INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 0000; **00**:1–25
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.4573

Full-complexity Polytopic Robust Control Invariant Sets for Uncertain Linear Discrete Time Systems

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SUMMARY

This paper presents an algorithm for the computation of full-complexity polytopic robust control invariant (RCI) sets, and the corresponding linear state feedback control law. The proposed scheme can be applied for linear discrete-time systems subject to additive disturbances and structured norm-bounded or polytopic uncertainties. Output, initial condition and performance constraints are considered. Arbitrary complexity of the invariant polytope is allowed to enable less conservative inner/outer approximations to the RCI sets, while the RCI set is assumed to be symmetric around the origin. The nonlinearities associated with the computation of such an RCI set structure are overcome through the application of Farkas' Theorem and a corollary of the Elimination Lemma to obtain an initial polytopic RCI set, which is guaranteed to exist under certain conditions. A Newton-like update, which is recursively feasible, is then proposed to yield desirable large/small volume RCI sets.

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KEY WORDS: Robust Control Invariant Sets; Full-complexity Polytope; Uncertain Systems; Optimization; LMIs

1. INTRODUCTION

Robust Control Invariant (RCI) sets determine a bounded region to which the system state can be confined, for all possible disturbances/uncertainties, through the application of a feedback control law [1, 2]. Therefore, RCI sets are widely used in the analysis and design of robust control schemes for uncertain systems. In particular, these sets are of primary importance in establishing the stability and recursive feasibility of Robust Model Predictive Control (RMPC) Schemes, see e.g. [3], [4] and the references therein. Invariant sets also form an important part of the tube-based MPC schemes [5], [6]. Furthermore, they serve as suitable target sets in robust time-optimal control schemes [7], [8].

Due to their widespread application, the problem of efficient computation of RCI sets has been studied extensively over the past few decades, see [1] and [2] for a comprehensive literature survey.

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The two main RCI set structures considered in the literature are polytopic and ellipsoidal [9]. For these structures, the problem of computing the minimal- as well as the maximal-volume RCI set is important and, in most cases, intractable. For example, it is shown in [9] that the exact computation of the minimal invariant set for uncertain systems involves the Minkowski's sum of infinitely many terms, which leads to intractability unless the system dynamics are nilpotent [8]. Therefore, most of the research has been focused on efficient computation of suitable (less conservative) inner/outer approximations to the maximal/minimal RCI set, which is discussed in the following paragraphs.

In [10], a method to compute an outer approximation of the minimal invariant set has been proposed for linear systems with additive disturbances. This approach was subsequently extended in [11] to the degenerate disturbance case. However, both these schemes employ a constant, pre-computed feedback control law which can lead to excessive conservatism. In [12] and [13], in contrast, methods are derived to compute control laws which render a fixed set invariant. It is clear that in order to optimize the size of the invariant set, the best approach is to simultaneously consider both the feedback control law and RCI set as decision variables in the optimization. In this regard, [14] presents a method to compute both the inner ellipsoidal approximation of the RCI set as well as the nonlinear control law for controllable linear system. However, as discussed in [1], generally the polytopic representation is not only less conservative than its ellipsoidal counterpart but also more naturally captures the physical constraints on state and control variables. Therefore, we focus on polytopic RCI sets in this paper.

In [15] and [16], the invariance conditions for the computation of polytopic RCI sets for discrete-time systems without uncertainty or disturbances have been presented. [17] presents a method of obtaining the maximal RCI set by iteratively imposing auxiliary constraints for linear systems with polytopic uncertainty and linear state feedback under the assumption that the system is robustly asymptotically stable. Algorithms for computing maximal RCI sets for stabilizable linear systems with polytopic uncertainty are also reported in [18] (for variable controllers) and [19] (for fixed controllers).

More recently, research in the literature has focused on the computation of low-complexity polytopic RCI sets, in which the the maximum number of faces of the polytope is equal to twice the dimension of the state–space. This is due to their computational advantages for the associated control schemes as well as their reduced conservatism in comparison to ellipsoidal RCI sets (see [20] for details). In [21], necessary and sufficient conditions are derived for the existence of a low-complexity polyhedral RCI set for discrete-time systems with uncertainty, and the set is computed, for a given feedback control law, by solving a quadratic optimization problem. A unified approach to determine the maximal RCI set for saturated linear system under some mathematical assumption is proposed in [22]. An algorithm that optimizes both the maximal polytopic RCI set and the feedback controller simultaneously is proposed in [23] for nonlinear systems, though the computation complexity for obtaining such an RCI set is substantially increased owing to the large number of variables involved. In [24], an efficient method was proposed to compute the maximal/minimal hyper-rectangular RCI set (which is a special case of low-complexity polytopic set) and the corresponding control law for linear discrete-time systems subject to bounded additive disturbances and polytopic input constraints in one step through a single LMI optimization problem. However, the hyper-rectangular set structure is, in general, conservative. In [20], a new algorithm to compute both the maximal low-complexity RCI set as well as the corresponding control law was

DOI: 10.1002/rnc.4573

proposed for systems with polytopic uncertainties. Finally, in [25], the authors developed an LMI-based algorithm, using a slack variable approach, to compute the maximal/minimal low-complexity RCI set for norm-bounded uncertain systems with additive disturbances. The results reported in this work demonstrated an improved set-volume in comparison to the existing schemes in the literature.

Although less conservative than the ellipsoidal set, the low-complexity polytopic RCI set structure is still restrictive due to the constraint on the number of faces of the polytope. To remedy this, [26] proposed a full-complexity RCI set, where an arbitrarily large number of faces could be specified for the polytope. However, the scheme considered linear discrete-time systems with only additive disturbances and state/input constraints and proposed a simple maximization update procedure.

In this paper, we substantially extend the work in [26] by proposing an algorithm to efficiently compute full-complexity RCI sets, for linear systems with structured norm-bounded/polytopic uncertainty as well as additive disturbances. Moreover, in addition to the state/input constraints, output constraints as well as initial condition and $\mathcal{H}_2$ performance constraints are also handled within the formulation in a unified framework. The proposed scheme enables the computation of inner/outer approximations to both maximal/minimal RCI sets (the scheme in [26] only considered the maximization problem). The proposed method computes an initial full-complexity inner/outer approximation to the maximal/minimal RCI set as well as the corresponding feedback gain through convex/LMI optimization. The volume of this initial invariant set is then iteratively optimized (minimized/maximized) based on a Newton-like update procedure. Through numerical examples, it is shown that the proposed algorithm can result in a substantially improved RCI set (volume-wise) as compared to the low-complexity set computation methods proposed in the literature. Furthermore, the iterative algorithm guarantees recursive feasibility and the existence of an initial solution under certain conditions. Finally, unlike the simple update procedure given in [26], the proposed iterations are based on a Newton-like update which results in an observed quadratic speed of convergence.

The efficiency of the proposed algorithm is compared with that in the literature in Section 7. The first example presents a 4th order nominal system, and our algorithm provides a larger maximal approximations of the RCI sets compared with the methods used in [27] and [28]. The second example demonstrates the efficiency of our proposed result (Theorem 4) to obtain an initial RCI set, and illustrates that the proposed algorithm can provide smaller minimal approximations of the RCI sets compared with the methods in [29] and [10]. The third and fourth examples show the efficiency of the proposed algorithm for systems with polyhedral and norm-bounded uncertainty, respectively.

Notation: The notation $A \succ 0$ ($A \prec 0$) denotes that the symmetric matrix A is positive (negative) definite. For an integer $m \geq 1$, we define $\mathcal{I}_m = \{1, \ldots, m\}$. The comparison between two vectors $a, b \in \mathbb{R}^n$ is taken element-wise with $a > b$ implying that $a_i > b_i, \ \forall i \in \mathcal{I}_n$ (similarly for $a < b$, $a \geq b$, and $a \leq b$). $\mathcal{D}_+^m$ denotes the set of positive semidefinite diagonal and $\mathcal{S}_+^m$ denotes the set of positive definite symmetric matrices of dimension $m \times m$. The $m \times m$ identity matrix is denoted as I_m and its ith column, where m is defined by the context, is denoted as e_i. The symbol $0_{m\times n}$ represents the $m \times n$ null matrix, with the dimensions omitted when defined by the context. The vector of ones is denoted as e with the dimension deduced from the context. The block diagonal matrix with the ith diagonal block A_i is denoted as $diag(A_1, \ldots, A_m)$. A congruence transformation T for the matrix inequality $A \prec 0$ implies pre- and post- multiplication by T and T^T for the inequality to deduce $TAT^T \prec 0$. A Schur complement corresponds to the result that

DOI: 10.1002/rnc.4573

$C - B^T A^{-1} B \prec 0$ is equivalent to $\begin{bmatrix} A & B \\ \star & C \end{bmatrix} \prec 0$ if $A = A^T \prec 0$ and $C = C^T$, where $\star$ denotes a term deduced from symmetry. For $P \in \mathbb{R}^{m \times n}$ and $0 < b \in \mathbb{R}^m$, we use the notation $\mathcal{P}(P, b)$ to donate the polytope $\{x \in \mathbb{R}^n : -b \leq Px \leq b\}$. For $Q \in \mathcal{S}_+^n$, we use the notation $\mathcal{E}(Q)$ to denote the ellipsoid $\{x \in \mathbb{R}^n : x^T Q x \leq 1\}$.

2. PROBLEM DESCRIPTION

This paper considers constrained, linear discrete-time systems. In this section, the structure of the disturbances and uncertainties of the system are stated, and some preliminary results for the RCI set computation are introduced.

2.1. System model

Consider a linear discrete time system with disturbances, model uncertainties and state feedback

$$\begin{bmatrix} x^+ \\ y \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} & B_w \\ C & D & D_w \end{bmatrix} \begin{bmatrix} x \\ u \\ w \end{bmatrix}, \qquad u = Kx, \tag{1}$$

with

$$\begin{bmatrix} x^+ \\ y \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^n \\ \mathbb{R}^{n_y} \end{bmatrix}, \quad \begin{bmatrix} \tilde{A} & \tilde{B} & B_w \\ C & D & D_w \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^{n \times n} & \mathbb{R}^{n \times n_u} & \mathbb{R}^{n \times n_w} \\ \mathbb{R}^{n_y \times n} & \mathbb{R}^{n_y \times n_u} & \mathbb{R}^{n_y \times n_w} \end{bmatrix}, \quad \begin{bmatrix} x \\ u \\ w \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^n \\ \mathbb{R}^{n_u} \\ \mathbb{R}^{n_w} \end{bmatrix},$$

where x is the current state, x^+ is the successor state and u and w denote the current control and disturbance signals, respectively. Here, $\tilde{A}$ and $\tilde{B}$ belong to the norm–bounded structured uncertainty set:

$$\Omega := \{(\tilde{A}, \tilde{B}) : [\tilde{A}\ \tilde{B}] = [A\ B] + B_p \Delta [C_q\ D_q],\ \|\Delta\| \leq 1,\ \Delta \in \boldsymbol{\Delta} \subseteq \mathbb{R}^{n_p \times n_q}\}, \tag{2}$$

where $A \in \mathbb{R}^n, B \in \mathbb{R}^{n \times n_u}, B_p \in \mathbb{R}^{n \times n_p}, C_q \in \mathbb{R}^{n_q \times n}$ and $D_q \in \mathbb{R}^{n_q \times n_u}$ are given matrices, and where it is assumed that the subspace $\mathcal{B}$ associated with the structured subspace $\boldsymbol{\Delta}$ in Lemma 1 below can be defined. We also consider polytopic model uncertainty; see Remark 7 below. We assume that the disturbance w belongs to a bounded polytope $\mathcal{W} := \mathcal{P}(V, d)$ where $V \in \mathbb{R}^{m_w \times n_w}$ and $0 < d \in \mathbb{R}^{m_w}$ are given. In the sequel, we consider two types of constraints: an output constraint related to the signal y in (1) and an $\mathcal{H}_2$-norm performance constraint.

2.2. Robust control invariant set

Standard procedures of calculating admissible RCI sets require a pre-defined structure for computational tractability [1]. A full-complexity polytopic structure, whose advantages have been shown in [26], will be considered in this paper. This has the form $\mathcal{P}(P, b) = \{x \in \mathbb{R}^n : -b \leq Px \leq$

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DOI: 10.1002/rnc.4573

$b\}$ where $0 < b \in \mathbb{R}^m$, $P \in \mathbb{R}^{m\times n}$ and $m \geq n$; m can be chosen based on the required accuracy. Note that for $m = n$, $\mathcal{P}(P, b)$ reduces to a low-complexity polytope (see e.g. [20, 25]).

The requirements on an RCI set [30] include invariance and output constraint satisfaction. For system (1), set $\mathcal{P}(P, b)$ and given $0 < \bar{y} \in \mathbb{R}^{n_y}$, these can be written as

$$\left\{\begin{array}{c} x \in \mathcal{P}(P,b) \\ w \in \mathcal{W} \\ (\tilde{A},\tilde{B}) \in \Omega \end{array}\right\} \Rightarrow x^+ \in \mathcal{P}(P,b) \qquad \text{(Invariance)} \tag{3}$$

and

$$\left\{\begin{array}{c} x \in \mathcal{P}(P,b) \\ w \in \mathcal{W} \end{array}\right\} \Rightarrow y \in \mathcal{Y} \qquad \text{(Output constraint)} \tag{4}$$

respectively, where $\mathcal{Y} := \mathcal{P}(I_{n_y}, \bar{y})$. Since RCI sets are in general not unique, the maximal RCI set is defined in terms of unions of all such sets, and the minimal RCI set under a specific control law is defined in terms of intersections of the RCI sets corresponding with that control law. For RCI sets of a pre–defined structure, this definition is modified to optimize the volume of these sets. Maximal RCI sets are associated with target sets in the state–space and are required to be large, since they are associated with switching from on–line to off–line control once the state is inside the set. Since a direct characterization of the volume of a polytope is not feasible when $m > n$, we introduce an inner bounding ellipsoid $\mathcal{E}(\underline{Q})$, require $\exists \underline{Q} \in \mathcal{S}^n_+$ such that

$$\mathcal{E}(\underline{Q}) \subset \mathcal{P}(P,b) \qquad \text{(Inner bounding ellipsoid)} \tag{5}$$

and maximize $\log\det(\underline{Q}^{-1})$. Maximal RCI sets are also required to be sufficiently small so that the performance is acceptable once the state is inside. While the volume is to some extent limited by the output constraint requirement, we further impose a performance constraint as follows. Consider the cost signal

$$z = C_2\bar{x} + D_2\bar{u}, \tag{6}$$

where $z \in \mathbb{R}^{n_z}$, $C_2 \in \mathbb{R}^{n_z\times n}$, $D_2 \in \mathbb{R}^{n_z\times n_u}$, and $\bar{x}^+ = \tilde{A}\bar{x} + \tilde{B}\bar{u},\ \bar{u} = K\bar{x}$ represent the nominal system of (1). With z_k denoting the current cost signal z and r a required performance level, we require

$$\left\{\begin{array}{c} \bar{x} \in \mathcal{P}(P,b) \\ (\tilde{A},\tilde{B}) \in \Omega \end{array}\right\} \Rightarrow J := \sum_{k=0}^{\infty} \|z_k\|^2 < r^2 \qquad (\mathcal{H}_2\text{−norm constraint}) \tag{7}$$

Note that the cost signal z is exclusive of additive disturbance.

Minimal RCI sets are associated with initial states, and are required to be small. To this end, we introduce an outer bounding ellipsoid $\mathcal{E}(\bar{Q})$, requiring $\exists \bar{Q} \in \mathcal{S}^n_+$ such that

$$\mathcal{P}(P,b) \subset \mathcal{E}(\bar{Q}) \qquad \text{(Outer bounding ellipsoid)} \tag{8}$$

DOI: 10.1002/rnc.4573

and minimize $\log\det(\bar{Q}^{-1})$. Minimal sets are also often required to include an initial state set [5] and so, for given $P_0 \in \mathbb{R}^{m_0 \times n}$ and $0 < b_0 \in \mathbb{R}^{m_0}$, we require

$$\mathcal{P}(P_0, b_0) \subseteq \mathcal{P}(P, b) \qquad \text{(Initial constraint)} \tag{9}$$

For given system (1) and (6), sets $\Omega, \mathcal{W}, \mathcal{Y}, \mathcal{P}(P_0, b_0)$, parameter r and $m \geq n$, and with $(P, b, K) \in \Psi := \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^{n_u \times n}$, we present convex algorithms, based on semidefinite programs (SDP), to solve the optimizations:

$$\begin{array}{ll} \max\limits_{P,b,K} & \log\det \underline{Q}^{-1}, \\ \textit{subject to} & (3), (4), (5), (7) \end{array} \qquad \begin{array}{ll} \min\limits_{P,b,K} & \log\det \bar{Q}^{-1}. \\ \textit{subject to} & (3), (4), (8), (9) \end{array} \tag{10}$$

A triple $(P, b, K) \in \Psi$ satisfying either of the constraints in (10) will be called admissible.

3. NONLINEAR FORMULATION

In this section, we derive conditions, in the form of nonlinear matrix inequalities (NLMIs), for the admissibility of $(P, b, K) \in \Psi$. While previous work uses Farkas' Lemma, we will use the following version of Farkas' Theorem instead since expressing the constraints in quadratic form will be shown to offer computational advantages.

Theorem 1 ([31])
Suppose that $\mathcal{C} \subseteq \mathbb{R}^r$ and $f_1, \ldots, f_s : \mathbb{R}^r \to \mathbb{R}$ are convex and satisfy the Slater condition (i.e. $\exists \bar{z} \in relint(\mathcal{C}) \Rightarrow f_j(\bar{z}) \leq 0, j = 1, \ldots, s$) [32]. Let $f_0 : \mathbb{R}^r \to \mathbb{R}$ and define the system

$$\mathcal{S} : \{f_0(z) < 0; f_j(z) \leq 0, j = 1, \ldots, s; z \in \mathcal{C}\}.$$

Consider the statements

(a) $\exists y_1, \ldots, y_s \geq 0 : f_0(z) + \sum_{j=1}^{s} y_j f_j(z) \geq 0, \forall z \in \mathcal{C}$.
(b) $\mathcal{S}$ is not solvable.

Then (a) $\Rightarrow$ (b). Furthermore, if f_0 is convex, then (a) $\Leftrightarrow$ (b).

The next well–known result is used for norm–bounded structured uncertainties.

Lemma 1 ([33])
Given $T_1 = T_1^T \in \mathbb{R}^{r \times r}, T_2 \in \mathbb{R}^{r \times n_p}, T_3 \in \mathbb{R}^{n_q \times r}$ and $\mathbf{\Delta} \subseteq \mathbb{R}^{n_p \times n_q}$. Define the subspace

$$\mathcal{B} = \{(S, T, R) \in \mathcal{S}_+^{n_p} \times \mathcal{S}_+^{n_q} \times \mathbb{R}^{n_p \times n_q} : S\Delta = \Delta T,\ \Delta R^T + R\Delta^T = 0\ \forall \Delta \in \mathbf{\Delta} \quad \},$$

and consider the statements:

(a) $\exists (S, T, R) \in \mathcal{B} : \begin{bmatrix} T_1 - T_2 S T_2^T & T_3^T + T_2 R \\ \star & T \end{bmatrix} \succ 0.$

(b) $T_1 + T_2 \Delta T_3 + (T_2 \Delta T_3)^T \succ 0\ \forall \Delta \in \mathbf{\Delta}, \|\Delta\| \leq 1.$

Then (a)$\Rightarrow$(b). Furthermore, if $n_p = n_q$ and $\mathbf{\Delta} = \mathbb{R}^{n_p \times n_p}$, then (a)$\Leftrightarrow$(b).

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DOI: 10.1002/rnc.4573

The following theorem uses the previous two results to derive conditions, in the form of NLMIs, for the existence of an admissible triple $(P, b, K) \in \Psi$ [26].

Theorem 2
Let all definitions be as above and denote

$$A^K := A + BK, \;\; C^K := C + DK, \;\; C_q^K := C_q + D_q K, \;\; C_2^K := C_2 + D_2 K.$$

Then for $(P, b, K) \in \Psi$ we have:

1. The invariance condition (3) is satisfied if (and only if when $n_p = n_q$ and $\mathbf{\Delta} = \mathbb{R}^{n_p \times n_p}$),

$$\begin{bmatrix} D_i \in \mathcal{D}_+^m \\ W_i \in \mathcal{D}_+^{m_w} \\ (S_i, T_i, R_i) \in \mathcal{B} \end{bmatrix}, \left[\begin{array}{c|c} 2e_i^T b - b^T D_i b - d^T W_i d & e_i^T P \begin{bmatrix} B_w & B_p S_i & B_p R_i & A^K \end{bmatrix} \\ \hline \star & \begin{bmatrix} V^T W_i V & 0 & 0 & 0 \\ \star & S_i & 0 & 0 \\ \star & \star & T_i & C_q^K \\ \star & \star & \star & P^T D_i P \end{bmatrix} \end{array}\right] \succeq 0, \forall i \in \mathcal{I}_m. \quad (11)$$

2. The output constraint condition (4) is satisfied if and only if,

$$\begin{bmatrix} E_j \in \mathcal{D}_+^m \\ G_j \in \mathcal{D}_+^{m_w} \end{bmatrix}, \begin{bmatrix} 2e_j^T \bar{y} - b^T E_j b - d^T G_j d & e_j^T D_w & e_j^T C^K \\ \star & V^T G_j V & 0 \\ \star & \star & P^T E_j P \end{bmatrix} \succeq 0, \forall j \in \mathcal{I}_{n_y}. \quad (12)$$

3. The inner bounding ellipsoid condition (5) is satisfied if and only if

$$\underline{Q} \in \mathcal{S}_+^n; \; \mu_i \geq 0, \begin{bmatrix} 2e_i^T b - \mu_i & e_i^T P \\ \star & \mu_i \underline{Q} \end{bmatrix} \succeq 0, \; \forall i \in \mathcal{I}_m. \quad (13)$$

4. The $\mathcal{H}_2$-norm constraint condition (7) is satisfied if

$$\begin{bmatrix} Q \in \mathcal{S}_+^n \\ D_z \in \mathcal{D}_+^m \\ (S, T, R) \in \mathcal{B} \end{bmatrix}, \begin{bmatrix} Q - B_p S B_p^T & 0 & B_p R & A^K \\ \star & rI & 0 & C_2^K \\ \star & \star & T & C_q^K \\ \star & \star & \star & Q^{-1} \end{bmatrix} \succ 0, \begin{bmatrix} Q & I \\ \star & P^T D_z P \end{bmatrix} \succ 0, r > b^T D_z b. \quad (14)$$

5. The outer bounding ellipsoid condition (8) is satisfied if

$$\begin{bmatrix} \bar{D} \in \mathcal{D}_+^m \\ \bar{Q} \in \mathcal{S}_+^n \end{bmatrix}, \quad P^T \bar{D} P - \bar{Q} \succ 0, \quad 1 > b^T \bar{D} b. \quad (15)$$

6. The initial constraint condition (9) is satisfied if and only if

$$F_i \in \mathcal{D}_+^{m_0}, \begin{bmatrix} 2e_i^T b - b_0^T F_i b_0 & e_i^T P \\ \star & P_0^T F_i P_0 \end{bmatrix} \succeq 0, \forall i \in \mathcal{I}_m. \quad (16)$$

DOI: 10.1002/rnc.4573

Hence solutions to the optimizations in (10) can be obtained by solving the nonlinear SDPs

$$\begin{array}{ll} \max\limits_{P,b,K,D_i,W_i,S_i,T_i,R_i,E_j,G_j,\underline{\mu}_i,\underline{Q},Q,D_z,S,T,R} \log\det Q^{-1}, & \min\limits_{P,b,K,D_i,W_i,S_i,T_i,R_i,E_j,G_j,\bar{D},\bar{Q},F_i} \log\det \bar{Q}^{-1}. \\ \mathit{subject\ to}\ (11),(12),(13),(14) & \mathit{subject\ to}\ (11),(12),(15),(16) \end{array} \tag{17}$$

Proof

The proof is an application of Lemma 1 and Farkas' Theorem. In more detail:

1. The invariance condition (3) is equivalent to the requirement that for all $i \in \mathcal{I}_m$ and for all $(\tilde{A}, \tilde{B}) \in \Omega$,

$$\left\{ \begin{array}{c} (e_j^T Px)^2 - (e_j^T b)^2 \leq 0, \forall j \in \mathcal{I}_m \\ (e_k^T Vw)^2 - (e_k^T d)^2 \leq 0, \forall k \in \mathcal{I}_{m_w} \end{array} \right\} \Rightarrow 2e_i^T \left(b - P((\tilde{A} + \tilde{B}K)x + B_w w) \right) \geq 0.$$

For each $i \in \mathcal{I}_m$, we use Theorem 1 with $\mathcal{C} = \mathbb{R}^r$, $r = n + n_w$, $s = m + m_w$, $z^T = \begin{bmatrix} x^T & w^T \end{bmatrix}$, and

$$\begin{aligned} f_0(z) &= 2e_i^T \left(b - P((\tilde{A} + \tilde{B}K)x + B_w w) \right) \\ &= 2e_i^T b - e_i^T P \begin{bmatrix} \tilde{A} + \tilde{B}K & B_w \end{bmatrix} z - z^T \begin{bmatrix} (\tilde{A} + \tilde{B}K)^T \\ B_w^T \end{bmatrix} P^T e_i; \\ f_j(z) &= z^T \begin{bmatrix} P^T \\ 0 \end{bmatrix} e_j e_j^T \begin{bmatrix} P & 0 \end{bmatrix} z - b^T e_j e_j^T b,\ j = 1, 2, \ldots, m; \\ f_j(z) &= z^T \begin{bmatrix} 0 \\ V^T \end{bmatrix} e_j e_j^T \begin{bmatrix} 0 & V \end{bmatrix} z - d^T e_j e_j^T d,\ j = m+1, m+2, \ldots, m + m_w. \end{aligned}$$

Note that in the last two expressions for $f_j(z)$, e_j has different dimensions which can be deduced from the context. Then we have $f_0(z) + \sum_{j=1}^{s} y_j f_j(x) = a^T N_i a$, where $a^T := \begin{bmatrix} -1 & w^T & x^T \end{bmatrix}$, and

$$N_i := \begin{bmatrix} 2e_i^T b - b^T D_i b - d^T W_i d & e_i^T P B_w & e_i^T P(\tilde{A} + \tilde{B}K) \\ \star & V^T W_i V & 0 \\ \star & \star & P^T D_i P \end{bmatrix},$$

with $D_i = diag(y_1, y_2, \ldots, y_m)$ and $W_i = diag(y_{m+1}, y_{m+2}, \ldots, y_{m+m_w})$. Since f_0 is linear, it is convex, and it follows from Theorem 1 that for $D_i \in \mathcal{D}_+^m$, $W_i \in \mathcal{D}_+^{m_w}$, and $(P, b, K) \in \Psi$, the invariance condition is equivalent to $N_i \succeq 0$.
Now for each $i \in \mathcal{I}_m$, let $N_i = T_1^i + T_2^i \Delta T_3^i + (T_2^i \Delta T_3^i)^T$ with $(T_2^i)^T = \begin{bmatrix} B_p^T P^T e_i & 0 & 0 \end{bmatrix}$ and $T_3^i = \begin{bmatrix} 0 & 0 & (C_q + D_q K) \end{bmatrix}$. Then following Lemma 1, we have $\exists (S_i, T_i, R_i) \in \mathcal{B}$:

$$\begin{bmatrix} 2e_i^T b - b^T D_i b - d^T W_i d - e_i^T P B_p S_i B_p^T P^T e_i & e_i^T P B_w & e_i^T P A^K & e_i^T P B_p R_i \\ \star & V^T W_i V & 0 & 0 \\ \star & \star & P^T D_i P & (C_q^K)^T \\ \star & \star & \star & T_i \end{bmatrix} \succeq 0$$

 DOI: 10.1002/rnc.4573

as a sufficient condition for $N_i \succeq 0$. Following a congruence transformation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix},$$

and a Schur complement for the $(1,1)$ entry provide the sufficiency of (11).

2. The proof is similar to Part 1 and follows from Theorem 1 and some manipulations.
3. The inner bounding ellipsoid condition (5) is equivalent to the requirement that for all $i \in \mathcal{I}_m$,

$$x^T \underline{Q} x - 1 \leq 0 \Rightarrow 2e_i^T(b - Px) \geq 0.$$

The result then follows from Theorem 1 using a procedure similar to Part 1.

4. For any $\bar{x} \in \mathcal{P}(P, b)$, a minor extension of the results in [34] (Theorem 1 in Section 3.1) gives the first inequality in the $\mathcal{H}_2$-norm constraint condition (14) and

$$r - \bar{x}^T Q^{-1} \bar{x} \geq 0 \tag{18}$$

as sufficient conditions for $J < r^2, \forall (\tilde{A}, \tilde{B}) \in \Omega$. Using a procedure similar to Part 1, Theorem 1 then gives the second and third inequalities in (14) as sufficient conditions for (18) to be satisfied for all $\bar{x} \in \mathcal{P}(P, b)$.

5. The outer bounding ellipsoid condition (8) is equivalent to the requirement that

$$(e_j^T Px)^2 - (e_j^T b)^2 \leq 0 \; \forall j \in \mathcal{I}_m \Rightarrow 1 - x^T \bar{Q} x \geq 0.$$

The result then follows from Theorem 1.

6. The initial constraint condition (9) is equivalent to the requirement that for all $i \in \mathcal{I}_m$,

$$(e_j^T P_0 x)^2 - (e_j^T b_0)^2 \leq 0 \; \forall j \in \mathcal{I}_{m_0} \Rightarrow 2e_i^T(b - Px) \geq 0.$$

The result then follows from Theorem 1.

Finally, (17) follows from 1–6 above. □

Remark 1

Although we give non-strict inequalities in (11)-(13), and (16) in order to emphasize the necessity and sufficiency of the statements, these will be replaced by strict inequalities in the sequel in order to avoid dealing with numerical difficulties associated with optimality. It follows that, in common with other LMI problems [35], the algorithms resulting from the use of Theorem 2 may become badly conditioned near optimality.

DOI: 10.1002/rnc.4573

4. LINEARIZATION AND INITIAL COMPUTATION

Nonlinearities of the conditions derived in Theorem 2 exist in the terms $e_i^T PH$ (where H denotes B_pS_i, B_pR_i, or A^K), b^THb (where H denotes D_i, E_j, or $\bar{D}$), P^THP (where H denotes D_i, E_j, D_z, or $\bar{D}$), and $\mu_i\underline{Q}$. Furthermore, the fact that matrix P is considered to be non-square prevents the direct application of the linearization procedure presented in [25]. In this section we propose a linearization algorithm, extending the basic ideas proposed in [26], that involves the computation of an initial solution. An update algorithm is then presented in the next section.

We set

$$\mathcal{P}(P,b) = \mathcal{P}(P_rX, b_r) = \{x \in \mathbb{R}^n : -b_r \leq P_rXx \leq b_r\}$$

as an initial full-complexity inner/outer approximation to the maximal/minimal RCI set, where b_r and P_r are given (see Remark 2), and where $X \in \mathbb{R}^{n\times n}$ is a variable used to reshape (rotate and scale) the polyhedral set defined by P_r.

The following is a corollary of the Elimination Lemma [36] and is used for the initial linear solution.

Corollary 1
[26] Given $T \in \mathcal{S}_+^n$, $E \in \mathbb{R}^{n\times p}$, $F \in \mathbb{R}^{p\times m}$, $Z \in \mathcal{S}_+^m$ and $\mathbb{Y} \subseteq \mathbb{R}^{p\times p}$. Consider the statements:

(a) $\begin{bmatrix} T & EY & 0 \\ \star & Y^T+Y & F \\ \star & \star & Z \end{bmatrix} \succ 0$ holds for some $Y \in \mathbb{Y}$.

(b) $\begin{bmatrix} T & EF \\ \star & Z \end{bmatrix} \succ 0.$

Then (a) $\Rightarrow$ (b). Furthermore, if $\mathbb{Y} = \mathbb{R}^{p\times p}$, then (a) $\Leftrightarrow$ (b).

The following result gives sufficient conditions for the admissibility of the triple (P_rX, b_r, K) in the form of LMIs by using Corollary 1.

Theorem 3
Let all the definitions be as above and let $P = P_rX$ and $b = b_r$, where $P_r \in \mathbb{R}^{m\times n}$ and $b_r \in \mathbb{R}^m$ are given and where $X \in \mathbb{R}^{n\times n}$. Denote

$$\hat{X} := X^{-1}, \qquad \hat{K} := KX^{-1}, \qquad \hat{A} := A\hat{X} + B\hat{K},$$
$$\hat{C} := C\hat{X}+D\hat{K}, \qquad \hat{C}_q := C_q\hat{X}+D_q\hat{K}, \qquad \hat{C}_2 := C_2\hat{X}+D_2\hat{K}.$$

Then

1. The invariance condition (11), hence (3), is satisfied if, with

$$\hat{l}_i(\lambda_i, \hat{D}_i, \hat{W}_i) := 2\lambda_i e_i^T b_r - b_r^T\hat{D}_ib_r - d^T\hat{W}_id,$$

DOI: 10.1002/rnc.4573

$$\begin{bmatrix}\lambda_i>0\\ \hat{D}_i\in\mathcal{D}_+^m\\ \hat{W}_i\in\mathcal{D}_+^{m_w}\\ (\hat{S}_i,\hat{T}_i,\hat{R}_i)\in\mathcal{B}\end{bmatrix},\begin{bmatrix}\hat{l}_i(\lambda_i,\hat{D}_i,\hat{W}_i) & \lambda_i e_i^T P_r & 0 & 0 & 0\\ \star & \hat{X}+\hat{X}^T-B_p\hat{S}_iB_p^T & B_w & B_p\hat{R}_i & \hat{A}\\ \star & \star & V^T\hat{W}_iV & 0 & 0\\ \star & \star & \star & \hat{T}_i & \hat{C}_q\\ \star & \star & \star & \star & P_r^T\hat{D}_iP_r\end{bmatrix}\succ 0,\forall i\in\mathcal{I}_m. \quad (19)$$

2. The output constraint condition (12), hence (4), is satisfied if and only if

$$\begin{bmatrix}E_j\in\mathcal{D}_+^m\\ G_j\in\mathcal{D}_+^{m_w}\end{bmatrix},\begin{bmatrix}2e_j^T\bar{y}-b_r^TE_jb_r-d^TG_jd & e_j^TD_w & e_j^T\hat{C}\\ \star & V^TG_jV & 0\\ \star & \star & P_r^TE_jP_r\end{bmatrix}\succ 0,\forall j\in\mathcal{I}_{n_y}. \quad (20)$$

3. The inner bounding ellipsoid condition (13), hence (5), is satisfied if

$$\underline{Q}^{-\frac{1}{2}}\in\mathcal{S}_+^n;\ \begin{bmatrix}\hat{\mu}_i>0\\ \gamma_i>0\end{bmatrix},\begin{bmatrix}2\gamma_ie_i^Tb_r-\hat{\mu}_i & \gamma_ie_i^TP_r & 0\\ \star & \hat{X}+\hat{X}^T & \underline{Q}^{-\frac{1}{2}}\\ \star & \star & \hat{\mu}_iI_n\end{bmatrix}\succ 0,\forall i\in\mathcal{I}_m. \quad (21)$$

4. The $\mathcal{H}_2$-norm constraint condition (14), hence (7), is satisfied if

$$\begin{bmatrix}\zeta>0\\ \hat{Q}\in\mathcal{S}_+^n\\ \hat{D}_z\in\mathcal{D}_+^m\\ (\hat{S},\hat{T},\hat{R})\in\mathcal{B}\end{bmatrix},\begin{bmatrix}\hat{Q}-B_p\hat{S}B_p^T & 0 & B_p\hat{R} & \hat{A}\\ \star & (2-\zeta)rI & 0 & \hat{C}_2\\ \star & \star & \hat{T} & \hat{C}_q\\ \star & \star & \star & \hat{X}+\hat{X}^T-\hat{Q}\end{bmatrix}\succ 0,$$
$$\begin{bmatrix}\hat{Q} & \hat{X}\\ \star & P_r^T\hat{D}_zP_r\end{bmatrix}\succ 0,\quad \zeta r>b_r^T\hat{D}_zb_r. \quad (22)$$

5. The outer bounding ellipsoid condition (15), hence (8), is satisfied if

$$\begin{bmatrix}\bar{D}\in\mathcal{D}_+^m\\ \bar{Q}^{-1}\in\mathcal{S}_+^n\end{bmatrix},\quad\begin{bmatrix}\bar{Q}^{-1} & \hat{X}\\ \star & P_r^T\bar{D}P_r\end{bmatrix}\succ 0,\quad 1>b_r^T\bar{D}b_r. \quad (23)$$

6. The initial constraint condition (16), hence (9), is satisfied if and only if

$$\begin{bmatrix}\nu_i>0\\ \hat{F}_i\in\mathcal{D}_+^{m_0}\end{bmatrix},\begin{bmatrix}2\nu_ie_i^Tb_r-b_0^T\hat{F}_ib_0 & \nu_ie_i^TP_r & 0\\ \star & \hat{X}+\hat{X}^T & I_n\\ \star & \star & P_0^T\hat{F}_iP_0\end{bmatrix}\succ 0,\forall i\in\mathcal{I}_m. \quad (24)$$

Hence initial solutions to the optimizations in (10) can be obtained by solving the convex SDPs

$$\max_{\hat{X},\hat{K},\lambda_i,\hat{D}_i,\hat{W}_i,\hat{S}_i,\hat{T}_i,\hat{R}_i,E_j,G_j,\underline{Q}^{-\frac{1}{2}},\hat{\mu}_i,\gamma_i,\zeta,\hat{Q},\hat{D}_z,\hat{S},\hat{T},\hat{R}} \log\det\underline{Q}^{-\frac{1}{2}}, \quad \min_{\hat{X},\hat{K},\lambda_i,\hat{D}_i,\hat{W}_i,\hat{S}_i,\hat{T}_i,\hat{R}_i,E_j,G_j,\bar{D},\bar{Q}^{-1},\nu_i,\hat{F}_i} trace(\bar{Q}^{-1}). \quad (25)$$
$$subject\ to\ (19),(20),(21),(22) \qquad subject\ to\ (19),(20),(23),(24)$$

DOI: 10.1002/rnc.4573

Proof

The proof consists in manipulating each of (11)-(16) into the form of statement (2) of Corollary 1 and then use the corollary to show that (19)-(24), after some manipulation, correspond to statement (1) with an appropriate Y, and are therefore sufficient conditions for (11)-(16) and hence for (3)-(9), respectively. In detail:

1. Applying Corollary 1 on the invariance condition (11) (with $E = e_i^T P_r X$ and $Y = \lambda_i X^{-1}$), effecting a Schur complement and the congruence $diag(\lambda_i^{\frac{1}{2}}, \lambda_i^{-\frac{1}{2}} I_n, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{-\frac{1}{2}} I_{n_q}, \lambda_i^{\frac{1}{2}} X^{-1})$ shows that (19) implies (11) upon the redefinitions

$$\hat{D}_i := \lambda_i D_i,\ \hat{W}_i := \lambda_i W_i,\ \hat{S}_i := \lambda_i^{-1} S_i,\ \hat{T}_i := \lambda_i^{-1} T_i,\ \hat{R}_i := \lambda_i^{-1} R_i. \tag{26}$$

2. Effecting the congruence $diag(1, I_{n_w}, X^{-T})$ on the output constraint condition (12) shows that it is equivalent to (20).
3. Applying Corollary 1 (with $E = e_i^T P_r X$ and $Y = \gamma_i X^{-1}$) on the inner bounding ellipsoid condition (13), implementing the congruence $diag(\gamma_i^{\frac{1}{2}}, \gamma_i^{-\frac{1}{2}} I_n, \gamma_i^{\frac{1}{2}} \underline{Q}^{-\frac{1}{2}})$ shows that (21) implies (13) upon the redefinition

$$\hat{\mu}_i := \gamma_i \mu_i. \tag{27}$$

4. Effecting the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{\frac{1}{2}} X^{-T})$ shows that the second inequalities of (14) and (22) are equivalent while the third inequality in (22) is ζ times the third in (14). Effecting the congruence $diag(I_n, 1, I_{n_q}, Q)$ and applying Corollary 1 on the first inequality in (14) (with $F = Q$ and $Y = \zeta X^{-1}$) followed by a Schur complement, the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{-\frac{1}{2}}, \zeta^{-\frac{1}{2}} I_{n_q}, \zeta^{-\frac{1}{2}} I_n)$ shows that (22) implies (14) since $\zeta^{-1} \geq 2 - \zeta$ for all $\zeta > 0$ upon the redefinitions

$$\hat{Q} := \zeta^{-1} Q,\ \hat{D}_z := \zeta D_z,\ \hat{S} := \zeta^{-1} S,\ \hat{R} := \zeta^{-1} R,\ \hat{T} := \zeta^{-1} T. \tag{28}$$

5. For the first inequality in the outer bounding ellipsoid condition (15), effecting the congruence X^{-T} and then using a Schur complement shows that (23) is equivalent to (15).
6. Applying Corollary 1 (with $E = e_i^T P_r X$ and $Y = \nu_i X^{-1}$) on the initial constraint condition (16) and implementing the congruence $diag(\nu_i^{\frac{1}{2}}, \nu_i^{-\frac{1}{2}} I_n, \nu_i^{\frac{1}{2}} I_n)$ shows that (24) implies (16) upon the redefinition

$$\hat{F}_i := \nu_i F_i.$$

Finally, (25) follows from 1–6 above and the fact that $\det(Z) \leq \left(\frac{\mathrm{trace}(Z)}{n}\right)^n$ for any $n \times n$ positive definite matrix Z. □

Remark 2

The conservatism introduced by the linearization in Theorem 3, compared to Theorem 2, can be traced back to the use of Corollary 1 and to the choice of P_r and b_r. Note that we restrict $\mathbb{Y}$ for a tractable solution. Although this restriction can be relaxed, the resulting optimization becomes nonlinear and this will not be pursued here. In our examples, we used the vector of ones for b_r and the regular polytope with $2m$ faces for P_r. Since the initial polytope is $\mathcal{P}(P_r X, b_r)$, then X provides

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DOI: 10.1002/rnc.4573

scaling and rotational degrees of freedom. A possible choice of the initial polytope $\mathcal{P}(P_r X, b_r)$, which is guaranteed to be feasible under some assumptions, is outlined in the next theorem.

Getting an initial RCI set is a very important step since the procedure of obtaining the maximal/minimal RCI set requires a feasible initial solution. In general, to guarantee the existence of a polytopic RCI set is difficult, see [1, 16] for more details. Based on an idea in [20], the following result gives conditions which guarantee the existence of an initial solution, under a mild assumption, satisfying the invariance and inner and outer bounding ellipsoidal conditions (3), (5), and (8) respectively, (which correspond to conditions (19), (21), and (23) in Theorem 3, respectively) in the special case of no disturbances or uncertainties.

Theorem 4

Suppose that all the uncontrollable eigenvalues of the pair (A, B) have absolute value less than $\frac{1}{\sqrt{n}}$. Let $m = ln$ where l is any integer greater than 0. Then there exist $P_r \in \mathbb{R}^{m\times n}$, $b_r \in \mathbb{R}^m$, $\hat{X} \in \mathbb{R}^{n\times n}$, $\hat{K} \in \mathbb{R}^{n\times n_u}$, $\bar{Q}^{-1}, \underline{Q}^{-1} \in \mathcal{S}_+^n$, $\bar{D} \in \mathcal{D}_+^m$ and $\lambda_i, \hat{\mu}_i, \gamma_i > 0$, $\hat{D}_i \in \mathcal{D}_+^m$, for every $i \in \mathcal{I}_m$, such that the invariance condition

$$L_i := \begin{bmatrix} 2\lambda_i e_i^T b_r - b_r^T \hat{D}_i b_r & \lambda_i e_i^T P_r & 0 \\ \star & \hat{X} + \hat{X}^T & A\hat{X} + B\hat{K} \\ \star & \star & P_r^T \hat{D}_i P_r \end{bmatrix} \succ 0, \tag{29}$$

and the, respectively, inner and outer bounding ellipsoid conditions (21) and (23) of Theorem 3 are satisfied.

Proof

Note first that condition (29) and the invariance condition (19) are identical in the absence of disturbances and uncertainties. Since all the uncontrollable eigenvalues of the pair (A, B) have absolute value less than $\frac{1}{\sqrt{n}}$, there exists $K \in \mathbb{R}^{n\times n_u}$ such that all the eigenvalues of $A + BK$ lie in the open disc with radius $\frac{1}{\sqrt{n}}$ centered on the origin of the complex plane. It follows from [37] that this is equivalent to

$$\begin{bmatrix} Q^{-1} & (A+BK)Q^{-1} \\ \star & \frac{1}{n}Q^{-1} \end{bmatrix} \succ 0, \tag{30}$$

for some $Q^{-1} \in \mathcal{S}_+^n$.

Let $b_r \in \mathbb{R}^m$ be the vector of ones and let $P_r = \begin{bmatrix} U_1^T \cdots U_l^T \end{bmatrix}^T Q^{-\frac{1}{2}} \in \mathbb{R}^{m\times n}$ for any orthogonal $U_j \in \mathbb{R}^{n\times n}$, $j = 1, \dots, l$. Let $\hat{X} = Q^{-1}$ and, for every $i \in \mathcal{I}_m$, define the unique integers l_i and n_i such that $i = (l_i - 1)n + n_i$ where $1 \le l_i \le l$ and $1 \le n_i \le n$ and let $\hat{D}_i := \frac{1}{n}(e_{l_i} e_{l_i}^T) \otimes I_n \in \mathcal{D}_+^m$ and $\lambda_i = 1$, where $\otimes$ denotes the Kronecker product. Substituting these definitions in the expression for L_i gives

$$L_i = \begin{bmatrix} 1 & e_{n_i}^T U_{l_i} Q^{-\frac{1}{2}} & 0 \\ \star & 2Q^{-1} & AQ^{-1} + B\hat{K} \\ \star & \star & \frac{1}{n}Q^{-1} \end{bmatrix},$$

DOI: 10.1002/rnc.4573

where $\hat{K} = KQ^{-1}$. Using an upper Schur complement, we have

$$L_i \succ 0 \quad \Leftrightarrow \quad \begin{bmatrix} Q^{-1} + Q^{-\frac{1}{2}} U_{l_i}^T (I - e_{n_i} e_{n_i}^T) U_{l_i} Q^{-\frac{1}{2}} & AQ^{-1} + B\hat{K} \\ \star & \frac{1}{n} Q^{-1} \end{bmatrix} \succ 0$$

$$\Leftrightarrow \quad \begin{bmatrix} Q^{-1} & (A+BK)Q^{-1} \\ \star & \frac{1}{n}Q^{-1} \end{bmatrix} + \begin{bmatrix} Q^{-\frac{1}{2}} U_{l_i}^T (I - e_{n_i} e_{n_i}^T) U_{l_i} Q^{-\frac{1}{2}} & 0 \\ \star & 0 \end{bmatrix} \succ 0.$$

Therefore $L_i \succ 0$ from (30).

Let $\gamma_i = \hat{\mu}_i = 1$ for every $i \in \mathcal{I}_m$. Let α be such that $0 < \alpha < 1$ and define $\underline{Q}^{-1} = \alpha Q^{-1}$. Substitute all the values defined above into the inner bounding ellipsoid condition (21), following an upper and a lower Schur complement, we have that (21) is equivalent to $(1-\alpha)Q^{-1} + Q^{-\frac{1}{2}} U_{l_i}^T (I - e_{n_i} e_{n_i}^T) U_{l_i} Q^{-\frac{1}{2}} \succ 0$, which is satisfied since $\alpha < 1, Q^{-1} \succ 0$ and $(I - e_{n_i} e_{n_i}^T) \succeq 0$.

Let β be such that $0 < \beta < 1$ and define $\bar{D} = \frac{\beta}{m} I_m \in \mathcal{D}_+^m$ and let $\bar{Q}^{-1}$ be any matrix in $\mathcal{S}_+^n$ satisfying $\bar{Q}^{-1} \succ \frac{n}{\beta} Q^{-1}$. The second inequality in (23) is satisfied since $b_r^T \bar{D} b_r = \beta < 1$ and a Schur complement shows that the first inequality in (23) is equivalent to $\bar{Q}^{-1} \succ \frac{n}{\beta} Q^{-1}$ which is satisfied from the definition of $\bar{Q}^{-1}$. □

Remark 3

Suppose that in addition, the initial polytope $\mathcal{P}(P_r X, b_r)$ is also required to guarantee that the output constraints in (20) are satisfied. Then, it can be shown that, in addition to (30), it is sufficient that, for all $j \in \mathcal{I}_{n_y}$

$$\begin{bmatrix} (e_j^T \bar{y})^2 & e_j^T (C+DK)Q^{-1} \\ \star & \frac{1}{n} Q^{-1} \end{bmatrix} \succ 0. \tag{31}$$

The proof is similar to that of Theorem 4 and is therefore omitted.

Remark 4

The theorem provides a choice of the initial polytope $\mathcal{P}(P_r X, b_r)$, where $\mathcal{P}(P_r, b_r)$ is not required to be an RCI set nor satisfy any of the constraints, that guaranteed the feasibility of Theorem 3 under the stated assumptions. The usefulness of this theorem is illustrated by the numerical examples in Section 7.

5. UPDATE COMPUTATION ALGORITHM

Once an admissible initial triple $(P, b, K) \in \Psi$ is obtained, this section presents an algorithm to update the solution based on the following result.

Lemma 2

Let $\boldsymbol{L}, L \in \mathbb{R}^{m \times n}$ and $\boldsymbol{D}, D \in \mathcal{S}_+^m$. Define

$$\mathcal{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D} := \boldsymbol{L}^T D^{-1} L + L^T D^{-1} \boldsymbol{L} - L^T D^{-1} \boldsymbol{D} D^{-1} L, \tag{32}$$

$$\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} := \boldsymbol{L}^T \boldsymbol{D}^{-1} \boldsymbol{L}. \tag{33}$$

DOI: 10.1002/rnc.4573

Then

$$\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} \succeq \mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}}, \tag{34}$$
$$\mathcal{N}_{L,D} = \mathcal{L}^{L,D}_{L,D}. \tag{35}$$

Hence, if $\mathcal{N}_{L,D} \succ 0$ for some $L \in \mathbb{R}^{m\times n}$ and $D \in \mathcal{S}^m_+$, then there exist $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{D} \in \mathcal{S}^m_+$ such that $\mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$. Furthermore, any $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{D} \in \mathcal{S}^m_+$ that satisfy $\mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$ also satisfy $\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} \succeq \mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$.

Proof
Consider the identity

$$\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} = \mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} + (\boldsymbol{L} - \boldsymbol{D}D^{-1}L)^T \boldsymbol{D}^{-1}(\boldsymbol{L} - \boldsymbol{D}D^{-1}L),$$

valid for all $\boldsymbol{L}, L \in \mathbb{R}^{m\times n}$ and $\boldsymbol{D}, D \in \mathcal{S}^m_+$. Then

$$\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} - \mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} = (\boldsymbol{L} - \boldsymbol{D}D^{-1}L)^T \boldsymbol{D}^{-1}(\boldsymbol{L} - \boldsymbol{D}D^{-1}L) \succeq 0,$$

since $\boldsymbol{D} \succ 0$, which proves (34). Furthermore,

$$\mathcal{N}_{L,D} - \mathcal{L}^{L,D}_{L,D} = (L - DD^{-1}L)^T D^{-1}(L - DD^{-1}L) = 0,$$

which proves (35). To prove the second part, suppose that $\mathcal{N}_{L,D} \succ 0$ for some $L \in \mathbb{R}^{m\times n}$ and $D \in \mathcal{S}^m_+$. Then it follows from (35) that there exist $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{D} \in \mathcal{S}^m_+$ (e.g., take $\boldsymbol{L} = L$ and $\boldsymbol{D} = D$) such that $\mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$. Finally, if $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{D} \in \mathcal{S}^m_+$ satisfy $\mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$, then $\mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} \succeq \mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$ from (34). □

Remark 5
Suppose that $\mathcal{M}_{L,D}$ is a linear matrix function of $L \in \mathbb{R}^{m\times n}$ and $D \in \mathcal{S}^m_+$. Then Lemma 2 states that if $L \in \mathbb{R}^{m\times n}$ and $D \in \mathcal{S}^m_+$ are solutions to the *nonlinear* matrix inequality $\mathcal{M}_{L,D} + \mathcal{N}_{L,D} \succ 0$, then there exist solutions $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{D} \in \mathcal{S}^m_+$ to the *linear* matrix inequality $\mathcal{M}_{\boldsymbol{L},\boldsymbol{D}} + \mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$ and, furthermore, these solutions also satisfy the original nonlinear matrix inequality $\mathcal{M}_{\boldsymbol{L},\boldsymbol{D}} + \mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} \succ 0$. Note also that the linear matrix equation $\mathcal{M}_{\boldsymbol{L},\boldsymbol{D}} + \mathcal{L}^{L,D}_{\boldsymbol{L},\boldsymbol{D}} = 0$ is the Newton update for the nonlinear matrix equation $\mathcal{M}_{\boldsymbol{L},\boldsymbol{D}} + \mathcal{N}_{\boldsymbol{L},\boldsymbol{D}} = 0$ from the initial approximation L, D.

The next result extends this idea to derive Newton–like updates for the nonlinear matrix inequalities of Theorem 2 starting from the initial approximations given in Theorem 3.

Theorem 5
With all definitions as above and $\mathcal{N}_{\cdot,\cdot}$ and $\mathcal{L}^{\cdot,\cdot}_{\cdot,\cdot}$ as defined in (32) and (33), respectively, let $(\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{K}) \in \Psi$. Then

1. Suppose that $(P, b, K, D_i, W_i, S_i, T_i, R_i), \forall i \in \mathcal{I}_m$ satisfy the invariance condition (11). Then

$$\begin{bmatrix} \boldsymbol{D}_i^{-1} \in \mathcal{D}^m_+ \\ \boldsymbol{W}_i \in \mathcal{D}^{m_w}_+ \\ (\boldsymbol{S}_i, \boldsymbol{T}_i, \boldsymbol{R}_i) \in \mathcal{B} \end{bmatrix}, \mathcal{M}_i + \mathcal{L}^{L_i(P,K,S_i,R_i),F_i(D_i^{-1})}_{L_i(\boldsymbol{P},\boldsymbol{K},\boldsymbol{S}_i,\boldsymbol{R}_i),F_i(\boldsymbol{D}_i^{-1})} \succ 0, \forall i \in \mathcal{I}_m, \tag{36}$$

DOI: 10.1002/rnc.4573

where

$$\mathcal{M}_i = \begin{bmatrix} \boldsymbol{D}_i^{-1} & \boldsymbol{b} & 0 & 0 & 0 & 0 & 0 \\ \star & 2e_i^T\boldsymbol{b} - d^T\boldsymbol{W}_i d & 0 & 0 & 0 & 0 & e_i^T\boldsymbol{P} \\ \star & \star & V^T\boldsymbol{W}_i V & 0 & 0 & 0 & B_w^T \\ \star & \star & \star & \boldsymbol{S}_i & 0 & 0 & \boldsymbol{S}_i B_p^T \\ \star & \star & \star & \star & \boldsymbol{T}_i & C_q^{\boldsymbol{K}} & \boldsymbol{R}_i^T B_p^T \\ \star & \star & \star & \star & \star & 0 & (A^{\boldsymbol{K}})^T \\ \star & \star & \star & \star & \star & \star & I_n \end{bmatrix},$$

$$L_i(P,K,S_i,R_i) = \begin{bmatrix} 0 & P^T e_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_w & B_p S_i^T & B_p R_i & A^K & 0 \\ 0 & 0 & 0 & 0 & 0 & P & 0 \end{bmatrix},$$

$$F_i(D_i^{-1}) = \mathrm{diag}(I_n, I_n, D_i^{-1}).$$

Furthermore, the invariance condition (11) and (3) are satisfied by

$$(P,b,K,D_i,W_i,S_i,T_i,R_i) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{K},\boldsymbol{D}_i,\boldsymbol{W}_i,\boldsymbol{S}_i,\boldsymbol{T}_i,\boldsymbol{R}_i).$$

2. Suppose that $(P,b,K,E_j,G_j), \forall j \in \mathcal{I}_{n_y}$ satisfy the output constraint condition (12). Then

$$\begin{bmatrix} \boldsymbol{E}_j^{-1} \in \mathcal{D}_+^m \\ \boldsymbol{G}_j \in \mathcal{D}_+^{m_w} \end{bmatrix}, \begin{bmatrix} \boldsymbol{E}_j^{-1} & \boldsymbol{b} & 0 & 0 \\ \star & 2e_j^T\bar{y} - d^T\boldsymbol{G}_j d & e_j^T D_w & e_j^T C^{\boldsymbol{K}} \\ \star & \star & V^T\boldsymbol{G}_j V & 0 \\ \star & \star & \star & \mathcal{L}_{\boldsymbol{P},\boldsymbol{E}_j^{-1}}^{P,E_j^{-1}} \end{bmatrix} \succ 0, \forall j \in \mathcal{I}_{n_y}. \quad (37)$$

Furthermore, the output constraint condition (12) and (4) are satisfied by

$$(P,b,K,E_j,G_j) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{K},\boldsymbol{E}_j,\boldsymbol{G}_j).$$

3. Suppose that $(P,b,\mu_i,\underline{Q}), \forall i \in \mathcal{I}_m$ satisfy the inner bounding ellipsoid condition (13). Then

$$\underline{\boldsymbol{Q}}^{-\frac{1}{2}} \in \mathcal{S}_+^n;\ \boldsymbol{\mu}_i > 0, \begin{bmatrix} 2e_i^T\boldsymbol{b} - \boldsymbol{\mu}_i & e_i^T\boldsymbol{P} & 0 \\ \star & 2\mu_i\underline{Q} & \mu_i\underline{Q}\underline{\boldsymbol{Q}}^{-\frac{1}{2}} \\ \star & \star & \boldsymbol{\mu}_i I_n \end{bmatrix} \succ 0, \forall i \in \mathcal{I}_m. \quad (38)$$

Furthermore, the inner bounding ellipsoid condition (13) and (5) are satisfied by

$$(P,b,\mu_i,\underline{Q}) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{\mu}_i,\underline{\boldsymbol{Q}}).$$

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4. Suppose that $(P, b, K, Q, D_z, S, T, R)$ satisfy the $\mathcal{H}_2$-norm constraint condition (14). Then

$$\begin{bmatrix} \boldsymbol{Q} \in \mathcal{S}_+^n \\ \boldsymbol{D}_z^{-1} \in \mathcal{D}_+^m \\ (\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{R}) \in \mathcal{B} \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{Q} - B_p \boldsymbol{S} B_p^T & 0 & B_p \boldsymbol{R} & A^{\boldsymbol{K}} \\ \star & rI & 0 & C_2^{\boldsymbol{K}} \\ \star & \star & \boldsymbol{T} & C_q^{\boldsymbol{K}} \\ \star & \star & \star & \mathcal{L}_{I,\boldsymbol{Q}}^{I,Q} \end{bmatrix} \succ 0,$$
$$\begin{bmatrix} \boldsymbol{Q} & I \\ \star & \mathcal{L}_{\boldsymbol{P},\boldsymbol{D}_z^{-1}}^{P,D_z^{-1}} \end{bmatrix} \succ 0, \qquad \begin{bmatrix} \boldsymbol{D}_z^{-1} & \boldsymbol{b} \\ \star & r \end{bmatrix} \succ 0. \tag{39}$$

Furthermore, the $\mathcal{H}_2$-norm constraint condition (14) and (7) are satisfied by

$$(P, b, K, Q, D_z, S, T, R) := (\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{K}, \boldsymbol{Q}, \boldsymbol{D}_z, \boldsymbol{S}, \boldsymbol{T}, \boldsymbol{R}).$$

5. Suppose that $(P, b, \bar{Q}, \bar{D})$ satisfy the outer bounding ellipsoid condition (15). Then

$$\begin{bmatrix} \bar{\boldsymbol{D}}^{-1} \in \mathcal{D}_+^m \\ \bar{\boldsymbol{Q}} \in \mathcal{S}_+^n \end{bmatrix}, \mathcal{L}_{\boldsymbol{P},\bar{\boldsymbol{D}}^{-1}}^{P,\bar{D}^{-1}} - \bar{\boldsymbol{Q}} \succ 0, \begin{bmatrix} \bar{\boldsymbol{D}}^{-1} & \boldsymbol{b} \\ \star & 1 \end{bmatrix} \succ 0. \tag{40}$$

Furthermore, the outer bounding ellipsoid condition (15) and (8) are satisfied by

$$(P, b, \bar{Q}, \bar{D}) := (\boldsymbol{P}, \boldsymbol{b}, \bar{\boldsymbol{Q}}, \bar{\boldsymbol{D}}).$$

6. Suppose that $(P, b, F_i), \forall i \in \mathcal{I}_m$ satisfy the initial constraint condition (16). Then

$$\boldsymbol{F}_i \in \mathcal{D}_+^{m_0}, \begin{bmatrix} 2e_i^T \boldsymbol{b} - b_0^T \boldsymbol{F}_i b_0 & e_i^T \boldsymbol{P} \\ \star & P_0^T \boldsymbol{F}_i P_0 \end{bmatrix} \succ 0, \forall i \in \mathcal{I}_m. \tag{41}$$

Furthermore, the initial constraint condition (16) and (9) are satisfied by

$$(P, b, F_i) := (\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{F}_i).$$

Hence,

$$\left(\begin{array}{ll} \max & \log\det \boldsymbol{Q}^{-\frac{1}{2}} \\ \multicolumn{2}{l}{\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{K}, \boldsymbol{D}_i^{-1}, \boldsymbol{W}_i, \boldsymbol{S}_i, \boldsymbol{T}_i, \boldsymbol{R}_i, \boldsymbol{E}_j^{-1}, \boldsymbol{G}_j, \boldsymbol{\mu}_i, \underline{\boldsymbol{Q}}^{-\frac{1}{2}}, \boldsymbol{Q}, \boldsymbol{D}_z^{-1}, \boldsymbol{S}, \boldsymbol{T}, \boldsymbol{R}} \\ \multicolumn{2}{l}{\textit{subject to } (36), (37), (38), (39)} \end{array} \right) \geq \log\det \underline{Q}^{-\frac{1}{2}}, \tag{42}$$

$$\left(\begin{array}{ll} \min & -\log\det \bar{\boldsymbol{Q}} \\ \multicolumn{2}{l}{\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{K}, \boldsymbol{D}_i^{-1}, \boldsymbol{W}_i, \boldsymbol{S}_i, \boldsymbol{T}_i, \boldsymbol{R}_i, \boldsymbol{E}_j^{-1}, \boldsymbol{G}_j, \bar{\boldsymbol{Q}}, \bar{\boldsymbol{D}}^{-1}, \boldsymbol{F}_i} \\ \multicolumn{2}{l}{\textit{subject to } (36), (37), (40), (41)} \end{array} \right) \leq -\log\det \bar{Q} \tag{43}$$

Proof

The proof is essentially an application of Lemma 2, congruences, Schur complements and some re–definitions to show that (11)-(16) imply (36)-(41), which in turn imply (11)-(16) (with bold variables) and therefore (3)-(9) (with bold variables), respectively, from Theorem 2. In more detail:

DOI: 10.1002/rnc.4573

1. Effecting an upper and a lower Schur complement on (11) and a manipulation show that (11) is equivalent to

$$\mathcal{M}_i + \mathcal{N}_{L_i(P,K,S_i,R_i),F_i(D_i^{-1})} \succ 0.$$

   The result follows by applying Lemma 2 (see also Remark 5) on the second term.
2. This follows by applying a Schur complement on $b^T E_j b$ and Lemma 2 on $\mathcal{N}_{P,E_j^{-1}} = P^T E_j P$ in (12).
3. This follows by applying Lemma 2 on $\mathcal{N}_{I,(\mu_i \underline{Q})^{-1}} = \underline{Q}\mu_i$ and taking a Schur complement in (13).
4. The result follows by applying Lemma 2 on $\mathcal{N}_{I,Q} = Q^{-1}$ and $\mathcal{N}_{P,D_z^{-1}} = P^T D_z P$ and a Schur complement on the third inequality, in (14).
5. The second inequalities in (40) and (15) are equivalent using a Schur complement argument. The result follows by applying Lemma 2 on the term $\mathcal{N}_{P,\bar{D}^{-1}} = P^T \bar{D} P$ in the first inequality in (15).
6. The result is trivially satisfied since (16) is linear in the variables.

Finally, (42) and (43) follow from 1-6 above. □

Remark 6
Note that taking $\boldsymbol{D}_i^{-1}, \boldsymbol{E}_j^{-1}, \boldsymbol{D}_z^{-1}, \bar{\boldsymbol{D}}^{-1}$ as variables allows us to use Lemma 2 to ensure recursive feasibility, that is, the volume of the updated inner/outer approximation to the maximal/minimal RCI set is at least as good (large for maximal and small for minimal sets) as that of the previous set. It also allows us to use $\boldsymbol{b}$ as a variable, thus improving the updated solution. Our numerical experience, part of which is reported below, as well as Remark 5, suggest quadratic convergence, although a formal proof of this is beyond the scope of this work.

6. SOLUTION ALGORITHM

The following algorithm summarizes our solution.

Algorithm 1
Given system (1) and (6), $\mathcal{Y} = \mathcal{P}(I_{n_y}, \bar{y})$ and sets $\Omega, \mathcal{W} = \mathcal{P}(V, d), \mathcal{P}(P_0, b_0)$ and parameter γ.

1. **Initial data**: Choose $m \geq n$, $P_r \in \mathbb{R}^{m\times n}$, $0 < b_r \in \mathbb{R}^m$ and tolerance level tol.
2. **Initial solution**
   (a) Use Theorem 3 to solve the convex SDPs in (25).
   (b) Define $D_i, W_i, S_i, T_i, R_i, Q, D_z, S, T, R$ and μ_i from (26)-(28) so that (11)-(16) are satisfied.
3. **Update** Solve the optimizations in (42) or (43).
4. **Stopping condition**
   (a) If $\det(\underline{\boldsymbol{Q}}^{-1}) - \det(\underline{Q}^{-1}) \leq tol$ (for maximization) or $\det(\bar{Q}^{-1}) - \det(\bar{\boldsymbol{Q}}^{-1}) \leq tol$ (for minimization), stop.

DOI: 10.1002/rnc.4573

(b) Else update $Z := \boldsymbol{Z}$, where $\boldsymbol{Z}$ denotes a variable in the optimizations in (42) or (43), and go to step 3.

5. **End**

Remark 7

Since our algorithms are linear, in the case of polyhedral uncertainty, that is, if $(\tilde{A},\tilde{B}) \in \Omega$ where

$$\Omega := \{(\tilde{A},\tilde{B}) : [\tilde{A}\ \tilde{B}] = \sum_{l=1}^{p} \alpha_i [A_l\ B_l], \sum_{l=1}^{p} \alpha_i = 1, \alpha_i \geq 0\},$$

and where $A_l \in \mathbb{R}^{n \times n}$ and $B_l \in \mathbb{R}^{n \times n_u}$ are given matrices for all $l \in \mathcal{I}_p$, all our algorithms are applicable except that (A, B) are replaced by (A_l, B_l) and the constraints need to be satisfied for all $l \in \mathcal{I}_p$. Note also that Theorem 4 can also be extended to provide an initial polytopic RCI set for systems subject to polyhedral uncertainty (with the controllability condition on (A, B) given in the theorem replaced by the existence of $K \in \mathbb{R}^{n \times n_u}$ such that all the eigenvalues of $A_l + B_l K$ lie in the disc with radius $\frac{1}{\sqrt{n}}$ centered on the origin of the complex plane, for all l), although this is not pursued here.

7. EXAMPLES

This section presents four examples that illustrate our results.

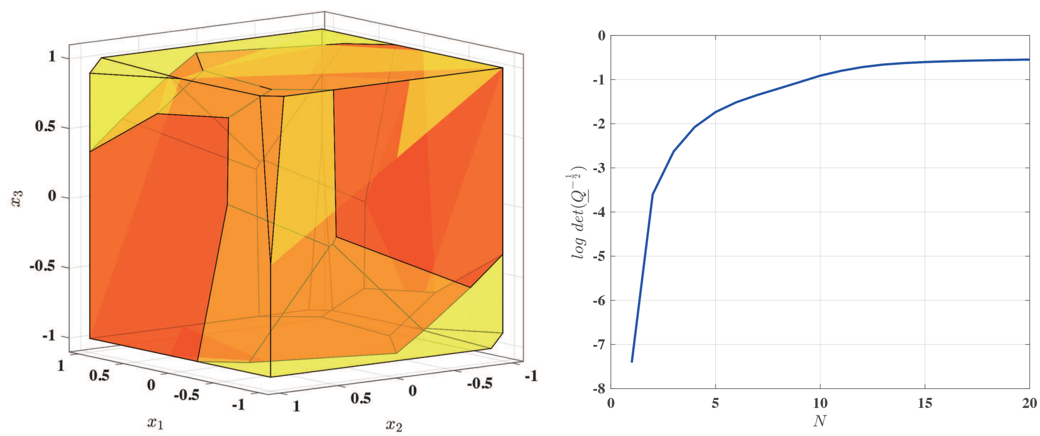
7.1. *Example 1: Nominal System with High Dimension*

Consider a fourth order system used in [27] with system matrices

$$A = \begin{bmatrix} 0.1 & 1 & 1 & 0.7 \\ 0 & 0.2 & 0.12 & 0.5 \\ 0 & 0.2 & 0.4 & 0.37 \\ 0.8 & -0.2 & 0.56 & 0.32 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.4 \\ 0.27 \\ 0.64 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We set $m = 12$ and starting from Theorem 4 with an initial polytope $\mathcal{P}(P_r, b_r)$. Applying Algorithm 1 gives the final inner approximations to the maximal RCI set in yellow colour shown in Figure 1a, with the final control law given as $K = [-0.68\ \ -0.02\ \ -0.44\ \ -0.17]$. Note that for display purposes, the RCI set plotting is cut through $x_4 = 0$. Our algorithm is implemented in MATLAB using CVX toolbox with platform 64-bit Intel Core i5-7600 at 3.5GHz with 4GB DDR4. The mean time to obtain an initial result is 0.78s and 20 iteration took 8.56s to give the final results. For the same example in [27], the maximal invariant set is calculated using the standard iterative algorithm in [28] with MPT3 toolbox in MATLAB. In the same platform, the referenced method used 1.12s to obtain the result shown in red colour in Figure 1a. It can be seen that the maximum approximation of the RCI set obtained using our proposed algorithm is larger than that obtained using the algorithm in [28]. Figure 1b displays the convergence of $\log\det \underline{\boldsymbol{Q}}^{-\frac{1}{2}}$ with the update times N, which illustrates that the objective value converges after 15 updates.

DOI: 10.1002/rnc.4573



(a) Final maximization results (in yellow) and results using method in [28] (in red)

(b) Convergence of the objective value $\log\det\underline{\boldsymbol{Q}}^{-\frac{1}{2}}$ with the iteration times N

Figure 1. Example 1

7.2. *Example 2: System with Disturbance*

Consider the system used in [29]:

$$x^{+}=\begin{bmatrix}1&1\\0&1\end{bmatrix}x+\begin{bmatrix}0.5\\1\end{bmatrix}u+w,$$

with $w\in\mathcal{W}:=\mathcal{P}(V,d)$, where

$$V=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ d=\begin{bmatrix}0.1\\0.1\end{bmatrix}.$$

We set $m=10$ and produce a random initial polytope $\mathcal{P}(P_r,b_r)$. Applying Algorithm 1 gives an initial solution in $0.92s$ as shown in Figure 2a in yellow colour with dashed border, and the final result in $6.69s$ as shown in red colour with solid border in Figure 2a. The corresponding feedback control law is $K=[-1\ \ -1.5]$. [29] provides a fast approach to compute the minimal RCI set, while the minimal results is not as good as that of [10]. The smallest volume set given in [29] and [10] is the ϵ-mRPI as shown in green colour with dotted border. The results illustrate that our proposed Algorithm 1 can provide smaller outer approximations of the minimal RCI set. Figure 2b displays the convergence of $\log\det\underline{\boldsymbol{Q}}^{-\frac{1}{2}}$ with the update times N, and illustrates that the objective value converges after 8 iterations.

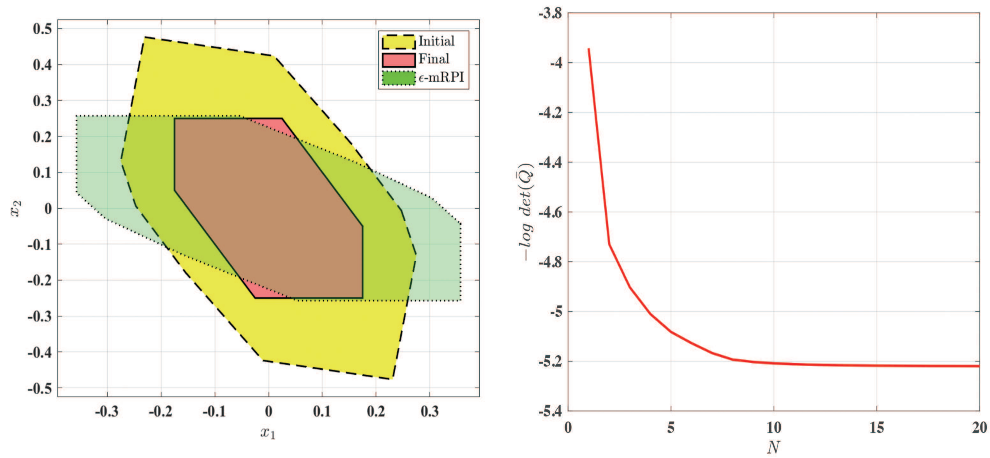
7.3. *Example 3: System with Polyhedral Uncertainty*

Consider the double integrator example in [17] and [18] with

$$A_1=\begin{bmatrix}1&0.1\\0&1\end{bmatrix},\ A_2=\begin{bmatrix}1&0.2\\0&1\end{bmatrix},\ B_1=\begin{bmatrix}0\\1\end{bmatrix},\ B_2=\begin{bmatrix}0\\1.5\end{bmatrix}.$$

We set $m=30$, $\mathcal{P}(P_r,b_r)$ a regular hexacontagon, and follow Remark 7 and Algorithm 1 to obtain the maximal approximations of the RCI set. Figure 3a shows the initial (in yellow, with solid

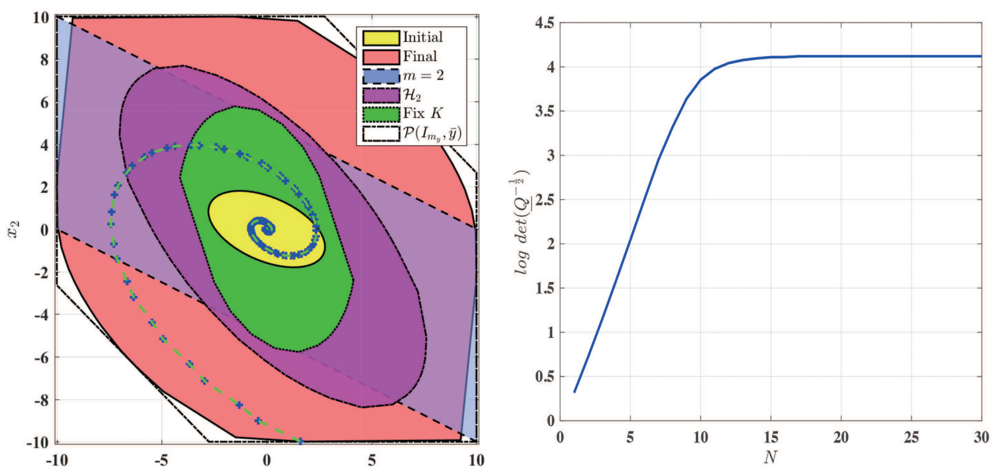
DOI: 10.1002/rnc.4573



(a) Initial (in yellow with dashed border) and final approximations (in red with solid border) and ϵ-mRPI results in [29] (in green with dotted border)

(b) Convergence of the objective value $-\log\det\bar{\boldsymbol{Q}}$ with the iteration times N

Figure 2. Example 2



(a) Maximization of RCI Sets

(b) Convergence of the objective value $\log\det\underline{\boldsymbol{Q}}^{-\frac{1}{2}}$ with the iteration times N

Figure 3. Example 3

border) and final (in red, with solid border) inner approximation to the RCI set, with the final control law given as $K = [-0.0794 \;\; -0.0781]$. The blue cross marks and the dashed green line shows the trajectory of the system states under the feedback control law (only one trajectory is shown for clarity). The white box with dash-dot border shows the output constraints. Figure 3b displays the convergence of $\log\det\underline{\boldsymbol{Q}}^{-\frac{1}{2}}$ with the update times N. The initial result is obtained in 1.47 seconds, then the updating result converge to a final value within 15 steps which takes 6.76 seconds. For comparison, the low-complexity ($m = n = 2$,) inner approximation to the RCI set is also shown in Figure 3a (in blue color and with dashed border). Note that considering a full-complexity RCI set leads to a much larger volume. [17] gives an optimal solution under the control

DOI: 10.1002/rnc.4573

gain $K = [-0.3 \ \ -0.1]$ as shown in green colour with dotted border in Figure 3a. Our result shows that the volume of the RCI set can be greatly increased by treating the feedback gain as a decision variable in the optimization.

Let

$$C_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}. \tag{44}$$

Using (14), the $\mathcal{H}_2$–performance level, defined in (7), for the final inner approximation to the maximal RCI set is given by $\gamma = 6.36$. We can improve the performance level by, for example, setting $\gamma = 3$ and incorporating the $\mathcal{H}_2$ constraint condition (14) in our algorithm. The final inner approximation to the RCI set with the improved performance requirement is shown in Figure 3a (in magenta color and with dash-dot border). This illustrates the compromise between the volume of the RCI set and the expected performance.

7.4. *Example 4: System with Norm-bounded Uncertainty*

Consider the example of a continuous-time DC motor system with norm-bounded structured uncertainty proposed in [20], which is defined by the following matrices:

$$A = \begin{bmatrix} -0.07 & -0.86(1+\omega_1) \\ 0.06(1+\omega_1) & -\omega_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{45}$$

where the uncertainty is defined by

$$\Omega = \{(\omega_1, \omega_2) | -0.2 \leq \omega_1 \leq 0.2, \ 0.0085 \leq \omega_2 \leq 0.5\}.$$

We discretize the system with a sampling period $T = 0.1s$ and express the discrete-time system in the form of (1) and the uncertainty in the form of (2) with appropriate nominal system (A, B), distribution matrices B_p, C_q and D_q and uncertainty set

$$\boldsymbol{\Delta} = \{diag(\delta_1 I_2, \delta_2), \ \delta_i \in \mathbb{R}, \ |\delta_i| \leq 1\}.$$

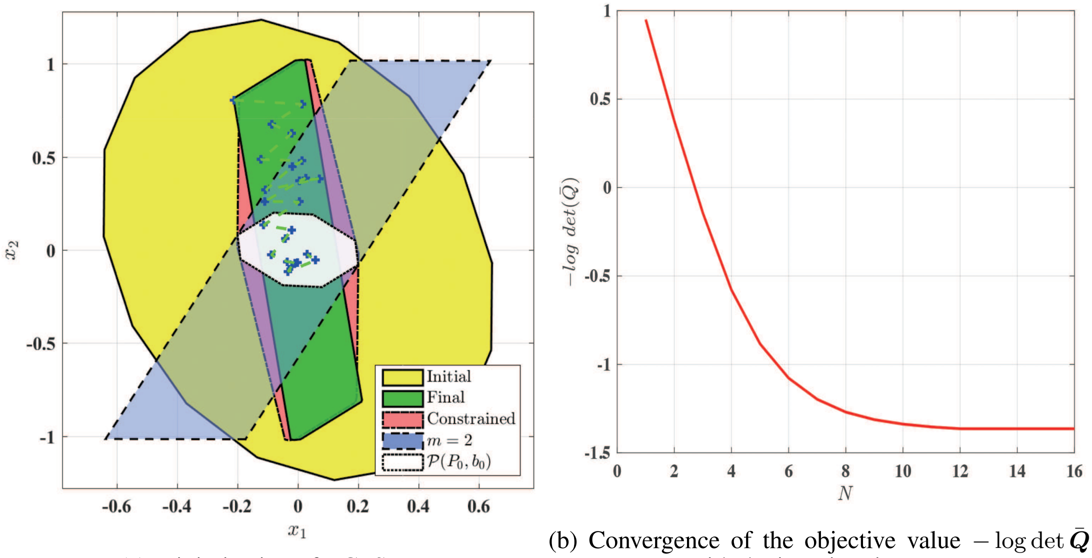
We also incorporate an additive disturbance and the state and input constraints are integrated into our output constraint by setting

$$B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$V = 1$, and d, and $\bar{y}$ as vectors of ones with appropriate dimensions.

We set $m = 8$ and $\mathcal{P}(P_r, b_r)$ a regular hexadecagon. Using Algorithm 1 to find an outer approximation to the minimal RCI set, we obtain the initial and final sets as shown in Figure 4a with the final control law as $K = [-9.9707 \ \ -0.0588]$. The convergence rate of $\log\det \bar{\boldsymbol{Q}}^{-1}$ with the update times N is shown in Figure 4b . The initial result is obtained in 0.95 seconds using the

DOI: 10.1002/rnc.4573



(a) Minimization of RCI Sets

(b) Convergence of the objective value $-\log\det\bar{\boldsymbol{Q}}$ with the iteration times N

Figure 4. Example 4

CVX toolbox with SDP solver in MATLAB, the update result converge to the final result in 20 steps in 7.42 seconds.

We can illustrate the effect of including the initial condition constraint (9) with

$$P_0 = \begin{bmatrix} 4.167 & 5.159 & 0.397 & 3.571 \\ 4.167 & 0.397 & 5.159 & -3.571 \end{bmatrix}^T,$$

and b_0 as the vector of ones with appropriate dimensions. The final outer approximation to the minimal RCI set with the initial constraint requirement is shown in Figure 4a (in red color and with dash-dot border; the initial constraint is shown in white color with dotted border). Under this requirement, the superiority of using full-complexity RCI set is obvious. For comparison, the outer approximation to the minimal low-complexity ($m = n = 2$) RCI set is also shown in Figure 4a (in blue color and with dashed border).

To illustrate the invariance condition, the blue cross marks and the dashed green line shows the trajectory of the system states under the feedback control law (only one trajectory is shown for clarity). The trajectory is representative since it starts from the edge of the set and is produced using the worst case disturbances and uncertainties.

8. CONCLUSION AND FUTURE WORK

We have proposed a novel scheme, based on convex/LMI optimizations, for the computation of full-complexity inner/outer approximations to polytopic maximal/minimal RCI sets and the corresponding feedback control law (K) for linear discrete–time systems subject to additive disturbances and model uncertainties as well as output, initial state and performance constraints.

DOI: 10.1002/rnc.4573

This paper first derives necessary and sufficient conditions for the existence of an admissible RCI set and feedback gain matrix, that are, in general, nonlinear and nonconvex. Farkas' Theorem is then used to relax the problem and obtain sufficient LMI conditions, thus rendering the optimization problem tractable. An initial invariant polytope, and the associated control law K, are first obtained and the set volume is then iteratively optimized by solving convex/LMI optimizations. These iterations are reminiscent of Newton updates which appears to promote good convergence speed. Furthermore, the proposed scheme is able to handle both structured norm–bounded as well as polytopic model uncertainties.

Unlike many of the schemes in the literature, the algorithm places no restriction on the complexity of the invariant polytope and allows for arbitrarily large values of m. This, coupled with the fact that K is treated as a variable of optimization, results in less conservative inner/outer approximations to the maximal/minimal RCI sets. This is reflected in the results from the numerical examples, which show that the proposed scheme can yield a polytopic RCI set with a substantially improved volume as compared to other schemes from the literature.

Nevertheless, the proposed algorithm assumes a constant linear state feedback control structure, and the disturbance, constraints and the RCI sets are considered to be symmetric to obtain a tractable solution. In practice, these assumption will result in conservatism. Furthermore, the computational complexity of proposed algorithm for large-scale systems requires further investigation. Future research directions include modifying our approach for more complex feedback control structures and reducing the conservatism further by considering asymmetric constraints and RCI sets.

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DOI: 10.1002/rnc.4573