THE GENERALISED VISCOELASTIC SPRING

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Abstract

A spring/rod model is presented that describes one-dimensional behaviour of solids susceptible to large or small viscoelastic deformation. Derivation of its constitutive equation is underpinned by the fact that the internal energy, which the elastic part of deformation stores into the spring, changes in time with the observed strain as well as with some, unknown part of the strain-rate. The latter emerges through the action of a viscous flow potential and is the source of inelastic deformation. Thus, unlike its conventional viscoelasticity counterparts, the model does not postulate *a priori* a rule that relates strain with viscous flow formation. Instead, it considers that such a rule, as well as other important features of combined elastic and inelastic material response, should become known *a posteriori* through the solution of a relevant, well-posed boundary value problem. This communication begins with considerations compatible with large viscoelastic deformations, and gradually progresses through simpler modelling situations. The latter also include the case of small viscoelastic strain that underpins formulation of classical, spring-dashpot viscoelastic models. In an example application, a relevant closed form solution is obtained for a spring undergoing small viscoelastic deformation under the influence of a viscous flow potential which is quadratic in the stress.

Keywords: Constitutive equations, Large viscoelastic deformations, Small viscoelastic deformations, Viscoelasticity, Viscoelastic rod, Viscoelastic spring.

1. Introduction

The subjects of Elasticity, Plasticity and Viscoelasticity are generally considered as parts of continuum solid mechanics that deal with different kinds of solid material behaviour. In this context, most of the existing standard models of either plastic or viscoelastic material behaviour begin with some postulation that presupposes the manner that the anticipated parts of elastic and inelastic deformation are assembled and enter the resulting constitutive rule. This observation may be felt more commonly associated with models met in the theory of plasticity. The model of multiplicative decomposition [1] is here referred to as one of the most successful relevant examples. Nevertheless, classical models in linear viscoelasticity also anticipate the manner that the observed strain and its time derivative are assembled. The well-known and long-established method of appropriate combinations of elastic springs and "inelastic" dampers/dashpots [2-4] is still considered a most successful such (e.g., [5, 6]).

A recent plasticity model [7] follows a different approach and perceives the work hardening stage of a stretched deformable cube as a regime of material response in which the anticipated elastic and inelastic parts of deformation are initially both unknown; along with the rule that these are finally combined and form the observed total deformation. The model considers instead that these important features of the observed material response should become known through the solution of a relevant, well posed mathematical boundary value problem. The manner in which this aim becomes possible in [7] is (i) by retaining the classical postulation that the inelastic part of deformation emerges through the reaction of a yield function,

which also acts as a flow potential, and (ii) by properly accounting for the evident fact that the change of the energy, W, stored in the material depends on the change of both the observed deformation and the observed material flow. It is recalled that, upon removing the yield function involvement, the plasticity model proposed in [7] reduces naturally to, and collapses into its own hyperelasticity framework.

The present communication aims to demonstrate that the outlined new postulations [7] are similarly relevant and applicable to classical viscoelasticity situations, and, hence, to indicate that all three subjects of Hyperelasticity, Plasticity and Viscoelasticity share a common theoretical background. With these aims in mind, attention here focuses onto one of the simplest possible models of one-dimensional deformable bodies, namely that of a cantilever viscoelastic cylindrical rod. Nevertheless, Figure 1 describes schematically the existing close relationship between the linearly elastic spring and a linearly elastic cantilever rod subjected to tension (e.g., Sects. 23 and 30 in [8]), and serves as a useful reminder of the implied well-known mathematical equivalence.

It thus becomes initially clear that, wherever the term "spring" is met in what follows, this may be replaced with the term "rod" without further consequences, at least within the small deformation regime. In cases of large deformation, the cross-sectional area of a cylindrical rod is naturally expected to change with the deformation in accordance with well-known relevant rules. However, such a change of the cross-sectional area is here regarded of minor importance, as it influences only the relationship between the force and the relevant traction distribution applied externally on the relevant end boundary of the cantilever rod.

Section 2 thus presents a new viscoelastic spring model that shares simultaneously properties of an elastic spring and an inelastic damper (dashpot). In strict terms, this essentially represents a one-dimensional deformable solid subjected to large viscoelastic deformation. Progressively simplifying considerations enable afterwards this communication to proceed through simpler modelling situations, and to finally establish the manner that the presented model relates to its well-known classical spring-dashpot counterparts.

Accordingly, Section 3 considers separately particular cases of infinitesimally small strain and succeeds to end with a relevant, geometrically linearised viscoelastic spring model. That model is applicable within the region of validity of the classical spring-dashpot models met in linear viscoelasticity. However, it is still substantially more general as, despite its small strain region of applicability, it makes use of a non-linear constitutive equation.

The implied material nonlinearity does not prevent Section 4 to proceed and specify a special case in which a corresponding boundary/initial value problem admits an exact, closed form solution. This is a case in which the internal energy function and the viscous flow potential of that geometrically linear viscoelastic spring are both quadratic functions of their arguments. Section 5 describes next the manner that further approximations enable that small strain version of the present model to connect with its well-known Kelvin-Voight counterpart met in classical viscoelasticity. Finally, Section 6 summarises important principal steps and considerations that underpin the model's novelty and originality, and briefly suggests directions for future relevant research and study.

2. The viscoelastic spring model

Figure 1 employs a single co-ordinate parameter, X, and illustrates schematically the known analogy between a linearly elastic spring, having elastic modulus K, and a linearly elastic cylindrical rod having Young's modulus E and cross-sectional area A. These elastic components are both considered of the same length, L, and are subjected to one-dimensional deformation through the action of the same external force. That force is having magnitude f and is applied externally on the free end of the component, where $X = L$, along the direction of the depicted co-ordinate axis.

Interest here is directed towards a corresponding situation in which the observed response of the externally loaded spring is viscoelastic. In this context, it is worth recalling that such a kind of viscoelastic material response is observed very often in nature. Most polymers behave in this viscoelastic manner, while the Mullins effect is here also referred to as a well-known relevant example of rubber-type material behaviour (e.g., [9-12]).

2.1 *Preliminaries and viscous flow considerations*

It is accordingly assumed that at its initial length L (at time $t = t_0$, say) the rod of interest is unloaded and unstressed. Loading from that initial configuration takes off through application of a force,

$$f(t) = T(t)A, \quad f(t_0) = T(t_0) = 0 \,. \tag{2.1}$$

This is considered uniformly distributed over the area of the free end of the rod and may vary slowly in time. Moreover, it gives rise to immediate viscoelastic material response which, at $t > t_0$, is described with the use of a current co-ordinate parameter, x; A is naturally expected to change at large deformation according to the known relevant rule. It is postulated that the observed one-dimensional deformation comprises simultaneous action and possible interaction of elastic deformation and viscous fluid flow.

The manner that those parts of elastic and inelastic material response are merged and compose the observed total deformation is initially considered unknown. However, the mass of the viscoelastic spring is required to obey the standard continuity equation

$$\dot{\rho} + \rho v_{,x} = 0 \,, \tag{2.2}$$

where ρ and v represent current mass density and velocity, respectively, a dot denotes total differentiation with respect to time and a comma differentiation with respect to x.

Denote with d^v the rate of the inelastic deformation gradient, and note that, in general, this is different to its total deformation counterpart, $v_{,x}$; namely $d^v \neq v_{,x}$. It is postulated that there exists a viscous flow potential $\phi(\sigma)$, where σ is the single Cauchy stress component acting along the spring/rod axis, and that this potential is the source of d^v. The relevant viscous flow rule may then be postulated in the direct manner

$$d^v = \frac{d\phi(\sigma)}{d\sigma} = \phi'(\sigma), \tag{2.3}$$

which is consistent with the general concept of a potential function; here, as well as in what follows, a prime denotes differentiation with respect to σ.

Alternatively, adopt relevant plasticity postulations and assume that $\phi(\sigma)$ is such that

$$\phi(\sigma) - \kappa = 0, \tag{2.4}$$

where κ is some non-negative parameter. Then, amend slightly a route employed already in classical plasticity [13, 14] as well as in non-linear visco-plasticity [15] by requiring from the rate of the dissipative work,

$$\psi = \frac{1}{2}\sigma_D d^v, \tag{2.5}$$

to be stationary. The implied alteration refers to the fact that the stress is split into two parts (e.g., [16]) as follows:

$$\sigma = \sigma_R + \sigma_D, \tag{2.6}$$

where the former part gives rise to recoverable work and the second contributes into the dissipative work rate ψ.

The imposed requirement necessitates minimisation of the expression

$$2\psi = \sigma_D d^v - \lambda\left[\phi(\sigma) - \kappa\right],$$

where λ is a Lagrange multiplier, thus leading to

$$d^v = \lambda\frac{d\phi}{d\sigma_D} = \lambda\frac{d\phi}{d\sigma}\frac{d\sigma}{d\sigma_D} = \lambda\frac{d\phi}{d\sigma} = \lambda\phi'(\sigma). \tag{2.7}$$

If λ is considered constant, then this flow rule is seen equivalent to (2.3) because, as becomes later more evident (see (2.18) and Section 3.2), that constant can be absorbed by $\phi(\sigma)$.

2.2. *Equilibrium and constitution*

Let
$$x = x(X, t) \tag{2.8}$$
represent the rule that governs the spring deformation. In the light of the adopted notation, consider that the initial length, L, of the spring changes into ℓ, so that $0 \leq x \leq \ell$. The following boundary conditions thus hold:
$$x\big|_{X=0} = 0, \quad \sigma\big|_{x=\ell} = T(t). \tag{2.9}$$

As is already mentioned, the elastic part of the deformation is initially considered indistinguishable from its inelastic counterpart. However, the rate of the internal energy that the deformation accumulates into the material is still governed by the usual rate-of-energy equation
$$\frac{d}{dt} \int_0^x \rho \left(\frac{1}{\rho_0} W + \frac{1}{2} v^2 \right) dx = \sigma v, \tag{2.10}$$
where W represents the internal energy per unit axial length stored into the implied, arbitrary part $[0, x]$ of the deformed spring and ρ_0 is the initial density of the spring, at $t = t_0$. The second term inside the integrand is the kinetic energy, while σ is evidently the only boundary traction acting externally on $[0, x]$.

It is now recalled (e.g., [17]) that the material time derivative appearing in the left-hand-side of (2.10) obeys the rule
$$\frac{d}{dt} \int_0^x \rho \left(\frac{1}{\rho_0} W + \frac{1}{2} v^2 \right) dx = \int_0^x \rho \frac{d}{dt} \left(\frac{1}{\rho_0} W + \frac{1}{2} v^2 \right) dx. \tag{2.11}$$

Introduction of (2.11) into (2.10), followed by differentiation with respect to x, leads to
$$\frac{\rho}{\rho_0} \dot{W} = (\sigma_{,x} - \rho \dot{v}) v + \sigma v_{,x}. \tag{2.12}$$

It is next observed that, due to the implied energy dissipation, W changes with, and therefore depends not only on the deformation and its gradient
$$F = \frac{\partial x}{\partial X}, \tag{2.13}$$
but also on the rate of the inelastic deformation gradient, d^v. In mathematical terms, this fact takes the form
$$W = W(F, d^v), \quad W(0, 0) = 0, \tag{2.14}$$
and leads to
$$\dot{W} = \frac{dW}{dt} = \frac{\partial W}{\partial F} \dot{F} + \frac{\partial W}{\partial d^v} \dot{d}^v = \left(\frac{\partial W}{\partial F} + \frac{\partial W}{\partial d^v} \frac{\dot{d}^v}{\dot{F}} \right) \dot{F} = \left(\frac{\partial W}{\partial F} + \frac{\partial W}{\partial d^v} \frac{\dot{d}^v}{\dot{F}} \right) \frac{\partial \dot{x}}{\partial x} \frac{\partial x}{\partial X} = \left(\frac{\partial W}{\partial F} + \frac{\partial W}{\partial d^v} \frac{\dot{d}^v}{\dot{F}} \right) F v_{,x}. \tag{2.15}$$

A comparison of this result with (2.12) leads to
$$(\sigma_{,x} - \rho \dot{v}) v + \left[\sigma - \frac{\rho}{\rho_0} F \left(\frac{\partial W}{\partial F} + \frac{\partial W}{\partial d^v} \frac{\dot{d}^v}{\dot{F}} \right) \right] v_{,x} = 0, \tag{2.16}$$

which, due to the arbitrariness of both v and $v_{,x}$, requires
$$\sigma_{,x} = \rho \dot{v}, \quad \sigma = \frac{\rho}{\rho_0} F \left(\frac{\partial W}{\partial F} + \frac{\partial W}{\partial d^v} \frac{\dot{d}^v}{\dot{F}} \right). \tag{2.17}$$

While (2.17a) is recognised as the standard one-dimensional equation of motion, (2.17b) emerges as the constitutive equation sought for the present viscoelastic spring model. On its own, the first term in the right-hand-side of (2.17b) represents the constitutive equation of the corresponding hyperelastic spring. In the present viscoelastic spring case though, that term refers to the change of W with respect to the total,

rather than to the still unknown elastic part of the deformation. In (2.17b), this is further accompanied by a term which is influenced heavily by the appearing features of inelastic deformation and viscous flow.

It is revealing in this context to note that, if the Lagrange multiplier parameter appearing in (2.7) is considered constant, one obtains

$$\dot{d}^v = \lambda \frac{\partial \dot{\phi}}{\partial \sigma} = \lambda \frac{\partial^2 \phi}{\partial \sigma^2} \dot{\sigma} = \lambda \phi'' \dot{\sigma} \,,$$

and (2.17b) thus reduces to

$$\sigma = \frac{\rho}{\rho_0} F \left(\frac{\partial W}{\partial F} + \lambda \phi'' \frac{\dot{\sigma}}{\dot{F}} \frac{\partial W}{\partial d^v} \right). \tag{2.18}$$

This makes it clearer that both (2.18) and its more general version (2.17b) are implicit in the stress and, also, differential rather than algebraic constitutive equations.

In summary, the presented viscoelastic spring model is alternatively regarded as an initial/boundary value problem that is described as follows: given the form of the flow potential introduced in (2.4) and the form of the internal energy density (2.14), determine three principal unknowns, namely the material density, ρ, the placement function, x, and the Cauchy stress, σ, in a manner that satisfies three principal equations, namely (2.2), (2.17a) and (2.18), subject to the boundary conditions (2.9) and the initial conditions

$$\rho|_{t=0} = \rho_0, \quad x|_{t=0} = X, \quad \sigma|_{t=0} = 0. \tag{2.19}$$

2.3 Energy and dissipation considerations

In view of the general form (2.14) of the internal energy density, it is anticipated that

$$W^e(F) = W(F,0) \geq 0, \tag{2.20}$$

represents the internal energy stored into a corresponding elastic spring subjected to same loading and deformation conditions; the equality sign holds in the absence of deformation ($F = 0$).

On the other hand, and in accordance with the definition (2.5), the energy dissipation is expected to be the following non-decreasing function of time:

$$D^w = \int_{t_0}^t \psi \, dt = \frac{1}{2} \int_{t_0}^t \sigma_D d^v \, dt = \frac{1}{2} \int_{t_0}^t \lambda \sigma_D \phi'(\sigma) \, dt \geq 0, \tag{2.21}$$

where σ_D is to be determined through the solution of the boundary value problem of interest, and the equality sign holds in the special case that the spring behaves elastically ($\sigma_D = 0$).

Hence, by proposing the following connection between the internal energy densities of a pair of corresponding elastic and viscoelastic springs:

$$W^e(F) = W(F,d^v) + D^w, \tag{2.22}$$

one arrives to the condition

$$W^e(F) \geq W(F,d^v) \geq 0. \tag{2.23}$$

This condition implies further that this generalised viscoelastic spring stops to represent a solid at the first instance, $t = t_F > t_0$, at which

$$D^w|_{t=t_F} = W^e|_{t=t_F} \quad \Leftrightarrow \quad W(F,d^v)|_{t=t_F} = 0. \tag{2.24}$$

In this general form, the developed viscoelastic model is highly nonlinear, and its connection with specific boundary value problems must confront the usual analytical hurdles met in corresponding hyperelasticity problems. Moreover, as is seen in the next Section, material nonlinearity is generally present even in the small strain regime. This additional mathematically difficulty is readily compatible with the viscoelastic nature of the solid of interest, as it is due to the fluid-type flow part of its material behaviour. Nevertheless, a further special case will later be identified as part of an application (Section 4), where both geometric and material nonlinearities can be considered negligible. In that case, the model obtains a

considerably simplified form that admits a closed form solution and facilitates its connection, as well as its comparison, with its conventional counterparts (Section 5).

3. Small strain

Potential connection of the present model with standard spring-dashpot models requires initially specialisation of the former within the region of applicability of the latter, namely the region of infinitesimally small strain. Such a specialisation begins with the introduction of the displacement

$$u = x - X ,$$ (3.1)

and the subsequent approximation

$$F = \frac{\partial x}{\partial X} = 1 + e \approx 1,$$ (3.2)

where the strain

$$e = u_{,X}$$ (3.3)

is considered much smaller than 1.

It is now recalled that in linear elasticity problems, where strains are considered infinitesimally small, the deformed and undeformed coordinate parameters are considered practically indistinguishable ($x \approx X$) and the current mass density, ρ, indistinguishable from its initial counterpart, ρ_0. However, in the present case, where the rod responds in a viscoelastic manner, potential differences between the current and the initial mass density might be found influential in applications. Therefore, such differences are not considered *a priori* negligible in what follows.

As (3.2) implies that

$$\dot{F} = \dot{e}, \quad \frac{\partial W}{\partial F} = \frac{\partial W}{\partial e}\frac{\partial e}{\partial F} = \frac{\partial W}{\partial e},$$ (3.4)

(2.14) is now approximated as follows:

$$W = W\left(e, d^v\right), \quad W\left(0,0\right) = 0 .$$ (3.5)

and the constitutive equation (2.18) takes the form

$$\sigma = \frac{\rho}{\rho_0}\left(\frac{\partial W}{\partial e} + \lambda\phi''\frac{\dot{\sigma}}{\dot{e}}\frac{\partial W}{\partial d^v}\right).$$ (3.6)

In general, this is still a non-linear differential equation for σ, and stays such even in cases that its solution might be sought and found independently of the solution of the continuity and the equilibrium equations (2.2) and (2.17a).

The following connection is now made between the rate of inelastic deformation and the rate of small strain due to viscous flow:

$$d^v = \dot{e}^v .$$ (3.7)

By virtue of (2.7), this implies that the inelastic part of small strain is

$$e^v = \int_{t_0}^{t} d^v dt = \lambda\int_{t_0}^{t} \phi'(\sigma)dt \cdot$$ (3.8)

Hence, the elastic parts of strain and displacement can respectively be calculated as follows:

$$e^e = e - \lambda\int_{t_0}^{t}\phi'(\sigma)dt,$$

$$u^e = \int_{0}^{X} e^e dX = \int_{0}^{X}\left[e - \lambda\int_{t_0}^{t}\phi'(\sigma)dt\right]dX.$$ (3.9)

It follows that the initial/boundary value problem set in the previous section for the case of large spring deformation, is now modified as follows: given the form of the flow potential appearing in (2.4) and

the form of the internal energy function (3.5), determine the three principal unknowns, ρ, u, and σ, in a manner that satisfies the three principal equations (2.2), (2.17a) and (3.6), subject to the boundary conditions

$$u\big|_{X=0} = 0, \quad \sigma\big|_{X=L} = T(t), \tag{3.10}$$

and the initial conditions

$$\rho\big|_{t=0} = \rho_0, \quad u\big|_{t=0} = \sigma\big|_{t=0} = 0. \tag{3.11}$$

Following the solution of this initial/boundary value problem, the total strain and its viscoelastic part may be evaluated, a posteriori, through use of (3.3) and (3.8), respectively, while (3.9) will lead to the determination (3.9) of the elastic parts of both the strain and the displacement.

3.1 *Quadratic form of the internal energy function*

Foundation of linearised small strain elasticity models is based on purely quadratic forms of W, so that the resulting constitutive equation relates the stress and the strain in a linear manner. In the present viscoelastic spring case, the most general, purely quadratic form of (3.5a) is

$$W\left(e, d^v\right) = \frac{1}{2} E e^2 + \eta e d^v + \frac{1}{2}\xi\left(d^v\right)^2, \tag{3.12}$$

where, E, η and ξ are appropriate material moduli whose values should enable consistency of the model with the internal energy conditions (2.20) and (2.23). It is seen that this form of W satisfies the initial condition (3.5b).

Moreover, the form (3.12) of W enables the constitutive equation (3.6) to acquire the more specific form

$$\sigma = \frac{\rho}{\rho_0}\left\{ E e + \eta d^v + \lambda\phi'' \frac{\dot{\sigma}}{\dot{e}}\left(\eta e + \xi d^v\right)\right\}, \tag{3.13}$$

or, by virtue of (2.7),

$$\sigma = \frac{\rho}{\rho_0}\left\{ E e + \lambda\eta\phi' + \lambda\phi'' \frac{\dot{\sigma}}{\dot{e}}\left(\eta e + \lambda\xi\phi'\right)\right\}. \tag{3.14}$$

It is noted that, in the special case that $\eta = \xi = 0$, (3.12) reduces to its standard linear elasticity form

$$\hat{W} = \frac{1}{2} E e^2. \tag{3.15}$$

In that case, the constitutive equation (3.14) resembles closely its linearly elastic spring/rod counterpart,

$$\sigma = \frac{d\hat{W}}{de} = E e, \tag{3.16}$$

which, based in the aforementioned linear elasticity postulate, neglects the influence of the continuity equation (2.2) and implies that the material density is not affected by the deformation ($\rho = \rho_0$). It is thus seen that if this linear elasticity postulate is applied when $\eta = \xi = 0$, then (3.14) and (3.16) become identical.

At the other extreme, by setting $E = \eta = 0$, one recognises a constitutive model that exhibits purely fluid material behaviour. Such an observation is fully compatible with a basic rheology concept, according which viscoelastic material description does not recognise or distinguish clear boundaries between solid and fluid material behaviour (e.g., [2-5]). That concept thus classifies the proposed generalised viscoelastic spring as a properly developed viscoelastic model, in the sense that at one extreme ($\eta = \xi = 0$) this resembles an elastic solid while at the other ($E = \eta = 0$) a viscous fluid. However, a study of the purely fluid case ($E = \eta = 0$) is not pursuit here, as it does not fulfil the energy and dissipation conditions set in Section 2.3, and thus is incompatible with the anticipated mechanical response of a solid spring.

It is also noted that the last term in the right-hand-side of either (3.13) or (3.14) represents a constitutive equation that still preserves the nonlinear differential equation features observed earlier in (3.6). However, additional postulations concerning the choice of the viscous flow potential may simplify the form

of this constitutive equation. Standard yield surfaces met in metal plasticity, for instance, are traditionally considered quadratic functions of their arguments.

3.2 *Quadratic form of the viscous flow potential*

In this regard, consider next the special case of a viscous flow potential which is quadratic in σ, namely

$$\phi(\sigma) = \frac{1}{2} G \sigma^2, \tag{3.17}$$

where G is some appropriate material modulus. Then, the viscous flow rule (2.7) yields

$$d^v = \lambda G \sigma = \hat{G} \sigma. \tag{3.18}$$

As was anticipated earlier in Section 2, introduction of the new parameter $\hat{G} = \lambda G$ enables incorporation of a constant value of λ into the form of the flow potential $\phi(\sigma)$, and, hence, validates in that case a direct postulation of the flow rule (2.3). The constitutive equation (3.14) thus reduces to

$$\sigma = \frac{\rho}{\rho_0} \left\{ Ee + \eta \hat{G} \sigma + \hat{G} \left(\eta e + \xi \hat{G} \sigma \right) \frac{\dot{\sigma}}{\dot{e}} \right\}, \tag{3.19}$$

which, due to involvement of the ratio $\dot{\sigma}/\dot{e}$ is differential with respect to time, as well as nonlinear in σ and e.

It is recalled that, in general, solution of (3.19) should be sought in conjunction with the kinematic relation (3.3), the equation of motion (2.17a) and the continuity equation (2.2), subject to the associated boundary conditions (3.10) and the initial conditions (3.11). Completion of the outlined solution will finally enable use of (3.8) and (3.9) and, hence, determination of the remaining unknown deformation features, namely

$$e^v = \int_{t_0}^{t} d^v dt = \hat{G} \int_{t_0}^{t} \sigma dt, \quad e^e = e - \hat{G} \int_{t_0}^{t} \sigma dt, \quad u^e = u - \hat{G} \int_{0}^{X} \int_{t_0}^{t} \sigma dt dX. \tag{3.20}$$

3.3. *Energy and dissipation considerations*

The energy and power dissipation considerations discussed in Section 2.3 can also be applied in *a posteriori* manner, as soon as decomposition of stress into its two-part form (2.6) enables a similar split the of the total work rate into its recoverable and dissipative parts. In more detail, introduction of (2.6) into (3.19) transforms the latter equation into the following:

$$\sigma_R + \sigma_D = \frac{\rho}{\rho_0} \left\{ E \left(e^e + e^v \right) + \eta \hat{G} \left(\sigma_R + \sigma_D \right) + \hat{G} \left[\eta \left(e^e + e^v \right) + \xi \hat{G} \left(\sigma_R + \sigma_D \right) \right] \frac{\dot{\sigma}_R + \dot{\sigma}_D}{\dot{e}^e + \dot{e}^v} \right\}. \tag{3.21}$$

A necessary second condition that leads to precise determination of the involved couple of unknowns, σ_R and σ_D, is next established by balancing separately all terms in (3.21) that contribute to recoverable work. This condition is postulated through the observation that (3.21) should also hold true at time instances or stages of the deformation that may be dissipation free ($\sigma_D = e^v = 0$).

Hence, a split of (3.21) into two parts containing terms that contribute to the creation of recoverable and dissipative work rate, respectively, leads to

$$\hat{G} \left(\xi \hat{G} \sigma_R + \eta e^e \right) \dot{\sigma}_R + \dot{e}^e \left(\eta \hat{G} - \frac{\rho_0}{\rho} \right) \sigma_R = -E e^e \dot{e}^e,$$

$$\tag{3.22}$$

$$\hat{G} \left(\xi \hat{G} \sigma_R + \eta e \right) \dot{\sigma}_D + \left[\dot{e} \left(\eta \hat{G} - \frac{\rho_0}{\rho} \right) + \xi \hat{G}^2 \sigma \right] \sigma_D = -E \left(e^e \dot{e}^v + e^v \dot{e}^e \right) - \eta \hat{G} \dot{\sigma}_R e^v.$$

It is noted that (3.22a) is a first order non-linear differential equation for σ_R, and its solution may be attempted independently of (3.22b). The latter is a first order linear differential equation for σ_D, and, hence,

its standard solution can be formed immediately after its first and last terms are fed by the solution of (3.22a).

After some suitable rearrangement, it can be verified that the forms of (3.19) and (3.22a) are essentially identical. This observation justifies, at least to a first approximation, the split of (3.21) that leads to (3.22) and suggests that σ and σ_R might be sought as two different solutions of the same time-dependent differential equation. However, such a search for relevant solutions should still take place in conjunction with the equation of motion (2.17a) and the continuity equation (2.2). Hence, while the density, ρ, and the total strain, e, are unknown during the solution process of (3.19), their counterparts ρ and e^e that emerge in equation (3.22a) are there considered determined and, therefore, known quantities.

4. Application: Quasi-static viscoelastic deformation at small strain

Despite the mathematically complicated form of the constitutive equation (3.19), a closed form solution of the outlined viscoelastic boundary value problem is still possible, at least in the case that small strain takes place in a quasi-static manner. Description of that solution begins with a displacement field of the form

$$u = \left[\alpha(t) - \alpha_0\right]X, \quad \alpha_0 = \alpha(t_0). \tag{4.1}$$

Here, the function $\alpha(t)$ is assumed to be such that the implied deformation is imposed in an adequately slow manner that justifies replacement of the equation of motion (2.17a) with its quasi-static version

$$\sigma_{,X} = 0. \tag{4.2}$$

By virtue of (3.3), (4.1) leads next to the following results:

$$e = \left[\alpha(t) - \alpha_0\right], \quad \dot{e} = \dot{u}_{,X} = v_{,X} = \dot{\alpha}(t). \tag{4.3}$$

These enable the continuity condition (2.2) to obtain the separable form

$$\dot{\rho} + \rho\dot{\alpha}(t) = 0, \tag{4.4}$$

and, hence, to give the material density of the viscoelastic spring as follows:

$$\rho / \rho_0 = \exp\left\{\alpha_0 - \alpha(t)\right\}. \tag{4.5}$$

The constitutive equation (3.19) attains then the following form of a first-order ordinary differential equation for σ:

$$\frac{\rho\hat{G}}{\rho_0\dot{\alpha}}\left[\xi\hat{G}\sigma + \eta(\alpha - \alpha_0)\right]\dot{\sigma} + \left(\frac{\rho\eta\hat{G}}{\rho_0} - 1\right)\sigma + \frac{\rho E}{\rho_0}(\alpha - \alpha_0) = 0, \tag{4.6}$$

which is nonlinear due to the first term that appears in the square bracket. It is noted that, as neither e nor $\dot{e}$ depend on X, equation (4.6) and, therefore, its potential solution for σ are also independent of X. Therefore, the quasi-static equation of motion (4.2) is already satisfied identically.

For small values of ξ, the non-linear differential equation (4.6) may be solved asymptotically with the use of some standard perturbation technique. The first approximation to the implied asymptotic solution is evidently obtained by setting $\xi = 0$ and represents a special case in which the rate of inelastic strain does not influence on its own the internal energy function (3.12). That solution represents the deformation and material response of a generalised linearly viscoelastic spring, and, hence, deserves separate mention on its own merit.

4.1 Particular case: The generalised linearly viscoelastic spring ($\xi = 0$)

When $\xi = 0$, the quadratic form (3.12) of the internal energy function reduces to

$$W\left(e, d^v\right) = \frac{1}{2}Ee^2 + \eta ed^v, \tag{4.7}$$

and is influenced by the rate of inelastic strain only though the coupling of the latter with the total strain. This form of W excludes effects associated with purely fluid constitutive behaviour but corresponds to a viscoelastic spring whose constitution equation (3.19) is generally still differential as well as non-linear at small strain.

However, the rather simple form of the displacement field (4.1) enables the reduced form (4.6) of the constitutive equation to acquire the following form of a linear first-order ordinary differential equation:

$$\dot{\sigma} + p(t)\sigma = q(t), \tag{4.8}$$

where

$$p(t) = \frac{\left(\eta\hat{G} - \alpha e^{\alpha(t)-\alpha_0}\right)\dot{\alpha}}{\hat{G}\eta(\alpha - \alpha_0)}, \quad q(t) = -\frac{E\dot{\alpha}}{\hat{G}\eta}. \tag{4.9}$$

Thus, when subjected to the initial condition (3.11c), equation (4.8) provides the stress in the following closed form:

$$\sigma = \frac{1}{I(t)}\int_{t_0}^{t} I(t)q(t)\,dt, \quad I(t) = \exp\left(\int p(t)\,dt\right), \tag{4.10}$$

where, however, the noted integrations are generally expected to require numerical evaluation regardless of the form of the function $\alpha(t)$.

4.2 *Strictly quasi-static spring deformation*

The linear form

$$\alpha(t) = t/t_0 \tag{4.11}$$

is the optimal choice of $\alpha(t)$, as it produces no acceleration and thus reduces the equation of motion (2.17a) into its quasi-static counterpart (4.2). When combined with (4.11), the displacement distribution (4.1) gives thus rise to a strictly quasi-static deformation, which also satisfies the displacement boundary condition (3.10a).

Moreover, solution of (2.2) suggests that the material density changes in time according to the rule

$$\rho/\rho_0 = \exp\left(1 - t/t_0\right). \tag{4.12}$$

When is based on the quadratic internal energy function (4.7) and the quadratic flow potential (3.17), the choice (4.11) still gives rise to the stress distribution (4.10), where, however, it is

$$p(t) = \frac{\left(\rho\eta\hat{G}t - \rho_0 t_0\right)}{\rho\hat{G}\eta t_0(t - t_0)}, \quad q(t) = -\frac{E}{\hat{G}\eta t_0}. \tag{4.13}$$

Thus, the implied displacement and stress distributions represent an exact solution of this well posed viscoelastic spring problem, provided that the external traction $T(t)$ that appears in the final, yet to be used boundary condition (3.10b) is applied according to the rule (4.10). The remaining unknown features of this viscoelastic deformation, namely e^v, e^e and u^e, may then be determined with the use of (3.20), where the noted integrations are expected to require numerical evaluation as well.

After e^e and e^v thus become known, the two-part decomposition (2.6) of σ becomes also possible through potential solution of the first-order ordinary differential equations (3.22). However, in the present case, where $\xi = 0$, both equations (3.22) attain linear forms; namely

$$\hat{G}\eta e^e \dot{\sigma}_R + \dot{e}^e\left(\eta\hat{G} - \frac{\rho_0}{\rho}\right)\sigma_R = -Ee^e\dot{e}^e,$$

$$\hat{G}\eta e\dot{\sigma}_D + \left[\dot{e}\left(\eta\hat{G} - \frac{\rho_0}{\rho}\right) + \xi\hat{G}^2\dot{\sigma}\right]\sigma_D = -E\left(e^e\dot{e}^v + e^v\dot{e}^e\right) - \eta\hat{G}\dot{\sigma}_R e^v. \tag{4.14}$$

Either of these is a first order linear ordinary equation and can evidently be brought into the standard form (4.8). Its closed form solution can thus be described in the standard manner (4.10). More practical future investigations will thus require considerable further computational work that will not only accompany the outlined solution but will also enlighten matters associated with the subsequent split of the work rate into its recoverable and dissipative parts.

5. Relevance to conventional linear viscoelastic solids

Conventional linear viscoelasticity models are based on the simplification

$$d^v = \dot{e}, \tag{5.1}$$

which implies that the rate of viscous strain and the rate of total strain are considered *a priori* identical. In this special case, the small strain constitutive equation (3.13) of the present model reduces to the following:

$$\sigma = \frac{\rho}{\rho_0} \left\{ Ee + \eta\dot{e} + \lambda\phi''\dot{\sigma}\left(\xi + \eta\frac{e}{\dot{e}} \right) \right\} . \tag{5.2}$$

Moreover, (2.7) implies that

$$\dot{e} = \lambda\frac{d\phi}{d\sigma}. \tag{5.3}$$

However, conventional viscoelasticity models do not make use of the continuity equation (2.2). They instead postulate *a priori* that the imposed small strain consideration is adequate to justify the approximation

$$\rho / \rho_0 = 1. \tag{5.4}$$

Moreover, classical models make no use of a viscous flow potential and, in this regard, do not make use of (5.3).

Hence, if (5.4) is assumed valid and, along with (5.3), the influence of the flow potential, ϕ, is also completely disregarded, then (5.2), reduces to

$$\sigma = Ee + \eta\dot{e}. \tag{5.5}$$

This is recognised as the constitutive equation of the Kelvin-Voight viscoelastic model shown schematically in Figure 2(a).

Nevertheless, in addition to the constitutive equation (5.5), the present model provides a corresponding internal energy function, namely

$$W\left(e,\dot{e}\right) = \frac{1}{2}Ee^2 + \eta e\dot{e}, \tag{5.6}$$

which is evidently obtained by inserting (5.1) into (4.7). The constitutive equation (5.5) is thus alternatively obtained with the use of

$$\sigma = \frac{\partial W}{\partial e}, \tag{5.7}$$

which is quoted through (3.6) in a similar manner. Namely, by imposing the approximation (5.4) and disregarding the noted influence of the flow potential, ϕ.

It is now recalled that the constitutive equations of a standard elastic spring or a viscous damper (dashpot) is obtained through (5.5) by simply setting $\eta = 0$ or $E = 0$, respectively. Constitutive equations of conventional viscoelastic solids which are more complicated than that of the Kelvin-Voight solid, may then still be built up in the manner employed in classical viscoelasticity. Namely, by (i) using appropriately several kinds of different combinations of one or more Kelvin-Voight solids with one or more springs and/or dashpots, and (ii) ignoring the concept of the internal energy function.

For instance, the well-known constitutive equation of the Kelvin's version of the standard, three-parameter viscoelastic solid depicted in Figure 2(b), namely

$$\sigma + \frac{\eta}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}e + \frac{E_2 \eta}{E_1 + E_2}\dot{e}, \qquad (5.8)$$

is evidently obtainable by combining in the usual well-known manner (e.g. [2, 3, 5]) the depicted pair of linear solids. Both solids make essentially use of the constitutive equation (5.5); for the first element E is replaced by E_1, while for the second E is replaced by E_2 and η is set equal to zero. However, the concept of the internal energy function must be ignored in this case, as there is no evident manner that a relevant form of W can be obtained and, in turn, lead the constitutive equation (5.8) through appropriate combination(s) of (5.7).

6. Conclusions

The generalised viscoelastic spring model presented in this communication is directly relevant to the behaviour of one-dimensional solid components susceptible to large or small viscoelastic deformation. In its most general form, the large deformation version of the model requires determination of three principal unknowns, namely the material density, ρ, the current placement function, x, and the Cauchy stress, σ, in a manner that satisfies a set of three principal differential equations; namely, the continuity equation (2.2), the equation of motion (2.17a) and the constitutive equation (2.18). The corresponding small strain version of the model is essentially described in the same manner, with the displacement, u, replacing the placement function, and the small strain constitutive equation (3.6) replacing (2.18). Even in the small strain case though, the constitutive equation is, in general, a differential rather than an algebraic equation.

In either case, derivation of the constitutive equation is underpinned by the fact that the internal energy stored in the viscoelastic spring material changes in time with, and therefore depends on both the observed actual strain and the unknown part of the strain-rate that is due to the anticipated viscous flow. Creation of the implied unknown part of the strain-rate, which is evidently the source of the observed inelastic deformation, is accordingly related with the action of a viscous flow potential, which, in turn, depends on the magnitude of stress.

In this manner, the presented generalised viscoelastic spring model makes successful simultaneous use of two different fundamental concepts met in the theories of Elasticity and Plasticity; namely, the concepts of the internal energy density/function and that of the yield/flow potential. Hence, along with [7], this model is regarded as another piece of evidence in support of the claim that hyperelastic, plastic and viscoelastic material behaviour have all the same source and can accordingly be based on and studied within a common theoretical framework.

It has indeed been shown that, unlike its classical viscoelasticity counterparts, the present generalised model does not need to postulate a priori the manner that the strain and its time rate are combined or related to form its constitutive equation. Instead, the model considers that this, as well as other important features of combined elastic and inelastic material behaviour, including the rate of energy dissipation, can become known *a posteriori* through the solution of a relevant, well-posed boundary value problem. The latter should always be completed by associating to the afore-mentioned set of three simultaneous differential equations an appropriate set of initial and boundary conditions.

This communication developed the presented viscoelastic model around the classical example of a cantilever spring. It accordingly completed description of the implied boundary value problem by employing the boundary conditions (2.9) and the initial conditions (2.19) in the general case of large deformation, or their counterparts (3.10) and (3.11), respectively, in the case that deformation is infinitesimally small. In this context, considerable attention was also given to the closed form solution obtained in Section 4, for a spring undergoing small viscoelastic strain under the influence of inelastic deformation underpinned by a quadratic form of the viscous flow potential.

That exact, closed form solution requires further consideration and study, and its computational implementation and treatment is regarded as a project of immediate priority and interest. Such a project should initially be expected to make better understood the relevance as well as the differences of this model

with its classical spring-dashpot counterparts and, hence, to expose practical aspects of its anticipated theoretical superiority. The latter is, in brief, underpinned by the facts that the proposed generalised viscoelastic spring model enables (i) *a posteriori* determination of the rule that its elastic and inelastic deformation parts are assembled, (ii) consideration of the levels of energy stored in its viscoelastic material, and (iii) through time account of corresponding levels of energy dissipation.

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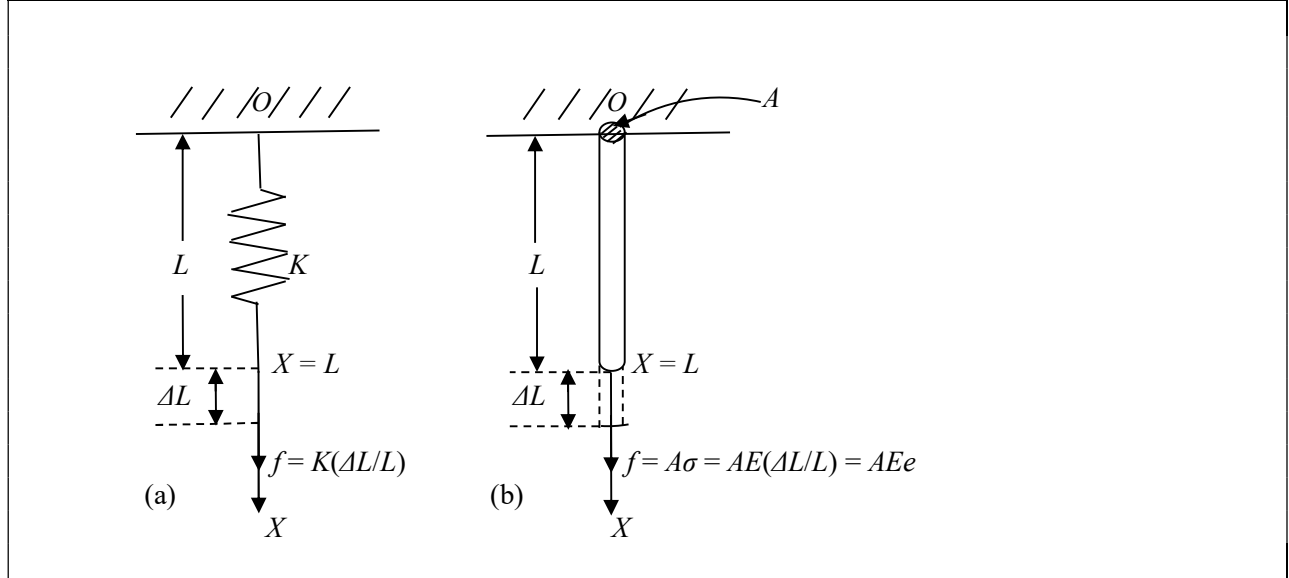17. Spencer, AJM, 1980. Continuum Mechanics. Longman, New York.

FIGURES

FIGURE 1: The known analogy between (a) a linearly elastic spring of elastic modulus K, and (b) a cantilever linearly elastic cylindrical rod of Young's modulus E, cross-sectional area A, and length L; f, σ and e represent externally applied force, axial normal stress and strain, respectively.
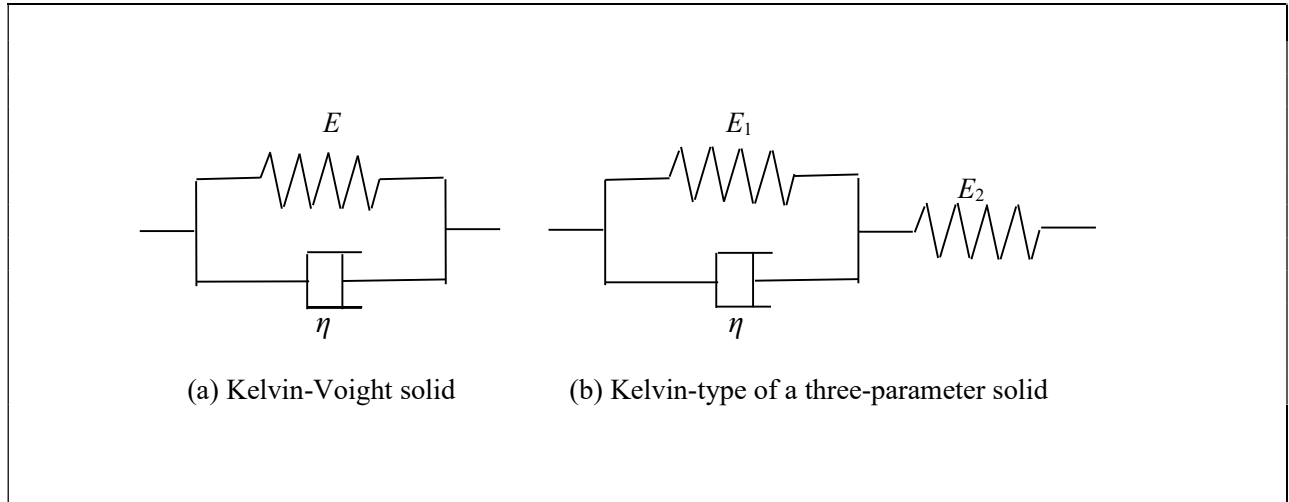

FIGURE 2: Two classical models of a viscoelastic solid met commonly in the classical viscoelasticity literature; E, E_1 and E_2 represent elastic spring stiffness moduli while η is the modulus of damper viscosity.

Media Summary:

THE GENERALISED VISCOELASTIC SPRING
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Summary
Existing viscoelasticity models presuppose the rule that relates elastic strain and viscous/inelastic flow during viscoelastic material deformation. The proposed generalised viscoelastic spring model describes one of the simplest relevant mathematical problems without employing *a priori* such a rule. Solution to this problem discovers that rule, and other characteristics of combined elastic and inelastic material behaviour, in *a posteriori* manner. This generalised model is underpinned by a novel combination of features and postulations used so far independently in the theories of elasticity and plasticity. It thus shows that all three subjects of Hyperelasticity, Plasticity and Viscoelasticity share a common theoretical background/framework.