ON THE REDUCTION THEORY OF BINARY FORMS

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1. Introduction

In [3], a reduction theory for binary forms of degree 3 and 4 with integer coefficients was developed in detail, the motivation in the case of quartics being to improve 2-descent algorithms for elliptic curves over $\mathbb{Q}$. In this paper we extend some of these results to forms of higher degree. One application of this is to the study of hyperelliptic curves, which are given by affine equations of the form

$$Y^2 = f(X)\,,$$

where $f(X)$ is a polynomial of degree $d \geq 5$; we will show how to reduce such an equation to one with smaller coefficients, via a unimodular transformation, in a systematic and (in a certain sense) optimal way. This is often useful, since the construction of such equations often results in polynomials with extremely large coefficients. For example, see [12], where rather *ad hoc* methods are used for reduction.

The goals of a reduction theory for binary forms (or for the corresponding polynomials) are two-fold, corresponding to two basic problems: first, given such a form defined over $\mathbb{R}$, find an equivalent one (with respect to integral unimodular transformations) with 'smaller' coefficients; second, for forms defined over $\mathbb{Z}$, enumerate (up to equivalence) all forms with a given discriminant, or a given set of invariants. Both these problems were studied for cubics and quartics over $\mathbb{Z}$ in [3]; in this paper we only consider the first, but for forms of arbitrary degree. The methods we use are inspired by Julia's treatise [8]: we observe, however, that Julia's results are only explicit for degrees 3 and 4.

The basic principle behind reduction in any set S on which the modular group $\mathrm{SL}(2,\mathbb{Z})$ acts (on the right), is to associate to each element $s \in S$ a covariant point $z(s)$ in the upper half-plane $\mathcal{H}$. Here, covariance means that for each $g \in \mathrm{SL}(2,\mathbb{Z})$ we have

$$z(s \cdot g) = g^{-1}(z(s))\,,$$

where $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathcal{H}$ in the usual way (on the left) via linear fractional transformations. Each $\mathrm{SL}(2,\mathbb{Z})$-orbit in $\mathcal{H}$ has a representative in the standard fundamental region $\mathcal{F}$ defined as follows:

$$\mathcal{F} = \{z \in \mathcal{H} : |z| \geq 1,\ -\tfrac{1}{2} \leq \mathrm{Re}(z) \leq \tfrac{1}{2}\}\,;$$

the representative in $\mathcal{F}$ is unique except if it is on the boundary of $\mathcal{F}$, when there are up to two representatives. We define $s \in S$ to be *reduced* if and only

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if $z(s) \in \mathcal{F}$. Note that there may be more than one way of defining the system of covariant points $s \mapsto z(s)$, in which case there will be more than one notion of 'reduced' for the set S. In such situations other considerations will determine which is best. In particular, this happens when S is the set of real binary forms of fixed degree d, where either $d \geq 5$, or $3 \leq d \leq 4$ and we fix a signature which is neither totally real nor totally imaginary (see below).

When S is the set of binary forms of fixed degree, the action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ on $F(X,Z)$ is by substitution: $(F \cdot g)(X,Z) = F(aX + bZ, cX + dZ)$.

An alternate viewpoint is to associate to each $s \in S$ a positive definite real quadratic form $Q(s)$ which is $\mathrm{SL}(2,\mathbb{Z})$-covariant, instead of a point $z(s) \in \mathcal{H}$. There is no essential difference, since each such form Q has a unique root in the upper half-plane, and conversely each point $z \in \mathcal{H}$ is the root of a positive definite real quadratic, unique up to multiplication by a positive constant. In this paper we will use both the language of covariant points and that of covariant quadratics, and make use of the hyperbolic geometry of $\mathcal{H}$ in some of our arguments.

This paper is organised as follows. After setting up some notation, we take up Gaston Julia's thesis [8], where he introduces an approach to reducing real (and also complex) forms of arbitrary degrees greater than two, which builds on earlier work of Hermite. Julia develops some of the theory in general, but only gives complete and explicit details for degrees three and four. We extend this to the general case, and show how this approach leads to a reduction algorithm. We then give some examples for the application of this algorithm, and finish with some additional results for forms with only real roots.

In a separate paper [10], we will discuss the question of whether the forms which are reduced, in the sense defined here, are in some sense the "smallest" representatives of their $\mathrm{SL}(2,\mathbb{Z})$-orbit.

2. Notation and Basics

In certain places below it is is useful to consider the upper half-plane $\mathcal{H}$ to be a vertical cross-section of hyperbolic 3-space or upper half-space $\mathcal{H}_3$. If we coordinatise $\mathcal{H}_3$ as

$$\mathcal{H}_3 = \{(z,u) \mid z \in \mathbb{C}, u \in \mathbb{R}_+\},$$

then $\mathcal{H} = \{(t,u) \mid t \in \mathbb{R}\}$, where we identify $(t,u) \in \mathcal{H}_3$ with $t + iu \in \mathcal{H}$. The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}$ is then compatible with the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathcal{H}_3$. This enlarged viewpoint was already used by Julia (following Hermite, Humbert, Bianchi and others), and allows the unification of several cases which otherwise have to be treated separately. In addition, this is the appropriate context in which to consider the reduction of complex (as opposed to real) forms, which is necessary in developing a reduction theory over number fields which are not totally real.

In this case, positive definite quadratic forms are replaced by positive definite Hermitian forms; the correspondence between them and points in $\mathcal{H}_3$ is as follows. A positive definite Hermitian form can be expressed as

$$Q(X,Z) = a\,|X|^2 + b\,X\bar{Z} + \bar{b}\,\bar{X}Z + c\,|Z|^2 = a(|X - t\,Z|^2 + u^2\,|Z|^2)$$

with $a, c, u > 0$ and $b, t \in \mathbb{C}$. The corresponding point in $\mathcal{H}_3$ is then (t, u).

In order to be able to treat the real and the complex cases in parallel later on, we set $\mathcal{H}_{\mathbb{R}} = \mathcal{H}$ and $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_3$. To avoid confusion, we denote by j the point $(0, 1) \in \mathcal{H}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{C}}$ (which corresponds to i when $\mathcal{H}$ is considered as a subset of $\mathbb{C}$) and also write $t + uj$ for the point $(t, u) \in \mathcal{H}_{\mathbb{C}}$. The symbol k will stand for either $\mathbb{R}$ or $\mathbb{C}$. Let $k[X, Z]_n$ be the space of forms of degree n in two variables with coefficients in k, and let $k[X, Z]'_n$ denote the subset of forms without repeated factors. If $k = \mathbb{R}$, we also use the notations $\mathbb{R}[X, Z]_{r,s}$ and $\mathbb{R}[X, Z]'_{r,s}$ for the space of forms and the subset of squarefree forms of signature (r, s), respectively.

To define a suitable covariant map $F \mapsto z(F)$ for binary forms F, we can forget that we are primarily interested in the action of $\mathrm{SL}(2, \mathbb{Z})$ on forms with integral coefficients, and consider $\mathrm{SL}(2, k)$, acting on forms with coefficients in k (of fixed degree, and maybe fixed signature when $k = \mathbb{R}$). Then we will require the stronger property that z be covariant with respect to the action of $\mathrm{SL}(2, k)$.

We further denote by $H(\mathbb{R})$ the set of positive definite binary quadratic forms and by $H(\mathbb{C})$ the set of positive definite binary Hermitian forms. We denote the canonical map $H(k) \longrightarrow \mathcal{H}_k$ by z (see above for its definition). The maps for $k = \mathbb{R}$ and $k = \mathbb{C}$ are compatible, so we can use the same name for both of them. There is an action of $\mathrm{SL}(2, \mathbb{C})$ on $H(\mathbb{C})$, defined by

$$Q(X, Z) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = Q(aX + bZ, cX + dZ)\,;$$

then $z : H(\mathbb{C}) \to \mathcal{H}_{\mathbb{C}}$ is covariant with respect to this action and the usual one on $\mathcal{H}_{\mathbb{C}}$.

Furthermore, for our purposes, we define the *discriminant* of the form $Q \in H(\mathbb{C})$, with coefficients (a, b, c) as above, (with $a, c \in \mathbb{R}_{>0}$ and $b \in \mathbb{C}$), by

$$\operatorname{disc} Q = 4\,(ac - |b|^2) \in \mathbb{R}_{>0}\,.$$

Then $z(Q) = (t, u)$ with $t = -\overline{b}/a$ and $u = \operatorname{disc}(Q)^{1/2}/(2a)$. It is easily seen that the discriminant is invariant under the $\mathrm{SL}(2, \mathbb{C})$-action. The same definition applies also to forms $Q \in H(\mathbb{R})$, which are characterised by $b \in \mathbb{R}$, and in this case the discriminant is the negative of the usual discriminant. (We use the negative here for notational convenience, since it is positive for positive definite forms.)

To summarise, we want to find an $\mathrm{SL}(2, k)$-covariant map

$$z : k[X, Z]'_n \longrightarrow \mathcal{H}_k \quad (\text{or} \longrightarrow H(k))$$

that is computable (in a practical sense), and has the property that a form F is 'small' if its image $z(F)$ is in the fundamental domain $\mathcal{F}$.

In the complex case, the map z should also be compatible with complex conjugation (acting on $\mathcal{H}_{\mathbb{C}}$ through the first coordinate). This implies that the restriction of the complex map to real polynomials has image in $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{C}}$ and thus also provides a suitable solution for the problem over $\mathbb{R}$.

We will use throughout the convention of using uppercase letters for binary forms $F(X, Z)$ of a given degree, and lowercase letters for the dehomogenised polynomials $f(X) = F(X, 1)$.

3. Julia's approach

In his thesis [8], Gaston Julia deals with the problem of how to define a good notion of being reduced for binary forms over $\mathbb{R}$ of degree larger than two, building on earlier work of Hermite [5], [6]. His approach (cast in slightly more modern language) is as follows. Let

$$F(X,Z) = a_0X^n + a_1X^{n-1}Z + a_2X^{n-2}Z^2 + \cdots + a_nZ^n$$

be a binary form of degree n; we suppose[1] that $a_0 \neq 0$. Then we can write

$$F(X,Z) = a_0(X-\alpha_1Z)(X-\alpha_2Z)\dots(X-\alpha_nZ)$$

with some complex numbers α_j. To obtain a representative point in the upper half-plane, we construct a positive definite quadratic form

$$Q(X,Z) = \sum_{j=1}^{n} t_j(X-\alpha_j\,Z)(X-\bar{\alpha}_j\,Z)\,,$$

where the t_j are positive real numbers that have to be determined.[2] Julia shows that the set of possible representative points is the convex hull (in hyperbolic geometry) of the roots α_j that lie in the upper half-plane or on the real axis. If we act on F by some element from $\mathrm{SL}(2,\mathbb{R})$, and simultaneously perform an appropriate operation on the t_j, then the resulting Q will be the result of acting on the original Q by the same substitution. Julia notes that the expression (first introduced by Hermite in [5])

$$\theta_0 = \frac{a_0^2\,(\mathrm{disc}(Q))^{n/2}}{t_1t_2\dots t_n}$$

is then an *invariant*. Furthermore, the leading coefficient of a form that has its representative point in the fundamental domain $\mathcal{F}$ can be bounded in terms of θ_0 (and the same is true for the other coefficients if a_0^2 is bounded below, as when we are considering forms with integral coefficients). Therefore he chooses the representative point that belongs to the quadratic Q that makes θ_0 minimal. We will see below that this gives indeed a well-defined point (i.e., there is a unique Q that minimises θ_0; Julia proves existence but not uniqueness). This then implies that this point (or the quadratic Q) is a *covariant* (under $\mathrm{SL}(2,\mathbb{R})$) of F, hence can be used to define a reduction theory.

Julia has solved the optimisation problem for degrees three and four. His results coincide with those obtained by one of us [3] by a different method. In [3] the problem is approached from a different direction, by looking for positive definite quadratic covariants of the given form. We now show why the results are necessarily the same (at least in the purely real and purely complex cases in degrees 3 and 4). The reason is that the presence of sufficiently many symmetries forces a unique covariant.

[1] This is not an essential restriction (the relevant quantities can be obtained by a suitable limiting process when $a_0 = 0$), but serves to simplify the exposition.

[2] Julia uses t_j^2 and u_j^2 to denote the positive real numbers t_j.

Lemma 3.1. *Let G be a group acting on two sets A, B. Suppose that for all $a \in A$, the stabiliser G_a of a in G has a unique fixed point $z(a) \in B$. Then $z : A \to B$ is the unique G-equivariant map from A to B.*

Proof: For definiteness, let us assume that G acts on the right on both sets. Let $a \in A$ and $g \in G$; then $G_{a\cdot g} = g^{-1}G_a g$, and therefore, $z(a) \cdot g$ is fixed by $G_{a\cdot g}$, whence $z(a \cdot g) = z(a) \cdot g$. So z is indeed equivariant. Now let $f : A \to B$ be any equivariant map, and let $a \in A$. Then for all $g \in G_a$, we have $f(a) \cdot g = f(a \cdot g) = f(a)$, hence $f(a)$ is fixed by G_a, so $f(a) = z(a)$ and $f = z$. □

We can apply this to forms of degrees 3 and 4, represented by the (unordered) set of their roots.

Lemma 3.2.

(1) *A set of three distinct points on the real line has exactly one* $\mathrm{SL}(2,\mathbb{R})$*-covariant point in the upper half-plane.*
(2) *A set of four distinct points on the real line has exactly one* $\mathrm{SL}(2,\mathbb{R})$*-covariant point in the upper half-plane.*
(3) *A set of two distinct points in the upper half-plane has exactly one* $\mathrm{SL}(2,\mathbb{R})$*-covariant point in the upper half-plane.*

Proof: We use the Poincaré disk model for the hyperbolic plane. In each case, we show that the stabiliser in $\mathrm{SL}(2,\mathbb{R})$ of the given configuration has a unique fixed point in $\mathcal{H}$. The claim then follows from Lemma 3.1.
(1) Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on sets of three real points, we can move the points such that they become the vertices of an equilateral triangle on the boundary of the disk. This shows that the set of three points has a stabiliser of order three (given by rotations of the disk) with a unique fixed point.
(2) The group $\mathrm{SL}(2,\mathbb{R})$ preserves the cyclic ordering of the four points. Hence the two diagonals of the ideal quadrilateral formed by the points are covariant, and so is their point of intersection. Conversely, we can move this intersection point to become the centre of the Poincaré disk; then the four points must be at the corners of a rectangle. This shows that the set of four points is stabilised by the rotation by π around the centre, which is the unique fixed point.
(3) A similar argument as in part (2) shows that the midpoint of the geodesic segment connecting the two points is the unique fixed point of the stabiliser. □

The lemma implies that to a real form of degree three with three real roots, or a real form of degree four with either four real roots or two pairs of conjugate complex roots, we can assign one and only one covariant point in the upper half-plane. Hence the methods of Julia in [8] and Cremona in [3] must obtain the same result in these cases. For all real cubics and quartics, Julia's covariant quadratic may be expressed as follows (with $n = 3$ or $n = 4$):

$$Q_0(F)(X,Z) = \sum_{j=1}^{n} \frac{1}{|f'(\alpha_j)|^{2/(n-2)}}(X - \alpha_j Z)(X - \bar{\alpha}_j Z)$$

(where, as usual, $f(X) = F(X,1)$); in fact, this expression gives a covariant for all degrees $n \geq 3$.

Lemma 3.3. *Q_0 is positive definite and a covariant of F for all $n \geq 3$.*

Proof: Positive definiteness is clear. Covariance with respect to translations is obvious, and covariance with respect to the inversion $(X, Z) \mapsto (Z, -X)$ follows from an easy calculation. □

For complex forms F, we define $Q_0(F) \in H(\mathbb{C})$ by

$$Q_0(F)(X, Z) = \sum_{j=1}^{n} \frac{|X - \alpha Z|^2}{|f'(\alpha_j)|^{2/(n-2)}} ;$$

then the same conclusions hold.

It follows from the uniqueness lemma that, for purely real cubics and purely real or purely complex quartics, Q_0 is the unique covariant quadratic (up to a scaling factor) and its root in the upper half-plane is the unique covariant point.

The lack of uniqueness in the mixed cases for degrees three and four is apparent in the literature; for real cubics with a single real root, Matthews [9] and Belabas [1] use the unique root in the upper half-plane as representative point, while both Julia and Cremona in [3] use a different choice, defined below, which depends on all three roots. Other choices are also possible. Similarly with mixed quartics, where Birch and Swinnerton-Dyer in [2] also use as covariant point the unique root in the upper half-plane.

However, if we enlarge our perspective (again following Julia) by considering the hyperbolic plane as embedded in hyperbolic three-space, so that we have an action of $\mathrm{SL}(2, \mathbb{C})$ on complex forms, we find similar uniqueness results for general forms of degrees three and four. Since $\mathrm{SL}(2, \mathbb{C})$ acts transitively on triples of points in $\mathbb{P}^1(\mathbb{C})$, the set consisting of the three roots of a form of degree three has stabiliser isomorphic to the symmetric group S_3, which fixes a unique point in $\mathcal{H}_3$. Similarly, the set of roots of a form of degree four has a Klein four group as stabiliser (coming from the symmetries of the cross-ratio), which again fixes a unique point in $\mathcal{H}_3$. If the given form has real coefficients, then this covariant point lies in the 'real' hyperbolic plane; it is again given by $Q_0(F)$ as above.

This enlarged perspective therefore eliminates the non-uniqueness of the covariant point in $\mathcal{H}$ for real forms F of degree 3 or 4 in the "mixed" cases. While in these cases there is not a unique $\mathrm{SL}(2, \mathbb{R})$-covariant point in $\mathcal{H}$, there *is* a unique $\mathrm{SL}(2, \mathbb{C})$-covariant point $z(F)$ in $\mathcal{H}_3$, which lies in $\mathcal{H}$. It is certainly a good idea to profit from the inherent symmetry of the situation by treating real and complex roots on an equal footing, all the more since this allows us to also set up a reduction theory for complex forms with respect to a subgroup of $\mathrm{SL}(2, \mathbb{C})$, e.g., $\mathrm{SL}(2, \mathbb{Z}[i])$. It therefore seems reasonable that this covariant $z(F)$ should be the best one to use for reduction.

For forms of degree five and higher, the stabiliser is usually trivial, and symmetry does not help to fix a covariant. In this case, we fix it by solving Julia's optimisation problem. As it turns out (see Corollary 4.7 below), this solution can also be characterised by a nice geometric property. This fact provides some additional justification for considering Julia's covariant as the 'best' one.

The root $z_0(F)$ of $Q_0(F)$ in $\mathcal{H}$ is a covariant for *any* real form F. This means that we can use it to define a reduction theory — we call a form F Q_0-*reduced* if $z_0(F)$ is in the usual fundamental domain $\mathcal{F}$, and we can Q_0-reduce a form by moving $z_0(F)$ into the fundamental domain by the action of $\mathrm{SL}(2,\mathbb{Z})$. The advantage of this definition is that it is easily implemented, since $Q_0(F)$ is easy to write down. But it does not give optimal results in general. In particular, it is *not* Julia's covariant if the degree is five or more. (See Section 6 below for an example.)

4. Implementing Julia's approach

Recall our notation:

$$F(X,Z) = a_0X^n + a_1X^{n-1}Z + a_2X^{n-2}Z^2 + \cdots + a_nZ^n$$

is a real binary form of degree $n \geq 3$; we suppose that $a_0 \neq 0$. Then we have

$$F(X,Z) = a_0(X-\alpha_1Z)(X-\alpha_2Z)\dots(X-\alpha_nZ)$$

with some complex numbers α_j. Unless otherwise specified, we restrict to forms without repeated factors, i.e., the α_j are assumed to be distinct.

We consider the positive definite quadratic form

$$Q(X,Z) = \sum_{j=1}^{n} t_j(X-\alpha_j\,Z)(X-\bar{\alpha}_j\,Z)\,,$$

where the t_j are positive real numbers, and we want to choose them in such a way as to minimise the quantity θ, where[3]

$$\theta = \frac{a_0^2(\operatorname{disc} Q)^{n/2}}{n^n\,t_1t_2\dots t_n}\,.$$

(Recall that with our definition of $\operatorname{disc} Q$, it is a positive real number.)

We first observe that if the t_j are minimising, then we must have $t_j = t_k$ if $\alpha_k = \bar{\alpha}_j$. This is because the two terms $t_j(X-\alpha_j)(X-\bar{\alpha}_j)$ and $t_k(X-\alpha_k)(X-\bar{\alpha}_k)$ are proportional, hence Q is unchanged if we replace (t_j,t_k) with $(t_j+\varepsilon, t_k-\varepsilon)$. Varying ε, the product in the denominator becomes maximal when $t_j = t_k$. We therefore can restrict the t_j to satisfy these equalities.

[3]Our θ differs by a factor of $(2/n)^n$ from Julia's θ_0.

If we write $Q(X,Z) = s((X - t\,Z)^2 + u^2\,Z^2)$ with $s, u > 0$ and $t \in \mathbb{R}$, we then find

$$\begin{aligned} s &= \sum_{j=1}^{n} t_j \ ; \\ st &= \sum_{j=1}^{n} \operatorname{Re}(\alpha_j) t_j \ ; \\ s(t^2+u^2) &= \sum_{j=1}^{n} |\alpha_j|^2 t_j \ ; \\ \tfrac{1}{4}\operatorname{disc} Q = s^2u^2 &= \sum_{j<k} |\alpha_j - \alpha_k|^2 t_j t_k \ . \end{aligned} \tag{4.1}$$

To deduce the fourth equation from the first three (which are obvious from the definition of Q), write

$$\begin{aligned} 4s^2u^2 &= 2s(t^2+u^2)\cdot s + 2s\cdot s(t^2+u^2) - 4(st)^2 \\ &= \sum_{j,k=1}^{n} \Big(2\alpha_j\bar\alpha_j + 2\alpha_k\bar\alpha_k - (\alpha_j+\bar\alpha_j)(\alpha_k+\bar\alpha_k)\Big) t_j t_k \\ &= \sum_{j,k=1}^{n} \Big((\alpha_j-\alpha_k)(\bar\alpha_j-\bar\alpha_k) + (\alpha_j-\bar\alpha_k)(\bar\alpha_j-\alpha_k)\Big) t_j t_k \\ &= \sum_{j,k=1}^{n} (|\alpha_j-\alpha_k|^2 + |\bar\alpha_j-\alpha_k|^2) t_j t_k \\ &= 2\sum_{j,k=1}^{n} |\alpha_j-\alpha_k|^2 t_j t_k \\ &= 4\sum_{j<k} |\alpha_j-\alpha_k|^2 t_j t_k \, . \end{aligned}$$

Note that we have used that $t_j = t_i$ if $\alpha_j = \bar\alpha_i$.

Now minimising θ is equivalent to minimising $\operatorname{disc} Q$ under the side condition that $t_1 t_2 \dots t_n$ is constant (equal to 1, say). Let V_0 denote the subspace of $\mathbb{R}^n$ given by $\sum_j x_j = 0$, and let $V_1 \subset V_0$ be the subspace defined by the additional restriction that $x_j = x_k$ if $\alpha_k = \bar\alpha_j$. Writing $t_j = \exp(x_j)$ where $x_j \in \mathbb{R}$, we then have to minimise

$$D(x) = \sum_{j<k} |\alpha_j - \alpha_k|^2 \exp(x_j + x_k)$$

on V_1. Now we have the following lemma.

Lemma 4.1. *If $n \geq 3$, then D is strictly convex from below on $\mathbb{R}^n$ (and hence the same is true on every linear subspace). If x varies in V_0 in such a way that $|x|$ tends to infinity, then $D(x)$ tends to infinity as well. In other words, the set $\{x \in V_0 \mid D(x) \leq C\}$ is compact.*

Hence D has a unique minimum on every subspace contained in V_0, and the minimising point is the only critical point of D in that subspace.

PROOF: For the first claim, consider a point $x \in \mathbb{R}^n$ and a line through it, parametrised as $y = x + \lambda u$, where $0 \neq u \in \mathbb{R}^n$. Then

$$\frac{d^2}{d\lambda^2} D(x+\lambda u)\big|_{\lambda=0} = \sum_{j<k} |\alpha_j - \alpha_k|^2 (u_j + u_k)^2 \exp(x_j + x_k) \geq 0\,.$$

If this expression vanishes, we must have $u_j + u_k = 0$ for all $j < k$ (since $\alpha_j \neq \alpha_k$ by assumption). This implies the contradiction $u = 0$ when $n \geq 3$. Hence the second derivative of D is positive definite, implying strict convexity. This already implies that there is at most one critical point in each subspace.

For the second claim, take some x in the subspace and assume that all $x_j + x_k \leq C$. By adding these inequalities over all k for a fixed j, we obtain $x_j \leq nC/(n-1)$. But then we also must have $x_j = -\sum_{k\neq j} x_k \geq -nC$, hence x is bounded.

The statement in the second paragraph is then clear. □

The preceding lemma guarantees us a *unique solution* to our minimisation problem. Since θ is an invariant, this implies that the (unique) minimising quadratic Q is a *covariant* of F under $\mathrm{SL}(2,\mathbb{R})$. This allows us to define a reduction theory. We let $z(F)$ be the point in the upper half-plane associated to Q (i.e., such that $Q(z(F),1) = 0$); then $z(F)$ is also a covariant of F, as explained in the Introduction.

Definition 4.2. A form $F(X,Z) \in \mathbb{R}[X,Z]$ of degree $n \geq 3$ is called *reduced* if $z(F)$ lies in the standard fundamental domain $\mathcal{F}$ of $\mathrm{SL}(2,\mathbb{Z})$, where $z(F)$ is the root in the upper half-plane $\mathcal{H}$ of the unique quadratic covariant $Q(X,Z)$ which minimises θ.

The covariance of $z(F)$ implies the following result.

Proposition 4.3. *Each* $\mathrm{SL}(2,\mathbb{Z})$*-orbit of binary forms of degree* $n \geq 3$ *contains at least one reduced form* F*.*

There will usually be exactly one reduced form in each orbit (up to sign when the degree is odd), unless $z(F)$ is on the boundary of the fundamental domain, when there may be two.

In order to find a reduced form in the orbit of a given form F, we can proceed as follows. Find $z(F)$; then use the usual algorithm to find $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2,\mathbb{Z})$ such that $S \cdot z(F) \in \mathcal{F}$. Then $F \cdot S^{-1} = F(dX - bZ, -cX + aZ)$ is reduced.

To make this practical, we need some means of actually finding $z(F)$, or equivalently, the minimising quadratic Q.

The first step is, of course, to write down the conditions for a critical point of D on V_0 (for reasons of symmetry, the unique critical point in V_0 must lie in V_1, so we can leave aside the conditions $x_j = x_k$ for conjugate roots). They are

$$\sum_{k=1}^{n} |\alpha_j - \alpha_k|^2 \exp(x_j + x_k) = \lambda \quad \text{for all } j\,,$$

where λ is a Lagrange multiplier. Going back to our original variables, this means

$$t_j \sum_{k=1}^{n} |\alpha_j - \alpha_k|^2 t_k = \lambda \quad \text{for all } j\,. \tag{4.2}$$

Using the formulas (4.1), we find that

$$\sum_{k=1}^{n} |\alpha_j - \alpha_k|^2 t_k = s((t-\alpha_j)(t-\bar{\alpha}_j) + u^2)\,.$$

Summing equations (4.2) over j, we obtain $2s^2u^2 = n\lambda$. Hence a set of minimising values of t_j must satisfy

$$t_j = \frac{2}{n}\,\frac{s\,u^2}{(t-\alpha_j)(t-\bar{\alpha}_j)+u^2}\,. \tag{4.3}$$

This shows that we can assume without loss of generality that $s = 1$. From $s = 1 = \sum_j t_j$ and $st = t = \sum_j \mathrm{Re}(\alpha_j)t_j$, we deduce that the following two equations must hold.

$$\begin{aligned} \sum_{j=1}^{n} \frac{u^2}{(t-\alpha_j)(t-\bar{\alpha}_j)+u^2} &= \frac{n}{2} \\ \sum_{j=1}^{n} \frac{t-\mathrm{Re}(\alpha_j)}{(t-\alpha_j)(t-\bar{\alpha}_j)+u^2} &= 0 \end{aligned} \tag{4.4}$$

Conversely, suppose that these two equations are satisfied for some $t \in \mathbb{R}$ and $u > 0$. We can then define positive t_j by formula (4.3) with $s = 1$. It is easily checked that we then have

$$\sum_{j=1}^{n} t_j (X - \alpha_j\, Z)(X - \bar{\alpha}_j\, Z) = (X - t\, Z)^2 + u^2\, Z^2$$

and that equations (4.2) are also satisfied with $\lambda = 2u^2/n$. Hence every solution to (4.4) gives rise to a critical point of D, which then must be *the unique minimising point*. We have therefore proved the following result.

Proposition 4.4. *The representative point $z(F)$ is given as $z(F) = t + ui \in \mathcal{H}$, where (t, u) is the unique solution (in $\mathbb{R} \times \mathbb{R}_+$) of the system (4.4).*

We can use this proposition to find $z(F)$ numerically, by performing a search for a solution of (4.4).

There is also another nice description of $z(F)$. We introduce the following polynomial in two variables associated to the form F.

$$\widetilde{F}(t,u) = a_0^2 \prod_{j=1}^{n} ((t-\alpha_j)(t-\bar{\alpha}_j) + u^2)\,.$$

Then it is easily verified that we obtain equations (4.4) by setting the logarithmic partial derivatives of $\widetilde{F}(t,u)/u^n$ equal to zero. Hence:

Proposition 4.5. *The representative point $z(F)$ is given as $z(F) = t + ui \in \mathcal{H}$, where (t, u) is the unique minimising point (in $\mathbb{R} \times \mathbb{R}_+$) of the function*

$$(t, u) \longmapsto \frac{\widetilde{F}(t, u)}{u^n}\,.$$

Moreover, the minimal value of θ is given by

$$\theta = \theta(F) = \min_{(t,u)} \frac{\widetilde{F}(t, u)}{u^n}\,.$$

If F splits over $\mathbb{R}$, we have $\widetilde{F}(t, u) = |F(t + ui, 1)|^2$, and hence $z(F)$ is the unique minimising point in the upper half-plane of

$$z \longmapsto |F(z, 1)| \cdot \operatorname{Im}(z)^{-n/2}\,.$$

Proof: We only have to prove the assertion about θ. This follows from $\operatorname{disc} Q = 4\,u^2$, equations (4.3) and the definition of $\widetilde{F}$. □

Remark: This description shows that $z(F)$ is still well-defined if F has multiple roots, as long as the multiplicity of the real roots is less than half the degree n of F (since in this case, $\widetilde{F}(t, u)/u^n$ is still unbounded for $u \to 0$ and $u \to \infty$). If there are two real roots of multiplicity $n/2$, then F is a power of an indefinite quadratic form, and there is no reasonable choice of $z(F)$ (any point on the geodesic joining the two roots could be taken). On the other hand, $\theta(F)$ is still defined and positive in this case (since $|F(z, 1)|^2/\operatorname{Im}(z)^n$ is constant on this geodesic). More generally, $\theta(F)$ is defined and positive if F has no real root of multiplicity larger than $n/2$. If there is a real root of multiplicity exceeding $n/2$, then $\theta(F)$ should be taken to be zero (and $z(F)$ should be taken to be that real root; of course this $z(F)$ is no longer in the upper half-plane, so we cannot use it for reduction purposes). In fact, in this case, the coefficients of F can be made arbitrarily small using suitable elements of $\mathrm{SL}(2, \mathbb{R})$, i.e., F is a "nullform" in the sense of Hilbert [7].

To obtain a nice geometric description in the general case, we consider again the upper half-space $\mathcal{H}_3$, and we think of the upper half-plane $\mathcal{H}$ as embedded in it (as described in Section 2). View the roots α_j of F as lying on the 'floor' or boundary of $\mathcal{H}_3$, identified with $\mathbb{C}$. Then an individual factor in the definition of $\widetilde{F}$ is the squared (Euclidean) distance from $t + uj$ to the root α, whereas

$$\log \frac{(t - \alpha)(\bar{t} - \bar{\alpha}) + u^2}{u}$$

measures the hyperbolic distance between $t + uj$ and $\alpha \in \mathbb{C}$, up to some arbitrary additive constant. More precisely, the difference of these distances for two points lying on a geodesic with α as a limit point is the same as their (oriented) hyperbolic distance. Furthermore, the points with the same 'distance' from α lie on a horosphere at α. So we have the following interpretation.

Proposition 4.6. *The representative point $z(F)$ is given as $z(F) = t + ui \in \mathcal{H}$, where $t + uj$ is the unique point in the upper half-plane contained in $\mathcal{H}_3$ such that the sum of its distances from all the roots of F is minimal.*

Remark: The same description is valid in the case of binary forms with complex coefficients (where we use $\mathrm{SL}(2,\mathbb{C})$ instead of $\mathrm{SL}(2,\mathbb{R})$); we only have to allow t to vary in all of $\mathbb{C}$ and not just in $\mathbb{R}$. The point $z(F)$ is then a point in $\mathcal{H}_3$, and can be used to reduce forms with respect to the action of a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$ such as the Bianchi group $\mathrm{SL}(2,\mathbb{Z}[i])$. (When the form is real, $z(F)$ will automatically be in the 'real' hyperbolic plane, and we obtain the same result as before.)

Note that these distances are not preserved by the action of $\mathrm{SL}(2,\mathbb{C})$ — the additive constant changes. If we add $2\log|a_0|$, then the sum of the distances becomes invariant (if we act on F by $S \in \mathrm{SL}(2,\mathbb{C})$ and on $t+uj$ by S^{-1}).

We can imagine a point in upper half-space that is drawn towards each of the roots by a force of equal magnitude in an attempt to minimise the total distance to the roots. The total distance will be at a minimum when the forces are in an equilibrium. This gives the following.

Corollary 4.7. *The point $z(F)$ (considered as a point in upper half space) is characterised by the property that the unit tangent vectors at $z(F)$ in the directions of the roots of F add up to zero.*

This property is obvious in the low degree cases $n=3$ and $n=4$; and it is this property that gives the correct generalisation to higher degrees.

There is a slightly more elegant way of formulating Proposition 4.5. In order to achieve this, we define the *resultant* of a binary form F and a Hermitian form Q by the rules

$$\mathrm{Res}(aX-bZ,Q) = Q(b,a) \quad \text{and} \quad \mathrm{Res}(F_1F_2,Q) = \mathrm{Res}(F_1,Q)\,\mathrm{Res}(F_2,Q)\,.$$

This is inspired by some of the properties of the usual resultant of two binary forms. Then it is easily seen that

$$\frac{\widetilde{F}(t,u)}{u^n} = \frac{2^n\,\mathrm{Res}(F,Q)}{(\operatorname{disc} Q)^{n/2}}$$

for $Q \in H(\mathbb{C})$, if $z(Q) = t+uj$. Hence the following holds.

Corollary 4.8. *For $F \in \mathbb{C}[X,Z]_n$, we have*

$$\theta(F) = \inf_{Q\in H(\mathbb{C})} \frac{2^n\,\mathrm{Res}(F,Q)}{(\operatorname{disc} Q)^{n/2}}\,.$$

When F has no root of multiplicity at least $n/2$, then the infimum is a minimum and is attained at a unique form Q, up to scaling, and we have $z(F) = z(Q)$.

A simple consequence of this is that $\theta(F_1F_2) \geq \theta(F_1)\theta(F_2)$, with equality if and only if $z(F_1) = z(F_2)$ (provided that both are defined).

5. The reduction algorithm

Given the definition of the covariant point $z(F)$ associated to each form F, the procedure to reduce F is standard; we recall it here and make some remarks of a practical nature.

Let F be a binary form of degree $n \geq 3$ with integral coefficients; we want to find a reduced form that is $\mathrm{SL}(2,\mathbb{Z})$–equivalent to it. We proceed as follows. First find $z := z(F)$. Repeat the following steps while z is outside the usual fundamental domain $\mathcal{F}$ for $\mathrm{SL}(2,\mathbb{Z})$.

1. Let m be the integer nearest to $\mathrm{Re}(z)$ and set $F(X,Z) := F(X+mZ,Z)$ and $z := z - m$.
2. If $|z| < 1$, then set $F(X,Z) := F(Z,-X)$ and $z := -1/z$.

After finitely many passes through the loop, z will be in $\mathcal{F}$, and F will be reduced.

For a practical implementation, a few remarks are useful.

Firstly, it may be a good idea to use $z_0(F)$ as given by $Q_0(F)$ instead of $z(F)$ to start with, since it is much more easily (and speedily) computed. When $z_0(F)$ is in $\mathcal{F}$, we expect that in most cases $z(F)$ will not be very far away from $\mathcal{F}$. This should make numerical methods easier to apply than when $z(F)$ is very close to the real axis. Furthermore, only a few extra steps will be necessary to move $z(F)$ into $\mathcal{F}$, so we will probably gain more than we lose by this slightly devious way of performing the reduction.

Secondly, in order to compute $z_0(F)$ or $z(F)$, we know of no better way than first to find all the complex roots of $F(X,1)$ numerically. The resulting value of z will have finite precision, and this precision will decrease during the computation. Therefore it seems advisable to recompute $z := z(F)$ (or $z_0(F)$) from time to time.

Thirdly, some care should be taken with the condition for leaving the loop. If taken literally, infinite looping can result from rounding errors when z is near the boundary of $\mathcal{F}$.

6. Examples

In this section, we give some examples that demonstrate how to use our approach to obtain smaller models for hyperelliptic curves over $\mathbb{Q}$. Such a hyperelliptic curve can be given by an affine equation of the form

$$y^2 = f(x)\,,$$

where $f(x)$ is a square-free polynomial with integral coefficients of degree $d \geq 5$ (we are excluding curves of genus less than 2; the genus of the curve above is $g = \lfloor (d-1)/2 \rfloor$). In order to obtain a smooth projective model, we write $f(x) = F(x,1)$ with a form $F(x,z)$ of *even* degree $n = 2\lceil d/2 \rceil = 2g+2$. The equation

$$y^2 = F(x,z)$$

then gives a smooth projective model of the curve, embedded in a weighted projective plane $\mathbb{P}^2_g$ (where x and z have weight 1 and y has weight $g+1$). Equivalently, we can glue together the two affine models

$$y^2 = F(x,1) \qquad \text{and} \qquad w^2 = F(1,z)$$

with the identifications $xz = 1$, $y = wx^{g+1}$. The modular group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{P}^2_g$ through its action on x and z; so we can use it to find a better model by

reducing the form F. In the examples below, we will make extensive use of our convention $f(x) = F(x, 1)$ (similarly for F_j and f_j).

The first example is taken from H.-J. Weber's 1996 Essen thesis [12], in which he considers certain hyperelliptic curves with modular Jacobians. Weber tries to simplify the models he obtains by a trial-and-error approach. One of his final models is given by

$$\begin{aligned} y^2 = f(x) = 19x^8 - 262x^7 + 1507x^6 - 4784x^5 + 9202x^4 - 10962x^3 \\ + 7844x^2 - 3040x + 475\,. \end{aligned}$$

(See [12] or [13, p. 284].) Let us follow the algorithm as applied to F in some detail. For the first reduction steps, we use $z_0(F)$. The roots of f are

$$\begin{gathered} 0.42798171, \quad 1.30152156, \quad 1.31947230, \quad 4.31651243, \\ 1.69098301 \pm 0.72287100\,i, \quad 1.52100984 \pm 0.12866975\,i\,. \end{gathered}$$

From the roots, we compute $Q_0(F)$ and its root

$$z = z_0(F) = 1.38323301 + 0.31233552\,i\,.$$

The integer m in the algorithm is 1, so we do a shift and replace f with

$$\begin{aligned} f_1(x) &= f(x+1) \\ &= 19x^8 - 110x^7 + 205x^6 - 180x^5 + 47x^4 + 40x^3 - 35x^2 + 10x - 1 \end{aligned}$$

and z with $z_1 = z - 1 = 0.38323301 + 0.31233552\,i$. Since we have $|z_1| < 1$, we invert f_1 to get

$$\begin{aligned} f_2(x) &= x^8 f_1(-1/x) \\ &= -x^8 - 10x^7 - 35x^6 - 40x^5 + 47x^4 + 180x^3 + 205x^2 + 110x + 19 \end{aligned}$$

and set $z_2 = -1/z_1 = -1.56792167 + 1.27785869\,i$. In the next pass through the loop, $m = -2$, so

$$f_3(x) = f_2(x-2) = -x^8 + 6x^7 - 7x^6 - 12x^5 + 27x^4 - 4x^3 - 19x^2 + 10x - 5$$

and $z_3 = z_2 + 2 = 0.43207833 + 1.27785869\,i$. Since $z_3 \in \mathcal{F}$, we see that F_3 is Q_0-reduced. Now we use Julia's covariant $z(F)$. We find the roots of f_3 and use some numerical method to compute

$$z_4 = z(F_3) = 0.64189877 + 1.18525166\,i\,.$$

This is not in $\mathcal{F}$, and $m = 1$ in our algorithm. Hence we set

$$f_5(x) = f_3(x+1) = -x^8 - 2x^7 + 7x^6 + 16x^5 + 2x^4 - 2x^3 + 4x^2 - 5$$

and $z_5 = z(F_5) = z_4 - 1 = -0.35810123 + 1.18525166\,i \in \mathcal{F}$, so F_5 is reduced.

To summarise, our algorithm produces after the first step (using $z_0(F)$) the model

$$y^2 = -x^8 + 6x^7 - 7x^6 - 12x^5 + 27x^4 - 4x^3 - 19x^2 + 10x - 5\,,$$

and after the second step (using $z(F)$)

$$y^2 = -x^8 - 2x^7 + 7x^6 + 16x^5 + 2x^4 - 2x^3 + 4x^2 - 5\,.$$

Incidentally, the fact that these two are distinct justifies our claim that $z_0(F)$ is in general not Julia's $z(F)$.

Another example is related to work by X. Wang, also in Essen.[4] This time, it concerns a genus 2 curve, and the initial model is

$$y^2 = x^6 + 30x^5 + 371x^4 + 2422x^3 + 8813x^2 + 16968x + 13524\,.$$

After Q_0-reduction, we obtain

$$y^2 = x^6 - 4x^4 + 2x^3 + 8x^2 - 12x + 9\,,$$

and finally

$$y^2 = x^6 + 6x^5 + 11x^4 + 6x^3 + 5x^2 + 4\,.$$

Here is a third example that shows that the Q_0-reduced and the reduced form do not always differ by a shift. Consider

$$f(x) = 6x^6 + 8x^5 - 10x^4 - 4x^3 + 10x^2 - 6x + 5\,.$$

This is Q_0-reduced ($z_0(F)$ is near i, and slightly above the unit circle), but in order to find the reduced representative, we have to invert (since $z(F)$ is also near i, but slightly below the unit circle).

7. More specific results in the totally real case

In this section, we prove the following result.

Proposition 7.1. *Let $F(X,Z)$ be a totally real form of degree $n \geq 3$ with distinct roots. Then $z(F)$ is the unique root in the upper half-plane of $G(X,1)$, where*

$$G(X,Z) = \frac{X\,F_X(-F_Z(X,Z),F_X(X,Z)) + Z\,F_Z(-F_Z(X,Z),F_X(X,Z))}{n\,F(X,Z)}$$

is a binary form of degree $(n-1)(n-2)$. (Here, F_X and F_Z denote partial derivatives).

Note that $G(X,Z)$ is indeed a polynomial. To see this, let

$$\widetilde{G}(X,Z) = X\,F_X(-F_Z(X,Z),F_X(X,Z)) + Z\,F_Z(-F_Z(X,Z),F_X(X,Z))$$

be the numerator of G. Let a be a root of F, so that $F(X,Z) = (X-aZ)H(X,Z)$. Then

$$\begin{aligned} F_X(U,V) &= H(U,V) + (U-aV)H_X(U,V) \\ F_Z(U,V) &= -aH(U,V) + (U-aV)H_Z(U,V)\,, \end{aligned}$$

so

$$\begin{aligned} &XF_X(U,V) + ZF_Z(U,V) \\ &\qquad = (X-aZ)H(U,V) + (U-aV)(XH_X(U,V) + ZH_Z(U,V))\,. \end{aligned}$$

The first term on the right is a multiple of $X - aZ$ for any U, V; so is the second when $U = -F_Z(X,Z)$ and $V = F_X(X,Z)$, since $nF = XF_X + ZF_Z$ implies

[4] Wang's work is described in [11]. The curve in the example is the curve of level 147; the model was communicated by Wang to the authors of [4].

$0 = aF_X(a,1) + F_Z(a,1)$, so $U - aV = -(F_Z + aF_X)$ is zero at $(a,1)$. Hence each linear factor of F divides $\widetilde{G}$, and since F has no repeated factors, $G = \widetilde{G}/(nF)$ is a polynomial.

The proof will now be in two steps. The first step is to show that $z(F)$ really is a root of $G(X,1)$. The second step is to show that $G(X,1)$, which is a polynomial of degree $(n-1)(n-2)$, has at least $n(n-3)$ real roots, leaving $z(F), \bar{z}(F)$ as the only possible pair of complex conjugate roots.

For the first step, recall that $z(F)$ is the point $z = t+iu$ in the upper half-plane minimising

$$\frac{\widetilde{F}(t,u)}{u^n} = \frac{f(z)f(\bar{z})}{(\operatorname{Im} z)^n}$$

(with the usual convention $f(z) = F(z,1)$; this equality is only valid when all the roots of f are real). Taking $z = t+iu$ and $\bar{z} = t - iu$ as new variables, the necessary conditions for the minimum can be written (after some simplification) as

$$(z-\bar{z})f'(z) = n\,f(z) \qquad \text{and} \qquad (z-\bar{z})f'(\bar{z}) = -n\,f(\bar{z})\,.$$

We can solve the first of these two equations for $\bar{z}$, obtaining

$$\bar{z} = z - n\frac{f(z)}{f'(z)}\,;$$

then we substitute this expression for $\bar{z}$ in the second equation. We get

$$f(z)\,f'\Big(z - n\frac{f(z)}{f'(z)}\Big) + f'(z)\,f\Big(z - n\frac{f(z)}{f'(z)}\Big) = 0\,. \tag{7.1}$$

Multiplying this by $f'(z)^{n-1}$ and re-writing the expression in terms of the homogeneous polynomial F (note that $f'(z) \neq 0$; otherwise $f(z)$ would have to vanish also, but f was assumed to be square-free), we get

$$\begin{aligned}
0 &= F(z,1)\,F_X(z\,F_X(z,1) - n\,F(z,1), F_X(z,1)) \\
&\quad + F(z\,F_X(z,1) - n\,F(z,1), F_X(z,1)) \\
&= F(z,1)\,F_X(-F_Z(z,1), F_X(z,1)) \\
&\quad + \tfrac{1}{n}\big(-F_Z(z,1)\,F_X(-F_Z(z,1), F_X(z,1)) + F_X(z,1)\,F_Z(-F_Z(z,1), F_X(z,1))\big) \\
&= \tfrac{1}{n}F_X(z,1)\big(z\,F_X(-F_Z(z,1), F_X(z,1)) + F_Z(-F_Z(z,1), F_X(z,1))\big)\,.
\end{aligned}$$

(We have again used the well-known relation $XF_X + ZF_Z = nF$.) This shows that $G(z(F),1) = 0$.

For the second step, we again use that if $f(x) \neq 0$, then

$$\frac{f'}{f}\Big(x - n\frac{f(x)}{f'(x)}\Big) = -\frac{f'(x)}{f(x)} \tag{7.2}$$

implies that $G(x,1) = 0$. We want to show that between any two consecutive zeroes of f (considered as lying on the circle $\mathbb{P}^1(\mathbb{R})$), there are at least $n-3$ real zeroes of $G(x,1)$. Since G is easily seen to be a covariant of F, we can assume that the two consecutive roots we are considering are 0 and 1. The rational function

f'/f has simple poles (with residue 1) at each of the roots of f and is monotonically decreasing (as can be seen from the partial fraction decomposition). Hence the right hand side of our equation grows monotonically from $-\infty$ to $+\infty$ in the open interval $(0,1)$. On the other hand, the function $x - nf(x)/f'(x)$ approaches zero from below when x approaches zero from above, it approaches 1 from above when x approaches 1 from below, and it has a unique (simple) pole of positive residue in the open interval $(0,1)$. This shows that when x goes from 0 to 1, the value of $x - nf(x)/f'(x)$ goes from 0 through $-\infty = \infty$ (on $\mathbb{P}^1(\mathbb{R})$) to 1. The function f'/f has $n-2$ simple poles outside the closed interval $[0,1]$, hence $(f'/f)(x-nf(x)/f'(x))$ has (at least) $n-2$ simple poles in the open interval $(0,1)$. Between any two consecutive of these poles, there must be a value of x satisfying equation (7.2). This shows that there are at least $n-3$ zeroes of $G(x,1)$ between two consecutive zeroes of f. Hence $G(x,1)$ has at least $n(n-3)$ real zeroes, as was to be shown.

When $n = 2$ one may quickly check that $G(X,Z)$ is identically zero. When $n = 3$, $G(X,Z)$ is the Hessian of F. Explicitly, if

$$F(X,Z) = aX^3 + bX^2Z + cXZ^2 + dZ^3\,,$$

then

$$G(X,Z) = (3ac - b^2)X^2 + (9ad - bc)XZ + (3bd - c^2)Z^2\,.$$

(This is minus the Hessian covariant as given in [3].)

When $n = 4$, $G(X,Z)$ is (up to a constant factor) the sextic covariant of F denoted g_6 in [3]. In both these cases it was noted in [3] that the unique root of $G(X,Z)$ in the upper half-plane was the appropriate covariant point with which to reduce a cubic or quartic with all its roots real.

In the special cases of degrees 3 and 4, we can express $\theta(F)$ explicitly as a root of a monic polynomial having rational invariants of F as its coefficients. Let $\Delta = \mathrm{disc}(F)$. Then if F is a binary cubic form splitting over $\mathbb{R}$, we have that $\theta(F)$ is the largest root of

$$T_3(x) = 3^3\,x^2 - 2^6\,\Delta$$

(cf.[8, p.51]; note that Julia's value of θ is $(3/2)^3$ times our value). If F is a binary quartic form splitting over $\mathbb{R}$, then $\theta(F)$ is the largest root of

$$T_4(x) = x^3 - 2I\,x^2 + I^2\,x - \Delta = x(x-I)^2 - \Delta\,,$$

where $I = 12\,a_0a_4 - 3\,a_1a_3 + a_2^2$ is the usual invariant. From this, we can easily deduce that $I < \theta(F) < \frac{4}{3}I$, noting that $T_4(0) < 0$, $T_4(\frac{1}{3}I) > 0$, $T_4(I) < 0$ and $T_4(\frac{4}{3}I) > 0$.

It would be interesting to investigate whether similar equations and inequalities are satisfied by θ in the higher degree cases.

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