance</th>
<th>Omitted factors variance</th>
<th>Covariance</th>
<th>Beta measurement error variance</th>
<th>Variance in addition to ( \sigma^2_{\hat{\alpha}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mkt adj. ret.</td>
<td>307.8</td>
<td>0</td>
<td>0</td>
<td>324.1</td>
<td>0</td>
<td>324.1</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>307.8</td>
<td>0.931</td>
<td>-0.032</td>
<td>234.3</td>
<td>32.4</td>
<td>235.2</td>
</tr>
<tr>
<td>NBRP</td>
<td>FF3</td>
<td>307.8</td>
<td>1.334</td>
<td>-0.018</td>
<td>50.4</td>
<td>33.5</td>
<td>51.7</td>
</tr>
<tr>
<td></td>
<td>FFC4</td>
<td>307.8</td>
<td>1.842</td>
<td>-0.010</td>
<td>18.2</td>
<td>36.8</td>
<td>20.0</td>
</tr>
<tr>
<td></td>
<td>FFC4+3 IND</td>
<td>307.8</td>
<td>1.945</td>
<td>0</td>
<td>0</td>
<td>56.7</td>
<td>1.94</td>
</tr>
<tr>
<td></td>
<td>Mkt adj. ret.</td>
<td>307.8</td>
<td>0.931</td>
<td>0.008</td>
<td>324.1</td>
<td>0</td>
<td>325.1</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>307.8</td>
<td>0</td>
<td>0</td>
<td>234.3</td>
<td>32.4</td>
<td>234.3</td>
</tr>
<tr>
<td>CAPM</td>
<td>FF3</td>
<td>307.8</td>
<td>0.579</td>
<td>0.193</td>
<td>50.4</td>
<td>33.5</td>
<td>51.2</td>
</tr>
<tr>
<td></td>
<td>FFC4</td>
<td>307.8</td>
<td>0.715</td>
<td>-0.001</td>
<td>18.2</td>
<td>36.8</td>
<td>18.9</td>
</tr>
<tr>
<td></td>
<td>FFC4+3 IND</td>
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<td>0</td>
<td>56.7</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>Mkt adj. ret.</td>
<td>307.8</td>
<td>1.334</td>
<td>0.014</td>
<td>324.1</td>
<td>0</td>
<td>325.5</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>307.8</td>
<td>0.579</td>
<td>-0.383</td>
<td>234.3</td>
<td>32.4</td>
<td>234.5</td>
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<tr>
<td>FF3</td>
<td>FF3</td>
<td>307.8</td>
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<td>0</td>
<td>50.4</td>
<td>33.5</td>
<td>50.4</td>
</tr>
<tr>
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<td>FFC4</td>
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<td>0.009</td>
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<td>18.6</td>
</tr>
<tr>
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<td>FFC4+3 IND</td>
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<td>0.504</td>
<td>0</td>
<td>0</td>
<td>56.7</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>Mkt adj. ret.</td>
<td>307.8</td>
<td>1.842</td>
<td>0.008</td>
<td>324.1</td>
<td>0</td>
<td>326.0</td>
</tr>
<tr>
<td></td>
<td>Market model</td>
<td>307.8</td>
<td>0.715</td>
<td>-0.033</td>
<td>234.3</td>
<td>32.4</td>
<td>235.0</td>
</tr>
<tr>
<td>FFC4</td>
<td>FF3</td>
<td>307.8</td>
<td>0.435</td>
<td>0.449</td>
<td>50.4</td>
<td>33.5</td>
<td>51.3</td>
</tr>
<tr>
<td></td>
<td>FFC4</td>
<td>307.8</td>
<td>0</td>
<td>0</td>
<td>18.2</td>
<td>36.8</td>
<td>18.2</td>
</tr>
</tbody>
</table>
This table presents the components of cross-sectional variance of multifactor model alpha estimates, decomposed as in Equation (24) for various combinations of hypothesized true asset pricing models \((K = 0, 1, 3, 4)\) and multifactor models in the simulated sample. Alphas are estimated as the Mkt adj. return, which is fund return minus market returns, and the other alphas are estimated using the indicated models. The hypothesized true asset pricing models are a model with no beta risk premium for any factors (NBRP), CAPM, FF3, and FFC4. The column \(\sigma_{\hat{a}}^2\) presents the cross-sectional variance of alphas estimated under the assumption that the true asset pricing model and true betas are known. The other columns present the variance as follows: (a) APM misspecification: unpriced factors used to compute alphas; (b) “omitted factors”: common factors excluded from the computation of alphas; (c) “covariance”: Covariance of betas on excluded unpriced factors and betas of included priced factors; and (d) “beta measurement error”: factor beta estimation errors. The results are based on 500 simulations.
### Table 9
Flow-performance relation in simulated sample

<table>
<thead>
<tr>
<th>True asset pricing model (K):</th>
<th>A. True betas used to estimate alphas</th>
<th>B. 60-month rolling window $\beta$s used to estimate alphas</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NBRP</td>
<td>CAPM</td>
</tr>
<tr>
<td>Alpha estimated using ($\eta$):</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Market adjusted ret</td>
<td>2.21</td>
<td>2.19</td>
</tr>
<tr>
<td>Market model</td>
<td>2.36</td>
<td>2.36</td>
</tr>
<tr>
<td>FF3</td>
<td>3.07</td>
<td>3.07</td>
</tr>
<tr>
<td>FFC4</td>
<td>3.27</td>
<td>3.28</td>
</tr>
<tr>
<td>FFC4+3 IND</td>
<td>3.48</td>
<td>3.50</td>
</tr>
</tbody>
</table>

This table presents the estimates of the slope coefficients of flow-alpha regression (14) in simulations with returns generated under the following models for expected returns: a model with no beta risk premium for any factors (NBRP), CAPM, FF3, and FFC4 models. The column headings identify the expected returns model. Alphas are computed with respect to the models indicated in the first column. Monthly flow is determined by the model as specified by Equation (12). Panel A presents the results using true betas to compute alphas, and panel B presents the results with factor betas estimated from the data. The table reports average coefficients across 500 repetitions of the simulations.
Table 10
Flow-performance relation with sign regressions in simulated sample

<table>
<thead>
<tr>
<th>True asset pricing model (K):</th>
<th>A. True betas used to estimate alphas</th>
<th>B. 60-month rolling window βs used to estimate alphas</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NBRP</td>
<td>CAPM</td>
</tr>
<tr>
<td>Alpha estimated using (η):</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Market adjusted ret</td>
<td>0.594</td>
<td>0.591</td>
</tr>
<tr>
<td>Market model</td>
<td>0.626</td>
<td>0.626</td>
</tr>
<tr>
<td>FF3</td>
<td>0.789</td>
<td>0.792</td>
</tr>
<tr>
<td>FFC4</td>
<td>0.846</td>
<td>0.851</td>
</tr>
<tr>
<td>FFC4+3 IND</td>
<td>0.950</td>
<td>0.968</td>
</tr>
</tbody>
</table>

This table presents the estimates of the slope coefficients of flow-alpha regression (32) in simulations with returns generated under the following models for expected returns: a model with no beta risk premium for any factors (NBRP), CAPM, FF3, and FFC4 models. The column headings identify the expected returns models. Alphas are computed with respect to the models indicated in the first column. Monthly flow is determined by the model as specified by Equation (12). Flow and alpha are assigned values of +1 when positive and -1 when negative. Panel A presents the results using true betas to compute alphas, and panel B presents the results with factor betas estimated from the data. The table reports average coefficients across 500 repetitions of the simulations.
Appendix A

This appendix presents the proofs of Propositions 1 and 2.

Proof of Proposition 1

At time \( t \) investors observe the history of net returns and fund size \( \{r_s, q_s\}_{s=0}^t \) over the life of each fund. Let \( \text{Age}_{p,t} \) denote the age of a fund as of time \( t \) and \( c_t(q) \) denote the cost per unit size of the fund. Because \( c_t(q) \) is in the investors’ information set, they back out the history of gross returns on each fund \( \{R_s\}_{s=0}^t \). Investors’ prior on managerial skill at \( t = 0 \) is given by \( N(\phi_0, 1/\nu) \). They use the gross returns history up to the end of period \( t \) and update their prior on managerial skill using an \( \eta \)-factor model benchmark to compute abnormal returns. Let \( X_{p,\eta,t} \) denote the benchmark adjusted gross returns for fund \( p \) at time \( t \), that is, \( X_{p,\eta,t} = \bar{a}_{p, \eta,t} + c_{t-1}(q_{p,t-1}) \), and \( \bar{X}_{p,\eta,t} \) denotes its sample mean over \( t \) periods. Note that, for each fund, the sample size of return history as of time \( t \) is equal to the age of the fund as of \( t \) (\( \text{Age}_{p,t} \)).

Under the assumption that all returns are all normally distributed, using theorem 1 of DeGroot (1970, p. 167), the posterior distribution is normal with mean \( \phi_{p,\eta,t} \) given by

\[
\phi_{p,\eta,t} = \frac{\nu \phi_0 + t \theta_{\bar{a},\eta} \bar{X}_{p,\eta,t}}{\nu + t \theta_{\bar{a},\eta}},
\]

and precision given by \( (\nu + t \theta_{\bar{a},\eta}) \), where \( \theta_{\bar{a},\eta} = \frac{1}{\sigma_{\bar{a},\eta}^2} \).

Equation (A.1) follows DeGroot and specifies investors’ cumulative update from time 0 to \( t \). This result, in conjunction with the competitive equilibrium condition yields the recursive update described by Equation (7) in Proposition 1. Under the competitive equilibrium condition in Equation (3):
\[ c_t(q_{p,t}) = E_t(\Phi_p) = \phi_{p,t} \]  \hspace{1cm} (A.2)

Rewriting \( t\bar{X}_{p,\eta,t} \) as \( (t-1)\bar{X}_{p,\eta,t-1} + X_{p,\eta,t} \) which can further be written as \( (t-1)\bar{X}_{p,\eta,t-1} + \bar{a}_{p,\eta,t} + c_{t-1}(q_{p,t-1}) \), Equation (A.1) becomes

\[
\phi_{p,\eta,t} = \frac{v \phi_0 + (t-1)\vartheta_{\tilde{a},\eta} \bar{X}_{p,\eta,t-1} + \vartheta_{\tilde{a},\eta} c_{t-1}(q_{p,t-1})}{v + t\vartheta_{\tilde{a},\eta}} + \frac{\vartheta_{\tilde{a},\eta}}{v + t\vartheta_{\tilde{a},\eta}} \bar{a}_{p,\eta,t} . \hspace{1cm} (A.3)
\]

From (A.2), the competitive equilibrium condition for period \( t - 1 \) will be \( c_{t-1}(q_{p,t-1}) = \phi_{p,\eta,t-1} \) and from (A.1), \( \phi_{p,\eta,t-1} = \frac{v \phi_0 + (t-1)\vartheta_{\tilde{a},\eta} \bar{X}_{p,\eta,t-1}}{v + (t-1)\vartheta_{\tilde{a},\eta}} \). Substituting these two results in (A.3) gives

\[
\phi_{p,\eta,t} = \frac{[v + (t-1)\vartheta_{\tilde{a},\eta}] \phi_{p,\eta,t-1} + \vartheta_{\tilde{a},\eta} \phi_{p,\eta,t-1}}{v + t\vartheta_{\tilde{a},\eta}} + \frac{\vartheta_{\tilde{a},\eta}}{v + t\vartheta_{\tilde{a},\eta}} \bar{a}_{p,\eta,t}.
\]

Further simplification yields

\[
\phi_{p,\eta,t} = \phi_{p,\eta,t-1} + \frac{\vartheta_{\tilde{a},\eta}}{v + t\vartheta_{\tilde{a},\eta}} \bar{a}_{p,\eta,t} . \hspace{1cm} (A.4)
\]

Finally, substituting the age of the fund as of time \( t \) \( (Age_{p,t}) \) for the sample size at \( t \), Equation (A.4) becomes

\[
\phi_{p,\eta,t} = \phi_{p,\eta,t-1} + \frac{\vartheta_{\tilde{a},\eta}}{v + Age_{p,t} \times \vartheta_{\tilde{a},\eta}} \bar{a}_{p,\eta,t} . \hspace{1cm} (A.5)
\]

which is the result in Proposition 1.

Note that Proposition 1 implies that the precision of the posterior after each \( t \) (which represents the prior for \( t + 1 \)) is \( v + Age_{p,t} \times \vartheta_{\tilde{a},\eta} \) which differs across \( \eta \). Therefore, although \( 1/v \) is the precision at \( t = 0 \), precision of priors differs across \( \eta \) for \( t > 0 \).
Using the competitive equilibrium condition in (A.2), we can also write a recursive relation for the cost function as

\[ c_t(q_{p,t}) = c_{t-1}(q_{p,t-1}) + \frac{\partial \hat{a}_t}{v + Ae_{p,t}} \hat{a}_{p,n,t}. \]  

(A.6)

Proof of Proposition 2

For part (a) of the proposition, to prove the “if” condition, substitute Equations (4) and (5) in Equation (7) to get

\[ E_t[\phi_{p,j,t} - \phi_{p,j,t-1}|f_{k,t}, \forall k, \tau \leq t] = \frac{\partial \hat{a}_j}{v + t \hat{a}_j} \times E_t[\xi_{p,t}|f_{k,t}, \forall k, \tau \leq t] = 0. \]

So if investors start with an unbiased prior that skill equals \( \phi_0 \) \( \forall p \) at time 0, then their posterior is unbiased in every subsequent period. To prove the “only if” condition, suppose the contrapositive

\[ E_t[\phi_{p,\eta,t}|f_{k,t}, \forall k, \tau \leq t] = \Phi_p \forall t \] is true for some \( \eta < j \). Consider a fund \( p \) with \( \beta_{p,k^*} \neq 0 \) for some \( k^* > \eta \). Substituting Equations (4) and (5) in Equation (7) we get

\[ E_t[\phi_{p,\eta,t} - \phi_{p,\eta,t-1}|f_{k,t}, \forall k, \tau \leq t] = \frac{\partial \hat{a}_\eta}{v + Ae_{t} \times \hat{a}_\eta} \times E_t \left[ \sum_{k=\eta+1}^{J} \beta_{p,k} f_{k,t} + \xi_{p,t}|f_{k,t}, \forall k, \tau \leq t \right] \neq 0. \]  

(A.7)

Therefore, if \( \phi_{p,\eta,t-1} \) is an unbiased estimate of \( \Phi_p \) then \( \phi_{p,\eta,t} \) is not an unbiased estimate because \( \beta_{p,k^*} f_{k^*,t} \neq 0 \), which leads to a contradiction of the contrapositive.

For part (b): From Equations (4) and (5), we get \( \hat{a}_{p,\eta,t} = \hat{a}_{p,j,t} + \sum_{k=\eta+1}^{j} \beta_{p,k} f_{k,t} \). Therefore,

\[ Var(\hat{a}_{p,\eta,t}) = Var(\hat{a}_{p,j,t}) + \beta'_{(\eta+1,j)} E[f_{(\eta+1,j)}] \beta_{(\eta+1,j)} \] where \( \beta_{(\eta+1,j)} \) and \( f_{(\eta+1,j)} \) are
vectors of betas and factors from $\eta + 1$ to $J$. Because no factor is redundant, $E[f_{(\eta+1,J)}']f_{(\eta+1,J)}]$ is positive definite and therefore $\beta_{(\eta+1,J)}' E[f_{(\eta+1,J)}']f_{(\eta+1,J)}] \beta_{(\eta+1,J)} > 0$ for any nonzero vector $\beta_{(\eta+1,J)}$. Therefore, $\text{Var}(\hat{\alpha}_{p,\eta,t}) > \text{Var}(\hat{\alpha}_{p,J,t})$. 
Appendix B

This appendix derives the result presented in Equation (18) for the covariance of flows with the empiricist’s alpha estimate. The relation between the empiricist’s estimate of alpha from an \( \eta \)-factor model and investors’ estimate of alpha from \( J \)-factor model is given by Equation (16). We rewrite this equation as \( \hat{\alpha}_{p,\eta,t}^E = \hat{\alpha}_{f,t} + v_{p,\eta,t} \), where \( v_{p,\eta,t} \) denotes the remaining terms on the RHS of Equation (16). Because \( \hat{\alpha}_{f,t} \) is uncorrelated with each individual term in \( v_{p,\eta,t} \), \( \text{Cov}(\hat{\alpha}_{f,t}, v_{p,\eta,t}) = 0 \). Rewrite Equation (12) as

\[
\Gamma_{p,t} = \mathcal{K}_{p,t} \times (1 + r_{p,t}) \times \hat{\alpha}_{p,f,t},
\]

where \( \mathcal{K}_{p,t} = \frac{\theta_{a,f}}{\nu + A e_{p,t} \times \theta_{a,f}} \times \frac{1}{\delta_{t-1} q_{p,t-1}} \).

With these notations:

\[
\text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,\eta,t}^E) = \text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,f,t}) + \text{Cov}(\mathcal{K}_{p,t} \hat{\alpha}_{p,f,t} (1 + r_{p,t}), v_{p,\eta,t}).
\]  

(B.1)

Because \( \mathcal{K}_{p,t} \) is a deterministic function of time, the second term on the right-hand side (RHS) becomes \( \overline{\mathcal{K}}_t \text{Cov}(\hat{\alpha}_{p,f,t} (1 + r_{p,t}), v_{p,\eta,t}) \), where the bar above represents the cross-sectional average. The covariance term can be equivalently written as \( \text{Cov}(\hat{\alpha}_{p,f,t} (1 + r_{p,t}), \bar{v}_{p,\eta,t}) \), where \( \bar{v}_{p,\eta,t} = v_{p,\eta,t} - \bar{v}_{\eta,t} \) and \( \bar{v}_{\eta,t} \) is the mean.

We can evaluate the covariance using the following identity:

\[
\text{Cov}(ab, c) = \text{Cov}(a, bc) - \text{Cov}(a, b)E(c) + E(a)\text{Cov}(b, c).
\]  

(B.2)

With \( a = \hat{\alpha}_{p,f,t}, b = (1 + r_{p,t}), c = \bar{v}_{p,\eta,t} \) and using \( (\hat{\alpha}_{p,f,t}) = 0, \text{Cov}(\hat{\alpha}_{p,f,t}, \bar{v}_{p,\eta,t}) = 0, \)

\( E(\bar{v}_{p,\eta,t}) = 0 \), we obtain

\[
\text{Cov}(\hat{\alpha}_{p,f,t} (1 + r_{p,t}), \bar{v}_{p,\eta,t}) = \text{Cov}(\hat{\alpha}_{f,t}, r_{p,t} \bar{v}_{p,\eta,t}).
\]  

(B.3)

From Equations (4) and (5), we obtain

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\[ r_{p,t} = \hat{\alpha}_{j,t} + \sum_{k=1}^{K} \beta_{k,p} E[F_{k,t}] + \sum_{k=1}^{J} \beta_{k,p} f_{k,t} \equiv \hat{\alpha}_{j,t} + \theta_{p,t}, \quad (B.4) \]

where \( \hat{\alpha}_{j,t} \) and \( \theta_{p,t} \) are independent.

From Equations (B.3) and (B.4), we get

\[ \text{Cov}(\hat{\alpha}_{j,t} (1 + r_{p,t}), \tilde{\nu}_{p,\eta,t}) = \text{Cov}(\hat{\alpha}_{j,t}, \tilde{\alpha}_{j,t} \tilde{\nu}_{p,\eta,t}) + \text{Cov}(\hat{\alpha}_{j,t}, \theta_{p,t} \tilde{\nu}_{p,\eta,t}). \quad (B.5) \]

Because \( \hat{\alpha}_{j,t} \) is independent of all factors and factor betas, the second term on the RHS of Equation (B.5) equals zero and the first term equals \( \text{Var}(\hat{\alpha}_{j,t}) E(\tilde{\nu}_{p,\eta,t}) \) using the identity in (B.2). Because \( E(\tilde{\nu}_{p,\eta,t}) = 0 \) in every time period, (B.5) equals zero.

Substituting this result in Equation (B.1), we get

\[ \text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,\eta,t}^E) = \text{Cov}(\Gamma_{p,t}, \hat{\alpha}_{p,j,t}). \]
Appendix C

This appendix presents the steps to empirically estimate various components of $\sigma^2_{\alpha_{ij}}$ conditional on the true asset pricing model. For brevity, the table below presents the components of $\sigma^2_{\alpha_{ij}}$ from Equation (24) with $J = 2$ and for $\eta = 0, 1, 2$ and $K = 0, 1$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\eta$</th>
<th>Variance due to</th>
<th>Covariance of omitted factors with APM misspecification</th>
<th>Beta measurement error</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\sigma^2_{\alpha_j}$ misspecification</td>
<td>Omitted factors</td>
<td>$\eta = J$ or $K$</td>
<td>$\eta = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>IV</td>
<td>0</td>
<td>V</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>IV</td>
<td>III</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>IV</td>
<td>III</td>
<td>V</td>
<td>VI</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>IV</td>
<td>0</td>
<td>V</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>IV</td>
<td>III</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>IV</td>
<td>III</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We first fill cells that are zero by definition. These cells are

a. APM misspecification, when $\eta = K$.

b. Omitted factors, when $\eta = J$.

c. Covariance of omitted factors with APM misspecification, when $\eta = J$ or $K$.

d. Beta measurement error, when $\eta = 0$.

Next, we fill the remaining columns of the above table using the sequence of steps discussed below. Numbers I to VI in the table correspond to the respective step numbers below and denote the order in which we estimate each component of the variance decomposition labeled in the column heading.
I. Using estimates from the time-series OLS regression (19), we compute \( \hat{\alpha}_{p,\eta,t}^E \) for each fund-month under each \( \eta \)-factor model and then compute the cross-sectional variance \( \sigma_{\hat{\alpha}_{\eta,t}}^2 \). We average this across months to get \( \sigma_{\hat{\alpha}_{\eta}}^2 = \frac{1}{T} \sum_t \sigma_{\hat{\alpha}_{\eta,t}}^2 \).

II. From the time-series OLS regression (19) for each fund-month and \( \eta \)-factor model, we get the covariance matrix of \( \hat{\beta} \) estimates: 

\[
\text{Cov} \left( \left[ \hat{\beta}_{(1,\eta),p,t} - \beta_{(1,\eta),p,t} \right]' , \left[ \hat{\beta}_{(1,\eta),p,t} - \beta_{(1,\eta),p,t} \right] \right) = \sigma_{\epsilon_p,t}^2 (X'_{(1,\eta),t} X_{(1,\eta),t})^{-1},
\]

where \( \beta_{(1,\eta),p,t} \) is the vector of true factor betas for fund \( p \) for month \( t \), \( X_{(1,\eta),t} \) is the data matrix of corresponding factors and \( \sigma_{\epsilon_p,t}^2 \) is the variance of residuals. We compute the variance due to beta measurement error component as

\[
\frac{1}{T} \sum_t \frac{1}{P_t} \sum_p F'_{(1,\eta),t} \text{Cov} \left( \left[ \hat{\beta}_{(1,\eta),p,t} - \beta_{(1,\eta),p,t} \right]' , \left[ \hat{\beta}_{(1,\eta),p,t} - \beta_{(1,\eta),p,t} \right] \right) F_{(1,\eta),t}, \tag{C.1}
\]

where \( P_t \) is the number of funds in the cross-section at time \( t \).

III. For \( \eta > K \), misspecification error variance = \( \bar{F}'_{(K+1,\eta)} \text{Cov}(\beta'_{(K+1,\eta)}, \beta_{(K+1,\eta)}) \bar{F}_{(K+1,\eta)} \), where \( \bar{F}_{(K+1,\eta)} \) is the sample mean of unpriced factors, \( \text{Cov}(\beta'_{(K+1,\eta)}, \beta_{(K+1,\eta)}) \) is the covariance matrix of true betas of corresponding factors. We estimate the covariance of true betas of an \( \eta \)-factor model as

\[
\text{Cov}(\beta'_{(1,\eta)}, \beta_{(1,\eta)}) = \frac{1}{T} \sum_t \left( \text{Cov}(\hat{\beta}'_{(1,\eta),t}, \hat{\beta}_{(1,\eta),t}) - \left( \frac{1}{P_t} \sum_p \sigma_{\epsilon_p,t}^2 (X'_{(1,\eta),t} X_{(1,\eta),t})^{-1} \right) \right). \tag{C.2}
\]

We use the submatrix of the above matrix starting at row \( K+1 \) to compute \( \bar{F}'_{(K+1,\eta)} \text{Cov}(\beta'_{(K+1,\eta)}, \beta_{(K+1,\eta)}) \bar{F}_{(K+1,\eta)} \). For \( \eta < K \), we use the covariance matrix of true betas from the case \( \eta \geq K \) for the corresponding factors, because these betas are not estimated in
this case. For example, with \( \eta = 1, K = 3 \), we use the covariance matrix of true betas for SMB, HML estimated for the case \( \eta = 3, K = 1 \).

IV. For \( \eta = J \), from Equation (24),

\[
\sigma^2_{\tilde{a}_{j,t}} = \sigma^2_{\tilde{a}^E_{\eta,t}} - \left( \tilde{F}'_{(K+1,\eta)} \left( \text{Cov}(\beta'_{(K+1,\eta)}, \beta_{(K+1,\eta)}) \right) \tilde{F}_{(K+1,\eta)} \right)
- \left( \tilde{F}'_{(1,\eta)} \text{Cov} \left( \left[ \tilde{\beta}_{(1,\eta),p,t} - \beta_{(1,\eta),p,t} \right], \left[ \tilde{\beta}_{(1,\eta),p,t} - \beta_{(1,\eta),p,t} \right] \right) \tilde{F}_{(1,\eta),t} \right).
\]

(C. 3)

We computed each term on the RHS of the equation using steps I, II, and III, and, hence, we can determine \( \sigma^2_{\tilde{a}_{j,t}} \), which is the variance of alphas computed by investors in the economy using their information set. Empiricists do not know the true asset pricing model, but, under the hypothesis that the \( K \)-factor model is the true asset pricing model, \( \sigma^2_{\tilde{a}_{j,t}} \) does not depend on the \( \eta \)-factor model used to estimate alpha. Therefore, \( \sigma^2_{\tilde{a}_{j,t}} \) is constant across all rows with the same \( K \). \( \sigma^2_{\tilde{a},j} \) is the time-series average of \( \sigma^2_{\tilde{a}_{j,t}} \).

V. To compute the variance due to omitted factors, let \( \eta = K \). From Equation (24), we get

\[
\begin{align*}
f'_{(\eta+1,J),t} \left( \text{Cov}(\beta'_{(\eta+1,J)}, \beta_{(\eta+1,J)}) \right) & f_{(\eta+1,J),t} + E[\beta_{(\eta+1,J)}] \left( f_{(\eta+1,J),t} f'_{(\eta+1,J),t} \right) E[\beta_{(\eta+1,J)}] \\
& = \sigma^2_{\tilde{a}^E_{\eta,t}} - \sigma^2_{\tilde{a}_{J,t}} - \left( \tilde{F}'_{(K+1,\eta)} \left( \text{Cov}(\beta'_{(K+1,\eta)}, \beta_{(K+1,\eta)}) \right) \tilde{F}_{(K+1,\eta)} \right)
- \left( \tilde{F}'_{(1,\eta),t} \text{Cov} \left( \left[ \tilde{\beta}_{(1,\eta),t} - \beta_{(1,\eta),t} \right], \left[ \tilde{\beta}_{(1,\eta),t} - \beta_{(1,\eta),t} \right] \right) \tilde{F}_{(1,\eta),t} \right).
\end{align*}
\]

We know all the variables on the RHS using steps I through IV, and, hence, we can compute the left-hand side (LHS). The terms on the LHS are a function of \( \eta \), true betas, and unexpected factor realizations, and they are not dependent on true \( K \). Therefore, the value we compute for

\[\text{footnote: We consider } \eta = J \text{ to compute } \sigma^2_{\tilde{a}_{J,t}} \text{ because the other remaining cells after steps I, II, and III are zero for this case.}\]

\[\text{footnote: We consider } \eta = K \text{ to estimate this term because the remaining cell after steps I through IV is zero for this case.}\]
\( \eta = K \) applies to all rows with the same \( \eta \). The variance due to omitted factors is the time-series average of the LHS in Equation (C.4).

VI. We have now computed all terms of Equation (24), except the covariance term, and, hence, we can now compute this term as well.
Appendix D

This appendix proves the result in Proposition 5 for the ordering of coefficients in the horse race regression with binary transformation. From regression (32), we have

$$B_\eta = E\left[ Q_{\Gamma_p} | Q_{\tilde{a}_{p,\eta}} \right] = \Pr(\Gamma_p \geq 0 | Q_{\tilde{a}_{p,\eta}}) - \Pr(\Gamma_p < 0 | Q_{\tilde{a}_{p,\eta}}).$$

When $Q_{\tilde{a}_{p,\eta}} = 1$, this term can be expanded as

$$E\left[ Q_{\Gamma_p} | Q_{\tilde{a}_{p,\eta}} = 1 \right]$$

$$= \{\Pr(\Gamma_p \geq 0 | \tilde{a}_{p,\eta} \geq 0, \tilde{a}_{p,J} \geq 0) - \Pr(\Gamma_p < 0 | \tilde{a}_{p,\eta} \geq 0, \tilde{a}_{p,J} \geq 0)\}$$

$$\times \Pr(\tilde{a}_{p,J} \geq 0 | \tilde{a}_{p,\eta} \geq 0) \quad (D.1)$$

$$+ \{\Pr(\Gamma_p \geq 0 | \tilde{a}_{p,\eta} \geq 0, \tilde{a}_{p,J} < 0) - \Pr(\Gamma_p < 0 | \tilde{a}_{p,\eta} \geq 0, \tilde{a}_{p,J} < 0)\}$$

$$\times \Pr(\tilde{a}_{p,J} < 0 | \tilde{a}_{p,\eta} \geq 0).$$

From Equation (12), the sign of flow is determined only by sign of $\tilde{a}_{p,J}$, because all other terms are positive. Therefore, $\tilde{a}_{p,\eta} \geq 0$ has no additional information about sign of flows being positive conditioning on sign of $\tilde{a}_{p,J}$. The model also implies that flow is always positive (negative) when $\tilde{a}_{p,J}$ is positive (negative). Substituting these conditions, we can simplify (D.1) as

$$E\left[ Q_{\Gamma_p} | Q_{\tilde{a}_{p,\eta}} = 1 \right]$$

$$= \Pr(\Gamma_p \geq 0 | \tilde{a}_{p,J} \geq 0) \Pr(\tilde{a}_{p,J} \geq 0 | \tilde{a}_{p,\eta} \geq 0) \quad (D.2)$$

$$- \Pr(\Gamma_p < 0 | \tilde{a}_{p,J} < 0) \Pr(\tilde{a}_{p,J} < 0 | \tilde{a}_{p,\eta} \geq 0).$$

From Equation (16), $\tilde{a}_{p,\eta} = \tilde{a}_{p,J} + \nu_{p,\eta}$ and $Var(\tilde{a}_{p,\eta}) = Var(\tilde{a}_{p,J}) + Var(\nu_{p,\eta})$. The average betas of funds on various factors are equal to betas of market portfolio by assumption, so the
unconditional average of the bias term in \( \nu_{\eta,t} \) is zero for all \( \eta \)-factor models considered in our study. Therefore, \( \hat{\alpha}_{p,\eta} \) is Normally distributed with mean zero for all \( \eta \), and \( \Pr(\hat{\alpha}_{p,\eta}^E \geq 0) = \Pr(\hat{\alpha}_{p,J} \geq 0) = 0.5 \). Using this along with Bayes’ rule gives

\[
\Pr(\hat{\alpha}_{p,J} \geq 0 | \hat{\alpha}_{p,\eta}^E \geq 0) = \Pr(\hat{\alpha}_{p,\eta}^E \geq 0 | \hat{\alpha}_{p,J} \geq 0),
\]

\[
\Pr(\hat{\alpha}_{p,J} < 0 | \hat{\alpha}_{p,\eta}^E \geq 0) = \Pr(\hat{\alpha}_{p,\eta}^E \geq 0 | \hat{\alpha}_{p,J} < 0).
\]

(D.3)

It can be easily verified that \( \Pr(\hat{\alpha}_{p,\eta}^E \geq 0 | \hat{\alpha}_{p,J} \geq 0) \) is decreasing in the variance of \( \nu_{p,\eta} \), while \( \Pr(\hat{\alpha}_{p,\eta}^E \geq 0 | \hat{\alpha}_{p,J} < 0) \) is increasing in the variance of \( \nu_{p,\eta} \). Therefore, from (D.2) and (D.3), we can infer that \( \mathbb{E}[Q_\Gamma_p | Q_{\hat{\alpha}_{p,\eta}^E} = 1] \) is decreasing with the variance of \( \hat{\alpha}_{p,\eta}^E \). We obtain similar inference with \( \mathbb{E}[Q_\Gamma_p | Q_{\hat{\alpha}_{p,\eta}^E} = -1] \).

Therefore, we can conclude that \( \sigma^2_{\hat{\alpha}_{p,\eta_1}^E} < \sigma^2_{\hat{\alpha}_{p,\eta_2}^E} \Rightarrow B_{\eta_1} > B_{\eta_2} \). We established earlier for the horse race regression (14) that \( \hat{b}_{\eta_1} > \hat{b}_{\eta_2} \Rightarrow \sigma^2_{\hat{\alpha}_{p,\eta_1}^E} < \sigma^2_{\hat{\alpha}_{p,\eta_2}^E} \). Therefore, \( \hat{b}_{\eta_1} > \hat{b}_{\eta_2} \Rightarrow B_{\eta_1} > B_{\eta_2} \).