

Dynamic Asset Allocation with Liabilities*

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Abstract

We develop an analytical solution to the dynamic multi-period portfolio choice problem of an investor with risky liabilities and time varying investment opportunities. We use the model to compare the asset allocation of investors who take liabilities into account, assuming time varying returns and a multi-period setting with the asset allocation of myopic ALM investors. In the absence of regulatory constraints on asset allocation weights, there are significant gains to investors who have access to a dynamic asset allocation model with liabilities. The gains are smaller under the typical funding ratio constraints faced by pension funds.

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1 Introduction

It is known since the work of [Samuelson \(1969\)](#) and [Merton \(1969; 1971; 1973\)](#) that long-term investors might judge risks very differently from short-term investors and hence hold different portfolios. Although [Merton \(1971; 1973\)](#) developed an analytical framework for understanding the portfolio demands of long-term investors, empirical implementation of his work lagged far behind due to the difficulty of solving Merton’s intertemporal model ([Campbell and Viceira, 2002](#)). The importance of liabilities in strategic asset allocation has also been long recognized by both practitioners and academics with analytical discrete-time solutions available only under static, one-period horizons. In this paper we develop an analytical solution to the dynamic multi-period portfolio choice problem of an investor under time-varying investment opportunities. We then use the model to compare the asset allocation of investors who ignore the time-varying properties of future asset risks and returns (myopic investors).

A brief taxonomy of multi-period portfolio choice solutions in an asset-only framework is provided in [Jurek and Viceira \(2011, JV hereafter\)](#). JV identify three types of approaches to optimal portfolio selection. First, optimal portfolios can be obtained as exact analytical solutions for special cases of the multi-period portfolio choice problem in a continuous time setting ([Kim and Omberg, 1996](#); [Sorensen, 1999](#); [Wachter, 2002](#); [Brennan and Xia, 2002](#); [Chacko and Viceira, 2005](#); [Liu, 2007](#)). Second, optimal portfolios can be determined with numerical methods ([Brennan et al., 1997](#); [Brandt, 1999](#); [Barberis, 2000](#); [Lynch, 2001](#); [Dammon et al., 2001](#); [Detemple et al., 2003](#); [Brandt et al., 2005](#); [Kojien et al., 2010](#)). Third, optimal portfolios can be computed in closed-form through approximate analytical methods ([Campbell and Viceira, 1999; 2001; 2002](#), [Campbell et al., 2003](#)). JV also discuss the shortcomings of these approaches and propose an analytical recursive approach which results in a closed-form solution for the dynamic multi-period portfolio choice problem.

Multi-period portfolio choice is particularly relevant for investors who invest with the objective to finance a long-term stream of liabilities. Best known examples of such investors are pension funds, foundations, and endowment funds. It is also relevant to individual investors who manage their own retirement funds or build portfolios to finance their children’s education. In this paper we study the portfolio choice problem of this type of investors, a problem that is generally referred to as Asset Liability Management (ALM hereafter).

[Leibowitz \(1987\)](#) and [Sharpe and Tint \(1990\)](#) are the first papers that incorporate liabilities in a single period context. Since then the literature on portfolio choice in a multi-period context in the presence of liabilities parallels developments in the asset-only portfolio choice

literature. In a continuous setting [Boulier et al. \(1995\)](#) formulate a dynamic programming model of pension fund behavior. [Sundaresan and Zapatero \(1997\)](#) formulate and solve a continuous time model for the strategic asset allocation of a defined benefit pension fund taking into account the marginal productivity of the employee under a constant investment opportunity set. [Rudolf and Ziemba \(2004\)](#) assume time-varying investment opportunities and develop a four-fund theorem for intertemporal surplus optimization.

A second strand of the literature, closer to asset-liability modeling practice,¹ uses stochastic programming techniques ([Cariño et al., 1998](#); [Cariño and Ziemba, 1998](#); [Geyer and Ziemba, 2008](#)) to find optimal asset weights. A third approach is based on simulation ([Binsbergen and Brandt, 2014](#)). These approaches are however subject to the criticisms expressed in JV in the context of asset-only portfolio choice. Stochastic programming techniques for example face the cost of tractability; the traditional numerical methods cannot handle more than a few state variables, magnifying the effects of estimation risk even further; and the ALM models carried out under a continuous time context have been mostly studied under a constant investment opportunity set.

Another strand of the ALM literature, closer in spirit to our paper, uses approximate analytical solutions and extends the asset-only models introduced by [Campbell and Viceira \(1999; 2001; 2002\)](#) and [Campbell et al. \(2003\)](#) by incorporating liabilities. [Hoevenaars et al. \(2008, HMSS hereafter\)](#) is a significant contribution in this literature. HMSS assume that an investor with risky liabilities chooses the optimal portfolio weights at the beginning of the investment horizon and rebalances to the initial allocation at a rebalancing time shorter than the investment horizon. [Barberis \(2000\)](#) describes the latter strategy as myopic rebalancing; it is myopic because the investor ignores the information available at the rebalancing time. Myopic investors maintain the initial static solution, through rebalancing, until the end of the horizon. In contrast, dynamic investors follow a dynamic investment strategy where the portfolio weights take into account changes in the investor's investment opportunity set. Using the myopic asset-liability model (M-ALM thereafter), HMSS find significant differences in the portfolio allocations of investors who ignore liabilities.

Our principal contribution in this paper is that we develop a model which results in a closed-form solution for the dynamic multi-period portfolio choice problem in the presence of liabilities (D-ALM thereafter). Our model nests the asset-only model of JV. It is analytically tractable and hence allows the study and interpretation of factors affecting optimal portfolio

¹ALM based on stochastic programming and used in practice include the Russel-Yasuda Kasai model described in [Cariño et al. \(1998\)](#) and the Towers Perrin-Tillinghast ALM approach presented in [Mulvey et al. \(2000\)](#).

choice in an ALM framework. The model we propose can accommodate any number of assets and state variables. To develop the proposed model we build on JV and HMSS. Our representative investor is endowed with power utility over the ratio of assets and liabilities, as opposed to over assets only, at a fixed horizon; she can invest in many risky and a riskless asset. The investment opportunity set is time-varying and captures the variation in risk premia, interest rates, and the state variables. The optimal investment policy that we derive has a simple structure; it is comprised of myopic and intertemporal hedging terms. The novel feature of our model is that we show analytically how the intertemporal hedging demand – which appears in the asset-only multi-period portfolio choice literature – is also driven by the correlation between the innovations of liabilities and the innovations in the changes of the state variables.

In our empirical investigation we take into account practical considerations. We incorporate short-sales or borrowing constraints, as in [Brandt et al. \(2009\)](#). We also consider a regulatory environment in which pension funds face minimum and maximum funding ratio constraints. The model we develop allows us to study the structure of optimal portfolios as well as the utility implications associated with holding these portfolios. In this respect, we provide new empirical insights that are useful both for ALM researchers but also for the practice of ALM. Furthermore, with the incorporation of inflation-indexed bonds in the list of potential investment assets, we are able to study the allocation to inflation-indexed bonds in the portfolios of long-term ALM investors. [Campbell and Viceira \(2002\)](#) argue that inflation-indexed bonds are important for long-term asset-only investors since both bonds and (rolling) short-term bills are subject to real interest rate variations over time. Inflation-indexed bonds are even more important for ALM investor with inflation sensitive liabilities.

Our empirical results can be summarized as follows. We find that optimal portfolios obtained using the D-ALM are different in many respects with optimal portfolios obtained for myopic ALM investors. Myopic investors hold portfolios which contain more equity and inflation-indexed bonds than D-ALM investors. The higher allocation to equities and indexed bonds is at the expense of a lower position in nominal bonds. This conclusion does not change when we impose short-sell or funding ratio constraints.

The differences in the composition of D-ALM optimal portfolios and M-ALM optimal portfolios have significant utility implications. In particular, we find that a M-ALM investor suffers large utility losses relative to the portfolios suggested by the D-ALM. The losses are increasing in the length of the investment horizon and decreasing in the degree of relative risk

aversion. The economic benefits of the D-ALM over the M-ALM strategy reflect the benefits from pursuing a dynamic investment strategy that exploits information about asset time-varying returns and risks and hedging abilities. The D-ALM provides the optimal strategic and tactical asset allocation portfolios for investors with multi-period horizons and liabilities. A moderate risk aversion M-ALM investor, assuming leverage is not allowed, is prepared to pay up to 1.14% per month to gain access to the D-ALM. The maximum management fee increases for more conservative investors. In the presence of minimum and maximum funding ratio constraints, like those faced by defined benefit pension funds in practice, the utility losses are smaller but remain significant. Imposing tight funding constraints reduces but does not eliminate the benefits from hedging dynamically the investment opportunity set and liabilities.

Comparing the composition of optimal portfolios obtained with the proposed model with optimal portfolios obtained for AM investors (as in JV), we find that the two approaches generate different asset allocations. While equity allocations are similar, nominal bond allocations tend to be lower for ALM investors compared to the allocations of AM investors. In contrast, ALM portfolios contain substantially more inflation-indexed bonds than AM portfolios, reflecting the liability hedging properties of inflation-indexed bonds. The differences in the composition of the optimal portfolios between the two approaches translate to substantial losses for a moderate and highly risk averse AM investor in the presence of liabilities which increase as the investment horizon increases.

The rest of the paper is organized as follows. Section 2 specifies the investment setting and derives the closed-form expression of asset weights for ALM optimal portfolios. Section 3 introduces the data and presents the results of the estimation of the dynamics of the available investment opportunities. Section 4 examines the structure of optimal portfolios obtained through the D-ALM and M-ALM strategies. Section 5 investigates the utility implications associated with the M-ALM strategy and Section 6 examines the implications for asset allocation and utility of funding ratio constraints and alternative liability benchmarks and we also benchmark key empirical results to JV. Finally, Section 7 concludes. The Appendix provides a detailed derivation of all the analytical results in the paper.

2 The setting

In this Section we outline the details of the dynamics of the investment opportunities that are available to our investor; an investor that determines her holdings such that she will be able

to finance a long-term stream of liabilities. We then define her multi-period portfolio choice problem. Finally we apply an analytical recursive approach which results in a closed-form solution and hence a closed-form expression for the optimal investment policy.

2.1 The dynamics of the available investment opportunities

We model the dynamics of the available investment opportunities through a VAR(1) specification as in [Campbell and Viceira \(2002\)](#).² We augment the investment opportunity set with the investor’s liabilities. Hence, our economy is one where expected returns, liabilities, and interest rates are time-varying and are jointly determined through:

$$\mathbf{z}_{t+1} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{z}_t + \mathbf{v}_{t+1} \quad (1)$$

In equation (1), \mathbf{z}_{t+1} denotes a column vector whose elements are the log returns on the asset classes we consider as potential investments, the liabilities (an appropriate proxy is considered as we discuss below), and the values of the state variables at time $t + 1$. $\mathbf{\Phi}_0$ is a vector of intercepts, $\mathbf{\Phi}_1$ is a square coefficient matrix, and \mathbf{v}_{t+1} is a vector of zero mean innovation process. We assume these innovations are homoskedastic and normally distributed, that is $\mathbf{v}_{t+1} \stackrel{i.i.d.}{\sim} N(0, \mathbf{\Sigma}_v)$. [Chacko and Viceira \(2005\)](#) argue that the persistence and volatility of risk do not affect the portfolio rule of long-term investors.

For ease of reference we can re-write the vector \mathbf{z}_{t+1} as:

$$\mathbf{z}_{t+1} = \begin{bmatrix} r_{tb,t+1} \\ \mathbf{x}_{A,t+1} \\ x_{L,t+1} \\ \mathbf{s}_{t+1} \end{bmatrix} \quad (2)$$

where $r_{tb,t+1}$ denotes the log real return on the T-bill; $\mathbf{x}_{A,t+1}$ is a vector of log real excess returns on all other assets (considered as potential investments) with respect to the real return on T-bills, that is $\mathbf{x}_{A,t+1} = r_{A,t+1} - r_{tb,t+1}$; $x_{L,t+1}$ is a vector of log excess returns on the liabilities with respect to the real return on T-bills, that is $x_{L,t+1} = r_{L,t+1} - r_{tb,t+1}$;

²[Campbell and Viceira \(2002\)](#) provide a comprehensive review of the applications of the VAR specification for modelling the intertemporal behaviour of asset returns and their application to the portfolio choice problem. Much of the dynamic portfolio choice literature utilizes a VAR(1) as a compromise between the size of the model and the estimation error. Other modelling approaches combine dynamic asset allocation with a regime-switching model which is beyond the scope of the paper (see for example [Ang and Bekaert \(2002\)](#) and [Graffund and Nilsson \(2003\)](#) among others).

and, finally, \mathbf{s}_{t+1} is a vector with the realizations of the state variables that are believed to be important in determining asset prices. All returns are defined in real terms in excess of realized inflation π_t . We can also write the variance–covariance matrix of the shocks \mathbf{v}_{t+1} as:

$$\Sigma_v = \begin{bmatrix} \sigma_{tb}^2 & \boldsymbol{\sigma}_{tbA}^\top & \boldsymbol{\sigma}_{tbL}^\top & \boldsymbol{\sigma}_{tbS}^\top \\ \boldsymbol{\sigma}_{tbA} & \boldsymbol{\Sigma}_{AA} & \boldsymbol{\sigma}_{AL}^\top & \boldsymbol{\Sigma}_{AS}^\top \\ \boldsymbol{\sigma}_{tbL} & \boldsymbol{\sigma}_{AL} & \sigma_L^2 & \boldsymbol{\sigma}_{LS}^\top \\ \boldsymbol{\sigma}_{tbS} & \boldsymbol{\Sigma}_{AS} & \boldsymbol{\sigma}_{LS} & \boldsymbol{\Sigma}_{SS} \end{bmatrix} \quad (3)$$

The elements on the main diagonal are σ_{tb}^2 , the variance of the real return on the T-bill; $\boldsymbol{\Sigma}_{AA}$, the variance–covariance matrix of unexpected excess asset returns; σ_L^2 , the variance of the unexpected excess liabilities returns; and $\boldsymbol{\Sigma}_{SS}$, the variance-covariance matrix of the shocks to the state variables. The off-diagonal elements are the covariances of the real return on the T-bill with excess returns on all other assets and liabilities, $\boldsymbol{\sigma}_{tbA}$ and $\boldsymbol{\sigma}_{tbL}$, respectively; the covariances of the real return on the T-bill and shocks to the state variables, $\boldsymbol{\sigma}_{tbS}$; the covariance of assets and liabilities $\boldsymbol{\sigma}_{AL}$; and the covariances of excess returns and liabilities with shocks to the state variables $\boldsymbol{\Sigma}_{AS}$ and $\boldsymbol{\sigma}_{LS}$, respectively.

2.2 The investor’s optimization problem

In the multi-period portfolio choice problem the investor targets a specific payoff at a given investment horizon and takes into account the changes in the anticipations for the distribution of asset returns or relevant factors. Investors that determine their holdings such that they will be able to finance a long-term stream of liabilities are concerned with the value of their assets relative to the value of their liabilities; not the absolute value of their assets. For example, a pension fund, should ensure that it covers future pension payments with their assets, without the need for further contribution. Therefore, within an ALM context, the investor’s utility optimization problem is expressed with respect to her funding ratio F_t at time t . The funding ratio is defined as the ratio of Assets (A) over Liabilities (L), i.e. the funding ratio at time t is $F_t = A_t/L_t$.³ The investor is assumed to have Constant Relative Risk Aversion (CRRA hereafter) preferences on the terminal funding ratio. We assume that at each decision point the investor’s liability has a constant maturity.⁴ Hence we specify her

³ Leibowitz et al. (1994), McCarthy and Miles (2013), Binsbergen and Brandt (2014), and HMSS are studies, among others in the ALM literature, that seek optimal asset allocations through maximization of the investor’s funding ratio utility.

⁴For pension funds that means that the fund is closed to new entrants and that the value of contributions is equal to the present value of the new liabilities created. Like HMSS we assume that the pension fund is in

utility function at the terminal date $t + K$ as:

$$u(F_{t+K}) = \begin{cases} \frac{1}{1-\gamma} F_{t+K}^{1-\gamma} & \text{for } \gamma \neq 1 \\ \log(F_{t+K}) & \text{for } \gamma = 1 \end{cases} \quad (4)$$

where γ is the relative risk aversion coefficient. We follow JV in that we assume that the investor chooses the portfolio weights $\alpha_{ALM,t+K-\tau}$ between times t and $t + K - 1$ so as to maximize the expected value of the utility function, that is:

$$\max E_t [u(F_{t+K})] \quad (5)$$

which we can re-write through equation (4) as:

$$\{\alpha_{ALM,t+K-\tau}\}_{\tau=K}^{\tau=1} = \operatorname{argmax} \frac{1}{1-\gamma} E_t [\exp \{(1-\gamma)r_{F,t \rightarrow t+K}\}], \text{ when } \gamma \neq 1 \quad (6)$$

$$\{\alpha_{ALM,t+K-\tau}\}_{\tau=K}^{\tau=1} = \operatorname{argmax} E_t [r_{F,t \rightarrow t+K}], \text{ when } \gamma = 1 \quad (7)$$

The subscript in the portfolio weight vector $\alpha_{ALM,t+K-\tau}$ in equations (6) and (7) corresponds to the time at which the portfolio weights are chosen and the superscript denotes the time remaining. $r_{F,t \rightarrow t+K}$ denotes the funding ratio log-return between time t and $t + K$, that is the logarithmic return of the asset portfolio $r_{A,t \rightarrow t+K}$ minus the logarithmic real return on the liabilities $r_{L,t \rightarrow t+K}$. To compute $r_{F,t+1}$, which is $r_{F,t+1} = r_{A,t+1} - r_{L,t+1}$ we use a standard log-linear approximation of $r_{A,t+1}$ following [Campbell et al. \(2003\)](#) and JV among others. [Campbell et al. \(2003\)](#) and JV use the following log-linear approximation to the portfolio return $r_{A,t+1}$, $r_{A,t+1} \approx \alpha_{AM,t}^\top (\mathbf{x}_{A,t+1} + \frac{1}{2}\sigma_A^2) - \frac{1}{2}\alpha_{AM,t}^\top \Sigma_{AA} \alpha_{AM,t} + r_{tb,t+1}$, where $r_{tb,t+1}$ denotes the T-bill rate, $\alpha_{AM,t}$ the allocation to risky assets and σ_A^2 is the vector consisting of the diagonal elements of the variance covariance matrix of the risky assets. The log return on the wealth portfolio is a linear combination of the excess returns, volatilities and covariances on the benchmark asset (i.e. T-bill rate) and the risky assets. [Campbell et al. \(2003\)](#) argue that the log-linear approximation to $r_{A,t+1}$ becomes increasingly accurate as the frequency of portfolio rebalancing increases and it is exact in the continuous limit. We subtract the return on liabilities (r_L) from both sides of the equation. Hence, the log-linear

a stationary state where the distribution of the age cohorts and the pension rights are constant over time.

approximation to the log funding ratio return $r_{F,t+1}$ can be expressed as:

$$r_{F,t+1} \approx \boldsymbol{\alpha}_{ALM,t}^\top \left(\mathbf{x}_{A,t+1} + \frac{1}{2} \boldsymbol{\sigma}_A^2 \right) - \frac{1}{2} \boldsymbol{\alpha}_{ALM,t}^\top \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t} - \mathbf{x}_{L,t+1} \quad (8)$$

where $\boldsymbol{\sigma}_A^2$ is the vector consisting of the diagonal elements of $\boldsymbol{\Sigma}_{AA}$. In equation (8) the benchmark asset is the investor's liability – hence the negative sign – whereas in [Campbell et al. \(2003\)](#) and JV the benchmark asset is the T-bill rate.

2.3 The optimal investment policy

We solve for the optimal dynamic optimization problem based on standard backwards recursion as in JV. The first step of this process involves the derivation of the portfolio rule (policy function) and the corresponding value function in the last period. In the second step we define the portfolio rule of the period preceding the last period as a function of the value and the policy function coefficients from the terminal period. The last step involves iterating this relationship to derive the solution to the ALM multi-period portfolio choice with dynamic rebalancing.

More specifically in the first step we define the optimal portfolio policy of an investor with one period remaining to the terminal date $t + K$:

$$\boldsymbol{\alpha}_{ALM,t+K-1}^{(1)} = \frac{1}{\gamma} \boldsymbol{\Sigma}_{AA}^{-1} \left(E_{t+K-1} [\mathbf{r}_{A,t+K} - r_{tb,t+K}] + \frac{1}{2} \boldsymbol{\sigma}_A^2 - (1 - \gamma) \boldsymbol{\sigma}_{AL} \right) \quad (9)$$

where

$$E_{t+K-1} [\mathbf{r}_{A,t+K} - r_{tb,t+K}] = \mathbf{H}_A (\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-1}) \quad (10)$$

and \mathbf{H}_A is a matrix operator that selects the rows corresponding to the vector of asset excess returns \mathbf{x}_A from the target matrix.

Equation (9) is the myopic or one-period mean-variance efficient portfolio rule. The optimal myopic portfolio has two components. A *speculative* component, which is the tangency portfolio, usually referred to as the Performance Seeking Portfolio (PSP):

$$\frac{1}{\gamma} \boldsymbol{\Sigma}_{AA}^{-1} \left(E_{t+K-1} [\mathbf{r}_{A,t+K} - r_{tb,t+K}] + \frac{1}{2} \boldsymbol{\sigma}_A^2 \right) \quad (11)$$

which depends on expected returns and the variance-covariance matrix of asset returns, and

a *hedging* component, the Liability Hedge Portfolio (LHP):⁵

$$\left(1 - \frac{1}{\gamma}\right) \Sigma_{AA}^{-1} \sigma_{AL} \quad (12)$$

which depends only on the variance–covariance matrix of asset returns and the covariance of asset returns with the liabilities.

Under the assumption of time-varying expected returns and homoskedasticity, the PSP changes with the investment opportunities in contrast to the LHP which remains constant across the investment opportunities. Log utility investors ($\gamma = 1$) hold only the PSP, while highly risk adverse investors ($\gamma \rightarrow \infty$) hold only the LHP.

In the second and third steps we derive the solution to the ALM multi–period portfolio choice with dynamic rebalancing (see ??):

$$\begin{aligned} \alpha_{ALM,t+K-\tau}^{(\tau)} &= \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(E_{t+K-\tau} [\mathbf{r}_{A,t+K-\tau+1} - r_{tb,t+K-\tau+1}] + \frac{1}{2} \sigma_A^2 - (1 - \gamma) \sigma_{AL} \right) \\ &- \left(1 - \frac{1}{\gamma}\right) \Sigma_{AA}^{-1} \Sigma_A \left(B_{ALM,1}^{(\tau-1)\top} + \left(B_{ALM,2}^{(\tau-1)} + B_{ALM,2}^{(\tau-1)\top} \right) E_{t+K-\tau} [\mathbf{z}_{t+K-\tau+1}] \right) \end{aligned} \quad (13)$$

where $\Sigma_A = \begin{bmatrix} \sigma_{tbA} & \Sigma_{AA} & \sigma_{AL}^\top & \Sigma_{AS}^\top \end{bmatrix}$ and $B_{ALM,1}^{(\tau-1)\top}$, $B_{ALM,2}^{(\tau-1)}$ and $B_{ALM,2}^{(\tau-1)\top}$ are functions of the remaining investment period, the coefficient risk aversion γ and the coefficients of the VAR system. The multi–period optimal portfolio obtained through equation (13) has two components too. The first component:

$$\frac{1}{\gamma} \Sigma_{AA}^{-1} \left(E_{t+K-\tau} [\mathbf{r}_{A,t+K-\tau+1} - r_{tb,t+K-\tau+1}] + \frac{1}{2} \sigma_A^2 - (1 - \gamma) \sigma_{AL} \right) \quad (14)$$

is the myopic component given by equation (9). The optimal myopic portfolio in equation (14) combines the PSP, i.e. equation (11), and the LHP, i.e. equation (12). It is a function of one–period expected returns and the conditional variance–covariance matrix of one–period returns of assets and liabilities.

The second component:

$$\left(1 - \frac{1}{\gamma}\right) \Sigma_{AA}^{-1} \Sigma_A \left(B_{ALM,1}^{(\tau-1)\top} + \left(B_{ALM,2}^{(\tau-1)} + B_{ALM,2}^{(\tau-1)\top} \right) E_{t+K-\tau} [\mathbf{z}_{t+K-\tau+1}] \right) \quad (15)$$

is the ALM–intertemporal hedging component. The ALM–intertemporal hedging component

⁵The corresponding portfolio in the AM framework is the Global Minimum Variance as defined in JV.

is the unique element of our model. It differs from the intertemporal hedging component which was first introduced by Merton (1969; 1971; 1973) under an AM context. It is the product of two factors. The first term $\Sigma_{AA}^{-1} \Sigma_A$ captures the ability of assets to hedge changes in the investment opportunities, taking also into account the instantaneous correlation with the liabilities, a correlation that it is not considered in the intertemporal hedging component of Merton. The second term, $\left(1 - \frac{1}{\gamma}\right) \Sigma_{AA}^{-1} \Sigma_A \left(B_{ALM,1}^{(\tau-1)\top} + \left(B_{ALM,2}^{(\tau-1)} + B_{ALM,2}^{(\tau-1)\top}\right) E_{t+K-\tau} [\mathbf{z}_{t+K-\tau+1}]\right)$, captures the effect of changes in investment opportunities on the investor's marginal utility. It corresponds to the ratio of the cross-partial derivative of the value function with respect to funding ratio and the vector of state variables, and, the product of funding ratio and the second derivative of the value function with respect to funding ratio.

For simplicity we can rewrite equation (13) as:

$$\boldsymbol{\alpha}_{ALM,t+K-\tau}^{(\tau)} = A_{ALM,0}^{(\tau)} + A_{ALM,1}^{(\tau)} \mathbf{z}_{t+K-\tau} \quad (16)$$

where $A_{ALM,0}^{(\tau)}$ and $A_{ALM,1}^{(\tau)}$ can be deduced from equation (13). Equation (16) shows that the solution to the ALM multi-period portfolio choice with dynamic rebalancing is an affine function of the vector of state variables $\mathbf{z}_{t+K-\tau}$. This suggests that as in JV $A_{ALM,0}^{(\tau)}$ and $A_{ALM,1}^{(\tau)}$ (and hence also the optimal investment policy), depend on the time remaining to the terminal horizon date but are themselves independent of time.

Some special cases of equation (16) deserve further attention. We show in ?? that for a log utility investor ($\gamma = 1$) or a constant investment opportunity set ($\Phi_1 = 0$) the dynamic portfolio choice problem converges to a myopic portfolio choice problem. In addition, an infinite risk averse investor ($\gamma \rightarrow \infty$) invests in the least-risky portfolio which is independent of the vector of unconditional expected returns; under a constant investment opportunity set the infinite risk averse investor holds the LHP.

One limitation of analytical approaches is that they cannot accommodate portfolio constraints explicitly. In our empirical analysis we adopt Brandt et al. (2009). We impose short-sales constraints by truncating the negative portfolio weights in equation (16) at zero and renormalizing the optimal weights in order to sum to one, that is:

$$\boldsymbol{\alpha}_{ALM,i,t}^+ = \frac{\max[0, \boldsymbol{\alpha}_{ALM,i,t}]}{\sum_{j=1}^N \max[0, \boldsymbol{\alpha}_{ALM,j,t}]} \quad (17)$$

3 Data and model calibration

In this Section, we discuss the investment opportunity set we consider and the data we use in our empirical analysis. We additionally discuss the estimation procedure and the estimates of the fitted dynamics of the available investment opportunities. Finally, we present the term structure of risk of the asset classes as well as the term structure of the correlation of the asset classes with liabilities.

3.1 Assets, liabilities and state variables

The framework we propose is a general one and hence its empirical implications can be investigated with any number of asset classes and state variables. In the empirical implementation of the model we present later we assume that the investor has access to equities, nominal bonds, cash and inflation-linked bonds.

The empirical implementation of the D-ALM portfolio choice problem also requires a measure of investors' liabilities. The liability structure however depends on the particular circumstances of the investor. From an individual investor's perspective for example the retirement portfolio's objective is to maintain the investor's standard of living after retirement. We can therefore assume that the investor aims for a positive real return and hence that her "liability" is similar to an inflation-indexed bond. From the perspective of a defined benefit pension fund we can assume that the age cohorts and vested pension rights per cohort are constant over time and future benefits are fully indexed to inflation. We also assume that future contributions equal the present value of the liability created. Under these conditions the liabilities of the pension fund could be proxied by the returns of an inflation-indexed portfolio with duration equal to the duration of the pension's liabilities. The choice of inflation-indexed bonds as a proxy for liabilities allows us to gain empirical insights with the proposed model in the context of a very general, yet representative, type of liability. The drawback of this choice is that risks that may affect the dynamics of liabilities, such as actuarial, longevity and demographic risks, are not taken into account. Future work should focus exclusively on the integration of such risks with market risks in the strategic asset allocation problem (see, e.g. [Bisetti et al. \(2016\)](#)).

When index-linked bonds do not exist or there is a short history of indexed bond prices, index-linked bond prices could be estimated from nominal bond yields. This is the approach taken in studies based on US data where indexed-bonds (TIPS) first appeared in 1998. Before the introduction of TIPS, [Campbell and Shiller \(1997\)](#) use short-term real interest

rates and the expectations theory of the term structure to create a hypothetical series for index-linked bond returns. [Kothari and Shanken \(2004\)](#), and HMSS use the observed yield on long term nominal bonds and assumptions about expected inflation and the inflation premium to calculate real yields for indexed bonds. This approach generates a long history of data which however it is subject to estimation errors.⁶ To isolate potential estimation errors of this kind, we use information about inflation-indexed bonds prices and yields from the UK. The first indexed-linked bond was introduced in the UK in the budget of 1981 and the first bond was traded in 1982. Prices and yields are available since then providing the longest history of real yields compared to other developed markets.

We source data from DataStream. All data are in pounds. Stock returns are represented by the UK Datastream value-weighted stock index (including distributions). We proxy for nominal bonds through the DataStream’s UK twenty-year constant maturity bond, for inflation-indexed bonds through the FTSE British Government Index Linked All Maturities index and for the cash we use the UK interbank 3-months rate.

As proxies of the state variables that determine the dynamics of stock excess returns, bond excess returns and interest rates as per equation (1) we use a set of common variables used in previous research.⁷ These include the dividend-price ratio (DY) on the UK value-weighted stock index; the short-term nominal interest rate; the term spread (the difference between the ten-year treasury constant maturity rate and the three-month T-Bill rate) and the ex-post real rate of return on a 3 month Treasury bill (the difference between the return on the three-month UK Treasury bills and the inflation rate). We consider the ex-post real rate as the real return on an investable asset (cash) as well as a state variable. JV find that the utility losses of dynamic investors who do not rebalance their portfolios on a monthly basis increase as the rebalancing frequency decreases. We therefore base our analysis on monthly data covering the period January 1982 to December 2014.

Table 1 presents the descriptive statistics of monthly nominal returns of the asset classes and state variables included in the VAR model for the entire period. The returns of the assets classes are expressed in nominal terms. The mean and the standard deviation are annualized. Equities achieved the highest annual nominal mean return (11.45%), albeit

⁶[Campbell and Viceira \(2009\)](#) study the differences between synthetic and actual index-linked returns. They find that synthetic yields are lower on average and less volatile compared to actual ones. Additionally, they observe a correlation equal to 0.70 for the period 1982–2008.

⁷Key papers in this literature are [Campbell and Shiller \(1988; 1991; 1998; 2005\)](#),[Fama and Schwert \(1977\)](#), [Campbell \(1987\)](#),[Glosten et al. \(1993\)](#),[Fama and Bliss \(1987\)](#), [Fama and French \(1989\)](#), [Campbell et al. \(2003\)](#), [Campbell and Viceira \(2005\)](#). The evidence on predictability are controversial ([Bossaerts and Hillion, 1999](#); [Welch and Goyal, 2008](#)).

only slightly higher than the return of nominal bonds (10.23%). During the period, the realized equity premium vis a vis nominal long-term bonds were significantly lower to the long-term historical average. Inflation-indexed bonds had a return of 7.54% per annum, lower than equities and nominal bonds. Equity returns are negatively skewed and have the highest kurtosis among all asset classes. Volatility is much higher for stocks (15.71%) than for nominal bonds (9.44%) and real bonds (6.89%); nominal bonds are more volatile than indexed bonds.

3.2 VAR estimates and term structure of risk and correlation

We present the estimates of the parameters of equation (1) in Table 2.⁸ Panel A of Table 2 shows the slope coefficient parameter estimates of the VAR system. Panel B of Table 2 reports the standard deviations (in the diagonal) and the correlations (off-diagonal elements) of the innovations. As in previous studies, we find that excess stock returns are predictable by the dividend-price ratio and the short term real interest rate. For the excess nominal bond return its lagged return and the term spread are important, statistically, variables that predict future bond premia. A high spread and nominal bond return are associated with higher future nominal bond returns. The short term nominal yield is a strong predictors of the real interest rate. Finally, an increase in short term nominal interest rate is associated with lower returns for indexed bonds and by implication liabilities. The R^2 for the equations of the excess market and bond as well as the liabilities returns are fairly low but typical of estimates reported in the predictability literature. [Campbell and Thompson \(2008\)](#) argue that while R^2 increases at lower frequencies, even low values are associated with economically meaningful outcomes. Evidence in [Rapach and Zhou \(2013\)](#) for stock returns suggests that even a small degree of predictability can produce substantial utility gains.

The dynamics of the state variables we document are consistent with the empirical literature ([Brennan et al., 1997](#); [Campbell and Viceira, 2002](#); [Campbell et al., 2003](#)). The coefficients of the dividend-price ratio, the nominal short interest rate, and the term spread suggest persistent autoregressive processes. The maximal eigenvalue of the coefficient matrix equals 0.9933, implying that the VAR is stationary with a half-life of about 7 years. The system is stable, but close to being integrated of order one. We do not make any adjustment in the coefficient estimates of the VAR system for possible biases related to near

⁸Incorporating parameter and model uncertainty is beyond the scope of this paper but results in [Hoevenaars et al. \(2014\)](#) suggest that taking this uncertainty into account will increase the allocation to nominal bonds at the expense of equity allocation as the investment horizon increases.

non-stationarity of some time series (Brennan et al., 1997; Campbell and Viceira, 2002; Campbell et al., 2003). Campbell and Yogo (2006) argue that the processes for the state variables are highly persistent rather than unit-root processes.

The results presented in Panel B of Table 2 suggest that the innovations to the dividend–price ratio are negatively correlated with unexpected excess returns on equities. The negative correlation between shocks to the dividend–price ratio and shocks to stock excess returns combined with the positive relation between the dividend–price ratio and subsequent stock returns is consistent with mean reversion in stock prices. A decrease in the dividend–price ratio following an increase in stock prices is associated with high returns today but followed by higher expected returns in the future. Mean reversion in stock prices is also indicated by the positive relation between the short term real rate and future stock returns and the negative correlation between shocks to returns and interest rates. The term spread reinforces the mean reversion of stock returns. The most important variables for nominal bond excess returns are the term spread and the short term real rate. The effects of these variables on the future behavior of bond returns are opposite. The term spread predicts positively future bond returns while shocks to the term spread and bond returns are negatively correlated. The combined effect is consistent with mean reversion in bond prices. In contrast, the positive relation between bond returns and real rates and the positive correlation between innovations to the real rate and bond returns suggest mean aversion. Which effect dominates is an empirical issue. The VAR estimates for the real interest rate are consistent with findings in the literature (see Campbell and Viceira (2005) and HMSS) and are indicative of increased risk with the horizon (mean aversion). Finally, future real bond returns and hence liabilities are positively related with short term real rates and negatively related to short term nominal rates. Shocks to indexed–bond returns are strongly negatively correlated with shocks to the nominal short yield.

The VAR model provides a convenient framework to calculate the conditional term structure of volatilities and correlation of asset returns, assuming that the VAR model captures correctly the dynamics of asset returns.

As shown in the Appendix, Figure B1 presents the term structure of risk of T-bills, equities, nominal and real bonds/liabilities. We find that equities are less risky in the long run, whilst nominal bond return volatility exhibits mild mean aversion as in Campbell and Viceira (2005). T-bills are riskier in the long run. Liability returns, as proxied by real bond returns, exhibit also a mean averting behavior, implying that liability risk is higher in the long run. The latter magnifies the importance of considering liabilities in a dynamic asset

allocation policy.

Figure B2 shows the correlation structure of asset returns with real returns on liabilities. In the short-term, equities are weakly correlated with liabilities; in the medium and long term the correlation declines to about zero. Real returns on nominal bonds with constant maturity are positively correlated with returns on liabilities at all investment horizons; at short horizons the correlation is about 55% and increases in the long run to 80%.⁹ Finally, T-bills are a bad liability hedge at all investment horizons, due to their negative correlation with liabilities.

4 Portfolio allocations in the presence of liabilities

In this Section we provide an investigation of the D-ALM implications for portfolio management. We are interested in particular at the differences between the optimal mean-variance portfolios generated by the dynamic asset liability model (D-ALM) and the optimal portfolios generated by a single period mean-variance approach that incorporates liabilities (the M-ALM). For each model, we derive two sets of optimal weights. One is derived through equation (16), and a second set of optimal weights is derived with short-selling constraints through equation (17).

To compute the allocations we assume a long-term expected real return on the T-bill equal to 2% per year, a return on the value-weighted stock index of 7% per year, a return on the nominal bond index of 4% per year, and a return on the real bond of 3.5% per year. We believe that these premia represent a reasonable compromise between historical averages (based on data covering the period January 1982 to December 2014) and historical/forward-looking estimates reported in [Dimson et al. \(2008; 2011\)](#). [Campbell and Viceira \(2002\)](#) make similar assumptions, i.e. riskless real interest rate of 2% per year, equity premium of 4% per year. We also assume an inflation risk premium of 0.5% based on [Kothari and Shanken \(2004\)](#). To generate the asset allocation weights we use the long term expected returns and the VAR model to generate time varying forecasts of returns, combined them with equation (16) to get one set of asset allocation weights per future time period for the dynamic asset-liability investor. In particular, we impose long-term views on asset returns and risk premia by adjusting the constant term of the proposed VAR model. The constant term can be derived as $\Phi_0^* = (\mathbf{I} - \Phi_1) \boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is a $(n \times 1)$ vector containing the long-term means of

⁹[Kothari and Shanken \(2004\)](#) find that the correlation between nominal and real bonds is 0.53 and 0.58 when nominal and real returns are used respectively.

the risk premia and state variables. Hence, the transformed constant Φ_0^* and the estimated coefficient matrix Φ_1 (shown in Table 2) are used to generate the asset allocation weights. Finally, the state variables are at their long run historical mean.

For both the M-ALM and the D-ALM cases we assume that portfolio decisions are made monthly. The main difference between M-ALM and D-ALM is that under the dynamic asset-liability setting the optimal weights today depend on the time-varying expectations of returns and risks. In contrast, under M-ALM portfolio weights today take into account the expected return, volatility and correlations to the horizon but assumes that the investor does not take into account monthly time varying opportunities. Dynamic investors take into account not only the short-term return and risk characteristics of assets but also their ability to hedge liabilities and provide a hedge against changes in the opportunity set.

Table 3 reports asset class weights for the D-ALM and M-ALM investor for different investment horizons (from 1 to 300 months) and different attitudes towards risk (different risk aversion parameters). Panel A of Table 3 tabulates the allocations for portfolios with no constraints on portfolio weights. Note that for the one month horizon both dynamic and myopic investors hold exactly the same portfolios. For reasonable risk aversion parameters, both myopic and dynamic investors would borrow to invest in the risky assets. The negative weight of cash reflects the increased volatility and the negative correlation of short term interest rates with liabilities.

Low and moderate risk averse dynamic investors invest in general less in equities than myopic investors. In contrast, dynamic investors invest more in nominal bonds for different risk aversion parameter and horizons. The allocation to inflation-indexed bonds tend to be higher for dynamic rather than myopic investors as the horizon increases.

For the myopic investor, an increase in the investment horizon is associated with increased demand for equities and nominal bonds and decreased demand for index linked bonds. For the myopic investor asset allocation weights depend on expected returns, risks and the correlation between asset and liabilities. The increasing weight of equities as the horizon lengthens reflects the decrease in equity volatility due to mean reversion of stock prices. The increasing weight in nominal bonds reflects two opposing forces: the good liability hedging properties of nominal bonds and the increased volatility of bonds at longer horizons. More conservative investors tend to put a large part of their portfolio to indexed bonds at the expense of both equities and nominal bonds, assets that are only partial hedges against liabilities.

Dynamic investors, in addition to their concerns to match liabilities, also take into account the predictability of asset returns. The main differences in portfolio weights compared to

the myopic investor is that as the investment horizon increases, the dynamic investor invests more in equities and nominal bonds financed by more borrowing than the myopic investor. The additional weight in bonds and equities reflects the ability of nominal bonds and stocks to hedge changes in the opportunity set. Indexed bond weights are almost stable at all investment horizons.

To gain further insights on the drivers of the differences in portfolio allocations and the role of state variables in their variation we turn our eye to Tables 2 and 3. For equities the most important driver of equity allocations is the dividend yield. An increase in the market dividend yield is associated with an increase in expected equity returns and an increase in the intertemporal hedging demand for equities. For nominal bonds, a negative innovation in the term spread reduces expected returns and increases the hedging demand for bonds

Panel B in Table 3 shows optimal weights for dynamic and myopic investors assuming borrowing is not allowed. Borrowing to invest may be unrealistic for most investors and in particular for pension funds. We observe that short-selling constraints have a significant impact on portfolio allocations. Both dynamic and myopic investors increase the weight of stocks as the horizon increases. Equities are attractive because volatility falls. For example dynamic investors with risk aversion equal to 5 increase the equity weights from 28% when the investment horizon is one month to 48% when the horizon is 300 months. For the myopic investors equity holdings increase from 28% to 63%. Myopic investors assume a lower volatility as a result of equity mean reversion than dynamic investors who use the full term structure of volatility estimates and therefore use on average higher volatility estimates in the asset allocation construction process.

Myopic investors with moderate risk aversion ($\gamma = 5$) allocate 29% in nominal bonds in the short term and the weight decrease to 17% for long term investors. Similarly, dynamic investors start with 29% at the monthly horizon and invest 25% at the 300 month horizon. The decrease in bond holdings as the horizon increases reflects the mean aversion properties of nominal bond prices discussed in Section 3. Myopic investors invest less than dynamic investors since they use, on average, a higher volatility estimate (mean aversion) in the asset allocation process. The asset allocation weight of indexed bonds is slightly over 40% and decreases to 20% at 25 years investment horizon. More conservative investors (higher risk aversion parameters) switch from equities and nominal bonds to indexed bonds as expected. The fall in bond portfolio weights as the horizon increases reflects the mean averting properties of nominal bonds whilst the increase in equity weights reflects the mean reversion properties of stocks.

Overall, the evidence presented in this Section suggests that dynamic optimal portfolios are different to myopic optimal portfolios primarily in their allocations to nominal bonds (for all investors and moderate to long investment horizons). Dynamic investors tend to allocate less in equities and more in indexed bonds. We study the economic implications of these differences in the next Section.

5 The economic value of dynamic rebalancing

The empirical results presented above suggest that a dynamic ALM investor holds a different portfolio compared with the portfolio held by a myopic ALM investor. In this Section we investigate the utility implications associated with these differences.

We measure the economic implications of different allocations through the certainty equivalent (CE hereafter) of the funding ratio F_t , which is defined as follows:

$$F_{CE} = u^{-1} \left(E \left[u \left(\widetilde{F}_T \right) \right] \right) \quad (18)$$

To compute the CE, we first simulate 10,000 paths of the VAR system for the monthly frequency. We compute the optimal asset allocation policy through equation (16), the realized terminal funding ratio, and the realized utility of the terminal funding ratio along each path. Finally, we compute the average realized utility across paths and based on equation (18) we obtain the CE. The value function with periods remaining is:

$$\max \frac{1}{1-\gamma} E_{t+K-\tau} \left[\exp \left\{ (1-\gamma) \sum_{i=1}^{\tau} r_{F,t+K-\tau} \left(\boldsymbol{\alpha}_{ALM,t+K+i-(\tau+1)}^{(\tau+1-i)} \right) \right\} \right] \quad (19)$$

The utility loss of a myopic investor who does not use the D-ALM is calculated as the percentage difference between the CE of the M-ALM strategy and the CE of the D-ALM strategy. The asset allocation of the myopic investor will be the same throughout a given investment horizon while the optimal asset mix of the dynamic investor will be time varying reflecting changes in expected returns and risks. The management fee quantifies the ‘profit’ or ‘loss’ that a dynamic achieves by holding a different portfolio to the constant mix adopted by the myopic investor.

We report the results of this analysis in Table 4, considering an investor who faces short selling constraints. Panel A reports the utility losses of the M-ALM strategy over D-ALM. Panel B reports the largest monthly management fee an investor who now uses the M-ALM

would be prepared to pay to gain access to the D-ALM. JV highlight that fees re-scale the utility losses to facilitate comparisons across investment horizons.

As Panel A shows there are significant utility losses from not using the D-ALM. Utility losses increase as the horizon lengthens but conservative investors lose more than more risk tolerant investors. Our analysis suggests that a M-ALM investor with experiences utility losses that range from -37.76% (5-year investment horizon) to -96.82% (25-year investment horizon). The differences in the CE we document translate to management fees that range from 0.79% (5-year investment horizon) to 1.14% (25-year investment horizon) per month (Panel B). Additionally, the management fee increases significantly as investor risk tolerance falls. For a twenty five year investment horizon the maximum monthly management fee ranges from 0.73% (for a risk aversion of 2) to 1.16% (for a risk aversion of 20). The management fees for M-ALM investors who allow leverage in order to gain access to the D-ALM strategy range from 1.83% (5-year investment horizon) to 6.49% (25-year investment horizon) per month.

Our analysis to this point considers inflation-indexed bonds as possible investment assets. In what we present below we relax this assumption. Investing in inflation-indexed bonds might not be possible or desirable for a number of reasons. First, inflation indexed bonds are less liquid than nominal bonds (D'Amico et al., 2014; Gürkaynak et al., 2010; Pflueger and Viceira, 2011). Second, despite the sharp growth of the linker market,¹⁰ a number of countries do not issue inflation-indexed bonds. Third, many investors might regard inflation indexed bonds unacceptable investments when the real yield to maturity is close to zero or even negative.¹¹

We report the results of this analysis in Table 5, considering an investor who faces short selling constraints. Panel A reports the utility losses of the M-ALM strategy over the D-ALM strategy. Panel B reports the largest monthly management fee an investor who now uses the M-ALM would be prepared to pay to gain access to the D-ALM. We observe significant utility losses from not using the D-ALM strategy. A moderate risk averse M-ALM investor for example with experiences utility losses that range from -22.74% (5-year investment horizon) to -89.43% (25-year investment horizon). These losses translate to a maximal monthly management fees that range from 0.43% (5- year investment horizon) to 0.75% (25-year investment horizon) per month.

¹⁰Based on the Barclays Capital World Government Inflation-Linked Bond Index (WGILB) in April 2012, the global market value of inflation-indexed bonds was \$2.0 trillion: US (\$866 billion), UK (\$549 billion), France (\$235 billion), Italy (\$132 billion).

¹¹As of May 2014 the real yield of 10-year TIPS is 0.37% and UK inflation-indexed bond -0.40%.

The evidence in this Section overall suggests that the economic benefits of pursuing optimal portfolios with the D-ALM model are substantial. The outperformance over the M-ALM strategy is due to the implicit pursuance of tactical asset allocation within the strategic asset allocation framework. The D-ALM model exploits the predictability in the investment opportunity set while the M-ALM does not. This predictability has been shown to provide sizeable economic benefits even in instances when it is marginal.

6 Additional Analysis

In this section we study the effects of minimum and maximum funding constraints and alternative liability benchmarks that include equities on asset allocation weights and utility for M-ALM and D-ALM investors. We also study the asset allocation and utility differences of a liability driven investor (dynamic ALM model) and an investor without liabilities (dynamic AM model, as in JV).

6.1 The economic value of dynamic rebalancing with funding ratio constraints

The empirical evidence presented in the previous section shows substantial utility benefits if an investor with liabilities switches from M-ALM to D-ALM. The benefits are reduced but remain significant when the investor operates under short-selling constraints. Investing under short-selling constraints represents the typical investment framework in which individual investors invest. Defined benefit pension funds face, in addition to short-selling constraints, constraints with respect to the minimum funding ratio. In many countries regulations or internal pension fund rules dictate that when the funding ratio of the pension fund is close to, or below, a minimum threshold the sponsor of the pension fund is asked to increase contributions until it reaches the target value. There are also regulations and customary practices in many countries which “force” pension funds whose assets are much higher than liabilities to cease contribution until the surplus disappears (contribution holiday). In this Section we investigate the utility losses associated with the presence of funding ratio constraints.

[Pugh \(2006\)](#) provides a comprehensive review on the regulations of the pension funds in the OECD countries. Minimum fund requirements whether imposed by legislation or not aim to protect plan member benefits. For example, for Dutch pension funds the minimum funding level is set at 105%, while the maximum (or target) funding ratio is on average 130%.

For UK pension plans there is no minimum funding level imposed by regulation but each pension fund is required to adopt a statutory funding objective. In the UK pension funds face a maximum funding level constrain which dictates that pension fund assets should not exceed 105% of accrued liabilities.

The imposition of minimum and maximum funding level constraints means that outcomes below the minimum funding level and above the maximum funding level are not possible. In our analysis we assume that the fund sponsor increases immediately contributions when the asset liability ratio reaches the minimum fund level.¹² Similarly when the asset to liability ratio reaches the maximum funding level we assume that the fund sponsor ceases contributions until the funding ratio falls below the maximum funding level. In the simulations we pursue to address these issues we assume (a) that the asset liability ratio should not fall below 1 and (b) that the target minimum funding level is 90% (asset/liability ratio of 90%). In both cases that contributions cease when the asset to liability ratio reaches 135%. We recognize that the nature of our model does not allow to internalize the funding ratio constraint in the optimal solution as in approaches that involve numerical analysis (see, e.g. [Berkelaar and Kouwenberg \(2010\)](#)).

The effect of minimum–maximum funding ratios on the distribution of funding ratios is significant. The median funding ratio for the M–ALM strategy increases from 97% in the five–year horizon to a modest 113% at the end of twenty five years. Increasing the minimum funding ratio from 90% to 100% and keeping the same maximum funding ratio (135%) leads as expected to better future funding ratios. The median funding ratio increases from 102% in the five–year horizon to 108% at the end of twenty five years (see Table B1, Panels B and C, Appendix). Imposing a min–max funding ratio constraint of 90%–135% reduces considerably the median funding ratios for the D–ALM investor too. At the end of twenty five years the median funding ratio is 129%. Similarly if the minimum funding ratio is 100% the median funding ratio at the twenty five year horizon is slightly higher at 130% (see Table B2, Panels B and C, Appendix).

In relative terms, the M–ALM strategy suffers from lower minimum funding ratios. For instance, the minimum funding ratio for the M–ALM strategy is 0.01 at the twenty five year horizon, while the minimum funding ratio for the D–ALM strategy is 0.91 over the same horizon. Moreover, in the absence of funding ratio constraints the median/mean funding ratio of the D–ALM strategy is always higher than the median/mean funding ratio of the

¹²In practice the pension fund with the sponsor and the advice of the pension fund’s actuary agree a multi–year plan of increased contributions or a contribution holiday until the deficit or surplus are eliminated.

M-ALM strategy, across all investment horizons. The latter are consistent with the results we present earlier, i.e. that the D-ALM strategy exhibits utility benefits over the M-ALM for a constrained investor in the absence of regulatory constraints. Imposing a min-max funding ratio constrain of 90% (100%) -135% reduces considerably the difference of the median/mean funding ratios between the two strategies.

To investigate the economic value of the D-ALM strategy in the presence of funding ratio constraints we calculate the utility loss from using the M-ALM (instead of the D-ALM) strategy as well as the associated management fee with results reported in Table 6 and Table 7 respectively. For example, a M-ALM investor with $\gamma = 5$ experiences utility losses that range between -14.70% (5-year investment horizon) and -17.25% (25-year investment horizon) for a min-max funding ratio constraint of 90% -135%. The utility losses are lower when the asset liability ratio is required to stay in the 100% - 135% range. Similarly, from the results reported in Table 7 we show that the largest monthly management fee an investor who currently uses the M-ALM would pay to gain access to the D-ALM ranges from 0.25% (5-year horizon) to to 0.06% (25-year horizon).

Collectively, the evidence in this Section suggests that imposing tight funding constraints reduces but does not eliminate the benefits from hedging dynamically the investment opportunity set and liabilities.

6.2 Asset Liability Management vs. Asset Management: Asset Allocation and Utility Implications

In this Section we examine the structure of the optimal portfolios and the utility implications of the proposed ALM model compared with the AM model developed by JV. The data inputs are the same as those described in section 4.

Table 8 reports the asset allocation at any investment horizon ranging from 1 to 300 months (25 years) and risk-aversion levels ranging from 2 to 500 ($\gamma = 2, 5, 10, 20, 500$) for both the dynamic ALM and AM models. Panel A of Table 8 tabulates the allocations for portfolios with no constraints on portfolio weights. Panel B in Table 8 presents the optimal weights assuming borrowing is not allowed. The 1-month mean percentage portfolio allocation corresponds to the one-period (myopic) asset allocation.

We begin our discussion with the case of unconstrained portfolio weights. For low and moderate levels of risk-aversion equity allocations are similar for both AM and ALM models. Equity allocations increase as the investment horizon increases and decrease as risk aversion increases. The allocation to equities when $\gamma = 5$ for example ranges from 0.56 (0.56) in 6

months to 1.61 (1.62) in 300 months for an ALM (AM) investor.

We observe significant differences in the allocations to inflation-indexed bonds. These differences are almost entirely attributable to the myopic portfolio. The myopic allocation to real bonds for an investor with for example is 73% and only increases to 89% for the same investor with a twenty five year investment horizon. On the other hand, the allocation to real bonds for the AM investor who ignores liabilities is very low (even negative) both across different levels of risk-aversion and investment horizons. For an ALM investor the allocation to inflation indexed bonds is increasing in the degree of risk aversion. [Brennan and Xia \(2002\)](#) also find that optimal real terminal wealth portfolios are tilted towards inflation-indexed bonds as risk-aversion increases, and are entirely invested in inflation-indexed bonds in the limit.

AM and ALM investors allocate similar amounts to nominal bonds for low levels of risk aversion and moderate levels up to five-year investment horizon, but the differences become material for relatively high risk aversion levels. ALM optimal portfolios comprise lower allocations to nominal bonds than AM optimal portfolios. The lower allocation to nominal bonds in the ALM optimal portfolios seems to come “at the expense” of higher allocation to inflation-indexed bonds relative to AM optimal portfolios. The increasing allocation to inflation indexed bonds reflects the myopic and intertemporal hedging demand for inflation-indexed bonds by an ALM investor to hedge real liabilities.

The allocation to T-bills is negative regardless of the length of the investment horizon and the level of risk aversion in ALM optimal portfolios; T-bills are a bad liability hedge asset class since their correlation with liabilities is negative at all investment horizons. On the other hand, AM optimal portfolios with moderate and high levels of risk aversion are heavily invested in T-bills, especially at short horizons.

Assuming for short-selling constraints, both dynamic ALM and AM investors increase their allocation to stocks as the investment horizon increases. ALM optimal portfolios comprise lower allocations to equities than AM optimal portfolios for moderate and high levels of risk aversion. For example dynamic ALM (AM) investors with risk aversion equal to 5 increase the equity weights from 28% (45%) when the investment horizon is one month to 48% (56%) when the horizon is 300 months.

AM investors with moderate risk aversion ($\gamma = 5$) allocate 45% in nominal bonds in the short term and the allocation decreases to 39% for long term investors. ALM investors start with 29% at the monthly horizon and invest 25% at the 300 month horizon. The asset allocation to real bonds for AM investors is almost negligible at all investment horizons and

levels of risk aversion. For ALM investors is slightly over 40% and remains constant over the horizon.

Our analysis so far finds that the composition of the optimal portfolios that we obtain when we use the proposed model is different from the composition of the optimal portfolios obtained through the model in JV. Hence, we need to investigate the utility implications that are associated with these differences. Our analysis utilizes the certainty equivalent of the funding ratio, as defined in section 5. For the AM strategy (JV), we compute the optimal allocation policy, the realized terminal funding ratio, and the realized utility of the terminal funding ratio along each path. We therefore consider that the AM investor has the same liabilities as the ALM investor.

Table 9 reports the utility losses of the D-AM strategy over D-ALM (Panel A) and the largest monthly management fee an investor who now uses the D-AM would be prepared to pay to gain access to the D-ALM (Panel B). We consider an investor who faces short selling constraints. The utility losses for investor with $\gamma = 5$ who allocates her wealth according to JV, range from -0.25% (1-month investment horizon) to -15.75% (300-months investment horizon). Accounting for liabilities, under short-sale constraints, makes little difference to the asset allocation and utility of AM and ALM investors with low to moderate risk aversion ($\gamma < 5$). For pension funds with young membership and few retired members, liabilities do not play an important role in their asset allocation decision. For more risk averse investors ($\gamma > 5$) liabilities play a very important role in the determination of the optimal asset mix. At an investment horizon of 300 months for example, the utility losses from disregarding liabilities range from -15.75% ($\gamma = 5$) to -64.95% ($\gamma = 20$). The differences in CE overall translate to management fees that range between 0.06% and 0.35% per month or between 0.72% and 4.20% per annum, respectively.

Collectively, we observe material differences between optimal ALM and optimal AM portfolios in inflation-indexed bonds and T-Bills (for all investors), and in equities and nominal bonds (for moderate to highly risk-averse investors). HMSS find also significant differences in the portfolio allocations of myopic asset-liability investors who ignore liabilities. Furthermore, the economic benefits of pursuing optimal portfolios with our model are significant for moderate and highly risk averse investors. The economic benefits over the JV strategy arise from the intertemporal liability hedging portfolio. This portfolio is a unique element of our model. As we highlight earlier, it takes into account the correlation between the innovations of liabilities and the innovations in the changes of the state variables.

6.3 Alternative Liability Benchmark

The asset allocation of investors with liabilities depends critically on the properties of the investor's liabilities. In our model liabilities are exogenously determined following the approach of other papers in the academic literature (Hoevenaars et al., 2008; Amenc et al., 2009; Binsbergen and Brandt, 2014). In practice a pension fund's liabilities will be a function of many parameters defined in the pension fund rules. Partial or full indexation of pensions in payment and the particular mix of active, deferred and retired members are two of the most important characteristics of a pension's liabilities.

Index-linked bonds will be the appropriate benchmark for a pension fund's fully indexed Accumulated Benefit Obligations (ABO). It will also be a good proxy for a mature pension fund's Projected Benefit Obligations (PBO). For defined benefit plans PBO liabilities will depend on economic productivity. To hedge part of PBO liabilities Black (1989), Lucas and Zeldes (2006; 2009), Waring (2011) among others suggest that equities might provide a better hedge assuming that wage growth and equities are positively correlated.¹³

For mature pension funds with full or partial indexation of pensions in payment a portfolio of index-linked bonds and equities would be a more appropriate liability benchmark. The exact mix of equities and index-linked bonds will depend on the particular characteristics of the pension fund but a 30% equity, 70% index-linked bonds portfolio represents a reasonable benchmark for a mature pension fund where wage growth does not affect the pensions of retired employees.

Table 10 shows the asset allocation of a dynamic and myopic investor with liabilities, proxied by a portfolio of 30% equities and 70% index-linked bonds, assuming short sales are not allowed. When equities are part of the liability benchmark the investor as expected allocates more to equities, at the expense of index-linked bonds compared with an investor whose liabilities are exclusively index-linked bonds. Both dynamic and myopic investors invest sizeable part of their portfolio in equities. The percentage allocated to equities decreases from over 50% for low risk aversion dynamic investors to 30% for very conservative investors. For myopic, low risk aversion, long-term investors the percentage invested in equities is close to 80%. For moderate risk aversion ($\gamma = 5, 10$) the weight invested in equities is between 42% ($\gamma = 5$, horizon 1 month) to 52% ($\gamma = 10$, horizon 300 months). Investors with 100% index-linked liabilities invest less in equities and more in index-linked bonds except when

¹³Empirical evidence on the relation between wages and equities are controversial. Benzoni et al. (2007) find that wages and equities are highly correlated over long horizons. Lustig and Van Nieuwerburgh (2008) report a negative relation.

risk aversion is low and therefore liabilities are not important in the asset allocation decision. The percentage allocated to nominal bonds is very similar under both liability benchmarks. In unreported results, we find that the new liability benchmark does not change significantly the utility gains of the D-ALM over the M-ALM strategy.

Excluding inflation-linked bonds from the investor's investment opportunity set reduces marginally the weight of equities but the main effect is a substitution of inflation-linked bonds with nominal bonds. An investor with average risk aversion ($\gamma = 5$) invests over 50% of her portfolio in nominal bonds (detailed allocation results available from the authors upon request).

7 Conclusions

In this paper we develop a closed-form solution for the dynamic multi-period portfolio choice problem in the presence of liabilities. The model is analytically tractable allowing the study and interpretation of factors affecting optimal portfolio choice in an ALM framework. The model can accommodate investment opportunity sets which comprise any number of assets and state variables. It also allows for time-varying risk premia and interest rates.

Investors who invest their portfolios with the objective to meet future liabilities, taking time-varying opportunities into account results in different portfolios compared with investors who do not. Access to a dynamic asset liability model provides significant utility benefits to a myopic investor. The benefits are reduced but not eliminated when the investor faces short-sell constraints and in particular the ability to borrow. The model should be of particular interest to individual investors with liabilities who invest under short-sell constraints.

Under pension fund regulation and internal rules, pension funds usually operate within funding ratio constraints. Pension fund strategic asset allocation in practice tend to follow a myopic asset allocation strategy that is re-visited every three to five years and usually ignores time-varying opportunities. Imposing funding ratio constraints reduces but do not eliminate the benefits of exploiting return predictability and hedging dynamically the investment opportunity set and liabilities.

We believe that there are several promising directions for future research. While we assume that our investor is endowed with a power utility of real end-of-period funding ratio for a finite horizon, it would be valuable to consider an ALM investor who takes into account intermediate consumption or labor income ([Campbell et al., 2003](#); [Rapach and Wohar,](#)

2009). Another extension would be to investigate the optimal asset allocation and the utility implications of a dynamic investor who faces different types of liabilities and longevity and demographic risks.

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Table 1: Summary Statistics

This Table reports summary statistics for the period 1982:01 to 2014:12 (396 data points). Data are sourced from DataStream. The Series Name corresponds to the mnemonic available from DataStream. Panel A present the characteristics of the *traditional asset classes* (log nominal return on 3- month T-bill (r_{tb}), log ex post real return on a 3 - month T-bill ($realr_{tb}$), log nominal return on a value weighted market portfolio (r_s), log nominal return on a 20 -year nominal bond (r_b) and the log nominal return on an inflation-indexed bond (r_{ib})). Panel B presents the *state variables* (term spread (ts), log dividend price ratio (log dp) and the nominal yield 3-month (y_{nom})) and Panel C presents the *inflation rate* (log cpi). The returns of the assets classes are expressed in nominal terms. The mean and standard deviation are annualized. The remaining statistics are on a monthly basis.

Panel A. Traditional Assets									
	Mean	S.D	Max	Min	Skew	Kurt	Source	Series Name	Initial Date
r_{tb}	6.61%	1.17%	1.37%	0.04%	0.28	-0.69	DataStream	LDNIB3M(RI)	29/01/1982
$realr_{tb}$	3.57%	1.67%	1.59%	-2.27%	-0.30	1.56	DataStream	-	29/01/1982
r_s	11.45%	15.71%	14.00%	-29.63%	-1.17	5.24	DataStream	TOTMKUK(RI)	29/01/1982
r_b	10.23%	9.44%	10.63%	-9.03%	0.14	0.72	DataStream	BMUK20Y(RI)	29/01/1982
r_{ib}	7.54%	6.89%	8.75%	-6.95%	0.52	2.36	DataStream	BGILALL(DSRI)	29/01/1982
Panel B. State Variables									
ts	2.38%	5.83%	3.51%	-4.87%	-0.46	-0.01	DataStream	-	29/01/1982
log dp	1.31	0.22	1.81	0.82	0.11	-0.73	DataStream	TOTMKUK(PE)	29/01/1982
y_{nom}	6.62%	4.05%	16.14%	0.50%	0.27	-0.71	DataStream	-	29/01/1982
Panel C. Other Variables									
log cpi	3.05%	1.49%	3.32%	-0.97%	1.13	7.78	DataStream	UKCONPRCF	29/01/1982

Table 2: VAR Estimation

Panel A reports parameter estimates of the VAR system $z_{t+1} = \Phi_0 + \Phi_1 z_t + v_{t+1}$ with variables: real returns of 3-month T-bill ($realr_{tb}$) and the excess real returns of 20-year bonds ($realx_b$), stocks ($realx_s$), inflation-indexed bonds ($realx_{ib}$), dividend-price ratio ($\log dp$), nominal 3-month T-bill (y_{nom}), term spread (ts). t-statistics are reported in parentheses and are computed using Newey-West standard errors. The last column contains the R^2 and the p-value of the F-statistic for no predictability in parentheses. The VAR estimates are obtained using monthly data from 1982:01 to 2014:12 (396 data points). Panel B reports the residual correlation matrix of the VAR system $z_{t+1} = \Phi_0 + \Phi_1 z_t + v_{t+1}$ with variables: 3-month T-bill, 20-year nominal bonds, stocks, inflation-indexed bonds, dp ratio, nominal 3-month T-bill, term spread. Diagonal entries correspond to monthly residual standard deviations; off-diagonal entries correspond to their correlations.

Panel A. VAR Estimates								
	$realx_{s,t}$	$realx_{b,t}$	$realx_{ib,t}$	ts_t	$\log dp_t$	$y_{nom,t}$	$realr_{tb,t}$	R^2
$realx_{s,t+1}$	0.0545 (0.794)	0.0700 (0.629)	-0.0501 (-0.312)	0.0000 (-0.007)	0.0432 (2.183)	-0.2138 (-1.425)	0.9964 (1.911)	4.04% (0.000)
$realx_{b,t+1}$	-0.0343 (-1.128)	0.1322 (1.882)	-0.1246 (-1.142)	0.0019 (1.925)	0.0080 (0.802)	-0.0023 (-0.028)	0.1638 (0.528)	4.77% (0.000)
$realx_{ib,t+1}$	0.0147 (0.397)	0.1447 (2.321)	-0.1308 (-1.479)	-0.0002 (-0.213)	0.0062 (0.724)	-0.1098 (-1.541)	0.0776 (0.338)	5.49% (0.000)
ts_{t+1}	-0.4131 (-0.950)	0.8124 (0.750)	-1.0740 (-0.634)	0.9172 (39.097)	0.4074 (3.118)	-3.5353 (-3.080)	-2.8244 (-0.534)	94.25% (0.000)
$\log dp_{t+1}$	-0.0595 (-0.771)	-0.1570 (-1.269)	0.1456 (0.749)	0.0010 (0.345)	0.9446 (38.414)	0.2975 (1.582)	-0.9558 (-1.687)	95.21% (0.000)
$y_{nom,t+1}$	0.0087 (2.003)	-0.0267 (-2.498)	0.0299 (1.544)	0.0007 (2.570)	-0.0049 (-3.493)	1.0311 (78.085)	-0.0031 (-0.062)	98.90% (0.000)
$realr_{tb,t+1}$	-0.0053 (-1.212)	0.0055 (0.534)	0.0080 (0.562)	0.0000 (0.209)	-0.0024 (-1.629)	0.0645 (6.271)	0.0347 (0.843)	23.88% (0.000)

Panel B. VAR Error Correlations and Standard deviations							
	$realx_s$	$realx_b$	$realx_{ib}$	ts	$\log dp$	y_{nom}	$realr_{tb}$
$realx_s$	0.0443	0.1246	0.2163	0.0007	-0.9138	-0.1397	-0.0050
$realx_b$		0.0264	0.5428	-0.3149	-0.1329	-0.3703	0.0583
$realx_{ib}$			0.0196	-0.2041	-0.2407	-0.2309	-0.0284
ts				0.4028	0.0152	-0.6518	-0.0997
$\log dp$					0.0485	0.1188	-0.0230
y_{nom}						0.0042	0.0888
$realr_{tb}$							0.0042

Table 3: Optimal Portfolio Choice; Dynamic ALM vs. Myopic Rebalancing ALM

Panel A shows the dynamic (D-ALM) and the myopic (M-ALM) strategy of an asset-liability investor who plans to invest in the four asset classes: T-bills, equities, nominal and inflation-indexed (I-I) Bonds for an investment horizon ranging from 1 to 300 months and for five levels of risk aversion (2, 5, 10, 20, 500), when portfolio weights are unconstrained. Panel B shows the dynamic (D-ALM) and the myopic (M-ALM) rebalancing strategy of an asset-liability investor who plans to invest in the four asset classes: T-bills, equities, nominal and inflation-indexed (I-I) Bonds for an investment horizon ranging from 1 to 300 months and for five levels of risk aversion (2, 5, 10, 20, 500), when portfolio weights are constrained. The case of the one-month investment horizon corresponds to the myopic investor. γ denotes the investor's coefficient of relative risk aversion. Weights may not add up to one due to rounding.

Panel A. Unconstrained Weighting Scheme										
D-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	-1.74	-1.89	-2.08	-2.39	-2.87	-3.19	-3.33	-3.38	-3.40
	Equities	1.19	1.32	1.48	1.76	2.13	2.20	2.21	2.22	2.22
	Nominal Bonds	1.22	1.15	1.16	1.17	1.28	1.52	1.64	1.68	1.70
	I-I Bonds	0.33	0.42	0.44	0.46	0.46	0.47	0.48	0.48	0.48
5	T-bills	-0.70	-0.79	-0.93	-1.18	-1.69	-2.09	-2.26	-2.32	-2.35
	Equities	0.47	0.56	0.68	0.91	1.35	1.53	1.58	1.60	1.61
	Nominal Bonds	0.49	0.44	0.44	0.44	0.49	0.69	0.79	0.83	0.85
	I-I Bonds	0.73	0.79	0.81	0.83	0.85	0.87	0.89	0.89	0.89
10	T-bills	-0.35	-0.40	-0.48	-0.64	-0.99	-1.30	-1.44	-1.50	-1.52
	Equities	0.24	0.29	0.35	0.50	0.82	0.99	1.04	1.06	1.07
	Nominal Bonds	0.24	0.22	0.22	0.21	0.23	0.35	0.43	0.46	0.47
	I-I Bonds	0.90	0.91	0.92	0.94	0.96	0.97	0.91	0.98	0.98
20	T-bills	-0.17	-0.20	-0.25	-0.33	-0.54	-0.74	-0.84	-0.88	-0.89
	Equities	0.12	0.14	0.18	0.26	0.46	0.58	0.62	0.63	0.64
	Nominal Bonds	0.12	0.11	0.11	0.10	0.11	0.18	0.22	0.24	0.25
	I-I Bonds	0.93	0.95	0.96	0.96	0.98	0.99	1.00	1.00	1.00
500	T-bills	-0.01	-0.01	-0.01	-0.01	-0.02	-0.03	-0.04	-0.04	-0.04
	Equities	0.00	0.01	0.01	0.01	0.02	0.03	0.03	0.03	0.03
	Nominal Bonds	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01
	I-I Bonds	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

M-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	-1.74	-1.73	-2.00	-2.44	-3.08	-3.99	-4.68	-5.17	-5.51
	Equities	1.19	1.35	1.61	2.13	3.19	4.10	4.57	4.85	5.05
	Nominal Bonds	1.22	1.33	1.40	1.45	1.37	1.35	1.35	1.34	1.35
	I-I Bonds	0.33	0.05	-0.01	-0.14	-0.48	-0.46	-0.23	-0.03	0.12
5	T-bills	-0.70	-0.69	-0.80	-0.98	-1.23	-1.59	-1.87	-2.07	-2.20
	Equities	0.47	0.54	0.64	0.85	1.28	1.64	1.83	1.94	2.02
	Nominal Bonds	0.49	0.53	0.56	0.58	0.55	0.54	0.54	0.54	0.54
	I-I Bonds	0.73	0.62	0.60	0.54	0.41	0.42	0.51	0.59	0.65
10	T-bills	-0.35	-0.35	-0.40	-0.49	-0.62	-0.80	-0.94	-1.03	-1.10
	Equities	0.24	0.27	0.32	0.43	0.64	0.82	0.91	0.97	1.01
	Nominal Bonds	0.24	0.27	0.28	0.29	0.27	0.27	0.12	0.27	0.27
	I-I Bonds	0.87	0.81	0.80	0.77	0.70	0.71	0.75	0.79	0.82
20	T-bills	-0.17	-0.17	-0.20	-0.24	-0.31	-0.40	-0.47	-0.52	-0.55
	Equities	0.12	0.13	0.16	0.21	0.32	0.41	0.46	0.48	0.50
	Nominal Bonds	0.12	0.13	0.14	0.14	0.14	0.13	0.13	0.13	0.13
	I-I Bonds	0.93	0.91	0.90	0.89	0.85	0.85	0.88	0.90	0.91
500	T-bills	-0.01	-0.01	-0.01	-0.01	-0.01	-0.02	-0.02	-0.02	-0.02
	Equities	0.00	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
	Nominal Bonds	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	I-I Bonds	1.00	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00

Panel B. Constrained Weighting Scheme										
D-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.43	0.46	0.48	0.52	0.55	0.53	0.51	0.51	0.50
	Nominal Bonds	0.45	0.40	0.38	0.35	0.33	0.36	0.38	0.38	0.39
	I-I Bonds	0.12	0.15	0.14	0.13	0.12	0.11	0.11	0.11	0.11
5	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.28	0.31	0.35	0.42	0.50	0.50	0.49	0.48	0.48
	Nominal Bonds	0.29	0.25	0.23	0.20	0.18	0.22	0.24	0.25	0.25
	I-I Bonds	0.43	0.44	0.42	0.38	0.32	0.28	0.27	0.27	0.27
10	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.18	0.20	0.24	0.31	0.41	0.43	0.43	0.43	0.43
	Nominal Bonds	0.18	0.15	0.15	0.13	0.11	0.15	0.17	0.18	0.19
	I-I Bonds	0.64	0.64	0.61	0.56	0.47	0.42	0.40	0.39	0.39
20	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.10	0.12	0.15	0.20	0.30	0.33	0.34	0.34	0.34
	Nominal Bonds	0.10	0.09	0.09	0.08	0.07	0.10	0.12	0.13	0.13
	I-I Bonds	0.79	0.79	0.77	0.72	0.63	0.57	0.54	0.53	0.53
500	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.00	0.01	0.01	0.01	0.02	0.03	0.03	0.03	0.03
	Nominal Bonds	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01
	I-I Bonds	0.99	0.99	0.99	0.99	0.98	0.97	0.96	0.96	0.96

M-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.43	0.49	0.54	0.60	0.70	0.75	0.77	0.78	0.78
	Nominal Bonds	0.45	0.49	0.46	0.40	0.30	0.25	0.23	0.22	0.21
	I-I Bonds	0.12	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.02
5	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.28	0.32	0.36	0.43	0.57	0.63	0.64	0.63	0.63
	Nominal Bonds	0.29	0.31	0.31	0.29	0.25	0.21	0.19	0.18	0.17
	I-I Bonds	0.43	0.37	0.33	0.28	0.18	0.16	0.18	0.19	0.20
10	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.18	0.20	0.23	0.29	0.40	0.46	0.47	0.48	0.48
	Nominal Bonds	0.18	0.20	0.20	0.19	0.17	0.15	0.14	0.13	0.13
	I-I Bonds	0.64	0.60	0.57	0.52	0.44	0.39	0.39	0.39	0.39
20	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.10	0.11	0.13	0.17	0.24	0.29	0.31	0.32	0.33
	Nominal Bonds	0.10	0.11	0.12	0.12	0.10	0.10	0.09	0.09	0.09
	I-I Bonds	0.79	0.77	0.75	0.71	0.65	0.61	0.60	0.59	0.59
500	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.00	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
	Nominal Bonds	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	I-I Bonds	0.99	0.99	0.99	0.99	0.98	0.98	0.98	0.98	0.97

Table 4: Utility Implications and Management Fee for Investors with Liabilities

Panel A presents the utility losses of the myopic (M-ALM) strategy relative to the dynamic (D-ALM) policy, in an constrained weighting scheme. The results are obtained by Monte Carlo simulation using 10,000 VAR paths based on the method of antithetic variates, with the same path set being used to evaluate the utility loss in all cases. The CE of the funding ratio is computed by evaluating the mean utility realized across the simulated paths and computing $F_{CE} = u^{-1} \left(E \left[u \left(\widetilde{F}_T \right) \right] \right)$. Panel B reports the largest monthly management fee which a constrained investor who behaves myopically within an ALM framework would pay to gain access to the dynamic asset-liability management strategy, when portfolio weights are constrained. In Panel A the utility losses are computed as the percentage difference between the CE of the myopic (M-ALM) and the CE of the dynamic (D-ALM) strategy (base strategy). γ denotes the investor's coefficient of relative risk aversion. In Panel B the management fee is computed as $fee = 1 - (1 + WelfareLoss)^{1/horizon}$.

Panel A. Utility Losses for Constrained M-ALM					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	-30.82%	-55.60%	-71.95%	-82.36%	-89.06%
5	-37.76%	-67.18%	-84.09%	-92.13%	-96.82%
10	-48.65%	-75.08%	-87.86%	-93.08%	-96.83%
20	-46.63%	-77.52%	-91.18%	-94.64%	-93.95%

Panel B. Maximal Monthly Fee for Constrained M-ALM					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	0.61%	0.67%	0.70%	0.72%	0.73%
5	0.79%	0.92%	1.02%	1.05%	1.14%
10	1.10%	1.15%	1.16%	1.11%	1.14%
20	1.04%	1.24%	1.34%	1.21%	1.16%

Table 5: Utility Implications and Management Fee for Investors with Liabilities when inflation indexed bonds are excluded from the asset universe list

Panel A presents the utility losses of the myopic (M-ALM) strategy relative to the dynamic (D-ALM) policy, in an constrained weighting scheme. The results are obtained by Monte Carlo simulation using 10,000 VAR paths based on the method of antithetic variates, with the same path set being used to evaluate the utility loss in all cases. The CE of the funding ratio is computed by evaluating the mean utility realized across the simulated paths and computing $F_{CE} = u^{-1} \left(E \left[u \left(\widetilde{F}_T \right) \right] \right)$. Panel B reports the largest monthly management fee which a constrained investor who behaves myopically within an ALM framework would pay to gain access to the dynamic asset-liability management strategy, when portfolio weights are constrained. In Panel A the utility losses are computed as the percentage difference between the CE of the myopic (M-ALM) and the CE of the dynamic (D-ALM) strategy (base strategy). γ denotes the investor's coefficient of relative risk aversion. In Panel B the management fee is computed as $fee = 1 - (1 + WelfareLoss)^{1/horizon}$. Inflation-indexed bonds are excluded from the asset universe list of the investor.

Panel A. Utility Losses for Constrained M-ALM					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	-24.02%	-45.92%	-62.86%	-74.87%	-83.19%
5	-22.74%	-51.58%	-74.99%	-83.88%	-89.43%
10	-31.94%	-61.85%	-73.42%	-82.58%	-88.78%
20	-21.96%	-61.53%	-69.06%	-78.27%	-79.21%

Panel B. Maximal Monthly Fee for Constrained M-ALM					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	0.46%	0.51%	0.55%	0.57%	0.59%
5	0.43%	0.60%	0.77%	0.76%	0.75%
10	0.64%	0.80%	0.73%	0.73%	0.73%
20	0.41%	0.79%	0.65%	0.63%	0.52%

Table 6: Utility Implications for Investors with Liabilities in the Presence of Maximum and Minimum Funding Ratio Constraints

Panel A presents the utility losses of the constrained myopic (M-ALM) strategy relative to the dynamic (D-ALM) policy, in the presence of minimum and maximum funding ratio constraints 90% and 135%, respectively. Panel B presents the utility losses of the constrained myopic (M-ALM) strategy relative to the dynamic (D-ALM) policy, in the presence of minimum and maximum funding ratio constraints 100% and 135%, respectively. The results are obtained by Monte Carlo simulation using 10,000 VAR paths based on the method of antithetic variates, with the same path set being used to evaluate the utility loss in all cases. The CE of the funding ratio is computed by evaluating the mean utility realized across the simulated paths and computing $F_{CE} = u^{-1} \left(E \left[u \left(\widetilde{F}_T \right) \right] \right)$. The utility losses are computed as the percentage difference between the CE of the myopic (M-ALM) and the CE of the dynamic (D-ALM) strategy (base strategy). γ denotes the investor's coefficient of relative risk aversion.

Panel A. Utility Losses for constrained M-ALM (90%-135%)					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	-14.58%	-15.95%	-16.20%	-16.24%	-16.26%
5	-14.70%	-16.96%	-17.27%	-17.26%	-17.25%
10	-14.05%	-17.97%	-18.57%	-18.45%	-18.39%
20	-11.48%	-17.37%	-18.80%	-18.63%	-18.50%

Panel B. Utility Losses for constrained M-ALM (100%-135%)					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	-10.21%	-10.98%	-11.15%	-11.16%	-11.18%
5	-10.25%	-11.38%	-11.52%	-11.49%	-11.47%
10	-9.93%	-12.07%	-12.24%	-12.10%	-12.04%
20	-8.13%	-12.26%	-13.06%	-12.79%	-12.59%

Table 7: Management Fee for Investors with Liabilities in the Presence of Maximum and Minimum Funding Ratio Constraints

Panel A reports the largest monthly management fee which an investor who behaves myopically within an ALM framework would pay to gain access to the dynamic asset-liability management strategy, in the presence of minimum and maximum funding ratio constraints 90% and 135%, respectively. Panel B reports the largest monthly management fee which an investor who behaves myopically within an ALM framework would pay to gain access to the dynamic asset-liability management strategy, in the presence of minimum and maximum funding ratio constraints 100% and 135%, respectively. The management fee is computed as $fee = 1 - (1 + WelfareLoss)^{1/horizon}$. γ denotes the investor's coefficient of relative risk aversion.

Panel A. Maximal Monthly Fee for constrained M- ALM (90%-135%)					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	0.26%	0.14%	0.10%	0.07%	0.06%
5	0.26%	0.15%	0.11%	0.08%	0.06%
10	0.25%	0.16%	0.11%	0.08%	0.07%
20	0.20%	0.16%	0.12%	0.09%	0.07%

Panel B. Maximal Monthly Fee for constrained M- ALM (100%-135%)					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	0.18%	0.10%	0.07%	0.05%	0.04%
5	0.18%	0.10%	0.07%	0.05%	0.04%
10	0.17%	0.11%	0.07%	0.05%	0.04%
20	0.14%	0.11%	0.08%	0.06%	0.04%

Table 8: Optimal Portfolio Choice; Dynamic ALM vs. Dynamic AM

Panel A shows the dynamic strategy of an asset-liability investor (D-ALM) and the dynamic strategy of an asset-only investor (D-AM) who plans to invest in the four asset classes: T-bills, equities, nominal and inflation-indexed (I-I) Bonds for an investment horizon ranging from 1 to 300 months and for five levels of risk aversion (2, 5, 10, 20, 500), when portfolio weights are unconstrained. Panel B shows the dynamic strategy of an asset-liability investor (D-ALM) and the dynamic strategy of an asset-only investor (D-AM) who plans to invest in the four asset classes: T-bills, equities, nominal and inflation-indexed (I-I) Bonds for an investment horizon ranging from 1 to 300 months and for five levels of risk aversion (2, 5, 10, 20, 500), when portfolio weights are constrained. The case of the one-month investment horizon corresponds to the myopic investor. γ denotes the investor's coefficient of relative risk aversion. Weights may not add up to one due to rounding.

Panel A. Unconstrained Weighting Scheme										
D-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	-1.74	-1.89	-2.08	-2.39	-2.87	-3.19	-3.33	-3.38	-3.40
	Equities	1.19	1.32	1.48	1.76	2.13	2.20	2.21	2.22	2.22
	Nominal Bonds	1.22	1.15	1.16	1.17	1.28	1.52	1.64	1.68	1.70
	I-I Bonds	0.33	0.42	0.44	0.46	0.46	0.47	0.48	0.48	0.48
5	T-bills	-0.70	-0.79	-0.93	-1.18	-1.69	-2.09	-2.26	-2.32	-2.35
	Equities	0.47	0.56	0.68	0.91	1.35	1.53	1.58	1.60	1.61
	Nominal Bonds	0.49	0.44	0.44	0.44	0.49	0.69	0.79	0.83	0.85
	I-I Bonds	0.73	0.79	0.81	0.83	0.85	0.87	0.89	0.89	0.89
10	T-bills	-0.35	-0.40	-0.48	-0.64	-0.99	-1.30	-1.44	-1.50	-1.52
	Equities	0.24	0.29	0.35	0.50	0.82	0.99	1.04	1.06	1.07
	Nominal Bonds	0.24	0.22	0.22	0.21	0.23	0.35	0.43	0.46	0.47
	I-I Bonds	0.90	0.91	0.92	0.94	0.96	0.97	0.91	0.98	0.98
20	T-bills	-0.17	-0.20	-0.25	-0.33	-0.54	-0.74	-0.84	-0.88	-0.89
	Equities	0.12	0.14	0.18	0.26	0.46	0.58	0.62	0.63	0.64
	Nominal Bonds	0.12	0.11	0.11	0.10	0.11	0.18	0.22	0.24	0.25
	I-I Bonds	0.93	0.95	0.96	0.96	0.98	0.99	1.00	1.00	1.00
500	T-bills	-0.01	-0.01	-0.01	-0.01	-0.02	-0.03	-0.04	-0.04	-0.04
	Equities	0.00	0.01	0.01	0.01	0.02	0.03	0.03	0.03	0.03
	Nominal Bonds	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01
	I-I Bonds	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

D-AM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	-1.24	-1.38	-1.58	-1.91	-2.43	-2.80	-2.96	-3.01	-3.04
	Equities	1.19	1.32	1.48	1.76	2.14	2.21	2.22	2.23	2.23
	Nominal Bonds	1.21	1.10	1.12	1.15	1.30	1.58	1.71	1.76	1.78
	I-I Bonds	-0.16	-0.04	-0.03	-0.01	0.00	0.01	0.02	0.03	0.03
5	T-bills	0.10	0.01	-0.13	-0.42	-1.04	-1.56	-1.77	-1.85	-1.88
	Equities	0.47	0.56	0.66	0.88	1.31	1.52	1.58	1.61	1.62
	Nominal Bonds	0.48	0.41	0.44	0.48	0.64	0.92	1.05	1.10	1.12
	I-I Bonds	-0.05	0.02	0.03	0.05	0.09	0.12	0.14	0.15	0.15
10	T-bills	0.55	0.50	0.41	0.22	-0.28	-0.74	-0.94	-1.02	-1.04
	Equities	0.24	0.28	0.33	0.45	0.76	0.97	1.04	1.07	1.08
	Nominal Bonds	0.23	0.20	0.23	0.29	0.45	0.66	0.76	0.80	0.82
	I-I Bonds	-0.02	0.02	0.03	0.04	0.08	0.11	0.13	0.14	0.14
20	T-bills	0.77	0.75	0.70	0.57	0.20	-0.17	-0.33	-0.39	-0.42
	Equities	0.12	0.14	0.15	0.20	0.38	0.55	0.62	0.64	0.65
	Nominal Bonds	0.11	0.10	0.13	0.20	0.36	0.53	0.60	0.63	0.64
	I-I Bonds	0.00	0.01	0.02	0.03	0.06	0.09	0.11	0.12	0.12
500	T-bills	0.99	0.99	0.98	0.93	0.75	0.55	0.47	0.44	0.43
	Equities	0.00	0.00	-0.02	-0.06	-0.08	-0.01	0.03	0.04	0.05
	Nominal Bonds	-0.01	0.01	0.04	0.12	0.30	0.40	0.43	0.44	0.45
	I-I Bonds	0.02	0.01	0.00	0.01	0.03	0.06	0.07	0.07	0.07

Panel B. Constrained Weighting Scheme										
D-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.43	0.46	0.48	0.52	0.55	0.53	0.51	0.51	0.50
	Nominal Bonds	0.45	0.40	0.38	0.35	0.33	0.36	0.38	0.38	0.39
	I-I Bonds	0.12	0.15	0.14	0.13	0.12	0.11	0.11	0.11	0.11
5	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.28	0.31	0.35	0.42	0.50	0.50	0.49	0.48	0.48
	Nominal Bonds	0.29	0.25	0.23	0.20	0.18	0.22	0.24	0.25	0.25
	I-I Bonds	0.43	0.44	0.42	0.38	0.32	0.28	0.27	0.27	0.27
10	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.18	0.20	0.24	0.31	0.41	0.43	0.43	0.43	0.43
	Nominal Bonds	0.18	0.15	0.15	0.13	0.11	0.15	0.17	0.18	0.19
	I-I Bonds	0.64	0.64	0.61	0.56	0.47	0.42	0.40	0.39	0.39
20	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.10	0.12	0.15	0.20	0.30	0.33	0.34	0.34	0.34
	Nominal Bonds	0.10	0.09	0.09	0.08	0.07	0.10	0.12	0.13	0.13
	I-I Bonds	0.79	0.79	0.77	0.72	0.63	0.57	0.54	0.53	0.53
500	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.00	0.01	0.01	0.01	0.02	0.03	0.03	0.03	0.03
	Nominal Bonds	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01
	I-I Bonds	0.99	0.99	0.99	0.99	0.98	0.97	0.96	0.96	0.96

D-AM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.49	0.54	0.57	0.60	0.62	0.58	0.56	0.56	0.55
	Nominal Bonds	0.51	0.46	0.43	0.40	0.38	0.42	0.43	0.44	0.44
	I-I Bonds	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01
5	T-bills	0.10	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.45	0.56	0.58	0.62	0.64	0.59	0.57	0.56	0.56
	Nominal Bonds	0.45	0.41	0.39	0.34	0.31	0.36	0.38	0.38	0.39
	I-I Bonds	0.00	0.02	0.03	0.04	0.04	0.05	0.05	0.05	0.05
10	T-bills	0.54	0.50	0.41	0.22	0.00	0.00	0.00	0.00	0.00
	Equities	0.23	0.28	0.33	0.45	0.59	0.55	0.54	0.53	0.53
	Nominal Bonds	0.23	0.20	0.23	0.29	0.35	0.38	0.39	0.40	0.40
	I-I Bonds	0.00	0.02	0.03	0.04	0.06	0.07	0.07	0.07	0.07
20	T-bills	0.77	0.75	0.70	0.57	0.20	0.00	0.00	0.00	0.00
	Equities	0.12	0.14	0.15	0.20	0.38	0.47	0.46	0.46	0.46
	Nominal Bonds	0.11	0.10	0.13	0.20	0.36	0.45	0.45	0.45	0.45
	I-I Bonds	0.00	0.01	0.02	0.03	0.06	0.08	0.08	0.08	0.08
500	T-bills	0.98	0.99	0.95	0.88	0.69	0.54	0.47	0.44	0.43
	Equities	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.04	0.05
	Nominal Bonds	0.00	0.01	0.04	0.11	0.27	0.40	0.43	0.44	0.45
	I-I Bonds	0.02	0.01	0.00	0.01	0.03	0.06	0.07	0.07	0.07

Table 9: Utility Implications and Management Fee for Investors who Ignore Liabilities

Panel A presents the utility losses of the dynamic asset-only (D-AM) strategy relative to the dynamic asset-liability (D-ALM) policy, in an constrained weighting scheme. The results are obtained by Monte Carlo simulation using 10,000 VAR paths based on the method of antithetic variates, with the same path set being used to evaluate the utility loss in all cases. The CE of the funding ratio is computed by evaluating the mean utility realized across the simulated paths and computing $F_{CE} = u^{-1} \left(E \left[u \left(\widetilde{F}_T \right) \right] \right)$. Panel B reports the largest monthly management fee which a constrained investor who behaves dynamically within an AM framework would pay to gain access to the dynamic asset-liability management strategy, when portfolio weights are constrained. In Panel A the utility losses are computed as the percentage difference between the CE of the dynamic asset-only (D-AM) and the CE of the dynamic asset-liability (D-ALM) strategy (base strategy).. γ denotes the investor's coefficient of relative risk aversion. In Panel B the management fee is computed as $fee = 1 - (1 + WelfareLoss)^{1/horizon}$.

Panel A. Utility Losses for Constrained D-AM					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	18.06%	36.21%	58.03%	81.95%	112.04%
5	-0.25%	-2.23%	-5.65%	-2.51%	-15.75%
10	-46.90%	-40.72%	-43.35%	-43.79%	-57.98%
20	-53.49%	-58.94%	-66.62%	-66.85%	-64.95%

Panel B. Management Fee for Constrained D-AM					
Investment Horizon (in months)					
γ	60	120	180	240	300
1	-	-	-	-	-
2	-0.28%	-0.26%	-0.25%	-0.25%	-0.25%
5	0.00%	0.02%	0.03%	0.01%	0.06%
10	1.05%	0.43%	0.32%	0.24%	0.29%
20	1.27%	0.74%	0.61%	0.46%	0.35%

Table 10: Optimal Portfolio Choice under an Alternative Liability Benchmark

This Table shows the dynamic (D-ALM) and myopic(M-ALM) strategy of an asset-liability investor who plans to invest in the four asset classes: T-bills, equities, nominal and inflation-indexed (I-I) Bonds for an investment horizon ranging from 1 to 300 months and for five levels of risk aversion (2, 5, 10, 20, 500), when portfolio weights are unconstrained. The case of the one-month investment horizon corresponds to the myopic investor. γ denotes the investor's coefficient of relative risk aversion. Weights may not add up to one due to rounding. The liability benchmark is represented by a mix of 30% equities and 70% index-linked bonds.

Constrained Weighting Scheme										
D-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.50	0.52	0.54	0.57	0.59	0.56	0.55	0.54	0.54
	Nominal Bonds	0.46	0.43	0.40	0.37	0.34	0.37	0.39	0.39	0.39
	I-I Bonds	0.04	0.05	0.06	0.06	0.06	0.06	0.06	0.06	0.06
5	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.43	0.45	0.48	0.53	0.59	0.57	0.55	0.55	0.55
	Nominal Bonds	0.30	0.27	0.25	0.22	0.19	0.23	0.25	0.25	0.26
	I-I Bonds	0.27	0.28	0.27	0.25	0.22	0.20	0.20	0.20	0.20
10	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.38	0.39	0.42	0.47	0.54	0.54	0.53	0.53	0.52
	Nominal Bonds	0.19	0.17	0.16	0.14	0.12	0.16	0.18	0.18	0.19
	I-I Bonds	0.43	0.43	0.42	0.39	0.33	0.30	0.29	0.29	0.29
20	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.34	0.35	0.37	0.41	0.48	0.49	0.48	0.48	0.48
	Nominal Bonds	0.11	0.10	0.10	0.09	0.08	0.11	0.12	0.13	0.13
	I-I Bonds	0.55	0.54	0.53	0.50	0.45	0.41	0.40	0.39	0.39
500	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.30	0.30	0.30	0.30	0.31	0.31	0.31	0.31	0.31
	Nominal Bonds	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	I-I Bonds	0.69	0.69	0.69	0.69	0.68	0.68	0.68	0.68	0.68

M-ALM										
Investment Horizon(in months)										
γ	Assets	1	6	12	24	60	120	180	240	300
2	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.50	0.52	0.54	0.60	0.70	0.76	0.77	0.79	0.79
	Nominal Bonds	0.46	0.48	0.46	0.40	0.30	0.24	0.23	0.21	0.21
	I-I Bonds	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.43	0.46	0.49	0.55	0.68	0.73	0.72	0.71	0.70
	Nominal Bonds	0.30	0.33	0.33	0.31	0.26	0.21	0.19	0.18	0.17
	I-I Bonds	0.27	0.22	0.19	0.14	0.06	0.06	0.09	0.11	0.12
10	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.38	0.40	0.42	0.47	0.56	0.61	0.61	0.61	0.61
	Nominal Bonds	0.19	0.21	0.21	0.21	0.18	0.16	0.14	0.14	0.13
	I-I Bonds	0.43	0.40	0.37	0.33	0.26	0.24	0.25	0.26	0.26
20	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.34	0.35	0.37	0.40	0.46	0.49	0.50	0.51	0.51
	Nominal Bonds	0.11	0.12	0.12	0.12	0.11	0.10	0.10	0.09	0.09
	I-I Bonds	0.55	0.53	0.51	0.48	0.43	0.40	0.40	0.40	0.40
500	T-bills	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Equities	0.30	0.30	0.30	0.30	0.30	0.31	0.31	0.31	0.31
	Nominal Bonds	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	I-I Bonds	0.69	0.70	0.69	0.69	0.69	0.68	0.68	0.68	0.68

Appendix A

DYNAMIC PORTFOLIO CHOICE FOR THE “ASSET LIABILITY” INVESTOR¹⁴

Portfolio Rule with one period remaining $t + K - 1$ ($\tau = 1$)

First we will define the policy and value function of the investor at the time with one period remaining.

Policy function in the one period remaining ($\tau = 1$)

The objective function of the investor with one period to the terminal date is to choose $\alpha_{ALM,t+K-1}^{(1)}$ such that $\max_{\alpha_{ALM,t+K-1}^{(1)}} E_{t+K-1} [\exp \{(1 - \gamma)r_{F,t+K}\}]$

We substitute the log-linear approximation to the funding ratio return

$$r_{F,t+1} = \alpha_{ALM,t}^\top \left(\mathbf{x}_{A,t+1} + \frac{1}{2} \sigma_A^2 \right) - \frac{1}{2} \alpha_{ALM,t}^\top \Sigma_{AA} \alpha_{ALM,t} - \mathbf{x}_{L,t+1} \quad (1)$$

into the previous objective function and we conclude to the following form of the objective function:

$$\begin{aligned} \max_{\alpha_{ALM,t+K-1}^{(1)}} E_{t+K-1} [\exp \{(1 - \gamma)r_{F,t+K}\}] &\equiv \\ \max_{\alpha_{ALM,t+K-1}^{(1)}} \exp \left\{ (1 - \gamma) E_{t+K-1} [r_{F,t+K}] + \frac{1}{2} (1 - \gamma)^2 \text{Var}_{t+K-1} [r_{F,t+K}] \right\} &\quad (2) \end{aligned}$$

We define the two first moments of the funding ratio log-return :

(a) The expected of the funding ratio log-return r_F is :

$$\begin{aligned} E_{t+K-1} [r_{F,t+K}] &= \alpha_{ALM,t+K-1}^{(1)\top} \mathbf{H}_A (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-1}) - \mathbf{H}_L (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-1}) \\ &\quad + \frac{1}{2} \alpha_{ALM,t+K-1}^{(1)\top} \left(\sigma_A^2 - \Sigma_{AA} \alpha_{ALM,t+K-1}^{(1)} \right) \quad (3) \end{aligned}$$

(b) The variance of the funding ratio log-return r_F is :

$$\text{Var}_{t+K-1} [r_{F,t+K}] = \sigma_L^2 - 2 \alpha_{ALM,t+K-1}^{(1)\top} \sigma_{AL} + \alpha_{ALM,t+K-1}^{(1)\top} \Sigma_{AA} \alpha_{ALM,t+K-1}^{(1)} \quad (4)$$

We substitute in the objective function the moments of the funding ratio log-return r_F and we obtain the following first order condition:

¹⁴For reasons of convenience and comparison , the structure of the Appendix is based on the corresponding Appendix (C) of [Jurek and Viceira \(2011\)](#).

$$\mathbf{H}_A (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-1}) + \frac{1}{2} \sigma_A^2 - \Sigma_{AA} \boldsymbol{\alpha}_{ALM,t+k-1}^{(1)} + (1 - \gamma) \left(\Sigma_{AA} \boldsymbol{\alpha}_{ALM,t+k-1}^{(1)} - \sigma_{AL} \right) = 0$$

Therefore, solving for $\boldsymbol{\alpha}_{ALM,t+K-1}^{(1)}$ the optimal weight of the investor with one period remaining is:

$$\boldsymbol{\alpha}_{ALM,t+K-1}^{(1)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(\mathbf{H}_A (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-1}) + \frac{1}{2} \sigma_A^2 - (1 - \gamma) \sigma_{AL} \right) \quad (5)$$

We can also write the optimal portfolio as an affine function of the:

$$\boldsymbol{\alpha}_{ALM,t+K-1}^{(1)} = A_{ALM,0}^{(1)} + A_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} \quad (6)$$

where

$$A_{ALM,0}^{(1)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 - (1 - \gamma) \sigma_{AL} \right) \quad (7)$$

$$A_{ALM,1}^{(1)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \Phi_1^A \quad (8)$$

Value function in the one period remaining ($\tau = 1$)

Now we proceed to the derivation of the value function of the investor at the time. Based on the conditional log normality of $r_{F,t+K}$ and equation (2), the value function at the time of the one period remaining depends both on the expected log return on the funding ratio and its variance. We substitute the optimal portfolio rule (5) into the two first moments of the funding ratio and we conclude that both expected return and the variance are quadratic functions of the vector \mathbf{z}_{t+K-1} :

$$\begin{aligned} E_{t+K-1} [r_{F,t+K}] &= \left(A_{ALM,0}^{(1)} + A_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} \right)^\top \left(\Phi_0^A + \Phi_1^A \mathbf{z}_{t+K-1} \right) - \left(\Phi_0^L + \Phi_1^L \mathbf{z}_{t+K-1} \right) \\ &\quad + \frac{1}{2} \left(A_{ALM,0}^{(1)} + A_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} \right)^\top \left(\sigma_A^2 - \Sigma_{AA} \left(A_{ALM,0}^{(1)} + A_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} \right) \right) \\ &= -\Phi_0^L + A_{ALM,0}^{(1)\top} \left(\Phi_0^A + \frac{1}{2} \left(\sigma_A^2 - \Sigma_{AA} A_{ALM,0}^{(1)} \right) \right) \\ &\quad + \mathbf{z}_{t+K-1}^\top A_{ALM,0}^{(1)\top} \left(\Phi_0^A + \frac{1}{2} \left(\sigma_A^2 - \Sigma_{AA} A_{ALM,0}^{(1)} \right) \right) \\ &\quad + \left(-\Phi_1^L + A_{ALM,0}^{(1)\top} \left(\Phi_1^A - \frac{1}{2} \Sigma_{AA} A_{ALM,1}^{(1)} \right) \right) \mathbf{z}_{t+K-1} \\ &\quad + \mathbf{z}_{t+K-1}^\top A_{ALM,1}^{(1)\top} \left(\Phi_1^A - \frac{1}{2} \Sigma_{AA} A_{ALM,1}^{(1)} \right) \mathbf{z}_{t+K-1} \end{aligned}$$

$$\begin{aligned} Var_{t+K-1} [r_{F,t+K}] &= \sigma_L^2 + A_{ALM,0}^{(1)\top} \left(\Sigma_{AA} A_{ALM,0}^{(1)} - 2\sigma_{AL} \right) + A_{ALM,0}^{(1)\top} \Sigma_{AA} A_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} + \\ &\quad + \mathbf{z}_{t+K-1}^\top A_{ALM,1}^{(1)\top} \left(\Sigma_{AA} A_{ALM,0}^{(1)} - 2\sigma_{AL} \right) + \mathbf{z}_{t+K-1}^\top A_{ALM,1}^{(1)\top} \Sigma_{AA} A_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} \end{aligned}$$

We can express the objective function in the following form:

$$\begin{aligned} E_{t+K-1} [\exp \{(1 - \gamma)r_{F,t+K}\}] &= \exp \left\{ (1 - \gamma)E_{t+K-1} [r_{F,t+K}] + \frac{1}{2}(1 - \gamma)^2 \text{Var}_{t+K-1} [r_{F,t+K}] \right\} \\ &= \exp \left\{ (1 - \gamma) \left[E_{t+K-1} [r_{F,t+K}] + \frac{1}{2} (1 - \gamma) \text{Var}_{t+K-1} [r_{F,t+K}] \right] \right\} \end{aligned} \quad (9)$$

We substitute the two moments in the previous objective function and we conclude that the value functions constitutes an exponential quadratic polynomial of the vector \mathbf{z}_{t+K-1}

$$\begin{aligned} E_{t+K-1} [\exp \{(1 - \gamma)r_{F,t+K}^*\}] &= \\ &= \exp \left\{ (1 - \gamma) \left(B_{ALM,0}^{(1)} + B_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top B_{ALM,2}^{(1)} \mathbf{z}_{t+K-1} \right) \right\} \end{aligned} \quad (10)$$

where:

$$\begin{aligned} B_{ALM,0}^{(1)} &= -\Phi_0^L + A_{ALM,0}^{(1)\top} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 \right) \\ &\quad + \frac{1 - \gamma}{2} \left(\sigma_L^2 - 2A_{ALM,0}^{(1)\top} \sigma_{AL} \right) - \frac{\gamma}{2} A_{ALM,0}^{(1)\top} \Sigma_{AA} A_{ALM,0}^{(1)} \end{aligned} \quad (11)$$

$$\begin{aligned} B_{ALM,1}^{(1)} &= -\Phi_1^L + A_{ALM,0}^{(1)\top} \left(\Phi_1^A - \gamma \Sigma_{AA} A_{ALM,1}^{(1)} \right) \\ &\quad + \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 - (1 - \gamma) \sigma_{AL} \right)^\top A_{ALM,1}^{(1)} \end{aligned} \quad (12)$$

$$B_{ALM,2}^{(1)} = A_{ALM,1}^{(1)\top} \left(\Phi_1^A - \frac{\gamma}{2} \Sigma_{AA} A_{ALM,1}^{(1)} \right) \quad (13)$$

Portfolio Rule with two periods remaining $t + K - 2$ ($\tau = 2$)

When the remaining horizon is two periods, the objective function of the investor is the following:

$$\begin{aligned} &\max_{\alpha_{t+K-2}^{(2)}, \mathbf{a}_{t+K-1}^{(1)}} \frac{1}{1 - \gamma} E_{t+K-2} [\exp \{(1 - \gamma) (r_{F,t+K-1} + r_{F,t+K})\}] \\ &= \max_{\alpha_{t+K-2}^{(2)}} \frac{1}{1 - \gamma} E_{t+K-2} \left[\exp \left\{ (1 - \gamma) \left(r_{F,t+K-1} + B_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top B_{ALM,2}^{(1)} \mathbf{z}_{t+K-1} \right) \right\} \right] \end{aligned} \quad (14)$$

The term inside the expectation is an exponential quadratic polynomial function of the vector of state variables \mathbf{z}_{t+K-1} . We will present an analytical expression for equation (10).

We can write the log-linear approximation to the funding ratio return as:

$$\begin{aligned} r_{F,t+K-1} &= \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)\top} \mathbf{x}_{A,t+K-1} - x_{L,t+K-1} + \frac{1}{2} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)\top} \left(\boldsymbol{\sigma}_A^2 - \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \right) \\ &= \begin{bmatrix} 0 & \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} & -1 & 0 \end{bmatrix} \mathbf{z}_{t+K-1} + \frac{1}{2} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)\top} \left(\boldsymbol{\sigma}_A^2 - \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \right) \end{aligned} \quad (15)$$

We substitute the above formula at the objective function (14)

$$\begin{aligned} E_{t+K-2} & \left[\left\{ (1-\gamma) \left(\frac{1}{2} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)\top} \left(\boldsymbol{\sigma}_A^2 - \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \right) + B_{ALM,0}^{(1)} + \right. \right. \right. \\ & \left. \left. \left. \left(B_{ALM,1}^{(1)} + \begin{bmatrix} 0 & \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} & -1 & 0 \end{bmatrix} \right) \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top B_{ALM,2}^{(1)} \mathbf{z}_{t+K-1} \right\} \right] \end{aligned} \quad (16)$$

Since the last two summands are random, we therefore focus on these summands.

$$E_{t+K-2} \left[\exp \left\{ (1-\gamma) \left(\left(B_{ALM,1}^{(1)} + \begin{bmatrix} 0 & \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} & -1 & 0 \end{bmatrix} \right) \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top B_{ALM,2}^{(1)} \mathbf{z}_{t+K-1} \right) \right\} \right] \quad (17)$$

We substitute (17) into (16) and we obtain the value function of the investor at the time $t+K-2$. The distribution of \mathbf{z}_{t+K-1} conditional on time $t+K-2$ shows that (17) computes the expectation of the exponential quadratic polynomial in a normal random variable.

For convenience, we introduce the following notation:

$$\mathbf{z}_{t+K-1} / \mathbf{z}_{t+K-2} \sim N(\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2}, \boldsymbol{\Sigma}_v)$$

For our convenience, we introduce the following notation:

$$\begin{aligned} C_{ALM,1}^{(2)} &= (1-\gamma) \begin{bmatrix} 0 \\ \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \\ -1 \\ 0 \end{bmatrix} + (1-\gamma) B_{ALM,1}^{(1)\top} \\ C_{ALM,2}^{(2)} &= (1-\gamma) B_{ALM,2}^{(1)} \end{aligned}$$

Based on the aforementioned notations we can express (17) as

$$\begin{aligned} E_{t+K-2} & \left[\exp \left\{ C_{ALM,1}^{(2)\top} \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top C_{ALM,2}^{(2)} \mathbf{z}_{t+K-1} \right\} \right] \\ &= E_{t+K-2} \left[\exp \left\{ C_{ALM,1}^{(2)\top} (\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2} \right. \right. \\ & \left. \left. + \mathbf{v}_{t+K-2}) + \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2} + \mathbf{v}_{t+K-2} \right)^\top C_{ALM,2}^{(2)} (\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2} + \mathbf{v}_{t+K-2}) \right\} \right] \end{aligned} \quad (18)$$

$$\begin{aligned}
E_{t+K-2} [\exp()] &= \exp \left(C_{ALM,1}^{(2)\top} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) + (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top C_{ALM,2}^{(2)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) \right) \\
&\quad \times E_{t+K-2} \left[\exp \left\{ C_{ALM,1}^{(2)\top} \mathbf{v}_{t+K-2} + \mathbf{v}_{t+K-2}^\top C_{ALM,2}^{(2)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) + \right. \right. \\
&\quad \left. \left. + (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top C_{ALM,2}^{(2)} \mathbf{v}_{t+K-2} + \mathbf{v}_{t+K-2}^\top C_{ALM,2}^{(2)} \mathbf{v}_{t+K-2} \right\} \right] \\
&= \exp \left(C_{ALM,1}^{(2)\top} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) + (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top C_{ALM,2}^{(2)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) \right) \\
&\quad \times E_{t+K-2} \left[\left\{ \exp C_{ALM,1}^{(2)\top} \mathbf{v}_{t+K-2} + \mathbf{v}_{t+K-2}^\top C_{ALM,2}^{(2)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) + \right. \right. \\
&\quad \left. \left. (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top C_{ALM,2}^{(2)} \mathbf{v}_{t+K-2} \right\} \times \exp \left\{ \mathbf{v}_{t+K-2}^\top C_{ALM,2}^{(2)} \mathbf{v}_{t+K-2} \right\} \right] \\
&= \exp \left(C_{ALM,1}^{(2)\top} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) + (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top C_{ALM,2}^{(2)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) \right) \\
&\quad \times E_{t+K-2} \left[\exp \left\{ \mathbf{v}_{t+K-2}^\top C_{ALM,2}^{(2)} \mathbf{v}_{t+K-2} \right\} \right. \\
&\quad \left. \times \exp \left\{ \left(C_{ALM,1}^{(2)\top} + 2(\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top \widetilde{C}_{ALM,2}^{(2)} \right) \mathbf{v}_{t+K-2} \right\} \right]
\end{aligned} \tag{19}$$

where

$$\widetilde{C}_{ALM,2}^{(2)} \equiv \frac{1}{2} \left(C_{ALM,2}^{(2)} + C_{ALM,2}^{(2)'} \right)$$

There are two approximations in order to evaluate the above expectation operator: the complementary and the exact approximation

Following the complementary approximation:

$$\lim_{\Delta t \rightarrow 0} \exp \left\{ \mathbf{v}_{t+K-2}^\top C_{ALM,2}^{(2)} \mathbf{v}_{t+K-2} \right\} / dt = \exp \left\{ tr \left(C_{ALM,2}^{(2)} \Sigma_v \right) \right\}$$

Thus equation (19):

$$\begin{aligned}
&\exp \left(C_{ALM,1}^{(2)\top} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) + (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top C_{ALM,2}^{(2)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) + tr(C_{ALM,2}^{(2)} \Sigma_v) \right) \\
&\quad \times E_{t+K-2} \left[\exp \left\{ \left(C_{ALM,1}^{(2)\top} + 2(\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top \widetilde{C}_{ALM,2}^{(2)} \right) \mathbf{v}_{t+K-2} \right\} \right]
\end{aligned} \tag{20}$$

where:

$$\begin{aligned}
&E_{t+K-2} \left[\exp \left\{ \left(C_{ALM,1}^{(2)\top} + 2(\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top \widetilde{C}_{ALM,2}^{(2)} \right) \mathbf{v}_{t+K-2} \right\} \right] = \\
&= \exp \left\{ \frac{1}{2} \left(C_{ALM,1}^{(2)\top} + 2(\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top \widetilde{C}_{ALM,2}^{(2)} \right)^\top \Sigma_v \left(C_{ALM,1}^{(2)\top} + 2(\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2})^\top \widetilde{C}_{ALM,2}^{(2)} \right) \right\}
\end{aligned} \tag{21}$$

Substituting equations (20) and (21) into (17) the equation in the continuous time limit:

$$\lim_{\Delta t \rightarrow 0} E_{t+K-2} \left[\exp \left\{ C_{ALM,1}^{(2)\top} \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top \right\} \right] = \exp \left\{ tr \left(C_{ALM,2}^{(2)} \boldsymbol{\Sigma}_v \right) + D_{ALM,0}^{(t+K-2)} + \frac{1}{2} D_{ALM,1}^{(t+K-2)} \boldsymbol{\Sigma}_v^{-1} D_{ALM,1}^{(t+K-2)\top} \right\} \quad (22)$$

where:

$$D_{ALM,0}^{(2)} = C_{ALM,1}^{(2)\top} (\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2}) + (\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2})^\top C_{ALM,2}^{(2)} (\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2}) \quad (23)$$

$$D_{ALM,1}^{(2)} = C_{ALM,1}^{(2)\top} + 2(\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_{t+K-2})^\top \widetilde{C}_{ALM,2}^{(2)} \quad (24)$$

Following the exact evaluation (Campbell et al., 2003) :

$$E_{t+K-2} \left[\exp \left\{ C_{ALM,1}^{(2)\top} \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top C_{ALM,2}^{(2)} \mathbf{z}_{t+K-1} \right\} \right] = \frac{|\boldsymbol{\Sigma}_v|^{-\frac{1}{2}}}{|\boldsymbol{\Sigma}_v^{-1} - 2D_{ALM,2}^{(t+K-2)}|^{-\frac{1}{2}}} \exp \left(D_{ALM,0}^{(t+K-2)} + \frac{1}{2} D_{ALM,1}^{(t+K-2)} (\boldsymbol{\Omega}^{(1)})^{-1} D_{ALM,1}^{(t+K-2)\top} \right) \quad (25)$$

where:

$$\boldsymbol{\Omega}^{(1)} \equiv \left(\boldsymbol{\Sigma}_v^{-1} - 2D_{ALM,2}^{(2)} \right)^{-1} = \left(\boldsymbol{\Sigma}_v^{-1} - 2(1-\gamma)B_{ALM,2}^{(1)} \right)^{-1} \quad (26)$$

We replace expectations (22) or (25) into the value function (17) :

$$\max_{\boldsymbol{\alpha}_{t+K-2}^{(2)}} \left\{ (1-\gamma) \left(\frac{1}{2} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)\top} \left(\boldsymbol{\sigma}_A^2 - \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \right) + B_{ALM,0}^{(1)} + D_{ALM,0}^{(t+K-2)} + \frac{1}{2} D_{ALM,1}^{(t+K-2)} \boldsymbol{\Omega}^{(1)} D_{ALM,1}^{(t+K-2)\top} \right) \right\} \quad (27)$$

The first order condition with respect to $\boldsymbol{\alpha}_{ALM,t+K-2}^{(2)}$ is the following

$$(1-\gamma) \left(\boldsymbol{\Phi}_0^A + \boldsymbol{\Phi}_1^A \mathbf{z}_{t+K-2} + \frac{1}{2} \boldsymbol{\sigma}_A^2 - \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \right) + \frac{1}{2} \left(\frac{\partial D_{ALM,1}^{(t+K-2)\top}}{\partial \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)}} \right)^\top (\boldsymbol{\Omega}^{(1)} + \boldsymbol{\Omega}^{(1)\top}) D_{ALM,1}^{(t+K-2)\top} = \\ = (1-\gamma) \left(\boldsymbol{\Phi}_0^A + \boldsymbol{\Phi}_1^A \mathbf{z}_{t+K-2} + \frac{1}{2} \boldsymbol{\sigma}_A^2 - \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \right) + \left(\frac{\partial D_{ALM,1}^{(t+K-2)\top}}{\partial \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)}} \right)^\top \widetilde{\boldsymbol{\Omega}}^{(1)} D_{ALM,1}^{(t+K-2)\top} = 0$$

In addition:

$$\left(\frac{\partial D_{ALM,1}^{(t+K-2)\top}}{\partial \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)}} \right)^\top = (1-\gamma) \mathbf{H}_A$$

Thus the first order condition takes the following form:

$$\begin{aligned}
& \Phi_0^A + \Phi_1^A \mathbf{z}_{t+K-2} + \frac{1}{2} \sigma_A^2 - \Sigma_{AA} \alpha_{ALM,t+K-2}^{(2)} + \mathbf{H}_A \tilde{\Omega}^{(1)} \left(C_{ALM,1}^{(2)} + 2 \widetilde{C}_{ALM,2}^{(2)} (\Phi_0^A + \Phi_1^A \mathbf{z}_{t+K-2}) \right) = \\
& = \Phi_0^A + \Phi_1^A \mathbf{z}_{t+K-2} + \frac{1}{2} \sigma_A^2 - \Sigma_{AA} \alpha_{ALM,t+K-2}^{(2)} + \\
& (1 - \gamma) \tilde{\Omega}_A^{(1)} \left(\begin{bmatrix} 0 \\ \alpha_{ALM,t+K-2}^{(2)} \\ -1 \\ 0 \end{bmatrix} + B_{ALM,1}^{(1)\top} + 2 \tilde{B}_{ALM,2}^{(1)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) \right) = \\
& = \Phi_0^A + \Phi_1^A \mathbf{z}_{t+K-2} + \frac{1}{2} \sigma_A^2 - \Sigma_{AA} \alpha_{ALM,t+K-2}^{(2)} + \\
& (1 - \gamma) \left(\tilde{\Omega}_{AA}^{(1)} \alpha_{ALM,t+K-2}^{(2)} - \tilde{\Omega}_{AL}^{(1)} + \tilde{\Omega}_A^{(1)} B_{ALM,1}^{(1)\top} + 2 \tilde{\Omega}_A^{(1)} \tilde{B}_{ALM,2}^{(1)} (\Phi_0 + \Phi_1 \mathbf{z}_{t+K-2}) \right) = \\
& = \left((1 - \gamma) \tilde{\Omega}_{AA}^{(1)} - \Sigma_{AA} \right) \alpha_{ALM,t+K-2}^{(2)} + \Phi_0^A + \\
& (1 - \gamma) \left(-\tilde{\Omega}_{AL}^{(1)} + \tilde{\Omega}_A^{(1)} \left(B_{ALM,1}^{(1)\prime} + 2 \tilde{B}_{ALM,2}^{(1)} \Phi_0 \right) \right) + \\
& + \frac{1}{2} \sigma_A^2 + \left(\Phi_1^A + 2(1 - \gamma) \tilde{\Omega}_A^{(1)} \tilde{B}_{ALM,2}^{(1)} \Phi_1 \right) \mathbf{z}_{t+K-2} = 0
\end{aligned}$$

Therefore, solving for $\alpha_{ALM,t+K-2}^{(2)}$, the optimal weight of the investor at time $t + K - 2$ is an affine function of \mathbf{z}_{t+K-2}

$$\alpha_{ALM,t+K-2}^{(2)} = A_{ALM,0}^{(2)} + A_{ALM,1}^{(2)} \mathbf{z}_{t+K-2} \quad (28)$$

where

$$A_{ALM,0}^{(2)} = \left(\Sigma_{AA} - (1 - \gamma) \tilde{\Omega}_{AA}^{(1)} \right)^{-1} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 + (1 - \gamma) \left(-\tilde{\Omega}_{AL}^{(1)} + \tilde{\Omega}_A^{(1)} \left(B_{ALM,1}^{(1)\top} + 2 \tilde{B}_{ALM,2}^{(1)} \Phi_0 \right) \right) \right) \quad (29)$$

$$A_{ALM,1}^{(2)} = \left(\Sigma_{AA} - (1 - \gamma) \tilde{\Omega}_{AA}^{(1)} \right)^{-1} \left(\Phi_1^A + 2(1 - \gamma) \tilde{\Omega}_A^{(1)} \tilde{B}_{ALM,2}^{(1)} \Phi_1 \right) \quad (30)$$

or (replacing $\tilde{\Omega}^{(1)}$ with Σ_v)

$$A_{ALM,0}^{(2)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 + (1 - \gamma) \left(-\Sigma_{AL} + \Sigma_A \left(B_{ALM,1}^{(1)\top} + 2 \tilde{B}_{ALM,2}^{(1)} \Phi_0 \right) \right) \right) \quad (31)$$

$$A_{ALM,1}^{(2)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(\Phi_1^A + 2(1 - \gamma) \Sigma_A \tilde{B}_{ALM,2}^{(1)} \Phi_1 \right) \quad (32)$$

Value function in the two periods remaining ($\tau = 2$)

Substituting the optimal portfolio policy at time $t + K - 2$, into the objective function (14), we define the value function with coefficients $B_{ALM,0}^{(2)}$, $B_{ALM,1}^{(2)}$ and $B_{ALM,2}^{(2)}$, which are

different from the coefficients $B_{ALM,0}^{(2)}$, $B_{ALM,1}^{(2)}$ and $B_{ALM,2}^{(2)}$ of the value function at time $t + K - 1$.

$$\begin{aligned}
& E_{t+K-2} \left[\exp \{ (1 - \gamma) r_{F,t+K-1} \} \exp \left\{ (1 - \gamma) \left(B_{ALM,0}^{(1)} + B_{ALM,1}^{(1)} \mathbf{z}_{t+K-1} + \mathbf{z}_{t+K-1}^\top B_{ALM,2}^{(1)} \mathbf{z}_{t+K-1} \right) \right\} \right] = \\
& = \exp \{ (1 - \gamma) \left(\frac{1}{2} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)\top} \left(\boldsymbol{\sigma}_A^2 - \boldsymbol{\Sigma}_{AA} \boldsymbol{\alpha}_{ALM,t+K-2}^{(2)} \right) + B_{ALM,0}^{(1)} \right) + \\
& \quad D_{ALM,0}^{(t+K-2)} + \frac{1}{2} D_{ALM,1}^{(t+K-2)} \boldsymbol{\Omega}^{(1)} D_{ALM,1}^{(t+K-2)\top} \} \\
& = \exp \left\{ (1 - \gamma) \left(B_{ALM,0}^{(2)} + B_{ALM,1}^{(2)} \mathbf{z}_{t+K-2} + \mathbf{z}_{t+K-1}^\top B_{ALM,2}^{(2)} \mathbf{z}_{t+K-2} \right) \right\}
\end{aligned} \tag{33}$$

where:

$$\begin{aligned}
B_{ALM,0}^{(2)} & = -\boldsymbol{\Phi}_0^L + A_{ALM,0}^{(1)\top} \left(\boldsymbol{\Phi}_0^A + \frac{1}{2} \boldsymbol{\sigma}_A^2 \right) + \frac{1-\gamma}{2} \left(\boldsymbol{\Omega}_{LL}^{(1)} - 2A_{ALM,0}^{(1)\top} \tilde{\boldsymbol{\Omega}}_{AL}^{(1)} \right) - \\
& - \frac{1}{2} A_{ALM,0}^{(1)\top} \left(\boldsymbol{\Sigma}_{AA} - (1 - \gamma) \tilde{\boldsymbol{\Omega}}_{AA}^{(1)} \right)^{-1} A_{ALM,0}^{(1)} + B_{ALM,0}^{(1)} + \left(B_{ALM,1}^{(1)} + \boldsymbol{\Phi}_0^\top B_{ALM,2}^{(1)} \right) \boldsymbol{\Phi}_0 + \\
& + (1 - \gamma) B_{ALM,1}^{(1)} \left(-\tilde{\boldsymbol{\Omega}}_L^{(1)} + \frac{1}{2} \boldsymbol{\Omega}^{(1)} B_{ALM,1}^{(1)\top} + \tilde{\boldsymbol{\Omega}}_A^{(1)} A_{ALM,0}^{(1)} \right) + \\
& + (1 - \gamma) \boldsymbol{\Phi}_0^\top \left(2\boldsymbol{\Lambda}^{(1)} \boldsymbol{\Phi}_0 - \boldsymbol{\Xi}_L^{(1)\top} + \boldsymbol{\Xi}_A^{(1)\top} A_{ALM,0}^{(1)} + \boldsymbol{\Gamma}^{(1)} B_{ALM,1}^{(1)\top} \right)
\end{aligned} \tag{34}$$

$$\begin{aligned}
B_{ALM,1}^{(2)} & = -\boldsymbol{\Phi}_1^L + A_{ALM,0}^{(1)\top} \left(\boldsymbol{\Phi}_1^A - \left(\boldsymbol{\Sigma}_{AA} - (1 - \gamma) \tilde{\boldsymbol{\Omega}}_{AA}^{(1)} \right) A_{ALM,1}^{(1)} \right) + \\
& \left(\boldsymbol{\Phi}_0^A + \frac{1}{2} \boldsymbol{\sigma}_A^2 - (1 - \gamma) \tilde{\boldsymbol{\Omega}}_{AL}^{(1)} \right)^\top A_{ALM,1}^{(1)} + \\
& + \left(B_{ALM,1}^{(1)} + 2\boldsymbol{\Phi}_0^\top \tilde{B}_{ALM,2}^{(1)} \right) \boldsymbol{\Phi}_1 + (1 - \gamma) \left(B_{ALM,1}^{(1)} \tilde{\boldsymbol{\Omega}}_A^{(1)\top} + \boldsymbol{\Phi}_0^\top \boldsymbol{\Xi}_A^{(1)\top} \right) A_{ALM,1}^{(1)} + \\
& + (1 - \gamma) \left(4\boldsymbol{\Phi}_0^\top \tilde{\boldsymbol{\Lambda}}^{(1)} - \boldsymbol{\Xi}_L^{(1)} + A_{ALM,0}^{(1)\top} \boldsymbol{\Xi}_A^{(1)} + B_{ALM,1}^{(1)} \boldsymbol{\Xi}^{(1)} \right) \boldsymbol{\Phi}_1
\end{aligned} \tag{35}$$

$$\begin{aligned}
B_{ALM,2}^{(2)} & = A_{ALM,1}^{(1)\top} \left(\boldsymbol{\Phi}_1^A - \frac{1}{2} \left(\boldsymbol{\Sigma}_{AA} - (1 - \gamma) \boldsymbol{\Omega}_{AA}^{(1)} \right) A_{ALM,1}^{(1)} \right) + \\
& + \boldsymbol{\Phi}_1^\top \left(B_{ALM,2}^{(1)} + 2(1 - \gamma) \boldsymbol{\Lambda}^{(1)} \right) \boldsymbol{\Phi}_1 + \\
& + (1 - \gamma) \boldsymbol{\Phi}_1^\top \boldsymbol{\Xi}_A^{(1)\top} A_{ALM,1}^{(1)}
\end{aligned} \tag{36}$$

and the auxiliary matrices $\boldsymbol{\Lambda}^{(1)} = \widetilde{B_{ALM,2}^{(1)}} \boldsymbol{\Omega}^{(1)} \widetilde{B_{ALM,2}^{(1)\top}}$, $\boldsymbol{\Gamma}^{(1)} = 2\widetilde{B_{ALM,2}^{(1)}} \tilde{\boldsymbol{\Omega}}^{(1)} \boldsymbol{\Xi}^{(1)} = \boldsymbol{\Gamma}^{(1)\top}$

or (replacing $\tilde{\boldsymbol{\Omega}}^{(1)}$ with $\boldsymbol{\Sigma}_v$)

$$\begin{aligned}
B_{ALM,0}^{(2)} & = -\boldsymbol{\Phi}_0^L + A_{ALM,0}^{(1)\top} \left(\boldsymbol{\Phi}_0^A + \frac{1}{2} \boldsymbol{\sigma}_A^2 \right) + \frac{1-\gamma}{2} \left(\boldsymbol{\Sigma}_{LL}^{(1)} - 2A_{ALM,0}^{(1)\top} \boldsymbol{\Sigma}_{AL}^{(1)} \right) - \frac{\gamma}{2} A_{ALM,0}^{(1)\top} \boldsymbol{\Sigma}_{AA} A_{ALM,0}^{(1)} + \\
& + B_{ALM,0}^{(1)} + \left(B_{ALM,1}^{(1)} + \boldsymbol{\Phi}_0^\top B_{ALM,2}^{(1)} \right) \boldsymbol{\Phi}_0 + (1 - \gamma) B_{ALM,1}^{(1)} \left(-\boldsymbol{\Sigma}_L^{(1)\top} + \frac{1}{2} \boldsymbol{\Sigma}_v B_{ALM,1}^{(1)\top} + \boldsymbol{\Sigma}_A^{(1)\top} A_{ALM,0}^{(1)} \right) + \\
& + (1 - \gamma) \boldsymbol{\Phi}_0^\top \left(2\boldsymbol{\Lambda}^{(1)} \boldsymbol{\Phi}_0 - \boldsymbol{\Xi}_L^{(1)\top} + \boldsymbol{\Xi}_A^{(1)\top} A_{ALM,0}^{(1)} + \boldsymbol{\Gamma}^{(1)} B_{ALM,1}^{(1)\top} \right)
\end{aligned} \tag{37}$$

$$\begin{aligned}
B_{ALM,1}^{(2)} &= -\Phi_1^L + A_{ALM,0}^{(1)\top} \left(\Phi_1^A - \gamma \Sigma_{AA} A_{ALM,1}^{(1)} \right) + \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 - (1 - \gamma) \Sigma_{AL} \right)^\top A_{ALM,1}^{(1)} + \\
&+ \left(B_{ALM,1}^{(1)} + 2\Phi_0^\top \tilde{B}_{ALM,2}^{(1)} \right) \Phi_1 + (1 - \gamma) \left(B_{ALM,1}^{(1)} \Sigma_A^\top + \Phi_0^\top \Xi_A^{(1)\top} \right) A_{ALM,1}^{(1)} + \\
&+ (1 - \gamma) \left(4\Phi_0^\top \tilde{\Lambda}^{(1)} - \Xi_L^{(1)} + A_{ALM,0}^{(1)\top} \Xi_A^{(1)} + B_{ALM,1}^{(1)} \Xi^{(1)} \right) \Phi_1
\end{aligned} \tag{38}$$

$$\begin{aligned}
B_{ALM,2}^{(2)} &= A_{ALM,1}^{(1)\top} \left(\Phi_1^A - \frac{\gamma}{2} \Sigma_{AA} A_{ALM,1}^{(1)} \right) + \Phi_1^\top \left(B_{ALM,2}^{(1)} + 2(1 - \gamma) \Lambda^{(1)} \right) \Phi_1 \\
&+ (1 - \gamma) \Phi_1^\top \Xi_A^{(1)\top} A_{ALM,1}^{(1)}
\end{aligned} \tag{39}$$

and the auxiliary matrices $\Lambda^{(1)} = \widetilde{B_{ALM,2}^{(1)}} \Sigma_v \widetilde{B_{ALM,2}^{(1)\top}}$, $\Gamma^{(1)} = 2\widetilde{B_{ALM,2}^{(1)}} \Sigma_v$, $\Xi^{(1)} = \Gamma^{(1)\top}$.

The previous formulas provide the link between the policy and value functions in any two successive periods. The general recursive solution is obtained by making the following replacements: (1) \rightarrow $(\tau - 1)$ and (2) \rightarrow $(\tau - 2)$.

Finally, the optimal dynamic portfolio choice literature converges to the following corner solutions:

1. Log utility investor ($\gamma = 1$)

Substituting $\gamma = 1$ into the general recursive solution, then the portfolio choice coefficients become

$$A_{ALM,0}^{(\tau)} = \Sigma_{AA}^{-1} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 \right) \quad \forall \tau$$

$$A_{ALM,1}^{(\tau)} = \Sigma_{AA}^{-1} \Phi_1^A \quad \forall \tau$$

Thus, the recursive equations (11), (12) and (13) become

$$B_{ALM,0}^{(1)} = -\Phi_0^L + A_{ALM,0}^{(1)\top} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 - \frac{1}{2} \Sigma_{AA} A_{ALM,0}^{(1)} \right) = -\Phi_0^L$$

$$\begin{aligned}
B_{ALM,1}^{(1)} &= -\Phi_1^L + A_{ALM,0}^{(1)\top} \left(\Phi_1^A - \Sigma_{AA} A_{ALM,1}^{(1)} \right) + \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 \right)^\top A_{ALM,1}^{(1)} \\
&= -\Phi_1^L + \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 \right)^\top \Sigma_{AA}^{-1} \Phi_1^A
\end{aligned}$$

$$B_{ALM,2}^{(1)} = A_{ALM,1}^{(1)\top} \left(\Phi_1^A - \frac{1}{2} \Sigma_{AA} A_{ALM,1}^{(1)} \right) = \frac{1}{2} \Phi_1^{A\top} \Sigma_{AA}^{-1} \Phi_1^A$$

and the recursive equations (37), (38) and (39) become

$$\begin{aligned}
B_{ALM,0}^{(\tau)} &= -\Phi_0^L + A_{ALM,0}^{(\tau-1)\top} \left(\Phi_0^A + \frac{1}{2}\sigma_A^2 - \frac{1}{2}\Sigma_{AA}A_{ALM,0}^{(\tau-1)} \right) + B_{ALM,0}^{(\tau-1)} + \left(B_{ALM,1}^{(\tau-1)} + \Phi_0^\top B_{ALM,2}^{(\tau-1)} \right) \Phi_0 \\
&= -\Phi_0^L + \left(\Phi_0^A + \frac{1}{2}\sigma_A^2 \right)^\top \Sigma_{AA}^{-1} \left(\Phi_0^A + \frac{1}{2}\sigma_A^2 \right) + B_{ALM,0}^{(\tau-1)} + \left(B_{ALM,1}^{(\tau-1)} + \Phi_0^\top B_{ALM,2}^{(\tau-1)} \right) \Phi_0
\end{aligned}$$

$$\begin{aligned}
B_{ALM,1}^{(\tau)} &= -\Phi_1^L + A_{ALM,0}^{(\tau-1)\top} \left(\Phi_1^A - \Sigma_{AA}A_{ALM,1}^{(\tau-1)} \right) + \left(\Phi_0^A + \frac{1}{2}\sigma_A^2 \right)^\top A_{ALM,1}^{(\tau-1)} + \left(B_{ALM,1}^{(\tau-1)} + 2\Phi_0^\top \tilde{B}_{ALM,2}^{(\tau-1)} \right) \Phi_1 \\
&= -\Phi_1^L + \left(\Phi_0^A + \frac{1}{2}\sigma_A^2 \right)^\top \Sigma_{AA}^{-1} \Phi_1^A + \left(B_{ALM,1}^{(\tau-1)} + 2\Phi_0^\top \tilde{B}_{ALM,2}^{(\tau-1)} \right) \Phi_1
\end{aligned}$$

$$B_{ALM,2}^{(\tau)} = A_{ALM,1}^{(\tau-1)\top} \left(\Phi_1^A - \frac{1}{2}\Sigma_{AA}A_{ALM,1}^{(\tau-1)} \right) + \Phi_1^\top B_{ALM,2}^{(\tau-1)} \Phi_1 = \frac{1}{2}\Phi_1^{A\top} \Sigma_{AA}^{-1} \Phi_1^A + \Phi_1^\top B_{ALM,2}^{(\tau-1)} \Phi_1$$

2. Investor with infinite risk aversion ($\gamma \rightarrow \infty$)

Substituting $\gamma \rightarrow \infty$ into the general recursive solution, we obtain the limits of the portfolio choice coefficients in equations (7) and (8) :

$$A_{ALM,0}^{(1)} \rightarrow \Sigma_{AA}^{-1} \sigma_{AL}$$

$$A_{ALM,1}^{(1)} \rightarrow 0$$

Thus, the recursive equations (11), (12) and (13) converge to

$$B_{ALM,0}^{(1)} \rightarrow -\infty$$

$$B_{ALM,1}^{(1)} \rightarrow -\Phi_1^L + \sigma_{AL}^\top \Sigma_{AA}^{-1} \Phi_1^A$$

$$B_{ALM,2}^{(1)} \rightarrow 0$$

Based on the aforementioned results, we can infer that $\Omega^{(1)}$ (see equation 26) will be $O(1)$ in the coefficient of risk aversion and, consequently, each iteration will result in a non-negligible adjustment to the covariance matrix of the shocks. Thus

$$\Omega^{(1)} \rightarrow \left(\Sigma_v^{-1} - \Phi_1^{A\top} \Sigma_{AA}^{-1} \Phi_1^A \right)^{-1}$$

Next, the equations (29) and (30) can be written in the following form, respectively

$$\begin{aligned}
A_{ALM,0}^{(2)} &= \\
&\left(\frac{1}{\gamma} \Sigma_{AA} - \left(\frac{1}{\gamma} - 1 \right) \tilde{\Omega}_{AA}^{(1)} \right)^{-1} \left(\frac{1}{\gamma} \Phi_0^A + \frac{1}{2\gamma} \sigma_A^2 + \left(\frac{1}{\gamma} - 1 \right) \left(-\tilde{\Omega}_{AL}^{(1)} + \tilde{\Omega}_A^{(1)} \left(B_{ALM,1}^{(1)\top} + 2\tilde{B}_{ALM,2}^{(1)} \Phi_0 \right) \right) \right)
\end{aligned}$$

$$A_{ALM,1}^{(2)} = \left(\frac{1}{\gamma} \Sigma_{AA} - \left(\frac{1}{\gamma} - 1 \right) \tilde{\Omega}_{AA}^{(1)} \right)^{-1} \left(\frac{1}{\gamma} \Phi_1^A + 2 \left(\frac{1}{\gamma} - 1 \right) \tilde{\Omega}_A^{(1)} \tilde{B}_{ALM,2}^{(1)} \Phi_1 \right)$$

Equations (29) and (30) converge to

$$A_{ALM,0}^{(2)} \rightarrow - \left(\tilde{\Omega}_{AA}^{(1)} \right)^{-1} \left(-\tilde{\Omega}_{AL}^{(1)} + \tilde{\Omega}_A^{(1)} B_{ALM,1}^{(1)} \right)$$

$$A_{ALM,1}^{(2)} \rightarrow 0$$

and equations (34), (35) and (36) converge to

$$B_{ALM,0}^{(2)} \rightarrow -\infty$$

$$B_{ALM,1}^{(2)} = -\Phi_1^L + A_{ALM,0}^{(1)\top} \Phi_1^A + B_{ALM,1}^{(1)} \Phi_1 - \gamma B_{ALM,1}^{(1)} \tilde{\Omega}_A^{(1)\top} A_{ALM,1}^{(1)} - \gamma \left(-\Xi_L^{(1)} + A_{ALM,0}^{(1)\top} \Xi_A^{(1)} + B_{ALM,1}^{(1)} \Xi^{(1)} \right) \Phi_1$$

$$B_{ALM,2}^{(2)} \rightarrow 0$$

since $\mathbf{\Lambda}^{(1)} \sim O(\gamma^{-2})$, $\mathbf{\Gamma}^{(1)} \sim O(\gamma^{-1})$ and $\mathbf{\Xi}^{(1)} \sim O(\gamma^{-1})$.

As the agent becomes infinitely risk averse the policy function becomes decoupled from the intercept vector Φ_0 of the vector autoregressive model. Furthermore, the portfolio allocation for an infinitely risk averse investor at $t + K - 1$ is the Liability Hedge Portfolio (LHP) for the myopic one-period problem.

3. No predictability ($\Phi_1 = 0$)

Substituting $\Phi_1 = 0$ into the general recursive solution, then the portfolio choice coefficients in equations (7) and (8) become

$$A_{ALM,0}^{(1)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 - (1 - \gamma) \sigma_{AL} \right)$$

$$A_{ALM,1}^{(1)} = 0$$

Thus, the recursive equations (11), (12) and (13) become

$$B_{ALM,0}^{(1)} = -\Phi_0^L + A_{ALM,0}^{(1)\top} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 \right) + \frac{1 - \gamma}{2} \left(\sigma_L^2 - 2A_{ALM,0}^{(1)\top} \sigma_{AL} \right) - \frac{\gamma}{2} A_{ALM,0}^{(1)\top} \Sigma_{AA} A_{ALM,0}^{(1)}$$

$$B_{ALM,1}^{(1)} = 0$$

$$B_{ALM,2}^{(1)} = 0$$

and equations (31) and (32) become

$$A_{ALM,0}^{(2)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 + (1 - \gamma) \left(-\Sigma_{AL} + \Sigma_A \left(B_{ALM,1}^{(1)\top} + 2\tilde{B}_{ALM,2}^{(1)} \Phi_0 \right) \right) \right)$$

$$A_{ALM,1}^{(2)} = \frac{1}{\gamma} \Sigma_{AA}^{-1} \left(\Phi_1^A + 2(1 - \gamma) \Sigma_A \tilde{B}_{ALM,2}^{(1)} \Phi_1 \right)$$

and the recursive equations (37), (38) and (39) become

$$B_{ALM,0}^{(2)} = -\Phi_0^L + A_{ALM,0}^{(1)\top} \left(\Phi_0^A + \frac{1}{2} \sigma_A^2 \right) + \frac{1 - \gamma}{2} \left(\Sigma_{LL}^{(1)} - 2A_{ALM,0}^{(1)\top} \Sigma_{AL}^{(1)} \right) - \frac{\gamma}{2} A_{ALM,0}^{(1)\top} \Sigma_{AA} A_{ALM,0}^{(1)} + B_{ALM,0}^{(1)}$$

$$B_{ALM,1}^{(2)} = 0$$

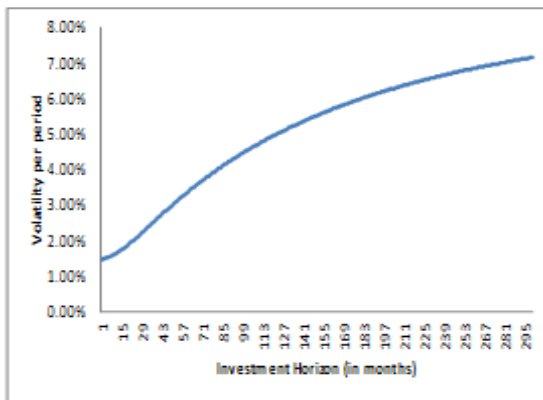
$$B_{ALM,2}^{(2)} = 0$$

In the case of no predictability, the portfolio allocation is constant across time and determined only by the coefficient $A_{ALM,0}$ and the value of the problem is determined by the coefficient $B_{ALM,0}$.

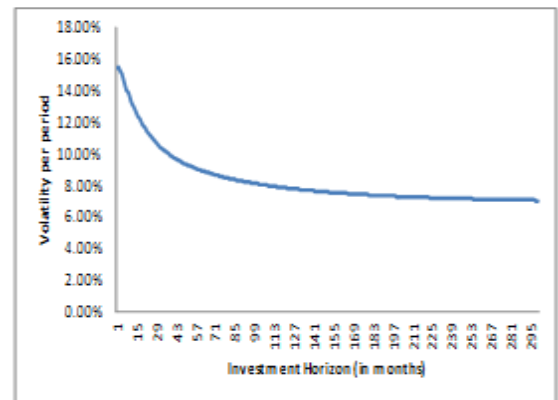
Appendix B

Figure B1: Term Structure of Risk of Assets and Liabilities

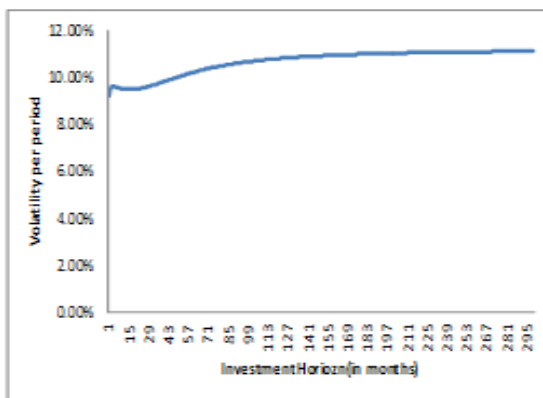
This Figure presents the annualized volatility per period of T-bills, equities, nominal bonds and inflation-indexed (I-I) Bonds (liabilities) in an investment horizon of 300 months. The framework comprises of four asset classes and state variables. The covariance matrix per period T is given by the formula $\Sigma(k) = \sum_{j=1}^T \left(\left(\sum_{i=0}^{j-1} \Phi_1^i \right) \Sigma \left(\sum_{i=0}^{j-1} \Phi_1^i \right)^\top \right)$ where the vectors $(z_{t+1} + z_{t+2} + \dots + z_{t+K})$ follow a VAR (1) process; Φ_1 and Σ denote the square coefficient matrix and variance covariance matrix of innovations, respectively.



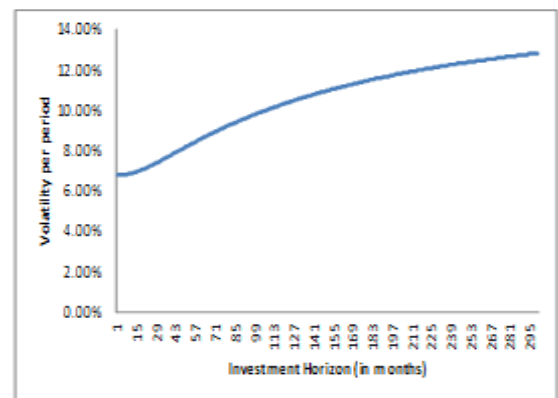
(a) T-bills



(b) Equities



(c) Nominal Bonds

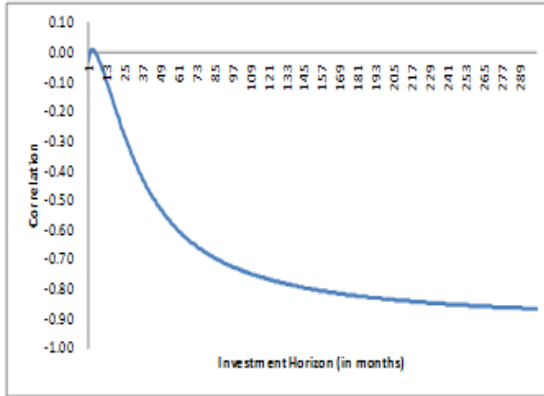


(d) Real Bonds

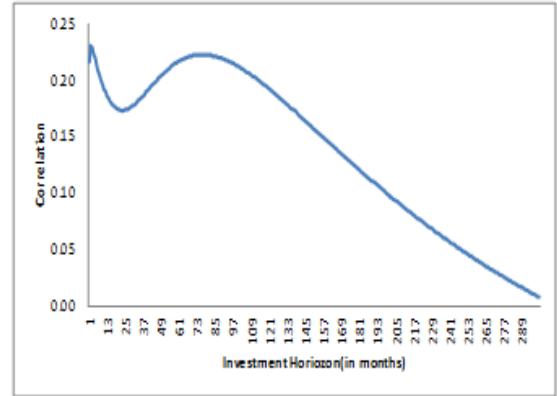
Figure B2: Term Structure of Correlation of Assets with Liabilities

This Figure presents the correlation per period of T-bills, equities and nominal bonds with inflation-indexed (I-I) Bonds (liabilities). The covariance matrix per period is given by the

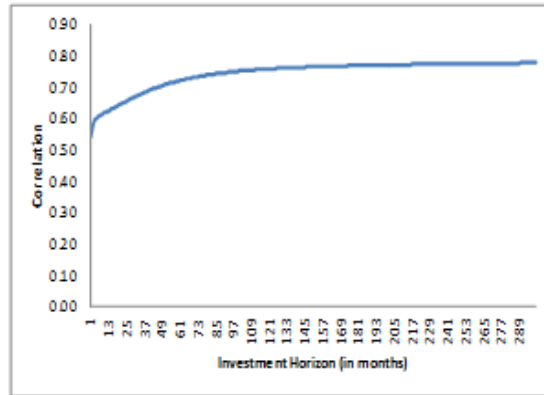
formula $\Sigma(k) = \sum_{j=1}^T \left(\left(\sum_{i=0}^{j-1} \Phi_1^i \right) \Sigma \left(\sum_{i=0}^{j-1} \Phi_1^i \right)^\top \right)$ where the vectors $(z_{t+1} + z_{t+2} + \dots + z_{t+K})$ follow a VAR (1) process; Φ_1 and Σ denote the square coefficient matrix and variance covariance matrix of innovations, respectively.



(a) T-bills



(b) Equities



(c) Nominal Bonds

Table B1: Distribution of the Terminal funding Ratio under a Myopic Asset Liability Management for Constrained Investor

Panel A reports the distribution of the terminal funding ratio of the myopic (M-ALM) strategy, in the absence of funding ratio constraints. Panel B reports the distribution of the terminal funding ratio of the myopic (M-ALM) strategy in the presence of minimum and maximum funding ratio constraints 90% and 135%, respectively. Panel C reports the distribution of the terminal funding ratio of the myopic (M-ALM) strategy in the presence of minimum and maximum funding ratio constraints 100% and 135%, respectively. By distribution we denote the minimum (min), the 2.5%, 25%, 50%, 75%, 97.5% quantiles, the maximum(max) and the average (mean) of the terminal funding ratio. γ denotes the investor's coefficient of relative risk aversion. The portfolio weighting scheme is constrained.

Panel A. Distribution of the funding ratio without funding ratio constraints								
$\gamma = 5$								
Horizon(months)	min	2.50%	25%	50%	75%	97.50%	max	mean
60	0.14	0.93	0.83	0.97	1.11	1.63	7.12	0.99
120	0.13	0.40	0.70	0.95	1.27	2.58	16.88	1.07
180	0.06	0.54	0.64	0.98	1.52	3.93	18.03	1.25
240	0.03	0.20	0.61	1.05	1.83	6.16	33.57	1.54
300	0.01	0.88	0.60	1.13	2.20	9.08	54.87	1.94

Panel B. Distribution of the funding ratio with minimum(90%) and funding ratio constraints (135%)								
$\gamma = 5$								
Horizon(months)	min	2.50%	25%	50%	75%	97.50%	max	mean
60	0.90	0.93	0.95	1.02	1.14	1.35	1.35	1.06
120	0.90	0.90	0.96	1.07	1.21	1.35	1.35	1.09
180	0.90	0.90	0.97	1.08	1.22	1.35	1.35	1.10
240	0.90	0.90	0.97	1.08	1.22	1.35	1.35	1.10
300	0.90	0.91	0.96	1.08	1.22	1.35	1.35	1.10

Panel C. Distribution of the funding ratio with minimum(100%) and funding ratio constraints(135%)								
$\gamma = 5$								
Horizon(months)	min	2.50%	25%	50%	75%	97.50%	max	mean
60	1.00	1.00	1.04	1.10	1.21	1.35	1.35	1.13
120	1.00	1.00	1.05	1.14	1.58	1.35	1.35	1.16
180	1.00	1.00	1.05	1.15	1.26	1.35	1.35	1.16
240	1.00	1.00	1.05	1.15	1.26	1.35	1.35	1.16
300	1.00	1.00	1.05	1.15	1.26	1.35	1.35	1.16

Table B2: Distribution of the Terminal Funding Ratio under a Dynamic Asset Liability Management for Constrained Investor

Panel A reports the distribution of the terminal funding ratio of the dynamic (D-ALM) strategy, in the absence of funding ratio constraints. Panel B reports the distribution of the terminal funding ratio of the dynamic (D-ALM) strategy in the presence of minimum and maximum funding ratio constraints 90% and 135%, respectively. Panel C reports the distribution of the terminal funding ratio of the dynamic (D-ALM) strategy in the presence of minimum and maximum funding ratio constraints 100% and 135%, respectively. By distribution we denote the minimum (min), the 2.5%, 25%, 50%, 75%, 97.5% quantiles, the maximum(max) and the average (mean) of the terminal funding ratio. γ denotes the investor's coefficient of relative risk aversion. The portfolio weighting scheme is constrained.

Panel A. Distribution of the funding ratio without funding ratio constraints								
$\gamma = 5$								
Horizon(months)	min	2.50%	25%	50%	75%	97.50%	max	mean
60	0.46	1.11	1.15	1.31	1.53	2.16	3.77	1.37
120	0.61	0.99	1.49	1.86	2.39	4.05	9.97	2.04
180	0.70	1.00	2.02	2.76	3.83	7.59	19.23	3.16
240	0.85	0.98	2.85	4.16	6.17	13.82	38.94	5.03
300	0.91	1.15	4.07	6.37	9.99	24.08	71.53	8.05

Panel B. Distribution of the funding ratio with minimum(90%) and funding ratio constraints (135%)								
$\gamma = 5$								
Horizon(months)	min	2.50%	25%	50%	75%	97.50%	max	mean
60	0.90	1.11	1.16	1.27	1.34	1.35	1.35	1.23
120	0.90	0.99	1.25	1.31	1.35	1.35	1.35	1.28
180	0.90	1.00	1.26	1.32	1.35	1.35	1.35	1.29
240	0.90	0.98	1.26	1.31	1.35	1.35	1.35	1.29
300	0.90	1.05	1.26	1.31	1.35	1.35	1.35	1.29

Panel C. Distribution of the funding ratio with minimum(100%) and funding ratio constraints(135%)								
$\gamma = 5$								
Horizon(months)	min	2.50%	25%	50%	75%	97.50%	max	mean
60	1.00	1.11	1.19	1.29	1.34	1.35	1.35	1.25
120	1.00	1.01	1.26	1.32	1.35	1.35	1.35	1.29
180	1.00	1.00	1.26	1.32	1.35	1.35	1.35	1.29
240	1.00	1.00	1.26	1.32	1.35	1.35	1.35	1.29
300	1.00	1.05	1.26	1.32	1.35	1.35	1.35	1.30