

Unit Root Testing under a Local Break in Trend using Partial Information on the Break Date*

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Abstract

We consider unit root testing allowing for a break in trend when partial information is available regarding the location of the break date. This takes the form of knowledge of a relatively narrow window of data within which the break takes place, should it occur at all. For such circumstances, we suggest employing a union of rejections strategy, which combines a unit root test that allows for a trend break somewhere within the window, with a unit root test that makes no allowance for a trend break. Asymptotic and finite sample evidence shows that our suggested strategy works well, provided that, when a break does occur, the partial information is correct. An empirical application to UK interest rate data containing the 1973 ‘oil shock’ is also considered.

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1 Introduction

When testing for a unit root in economic and financial time series, it is now a matter of regular practice to allow for a possible break in linear trend in the underlying deterministic specification. In doing so, it is almost always the case that the timing of the potential break is treated as an unknown quantity, and a variety of methods have been proposed to deal with this uncertainty. Popular approaches typically either directly estimate a break date endogenously from the data, then subsequently apply a unit root test conditional on a break at this estimated point, or simply take an infimum functional of unit root

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statistics applied at each candidate break point. Examples of the conditional approach include the OLS-based tests presented in Banerjee *et al.* (1992), Perron and Vogelsang (1992), Perron (1997). Examples of the infimum approach include the OLS-based tests of Zivot and Andrews (1992), Harvey *et al.* (2012c) and Harvey and Leybourne (2012); of these three tests only the latter does not suffer from over-sizing when a break is present under the unit root null hypothesis. However, none of these OLS-based tests exploits the power gains afforded by GLS detrending. Perron and Rodríguez (2003) [PR] suggest GLS-based variants of both conditional and infimum tests, recommending the latter on the grounds of superior power. Harvey *et al.* (2011) [HLT] further demonstrate that PR's infimum GLS-based test is, importantly, not over-sized when a break is present under the unit root null.

A potentially unattractive feature of all tests discussed above is that since they always allow for a break, in a situation where no break occurs power is forfeited relative to the corresponding test that excludes the trend break component in the deterministic specification. In response to this, Kim and Perron (2009), Carrion-i-Silvestre *et al.* (2009) and Harris *et al.* (2009) have proposed alternative procedures based on a prior break detection step before the implementation of either a with-trend break or without-trend break unit root test (the last two of these are GLS-based tests, while the first of the three is OLS-based, and so is less powerful, other things being equal).

A major concern with the procedures that rely on break detection is that unless the trend break magnitude is fairly large, the break detection steps can easily fail to detect it, resulting in the incorrect application of without-trend break unit root tests. As a consequence, efforts to recover some extra unit root test power in the no-break case can have the unpleasant side effect of surrendering power when a break is present but not detected, as documented in Harvey *et al.* (2012b) in a doubly-local asymptotic power analysis (i.e. a local to unit root alternative combined with a local to zero assumption regarding the break magnitude). Indeed, HLT show that the extent of such asymptotic power losses are sufficiently severe that the straightforward infimum unit root test of PR arguably provides the more robust inference overall, and certainly so when extended to a multiple trend break environment where break detection failures can pose even more of an issue.

What all of the aforementioned unit root test procedures have in common is that they treat the location of the break as unrestricted, other than making various arbitrary assumptions to exclude a common proportion of break dates at the beginning and end of the sample period (so-called trimming). However, it is often the case that a practitioner will have some degree of confidence as to the approximate location of a putative break, despite not knowing it precisely. Andrews (1993), in the context of testing for general structural instability, introduces this possibility motivated by two sets of examples: (i) where a political or institutional event has occurred during a defined time-frame (e.g. a war) but it is unknown exactly when any change-point takes effect; (ii) where an event occurs at a known date but its effect is either anticipated or occurs after a delay. In each case, an analyst has information on the approximate timing of any break, but remains unsure over its exact date and its magnitude, or indeed its presence at all.

In this paper, we consider the case where true partial information of this form is available. In this context it makes sense to follow the spirit of the Andrews (1993) approach and restrict the search

region for the trend break to an appropriately narrow window of possible break dates. All of the above unit root test procedures could be adapted to a restricted search set of this form; however, in this paper we analyse the infimum unit root test recommended by PR and HLT, on the grounds of its appealing power properties amongst those procedures that do not rely on break detection, and its relative robustness compared to those that do.

Taking the infimum of GLS-detrended unit root statistics over a restricted rather than unrestricted region of trend break dates would be expected, *prima facie*, to yield improvements in test power by reducing uncertainty about the break location, should one be present. However, this procedure on its own includes no mechanism for capturing the additional power that a without-trend break unit root test can offer when no break occurs. We attempt to overcome this impediment by suggesting a *union of rejections* strategy whereby the unit root null is rejected if either the restricted-range with-trend break infimum unit root test or the without-trend break unit root test rejects. This approach builds on the ideas in Harvey *et al.* (2009, 2012a) and Hanck (2012), who suggested accounting for uncertainty regarding the presence of a fixed linear trend in the data by taking a union of rejections of with- and without-trend unit root tests as an alternative to pre-testing for a linear trend prior to unit root testing. In the current context, the union of rejections over with-trend break infimum unit root tests and without-trend break unit root tests obviates the need for prior trend break detection (which, as noted above, can seriously compromise unit root test power). We find that the combination of the restricted-range approach, together with application of a union of rejections strategy, provides a unit root test with very attractive properties in terms of size and power.

The plan of the paper is as follows. In section 2 we present the trend break model and describe our union of rejections testing strategy in detail. Section 3 details the large sample distributions of the union strategy under local-to-zero trend breaks and for a local-to-unity autoregressive root; asymptotic critical values are given for our procedure for a selection of window widths and break locations. We then examine the asymptotic size and power of our procedure across different local trend break magnitudes, including situations where the break exists outside of the restricted region. The finite sample behaviour of our strategy is considered in section 4 and an empirical illustration using UK bond market data is given in section 5. Some conclusions are offered in section 6.

In the following ‘ $[\cdot]$ ’ denotes the integer part of its argument, ‘ \Rightarrow ’ denotes weak convergence, ‘ $x := y$ ’ (‘ $x =: y$ ’) indicates that x is defined by y (y is defined by x), $\mathbb{I}_x^y := 1(y > x)$, and ‘ $1(\cdot)$ ’ denotes the indicator function.

2 The Model and Test Statistic

We consider a time series $\{y_t\}$ to be generated according to the following DGP,

$$y_t = \mu + \beta t + \gamma_T DT_t(\tau_0) + u_t, \quad t = 1, \dots, T \quad (1)$$

$$u_t = \rho_T u_{t-1} + \varepsilon_t, \quad t = 2, \dots, T \quad (2)$$

where $DT_t(\tau) := 1(t > \lfloor \tau T \rfloor)(t - \lfloor \tau T \rfloor)$. In this model τ_0 is the (unknown) putative trend break fraction, with γ_T the associated break magnitude parameter; a trend break therefore occurs in $\{y_t\}$ at time $\lfloor \tau_0 T \rfloor$ when $\gamma_T \neq 0$. The break fraction is assumed to be such that $\tau_0 \in \Lambda$ where Λ is a closed subset of $(0, 1)$. It would also be possible to consider a second model which allows for a simultaneous break in the level of the process at time $\lfloor \tau_0 T \rfloor$ in the model in (1)-(2). However, as argued by Perron and Rodríguez (2003, pp.2,4), we need not analyze this case separately because a change in intercept is an example of a slowly evolving deterministic component (see Condition B of Elliott *et al.*, 1996, p.816) and, consequently, does not alter any of the large sample results presented in this paper.¹

In (2), $\{u_t\}$ is an unobserved mean zero stochastic process, initialized such that $u_1 = o_p(T^{1/2})$. The disturbance term, ε_t , is taken to satisfy the following conventional stable and invertible linear process-type assumption:

Assumption 1 *Let $\varepsilon_t = C(L)\nu_t$, $C(L) := \sum_{i=0}^{\infty} C_i L^i$, $C_0 := 1$, with $C(z) \neq 0$ for all $|z| \leq 1$ and $\sum_{i=0}^{\infty} i|C_i| < \infty$, and where ν_t is an independent and identically distributed (IID) sequence with mean zero, variance σ^2 and finite fourth moment. We also define the short-run and long-run variances of ε_t as $\sigma_\varepsilon^2 := E(\varepsilon_t^2)$ and $\omega_\varepsilon^2 := \lim_{T \rightarrow \infty} T^{-1}E(\sum_{t=1}^T \varepsilon_t)^2 = \sigma^2 C(1)^2$, respectively.*

Our interest in this paper centres on testing the unit root null hypothesis $H_0 : \rho_T = 1$, against the local alternative, $H_c : \rho_T = 1 - c/T$, $c > 0$. In order to appropriately model the case where uncertainty exists as to the presence of a break, we assume that the trend break magnitude is local-to-zero, i.e. $\gamma_T = \kappa \omega_\varepsilon T^{-1/2}$ where κ is a finite constant, thereby adopting the appropriate Pitman drift for a trend break in a local-to-unit root process.² Given a degree of prior information concerning the location of the putative break, we assume that $\tau_0 \in \Lambda(\tau_m, \delta)$, where $\Lambda(\tau_m, \delta) := [\tau_m - \delta/2, \tau_m + \delta/2]$; here, $\delta > 0$ defines the width of the window containing all permissible break fractions and τ_m denotes the window mid-point. For cases where $\tau_m - \delta/2 < 0$ or $\tau_m + \delta/2 > 1$, we use the truncated windows $[\epsilon, \tau_m + \delta/2]$, or $[\tau_m - \delta/2, 1 - \epsilon]$, respectively, for some small $\epsilon > 0$.

For an arbitrary break fraction τ , let $DF^{GLS}(\tau)$ denote the PR unit root test, that is, the standard t -ratio associated with $\hat{\pi}$ in the fitted ADF-type regression

$$\Delta \tilde{u}_t = \hat{\pi} \tilde{u}_{t-1} + \sum_{j=1}^k \hat{\psi}_j \Delta \tilde{u}_{t-j} + \hat{e}_{t,k}, \quad t = k+2, \dots, T \quad (3)$$

with $\tilde{u}_t := y_t - \tilde{\mu} - \tilde{\beta}t - \tilde{\gamma}DT_t(\tau)$, where $[\tilde{\mu}, \tilde{\beta}, \tilde{\gamma}]'$ is obtained from a local GLS regression of $\mathbf{y}_{\bar{\rho}} := [y_1, y_2 - \bar{\rho}y_1, \dots, y_T - \bar{\rho}y_{T-1}]'$ on $\mathbf{Z}_{\bar{\rho}, \tau} := [\mathbf{z}_1, \mathbf{z}_2 - \bar{\rho}\mathbf{z}_1, \dots, \mathbf{z}_T - \bar{\rho}\mathbf{z}_{T-1}]'$, $\mathbf{z}_t := [1, t, DT_t(\tau)]'$ with $\bar{\rho} := 1 - \bar{c}/T$. Following HLT, we set $\bar{c} = 17.6$.³ The without-break version of this test, which we denote by DF^{GLS} ,

¹It should, however, be noted that a change in level can have a significant impact in finite samples; see Rodríguez (2007). For that reason a simultaneous level shift dummy, $DU_t(\tau) := 1(t > \lfloor \tau T \rfloor)$, might also be added to the deterministic vector \mathbf{z}_t in the with-break case in what follows without altering the stated large sample properties of the resulting tests.

²Scaling the trend break by ω_ε is merely a convenience device allowing it to be factored out of the limit distributions that arise later.

³We suppress the dependence of quantities such as \tilde{u}_t on τ for notational economy.

is computed in exactly the same way as $DF^{GLS}(\tau)$, but with $\mathbf{z}_t := [1, t]'$ and using $\bar{c} = 13.5$ as in Elliott *et al.* (1996). It is assumed that the lag truncation parameter, k , is chosen according to an appropriate model selection procedure, such as the modified Akaike information criterion (MAIC) procedure of Ng and Perron (2001) and Perron and Qu (2007), starting from a maximum lag truncation, k_{\max} , which satisfies the usual condition that $1/k_{\max} + k_{\max}^3/T \rightarrow 0$ as $T \rightarrow \infty$.

For given choices of the break window parameters τ_m and δ , the infimum GLS detrended Dickey-Fuller statistic that we consider is

$$MDF(\tau_m, \delta) := \inf_{\tau \in \Lambda(\tau_m, \delta)} DF^{GLS}(\tau)$$

Clearly, the test recommended by HLT based on an unrestricted search set for τ , but using 15% trimming (denoted by MDF_1 in their paper) is a special case of the above, equivalent to $MDF(0.5, 0.7)$.

Our union of rejections strategy is then given by the following decision rule:

$$U(\tau_m, \delta) := \text{Reject } H_0 \text{ if } \{DF^{GLS} < \lambda cv_{DF} \text{ or } MDF(\tau_m, \delta) < \lambda cv_{MDF}\}$$

where cv_{DF} denotes the asymptotic null critical value associated with DF^{GLS} , and cv_{MDF} denotes the asymptotic critical value for $MDF(\tau_m, \delta)$ obtained under the null with no break in trend ($\gamma = 0$). The parameter $\lambda (> 1)$ is a scaling constant applied to both cv_{DF} and cv_{MDF} , chosen so that the asymptotic size of $U(\tau_m, \delta)$ is correctly controlled when $\gamma = 0$; the value of λ will of course depend on the significance level at which DF^{GLS} and $MDF(\tau_m, \delta)$ are conducted, as well as the window parameters τ_m and δ . Observe that $U(\tau_m, \delta)$ can alternatively be expressed as

$$U(\tau_m, \delta) := \text{Reject } H_0 \text{ if } \left\{ DF_U^{GLS}(\tau_m, \delta) := \min \left(DF^{GLS}, \frac{cv_{DF}}{cv_{MDF}} MDF(\tau_m, \delta) \right) < \lambda cv_{DF} \right\} \quad (4)$$

a form that proves useful for determining λ , as outlined in the next section.

3 Asymptotic Behaviour

In this section, we begin by stating the large sample properties of DF^{GLS} and $MDF(\tau_m, \delta)$, the proof of which follows directly from Harvey *et al.* (2012b) and HLT, for parts (i) and (ii) respectively.

Theorem 1 *Let y_t be generated according to (1) and (2) under Assumption 1. Let $H_c : \rho_T = 1 - c/T$, $c \geq 0$ hold, and let $\gamma_T = \kappa \omega_\varepsilon T^{-1/2}$. Then, for any τ_0 ,*

$$(i) \quad DF^{GLS} \Rightarrow \frac{K_{c, \bar{c}}(1, \tau_0, \kappa)^2 - 1}{2 \sqrt{\int_0^1 K_{c, \bar{c}}(r, \tau_0, \kappa)^2 dr}} =: \mathcal{D}_{c, \bar{c}}^{DF}(\tau_0, \kappa)$$

where

$$K_{c, \bar{c}}(r, \tau_0, \kappa) := W_c(r) + \kappa(r - \tau_0) \mathbb{I}_{\tau_0}^r - \{b_{c, \bar{c}} + \kappa f_{c, \bar{c}}(\tau_0)\} r / a_{\bar{c}}$$

with

$$\begin{aligned} b_{c,\bar{c}} &:= (1 + \bar{c})W_c(1) + \bar{c}^2 \int_0^1 sW_c(s)dr \\ f_{c,\bar{c}}(\tau_0) &:= (1 - \tau_0) \{a_{\bar{c}} - \bar{c}^2\tau_0(1 + \tau_0)/6\} \\ a_{\bar{c}} &:= 1 + \bar{c} + \bar{c}^2/3 \end{aligned}$$

and $W_c(r) := \int_0^r e^{-(r-s)c}dW(s)$, $W(r)$ a standard Wiener process.

(ii)

$$MDF(\tau_m, \delta) \Rightarrow \inf_{\tau \in \Lambda(\tau_m, \delta)} \frac{L_{c,\bar{c}}(1, \tau_0, \tau, \kappa)^2 - 1}{2\sqrt{\int_0^1 L_{c,\bar{c}}(r, \tau_0, \tau_0, \kappa)^2 dr}} =: \mathcal{D}_{c,\bar{c}}^{MDF}(\tau_m, \delta, \tau_0, \kappa) \quad (5)$$

where

$$\begin{aligned} L_{c,\bar{c}}(r, \tau_0, \tau, \kappa) &:= W_c(r) + \kappa(r - \tau_0)\mathbb{I}_{\tau_0}^r \\ &\quad - \begin{bmatrix} r \\ (r - \tau)\mathbb{I}_{\tau}^r \end{bmatrix}' \begin{bmatrix} a_{\bar{c}} & m_{\bar{c}}(\tau) \\ m_{\bar{c}}(\tau) & d_{\bar{c}}(\tau) \end{bmatrix}^{-1} \begin{bmatrix} b_{c,\bar{c}} + \kappa f_{c,\bar{c}}(\tau_0) \\ b_{c,\bar{c}}(\tau) + \kappa f_{c,\bar{c}}(\tau_0, \tau) \end{bmatrix} \end{aligned}$$

with

$$\begin{aligned} m_{\bar{c}}(\tau) &:= a_{\bar{c}} - \tau(1 + \bar{c} + \bar{c}/2 - \bar{c}\tau^2/6) \\ d_{\bar{c}}(\tau) &:= a_{\bar{c}} - \tau(1 + 2\bar{c} - \bar{c}\tau + \bar{c} - \bar{c}\tau + \bar{c}\tau^2/3) \\ b_{c,\bar{c}}(\tau) &:= (1 + \bar{c} - \bar{c}\tau)W_c(1) - W_c(\tau) + \bar{c} \int_{\tau}^1 (s - \tau)W_c(s)ds \\ f_{c,\bar{c}}(\tau_0, \tau) &:= (1 - \tau_0)\{a_{\bar{c}} - \bar{c}\tau - \bar{c}\tau(1 - \tau_0)/2 - \bar{c}\tau_0(1 + \tau_0)/6\} \\ &\quad - (\tau - \tau_0)\{1 - \bar{c}(\tau - \tau_0)^2/6\}\mathbb{I}_{\tau_0}^r \end{aligned}$$

Given the results of Theorem 1, it is straightforward to establish the asymptotic behaviour of the union of rejections statistic DF_U^{GLS} in (4). Under the conditions of Theorem 1, application of the continuous mapping theorem gives

$$DF_U^{GLS}(\tau_m, \delta) \Rightarrow \min \left(\mathcal{D}_{c,\bar{c}}^{DF}(\tau_0, \kappa), \frac{cv_{DF}}{cv_{MDF}} \mathcal{D}_{c,\bar{c}}^{MDF}(\tau_m, \delta, \tau_0, \kappa) \right) \quad (6)$$

Remark 1 Notice that when $\kappa = 0$, the limit distributions $\mathcal{D}_{c,\bar{c}}^{DF}(\tau_0, \kappa)$ and $\mathcal{D}_{c,\bar{c}}^{MDF}(\tau_m, \delta, \tau_0, \kappa)$, and therefore the limit of $DF_U^{GLS}(\tau_m, \delta)$, do not depend on τ_0 , since no break occurs.

We next obtain asymptotic null critical values for $MDF(\tau_m, \delta)$ (the critical values for DF^{GLS} are well known, e.g. -2.85 at the nominal 0.05 significance level) for a range of window locations τ_m and the window width settings $\delta = \{0.05, 0.10, 0.15, 0.20\}$.⁴ These are obtained for the case of no break in trend (an investigation into the size properties of the tests for non-zero breaks is considered in the next sub-section), and so we simulate (5) with $c = 0$ and $\kappa = 0$. Table 1 reports critical values for the nominal 0.10, 0.05 and 0.01 significance levels; here and throughout the paper, we used 50,000

⁴Here we set $\epsilon = 0.001$.

Monte Carlo replications, and approximated the Wiener processes in the limiting functionals using $NIID(0,1)$ random variates, with the integrals approximated by normalized sums of 1,000 steps.

Given choices of τ_m and δ , together with the corresponding critical values cv_{DF} and cv_{MDF} , the appropriate constant λ to be used in (4) can be determined, so as to ensure $U(\tau_m, \delta)$ has the correct asymptotic size when no break occurs. These values can be obtained by simulating the limit distribution of DF_U^{GLS} in (6), calculating the asymptotic critical value for this empirical distribution at the desired significance level, say cv_U , and then computing $\lambda := cv_U/cv_{DF}$. We obtained constants in this way at the 0.10, 0.05 and 0.01 nominal significance levels, and the results are presented in Table 2, for the same combinations of τ_m and δ as considered in Table 1.

Asymptotic Size

We now consider the asymptotic sizes of $MDF(\tau_m, \delta)$ and $U(\tau_m, \delta)$ under a local break in trend, for the four window widths $\delta = \{0.05, 0.10, 0.15, 0.20\}$, and setting $\tau_m = \tau_0$, so we assume the true break fraction coincides with the mid-point of the window. In Figure 1 we show the sizes, as functions of the local trend break magnitude $\kappa = \{0, 0.2, \dots, 15\}$, for break fractions $\tau_0 = \{0.3, 0.5, 0.7\}$ at the nominal 0.05 level; for purposes of comparison, we also show the asymptotic sizes of DF^{GLS} and the MDF_1 procedure of HLT (i.e. $MDF(0.5, 0.7)$).

The general pattern of results is as follows. Firstly, DF^{GLS} displays the predictable feature of its size rapidly decaying from 0.05 towards zero as κ increases, since a local trend break of growing magnitude is being omitted from its underlying deterministic specification. The $MDF(\tau_m, \delta)$ statistics have sizes which demonstrate quite different behaviour; their sizes increase from 0.05 as κ increases, before levelling off then decreasing slightly. The maximum size is reached more rapidly in κ the larger is δ ; but the maximum attained is higher the smaller is δ .⁵ The corresponding $U(\tau_m, \delta)$ testing strategies demonstrate sizes that are fairly close to 0.05 across all κ - they are all slightly undersized for $\kappa > 0$, though with less undersizing the smaller is δ . What is happening here is that, as κ is increasing, any over-sizing inherent in $MDF(\tau_m, \delta)$ is being counteracted by the diminishing size of DF^{GLS} .⁶ The sizes of the $U(\tau_m, \delta)$ strategies are pretty much in line with those of the HLT test MDF_1 when $\tau_0 = 0.3$ and $\tau_0 = 0.5$, but are less undersized when $\tau_0 = 0.7$.

By way of a more comprehensive check on the asymptotic size properties of the $U(\tau_m, \delta)$ strategies, we simulated the limit distribution of DF_U^{GLS} in (6) using the λ values of Table 2, that, is those calculated for $\kappa = 0$, across a grid of values $\kappa = \{0, 1, 2, \dots, 15\}$ and $\tau_0 = \{0.025, 0.050, \dots, 0.975\}$ for every case considered in Table 2 (i.e. each τ_m , δ and significance level combination). This allows us to examine size in cases where (i) $\tau_m = \tau_0$, (ii) $\tau_0 \in \Lambda(\tau_m, \delta)$, $\tau_m \neq \tau_0$ such that the partial information on break location remains true but the break no longer occurs at the mid-point of the window, and (iii) $\tau_0 \notin \Lambda(\tau_m, \delta)$ so that the partial information is entirely wrong. The pertinent finding of this analysis is that, for any given significance level, the maximum asymptotic size obtained across all

⁵For $\delta = 0.05$, the maximum size is not obtained until $\kappa = 30$ at which point the size is 0.086.

⁶Note that although the size of U for $\delta = 0.05$ is still rising slightly when $\kappa = 15$, its size reaches a ceiling of 0.049 when $\kappa = 30$.

τ_m, δ combinations considered was always within 0.0009 of the nominal level, thereby confirming the suitability of the λ values of Table 2 to deliver what is, to all intents and purposes, a size-controlled procedure.⁷

Asymptotic Power

Figure 2 plots the asymptotic local power functions of nominal 0.05 level tests across $\kappa = \{0, 0.2, \dots, 15\}$, with $\tau_m = \tau_0 = \{0.3, 0.5, 0.7\}$, for the two alternatives $c = 20$ and $c = 25$. We do not report plots for the $MDF(\tau_m, \delta)$ statistics alone (other than MDF_1), since these are oversized. The results may be summarized as follows. The power of DF^{GLS} starts at a high level, but then decays towards zero as κ increases, due to the increasing magnitude of the unattended break. As regards the $U(\tau_m, \delta)$ testing strategies, for all four δ settings considered we observe that they are all able to harness most of the high power available from DF^{GLS} for zero or small values of κ , as well as power arising from $MDF(\tau_m, \delta)$. Then, as κ further increases, their power drops down to a lower level since the power contribution of DF^{GLS} is declining towards zero and all their power is obtained from the $MDF(\tau_m, \delta)$ statistics alone. Outside of small values of κ , that is, in the region where $MDF(\tau_m, \delta)$ and not DF^{GLS} dominates the power profile, it becomes quite clear that the smaller is δ , the higher is the power which is achieved.

The non-union test MDF_1 cannot access any of the high power associated with DF^{GLS} for zero or very small values of κ . For the larger values of κ , when $\tau_0 = 0.3$ and $\tau_0 = 0.5$, the $U(\tau_m, \delta)$ strategies are more powerful than MDF_1 apart from when $\delta = 0.20$, where the power is marginally lower. When $\tau_0 = 0.7$, all the $U(\tau_m, \delta)$ strategies have substantially higher power than MDF_1 , for large (as well as small) κ .

We therefore see that the more accurate the true partial information available regarding the location of the trend break (i.e. the smaller is δ), the higher the power that can be achieved with $U(\tau_m, \delta)$. As we might expect, the level of information accuracy has little effect on power when the local break does not occur, or is small, but allows for very decent power gains otherwise.

Asymptotic Behaviour when $\tau_m \neq \tau_0$

Finally, we jointly investigate the size and power properties of $U(\tau_m, \delta)$ when $\tau_m \neq \tau_0$. Here we set $\tau_0 = 0.57$ and $\tau_m = 0.50$ such that $\delta = 0.15$ and $\delta = 0.20$ correspond to case (ii) discussed in the first sub-section above, with the break being "least central" for $\delta = 0.15$; and such that $\delta = 0.05$ and $\delta = 0.10$ correspond to case (iii), with the partial information being "most in error" for $\delta = 0.05$.

Asymptotic sizes for $U(\tau_m, \delta)$ are given in Figure 3(a). The sizes for $U(\tau_m, \delta)$ with $\delta = \{0.15, 0.20\}$ across κ are always very close to their counterparts for $\tau_m = \tau_0 = 0.5$ in Figure 1, indicating that the issue of whether or not the break occurs towards the window midpoint has little effect on size. For $\delta = \{0.05, 0.10\}$, the sizes approach zero in κ , and this occurs particularly quickly for $\delta = 0.05$, where the partial information is the more incorrect. In Figure 3(b) we present the corresponding powers for

⁷The full results of these simulation experiments are available from the authors on request.

$c = 20$. Once more we see that the power profiles for $U(\tau_m, \delta)$ with $\delta = \{0.15, 0.20\}$ lie very close to their counterparts for $\tau_m = \tau_0 = 0.5$ in Figure 2, while the powers for $\delta = \{0.05, 0.10\}$ are decreasing in κ , reaching zero for $\delta = 0.05$.

From this analysis we make two observations. First, that the asymptotic size and power are little affected by the placement of a break within a given window, which endows the $U(\tau_m, \delta)$ strategy with a degree of robustness. Secondly, and not surprisingly, that size and power are both driven towards zero by false partial information (more quickly when the more in error is that information). While this may be considered a negative, we can at least rest assured that our strategy is very unlikely to yield spurious rejections of the unit root null when one is present but the trend break exists outside of the chosen window.

4 Finite Sample Results

In this section we present finite sample simulation results for the empirical size and power of the $U(\tau_m, \delta)$ tests. We set $T = 200$ and simulate the DGP given by (1) and (2) with $\mu = \beta = 0$ (without loss of generality), $\varepsilon_t \sim IIDN(0, 1)$ and $u_1 = \varepsilon_1$. We use the same local trend break magnitudes as in the previous section, and the same choices for τ_0 . As in the first two sub-sections of section 3 we set $\tau_m = \tau_0$ and consider the four window widths $\delta = \{0.05, 0.10, 0.15, 0.20\}$. The Dickey-Fuller regressions are implemented with k set to zero in (3), thereby abstracting from issues of lag selection. Figures 4 and 5 report, respectively, the size ($c = 0$) and local power ($c = 20, 25$) of the $U(\tau_m, \delta)$ testing strategies, with the tests conducted using nominal 0.05 level asymptotic critical values and the corresponding union of rejections scaling constants.

We find that the reliable asymptotic size performance of the $U(\tau_m, \delta)$ strategies is largely replicated in the finite sample situation. The relative size profiles across κ for the four settings of δ follow the same pattern as was observed for the limit case in Figure 1. The sizes for $T = 200$ are a little above the nominal level for some values of κ , but such oversizing is very moderate, never being in excess of 0.066.

Turning now to the finite sample power results of Figure 5, we again observe the same relative power rankings among the four union of rejections strategies as in the limit, i.e. very similar power for all values of δ when the local break magnitude is zero or very small, but then for larger κ , power increases as the window width narrows. The level of finite sample power for each strategy is a little higher than the corresponding local asymptotic power level, as expected given the moderate degree of finite sample oversize observed in Figure 4. Overall then we find that the attractive asymptotic properties of the $U(\tau_m, \delta)$ testing strategies carry over to the finite sample context, reinforcing the case for use of the union of rejections approach in practical applications.

5 An Empirical Illustration

By way of an illustration of the how the $U(\tau_m, \delta)$ strategies might work in practice, we applied them to UK interest rate data. The interest rate series we consider is the natural log of the monthly gross flat yield on UK government 2.5% Consols for the period January 1954 to November 1994.⁸ The series, comprising 491 observations, is shown in Figure 6. Observation 238, located near the middle of the series (and also indicated in Figure 6), represents October 1973, the month in which the Middle East Oil Crisis began. Few would argue that the period around the Oil Crisis should not be considered a serious candidate for the location of a potential trend break in the long-term behaviour of economic and financial markets. Arguably, it was the most seismic economic event of the twentieth century, aside from the Great Depression.

In what follows all ADF statistics are implemented with the MAIC lag selection procedure of Ng and Perron (2001) and Perron and Qu (2007), using $k_{\max} = \lfloor 12(T/100)^{1/4} \rfloor = 17$. All test procedures are carried out at the nominal 0.05 level, using asymptotic critical values and scaling constants.

As a benchmark, we took the view that a trend break did occur in October 1973. That is, we assumed complete knowledge of the break location. The appropriate unit root statistic is then $DF^{GLS}(\tau)$ with $\tau = 238/491 = 0.485$. We found that this statistic does not yield a rejection of the unit root null. Moving to the other extreme, we eschewed all relevant information on break location around the occurrence of the Oil Crisis and calculated the MDF_1 test of HLT, i.e. $MDF(0.5, 0.7)$. This test did not reject the unit root null either.⁹

To examine whether taking a partial information position might uncover evidence of stationarity around a broken trend, we then calculated $U(\tau_m, \delta)$ for $\delta = \{0.05, 0.10, 0.15, 0.20\}$. Intuition led us to set $\tau_m = 0.485$ such that each break window midpoint is fixed at October 1973, thereby incorporating the effects of uncertainty caused by both anticipation (before) and delayed reactions (after) this date. However, we also calculated $U(\tau_m, \delta)$ setting $\tau_m - \delta/2 = 0.485$, so that October 1973 is now always the earliest date in the window (the midpoint now varies) to allow for effects of delayed reactions alone (in keeping with the the Oil Crisis being termed a “shock” in the vernacular).

The results are shown in Table 3, where NR and R denote non-rejection and rejection, respectively, of the unit root null at the 0.05 level. In analyzing these results, it should be borne in mind that the DF^{GLS} component of each $U(\tau_m, \delta)$, i.e. the without break unit root test, is never responsible for triggering a rejection; its value is -0.90 , compared with a critical value of -2.85 , even before any λ scaling is applied. Therefore, the pattern of rejections in Table 3 is solely within the purview of the corresponding with-break $MDF(\tau_m, \delta)$ component of $U(\tau_m, \delta)$.

If we set $\tau_m = 0.485$ then only the $\delta = 0.15$ window yields a rejection. It seems reasonable to argue, informally at least, that this arises because the $\delta = \{0.05, 0.10\}$ windows are excluding a true break point which is contained in the $\delta = 0.15$ window. Of course, this break point is also included

⁸This same data set was examined by Leybourne *et al.* (1998) in the context of unit root tests allowing for smooth transition trend deterministics.

⁹Non-rejection was also found using a Zivot-Andrews-type OLS detrended variant of $MDF(0.5, 0.7)$.

in the $\delta = 0.20$ window, but the price here is a more left-shifted critical value, which overturns the rejection. Once we set $\tau_m - \delta/2 = 0.485$ we no longer consider pre-October 1973 dates as candidate break points for any window. The $\delta = 0.15$ window still produces a rejection but is now joined by rejections for the $\delta = \{0.05, 0.10\}$ windows, which leads us to presume that both of these now include the trend break point. Interestingly, the $\delta = 0.20$ window also now yields a rejection. Essentially, this is because the rightward shift of the window midpoint (relative to $\tau_m = 0.485$) means the critical value is less left-shifted.¹⁰

Our findings lead us to conclude that this interest rate series is stationary, around a trend break, though it would seem that the break point appears much less likely to have occurred before the onset of the Oil Crisis than at some stage shortly thereafter. It therefore seems doubtful that the Oil Crisis was anticipated by the bond market to any appreciable extent. In fact, our results with window width $\delta = 0.05$ suggest that the trend break occurred between observations 251 and 262; that is, between November 1974 and October 1975.¹¹ Visual inspection of Figure 6 would not appear to contradict this suggestion.

6 Conclusions

In this paper we have proposed a union of rejections-based approach to testing for a unit root when partial information is available regarding the location, but not necessarily the presence, of a break in linear trend. The union of rejections approach, comprised as it is of both a with-break and without-break unit root test, allows the capture of most of the high power available when no break is actually present, while also ensuring a reliable level of power should a break occur. Our recommended approach relies on the user specifying a window within which the putative trend break must lie, and the narrower this window width is, the greater is the level of power achievable for non-zero break magnitudes. Our results demonstrate that this new approach can outperform existing approaches and delivers a reliable procedure for testing for a unit root when such partial information is available. Of course, the decision regarding the window width and location lies with the practitioner, and should reflect their degree of belief regarding the approximate date of any trend break that might occur. While the obvious caveat exists that a mis-specified window choice could result in low power, we think that this new procedure has attractive properties, and should prove desirable to users. Our procedure can be seen as bridging the gap between an often unrealistic assumption that the putative trend break date is known with complete certainty, and the highly conservative assumption of no knowledge whatsoever regarding the break location; both extremes being rather unappealing for many events likely to generate trend breaks in economic or financial data.

¹⁰This window midpoint is actually now $\tau_m = 0.585$ instead of 0.485. Table 1 shows that the critical value of $MDF(\tau_m, \delta)$ changes to -3.59 from -3.62 . In addition, from Table 2, the λ scaling reduces to 1.062 from 1.066.

¹¹This interval is calculated by noting that with $\delta = 0.05$, $MDF(\tau_m, \delta)$ involves potential break points between observations 225 and 250 when $\tau_m = 0.485$, and observations 238 to 262 when $\tau_m - \delta/2 = 0.485$.

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TABLE 1
Asymptotic ξ level critical values for MDF(τ_m, δ) tests

τ_m	$\xi = 0.10$					$\xi = 0.05$					$\xi = 0.01$				
	$\delta = 0.05$	$\delta = 0.10$	$\delta = 0.15$	$\delta = 0.20$	$\delta = 0.20$	$\delta = 0.05$	$\delta = 0.10$	$\delta = 0.15$	$\delta = 0.20$	$\delta = 0.20$	$\delta = 0.05$	$\delta = 0.10$	$\delta = 0.15$	$\delta = 0.20$	$\delta = 0.20$
	0.025	-2.96	-3.01	-3.06	-3.10	-3.10	-3.26	-3.31	-3.36	-3.40	-3.40	-3.81	-3.88	-3.93	-3.97
0.050	-3.01	-3.06	-3.10	-3.14	-3.14	-3.31	-3.36	-3.40	-3.43	-3.43	-3.88	-3.93	-3.97	-4.00	-4.00
0.100	-3.09	-3.13	-3.17	-3.20	-3.20	-3.38	-3.43	-3.47	-3.50	-3.50	-3.95	-3.99	-4.03	-4.07	-4.07
0.200	-3.18	-3.23	-3.27	-3.31	-3.31	-3.46	-3.51	-3.55	-3.59	-3.59	-4.02	-4.07	-4.12	-4.15	-4.15
0.300	-3.22	-3.27	-3.31	-3.35	-3.35	-3.50	-3.55	-3.59	-3.63	-3.63	-4.05	-4.10	-4.15	-4.18	-4.18
0.400	-3.21	-3.26	-3.31	-3.35	-3.35	-3.49	-3.55	-3.59	-3.63	-3.63	-4.05	-4.11	-4.16	-4.19	-4.19
0.500	-3.20	-3.26	-3.30	-3.34	-3.34	-3.49	-3.54	-3.58	-3.62	-3.62	-4.02	-4.09	-4.14	-4.17	-4.17
0.600	-3.17	-3.22	-3.26	-3.30	-3.30	-3.45	-3.50	-3.55	-3.59	-3.59	-3.99	-4.05	-4.09	-4.13	-4.13
0.700	-3.10	-3.15	-3.19	-3.23	-3.23	-3.40	-3.45	-3.49	-3.53	-3.53	-3.93	-3.99	-4.03	-4.07	-4.07
0.800	-3.02	-3.06	-3.10	-3.14	-3.14	-3.30	-3.35	-3.40	-3.44	-3.44	-3.85	-3.90	-3.94	-3.98	-3.98
0.900	-2.89	-2.93	-2.97	-3.01	-3.01	-3.17	-3.22	-3.27	-3.31	-3.31	-3.74	-3.79	-3.83	-3.86	-3.86
0.950	-2.80	-2.85	-2.89	-2.93	-2.93	-3.09	-3.14	-3.18	-3.23	-3.23	-3.66	-3.70	-3.75	-3.80	-3.80
0.975	-2.74	-2.80	-2.85	-2.89	-2.89	-3.03	-3.09	-3.14	-3.18	-3.18	-3.60	-3.66	-3.70	-3.75	-3.75

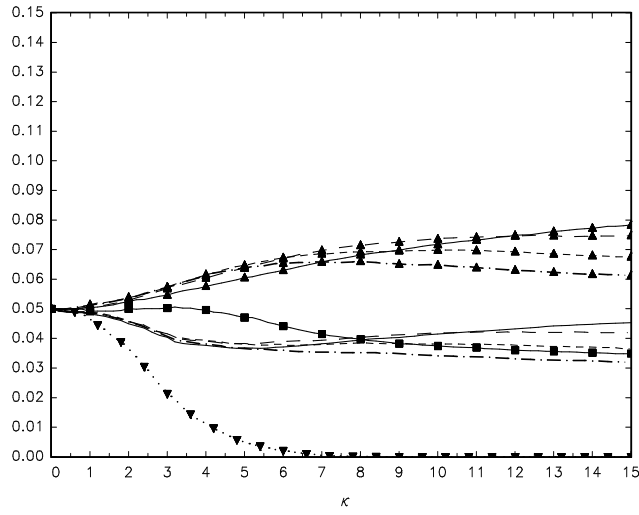
TABLE 2
Asymptotic λ scaling constants for $U(\tau_m, \delta)$ tests

τ_m	$\xi = 0.10$					$\xi = 0.05$					$\xi = 0.01$				
	$\delta = 0.05$	$\delta = 0.10$	$\delta = 0.15$	$\delta = 0.20$	$\delta = 0.20$	$\delta = 0.05$	$\delta = 0.10$	$\delta = 0.15$	$\delta = 0.20$	$\delta = 0.20$	$\delta = 0.05$	$\delta = 0.10$	$\delta = 0.15$	$\delta = 0.20$	$\delta = 0.20$
	0.025	1.055	1.060	1.063	1.066	1.066	1.050	1.052	1.056	1.058	1.058	1.037	1.038	1.040	1.041
0.050	1.060	1.063	1.066	1.069	1.069	1.052	1.056	1.058	1.059	1.059	1.038	1.040	1.041	1.041	1.043
0.100	1.066	1.069	1.070	1.071	1.071	1.057	1.058	1.060	1.062	1.062	1.041	1.044	1.044	1.044	1.045
0.200	1.070	1.072	1.074	1.075	1.075	1.062	1.063	1.064	1.064	1.064	1.045	1.045	1.047	1.047	1.048
0.300	1.070	1.074	1.075	1.076	1.076	1.062	1.063	1.064	1.065	1.065	1.046	1.047	1.046	1.045	1.045
0.400	1.072	1.074	1.075	1.076	1.076	1.062	1.064	1.065	1.065	1.065	1.044	1.044	1.043	1.044	1.044
0.500	1.071	1.073	1.074	1.076	1.076	1.062	1.065	1.065	1.066	1.066	1.045	1.044	1.044	1.045	1.045
0.600	1.070	1.072	1.073	1.074	1.074	1.060	1.061	1.062	1.062	1.062	1.045	1.044	1.046	1.046	1.046
0.700	1.068	1.070	1.073	1.073	1.073	1.058	1.059	1.060	1.061	1.061	1.042	1.044	1.046	1.046	1.046
0.800	1.060	1.063	1.066	1.068	1.068	1.052	1.056	1.057	1.057	1.057	1.037	1.041	1.042	1.043	1.043
0.900	1.048	1.052	1.054	1.056	1.056	1.042	1.044	1.047	1.050	1.050	1.030	1.032	1.034	1.037	1.037
0.950	1.037	1.043	1.047	1.051	1.051	1.034	1.038	1.041	1.044	1.044	1.024	1.026	1.028	1.032	1.032
0.975	1.030	1.037	1.042	1.047	1.047	1.026	1.034	1.037	1.041	1.041	1.021	1.025	1.026	1.028	1.028

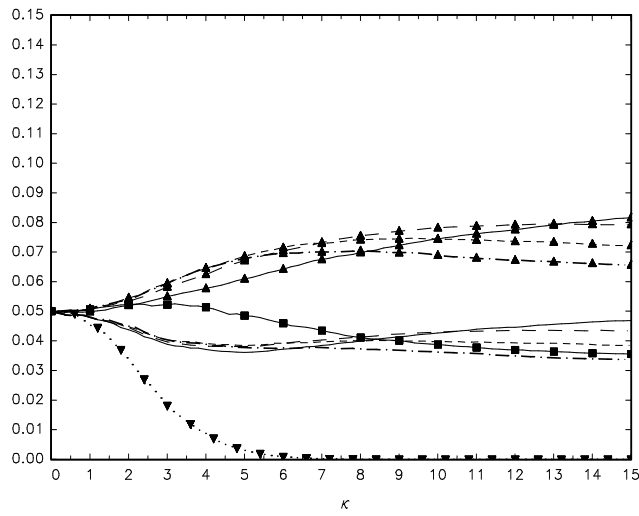
TABLE 3

Application of $U(\tau_m, \delta)$ at the nominal 0.05-level to yield on UK 2.5% Consols

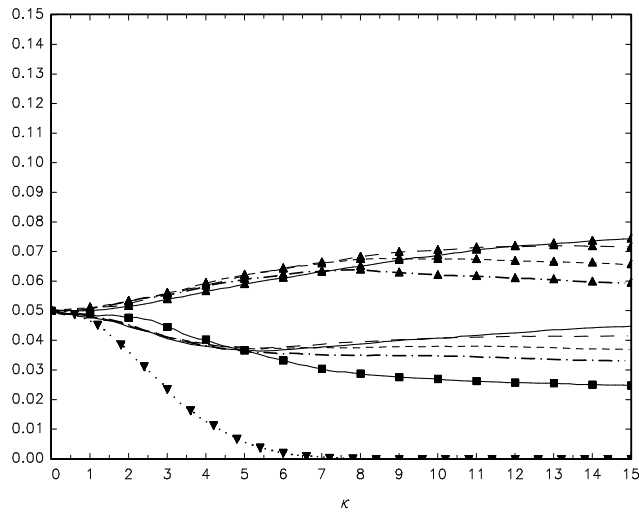
δ	$\tau_m = 0.485$	$\tau_m - \delta/2 = 0.485$
0.05	<i>NR</i>	<i>R</i>
0.10	<i>NR</i>	<i>R</i>
0.15	<i>R</i>	<i>R</i>
0.20	<i>NR</i>	<i>R</i>



(a) $\tau_0 = 0.3$

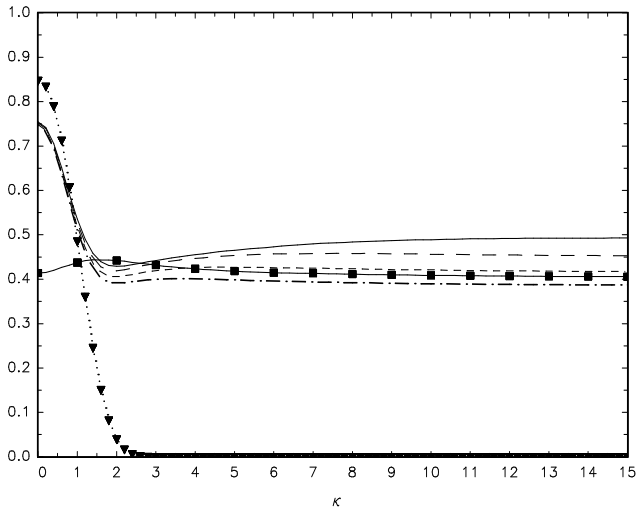


(b) $\tau_0 = 0.5$

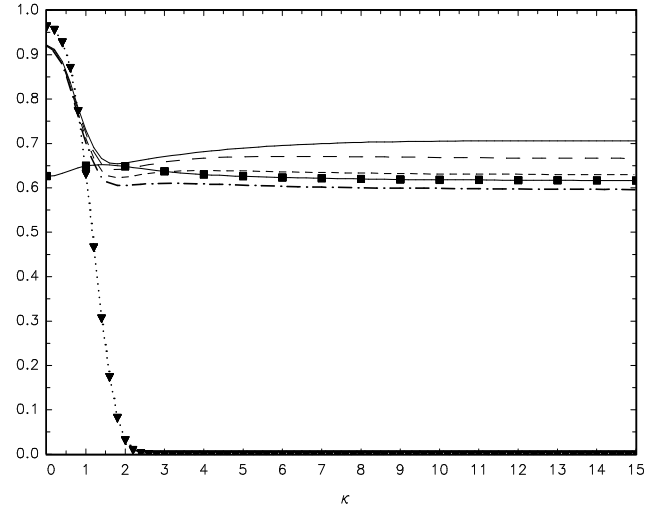


(c) $\tau_0 = 0.7$

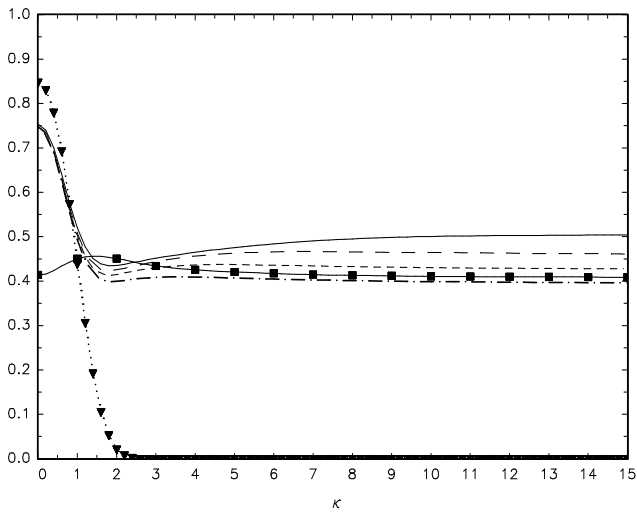
Figure 1. Asymptotic size: DF^{GLS} : $\dots\blacktriangledown\dots$, MDF_1 : \blacksquare ;
 $U(\tau_0, 0.05)$: — , $U(\tau_0, 0.10)$: -- -- , $U(\tau_0, 0.15)$: - - - , $U(\tau_0, 0.20)$: $\text{-} \cdot \text{-}$;
 $MDF(\tau_0, 0.05)$: $\text{—}\blacktriangle$, $MDF(\tau_0, 0.10)$: $\text{-}\blacktriangle$, $MDF(\tau_0, 0.15)$: $\text{- -}\blacktriangle$, $MDF(\tau_0, 0.20)$: $\text{-} \cdot \blacktriangle$



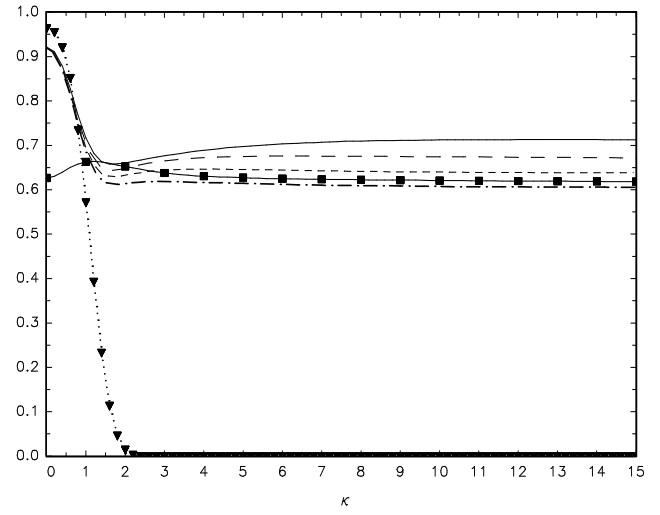
(a) $c = 20, \tau_0 = 0.3$



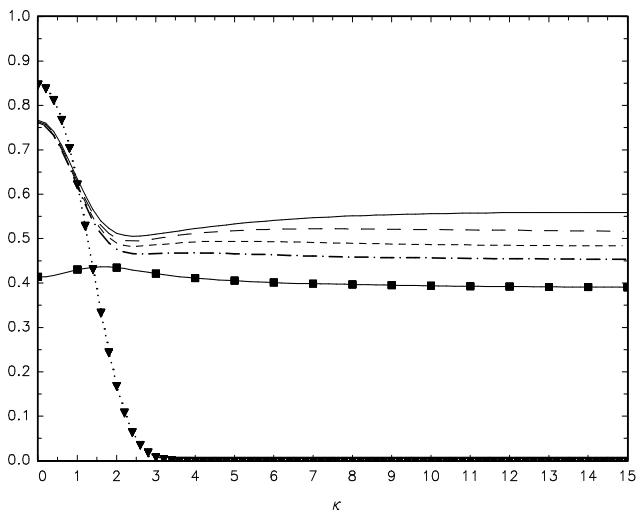
(b) $c = 25, \tau_0 = 0.3$



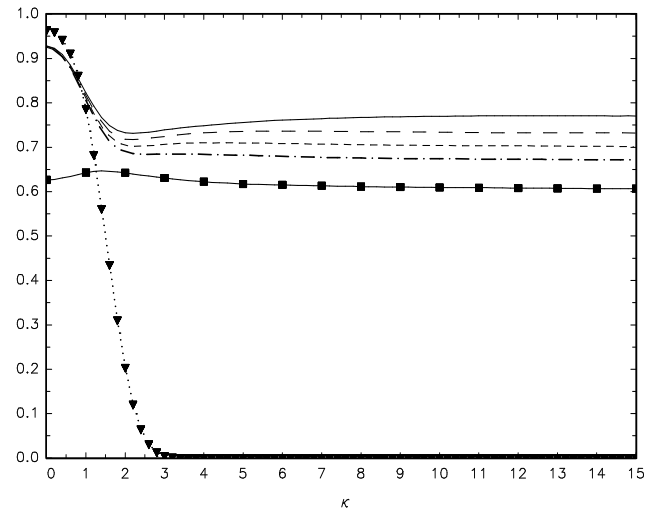
(c) $c = 20, \tau_0 = 0.5$



(d) $c = 25, \tau_0 = 0.5$

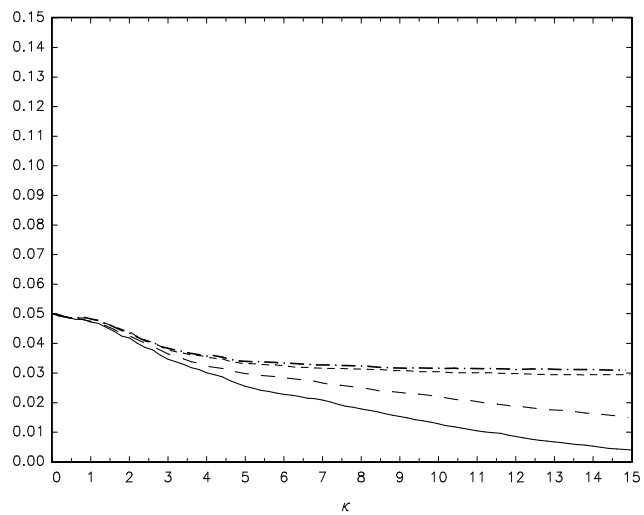


(e) $c = 20, \tau_0 = 0.7$

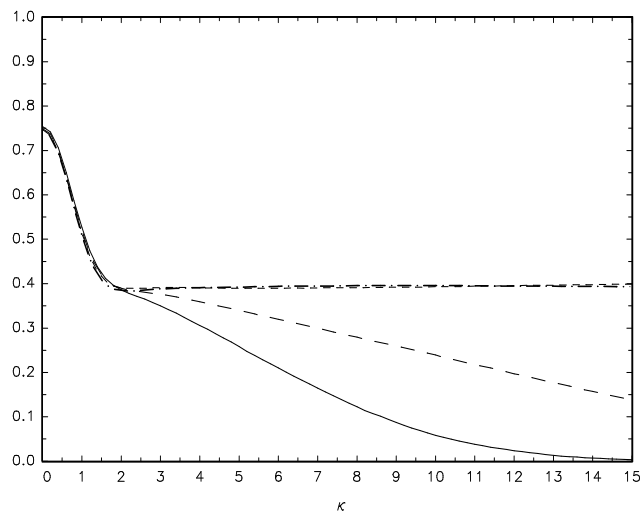


(f) $c = 25, \tau_0 = 0.7$

Figure 2. Asymptotic local power: DF^{GLS} : $\cdots\blacktriangledown\cdots$, MDF_1 : $-\blacksquare-$; $U(\tau_0, 0.05)$: $—$, $U(\tau_0, 0.10)$: $- -$, $U(\tau_0, 0.15)$: $- - -$, $U(\tau_0, 0.20)$: $- \cdot -$

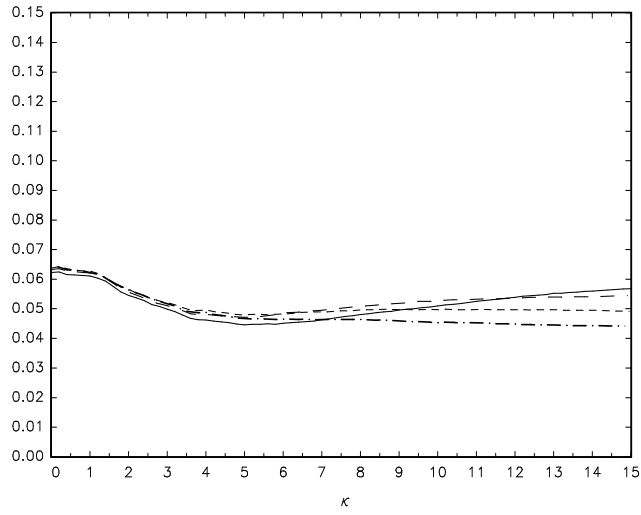


(a) $c = 0$

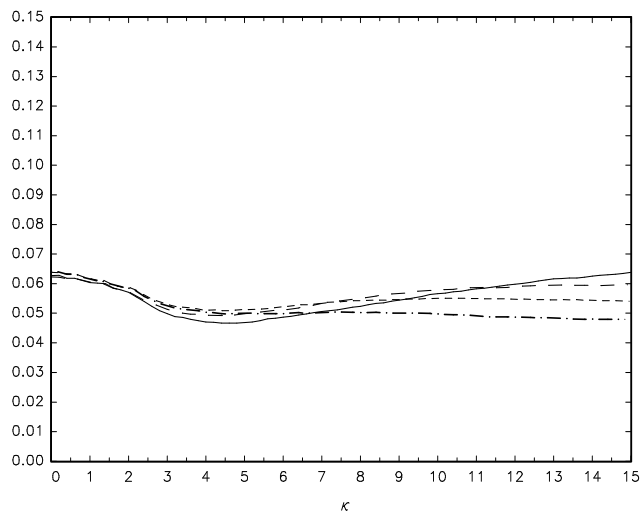


(b) $c = 20$

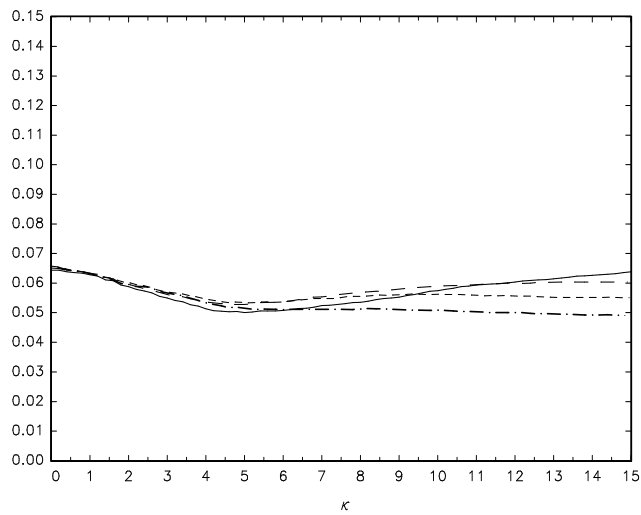
Figure 3. Asymptotic size and local power, $\tau_0 = 0.57$:
 $U(0.5, 0.05)$: —, $U(0.5, 0.10)$: --, $U(0.5, 0.15)$: - · -, $U(0.5, 0.20)$: - · · -



(a) $\tau_0 = 0.3$

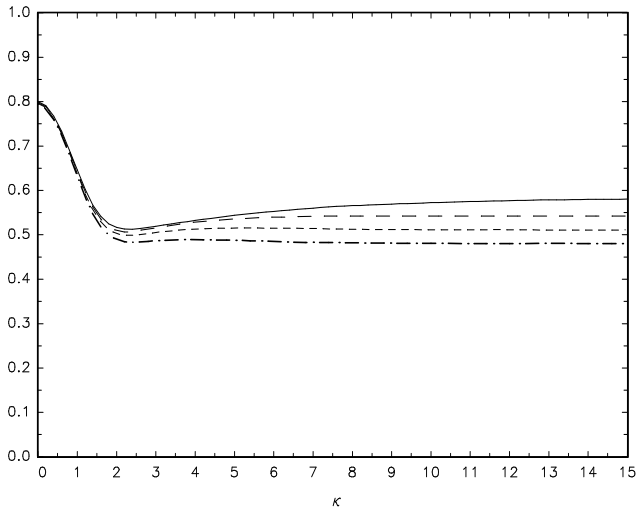


(b) $\tau_0 = 0.5$

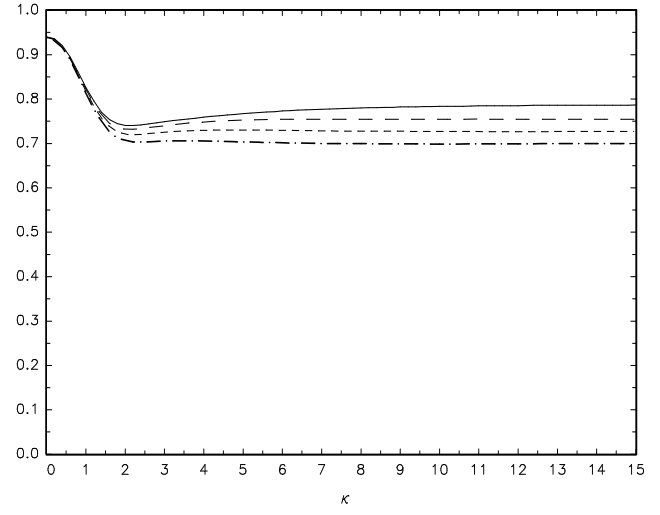


(c) $\tau_0 = 0.7$

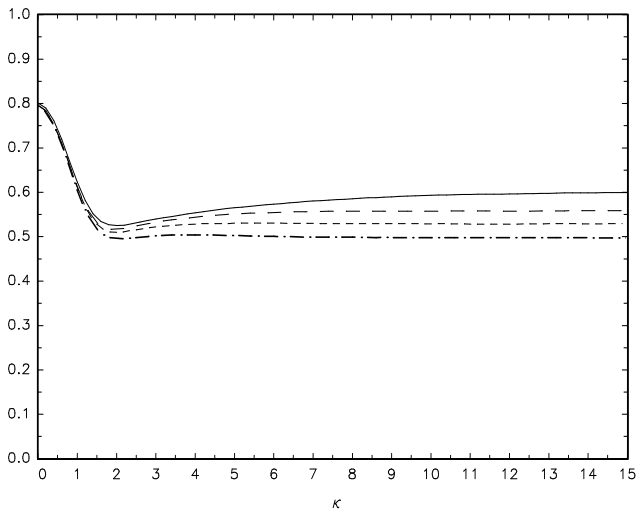
Figure 4. Finite sample size, $T = 200$, $\rho_T = 1$, $\gamma_T = \kappa T^{-1/2}$:
 $U(\tau_0, 0.05)$: —, $U(\tau_0, 0.10)$: --, $U(\tau_0, 0.15)$: - - -, $U(\tau_0, 0.20)$: - . -



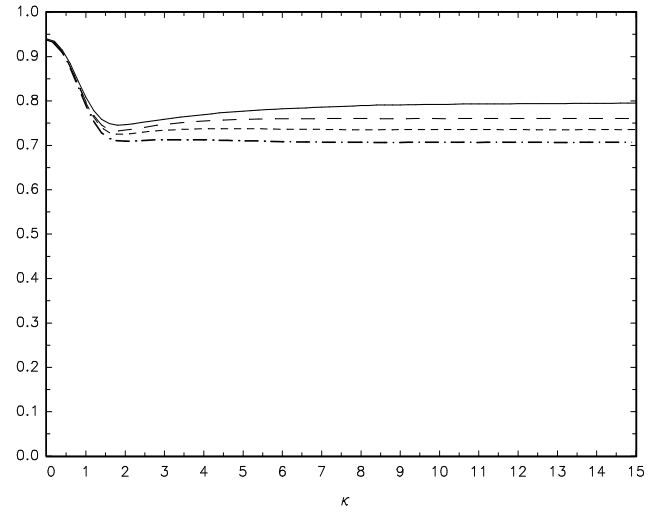
(a) $c = 20, \tau_0 = 0.3$



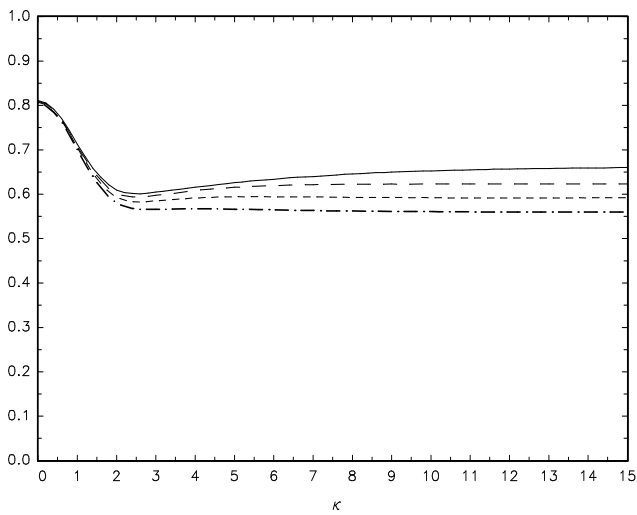
(b) $c = 25, \tau_0 = 0.3$



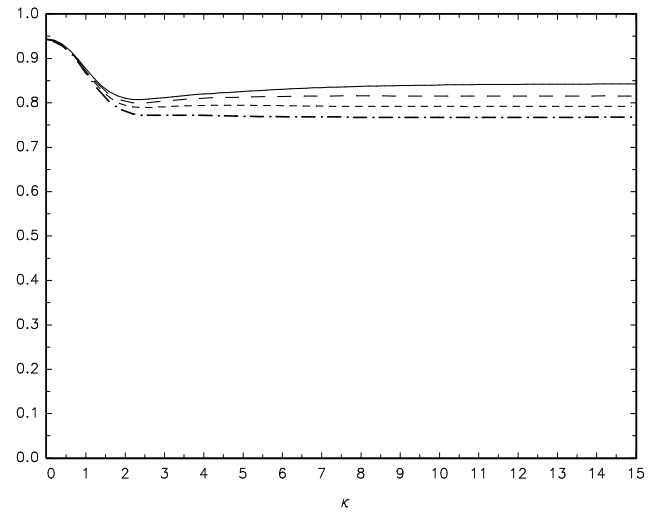
(c) $c = 20, \tau_0 = 0.5$



(d) $c = 25, \tau_0 = 0.5$



(e) $c = 20, \tau_0 = 0.7$



(f) $c = 25, \tau_0 = 0.7$

Figure 5. Finite sample power, $T = 200$, $\rho_T = 1 - c/T$, $\gamma_T = \kappa T^{-1/2}$:
 $U(\tau_0, 0.05)$: —, $U(\tau_0, 0.10)$: --, $U(\tau_0, 0.15)$: - - -, $U(\tau_0, 0.20)$: - · -

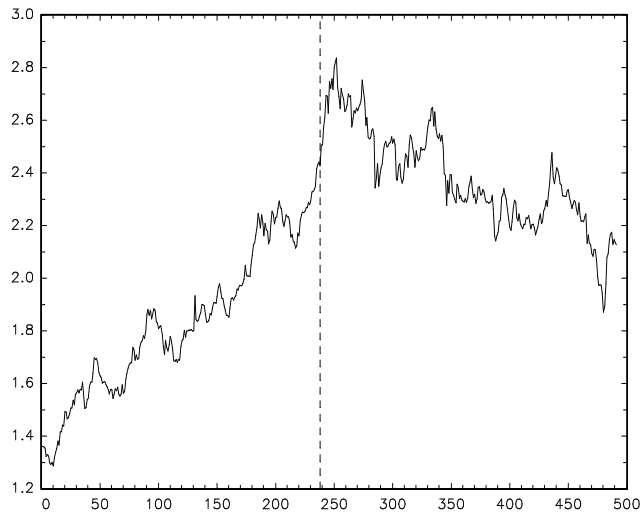


Figure 6. Yield on UK 2.5% Consols: January 1954–November 1994