Evolution and spherical collapse in Einstein-Æther theory and Hořava gravity

Jishnu Bhattacharyya,[1] Andrew Coates,[1] Mattia Colombo,[1] and Thomas P. Sotiriou[1,2]

[1]*School of Mathematical Sciences, University of Nottingham, University Park,*
Nottingham NG7 2RD, United Kingdom
[2]*School of Physics and Astronomy, University of Nottingham, University Park,*
Nottingham NG7 2RD, United Kingdom

We compare the initial value formulation of the low-energy limit of (nonprojectable) Hořava gravity to that of Einstein-æther theory when the æther is assumed to be hypersurface orthogonal at the level of the field equations. This comparison clearly highlights a crucial difference in the causal structure of the two theories at the nonperturbative level: in Hořava gravity evolution equations include an elliptic equation that is not a constraint relating initial data but needs to be imposed on each slice of the foliation. This feature is absent in Einstein-æther theory. We discuss its physical significance in Hořava gravity. We also focus on spherical symmetry, and we revisit existing collapse simulations in Einstein-æther theory. We argue that they have likely already uncovered the dynamical formation of a universal horizon and that they can act as evidence that this horizon is indeed a Cauchy horizon in Hořava gravity.

I. INTRODUCTION

Lorentz violating gravity theories have garnered interest in recent years. One key reason is that obtaining quantitative infrared constraints on Lorentz violation in the gravity sector requires a consistent parametrization of deviations from Lorentz invariance in terms of a Lorentz-violating gravity theory [1]. Additional recent motivation came from a concrete realization of the idea that abandoning Lorentz symmetry can lead to improved ultraviolet behavior in gravity [2].

Lorentz symmetry is clearly intimately related with the causal structure of general relativity. The most extreme manifestation of this is the notion of a black hole, the existence of which one tends to associate with the behavior of light rays in the vicinity of a horizon. Lorentz-violating gravity theories tend to include superluminal excitations—in fact, constraints from vacuum Čerenkov radiation seem to not leave much of an alternative [3]. One then faces the question of whether a black hole can be said to exist in a meaningful way. And if so, what is the appropriate definition? The precise answer to these questions will actually depend on the theory under scrutiny.

Einstein-æther theory [1] is a vector-tensor theory where the vector, referred to as the æther, is constrained to be unit timelike. Because of this constraint the æther cannot vanish even in flat spacetime and defines through its trajectories a preferred set of timelike curves that thread the spacetime. The theory propagates the usual spin-2 graviton as well as a spin-1 mode and a spin-0 mode. The propagation of these modes can be described by following null rays of separate metrics, $\tilde{\mathsf{g}}_{ab}^{(i)} = \mathsf{g}_{ab} - (s_i^2 - 1)u_a u_b$, where s_i is the speed of the spin-i mode. Here g_{ab} is the metric that minimally couples to matter and whose null rays are photon trajectories. All modes generically have different propagation speeds from each other and can be superluminal. A stationary black hole in this theory is then identified with a region cloaked by a succession of Killing horizons for g_{ab} and each of the metrics $\tilde{\mathsf{g}}_{ab}^{(i)}$ [4]. These horizons are referred to as spin-i horizons.

Hořava gravity is a theory with a preferred foliation. The purpose of having a preferred foliation is to introduce higher order spatial derivatives that modify the propagator in the ultraviolet and render the theory power-counting renormalizable [2]. It is precisely this feature of the theory that leads to dispersion relations of the type $\omega^2 \propto k^6$ as $k \to \infty$. This seems to present a challenge to defining black holes as there is no upper limit for the speed of perturbations. A less evident but even more worrisome feature when it comes to black holes is that, even at the low-energy limit where the dispersion relations become linear, there is still instantaneous propagation, in the form of an elliptic mode [5].

Perhaps surprisingly, black holes can still be defined in a meaningful way. The boundary of the causally disconnected region is called a "universal horizon" [5,6], owing to the fact that no signals of any speed can leave it. In terms of the preferred foliation this horizon corresponds to a leaf which is disconnected from spatial infinity [7]. Since propagation into the future means crossing the leaves of the foliation in a given direction, if such a leaf exists no signal emitted inside it can ever cross it toward spatial infinity without traveling toward the past. Universal horizons were initially found in static, spherically symmetric black holes [5,6,8] but they seem to be a more generic

feature, as they have also been shown to exist in slowly rotating black holes [9] and in lower-dimensional rotating black holes [10].

Because of the close relation between Einstein-æther theory and the low energy limit of Hořava gravity [11], which we will discuss thoroughly in the next section, the known static, spherically symmetric, asymptotically flat solutions of the former are also solutions of the latter and vice versa [12]. This is no longer true for slowly rotating solutions [9,12]. Moreover, it is not known if nonstatic, spherically symmetric solutions in the two theories match, so it is not clear if spherical collapse in the two theories is identical or simply leads to the same end state.

This is one of the questions we are seeking to answer. More generally, we will perform a comparison of the initial value formulation of Hořava gravity and that of Einstein-æther theory, under the additional assumption that the æther in the latter is hypersurface orthogonal at the level of the field equations. Under this assumption the æther defines a foliation (as opposed to just a preferred frame), and this makes the two theories resemble each other. Hence, making this assumption suppresses some obvious differences, such as the existence of propagating spin-1 modes in Einstein-æther theory, and allows one to focus on some more subtle ones, such as the different standings of the preferred foliation in each theory and the dependence of evolution on (future) boundary data in Hořava gravity.

We will also consider the special case of spherical symmetry in more detail. Spherical collapse in Einstein-æther theory has been considered in Ref. [13]. To date there are no collapse simulations in Hořava gravity (see, however, Ref. [14]). Moreover, Ref. [13] predates the introduction of the notion of a universal horizon, which was later understood to be one of the most prominent features of spherical black holes in both Einstein-æther and Hořava gravity [6]. Clearly, whether universal horizons actually form from collapse is a key question. Additionally, it is important to verify that the known static solutions that harbor them are indeed the end points of gravitational collapse in Lorentz violating theories. We will revisit the results of Ref. [13] and attempt to understand whether dynamical formation of universal horizons had indeed been found. We will also discuss possible improvements in future simulations that could answer this question unambiguously. Our analysis will also shed light on whether these simulations can be seen as modeling collapse in Hořava gravity as well.

II. THE THEORIES

By introducing a Stuckelberg field, Hořava gravity can be reformulated in a manifestly covariant manner [11] as a scalar-tensor theory, where the tensor degree of freedom is the metric g_{ab} as usual, while the "nonmetric" degree of freedom is a scalar field T. The level surfaces of this scalar are the leaves of the preferred foliation; hence T is

constrained to have a *timelike gradient everywhere*. Therefore, one may introduce a timelike unit one-form u_a such that

$$u_a = -N\nabla_a T, \qquad (1)$$

by virtue of being orthogonal to the constant T hypersurfaces, and

$$u_a u_a \mathsf{g}^{ab} = u \cdot u = -1, \qquad (2)$$

on account of being unit timelike. Combining the two conditions yields

$$N^{-2} = -\mathsf{g}^{ab}(\nabla_a T)(\nabla_b T). \qquad (3)$$

In accordance with common practice, we will call the normalized gradient of T, i.e. u_a, the æ*ther*.

T plays the role of a preferred time in Hořava theory, but it should be noted that the theory is invariant under reparametrizations of the type

$$T \mapsto \tilde{T} = \tilde{T}(T), \qquad (4)$$

where $\tilde{T}(T)$ is some arbitrary function of T. That is, T defines an ordered preferred slicing but does not introduce a preferred labeling of the slices. Under such reparametrizations, the function N is required to transform as

$$N \mapsto \tilde{N} = (\mathrm{d}\tilde{T}/\mathrm{d}T)^{-1}N, \qquad (5)$$

in accordance with Eq. (3) such that u_a as well as quantities built out of u_a and its covariant derivatives are invariant under the above reparametrizations.

Consider the action

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-\mathsf{g}}(R + \mathcal{L}), \qquad (6)$$

where G is a coupling constant with suitable dimensions, R is the four-dimensional curvature scalar, and $\mathcal{L}$ is given in terms of the derivatives of the æther as

$$\mathcal{L} = -Z^{abcd}(\nabla_a u_c)(\nabla_b u_d), \qquad (7)$$

so that everything is manifestly invariant under the reparametrizations (4). The tensor Z^{abcd} defining the Lagrangian (7) is given by

$$Z^{abcd} = c_1 \mathsf{g}^{ab}\mathsf{g}^{cd} + c_2 \mathsf{g}^{ac}\mathsf{g}^{bd} + c_3 \mathsf{g}^{ad}\mathsf{g}^{bc} - c_4 u_e u_f \mathsf{g}^{ae}\mathsf{g}^{bf}\mathsf{g}^{cd}, \qquad (8)$$

with coupling constants $c_1, \ldots, c_4$ allowing for the most general two-derivative action for a unit one-form field. Adopting T as a time coordinate, on the premises that its gradient is always timelike, introduces a foliation defined

by the constant-T surfaces. Action (6) then becomes the second derivative truncation of Hořava gravity [11], i.e. the low-energy part of the theory. Once one has adopted T as a time coordinate, the residual symmetry is invariance under diffeomorphisms that preserve the foliation. Indeed, the full action of Hořava gravity includes all the terms that respect this symmetry and contain up to sixth order spatial derivatives in the preferred foliations [15].[1]

Here we will only consider the low-energy part of Hořava gravity so we will not discuss these higher-order terms any further. However, some remarks are in order. These terms are higher order in spatial derivatives, and they do not contain any time derivatives. This underscores the existence of a preferred foliation in Hořava gravity. Even though these terms can be written in a manifestly covariant way in the same fashion as the low-energy part of the action, in such a covariant formulation the full theory would appear highly fine-tuned (as higher-order time derivatives would have to cancel out) [23]. Moreover, discarding the higher-order terms does not mean that the preferred foliation ceases to be preferred. As we noted above, even in the low-energy theory that can be described in a covariant manner by action (6), T has to be nonzero and have a timelike gradient in *every* solution, thereby signifying that every solution comes with a special foliation. Additionally, action (6) actually contains more than two derivatives of T, which is an indication that the theory will not satisfy second order differential equations in a generic foliation.

Indeed, a variation of action (6) (up to boundary terms) gives

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[E^H_{ab}\delta g^{ab} + 2(\nabla_a[N\vec{\mathcal{Æ}}^a])\delta T], \quad (9)$$

where Eq. (1) has been taken into account (recall that T and not the æther is the fundamental field here). The tensor E^H_{ab} is defined as

$$E^H_{ab} = G_{ab} - T^H_{ab}, \quad (10)$$

where G_{ab} is the four-dimensional Einstein tensor and T^H_{ab} is T's stress-energy tensor. $\mathcal{Æ}^a$ is the functional derivative of the æther Lagrangian Eq. (7) with respect to the æther,

$$\vec{\mathcal{Æ}}^a \equiv p^a{}_b \mathcal{Æ}^b, \quad (11)$$

and p_{ab} is the projector onto the constant khronon leaves

$$p_{ab} = g_{ab} + u_a u_b, \quad (12)$$

also acting as the induced metric on the leaves of the preferred foliation. Therefore, $\vec{\mathcal{Æ}}^a$ is manifestly orthogonal

to the æther by construction. From Eq. (9), the equations of motion of Hořava gravity are then

$$E^H_{ab} = 0, \qquad \nabla_a[N\vec{\mathcal{Æ}}^a] = 0. \quad (13)$$

$\vec{\mathcal{Æ}}^a$ already contains the second derivative of the æther, which implies that Eq. (13) contains third order time derivatives in an arbitrary foliation. However, the fact that $\vec{\mathcal{Æ}}^a$ is orthogonal to the æther implies that (only) in the preferred foliation defined by T, the divergence in Eq. (13) is purely spatial and there are only two time derivatives [11].

The other theory we will consider, namely Einstein-æther theory [1], is a true vector-tensor theory. The fundamental fields are the metric and the æther. The æther was treated as a vector in the original formulation [1] of the theory, but treating it as a one-form (i.e. keeping the variation of u_a fixed on the boundary) leads to the same theory (see also [9] for a discussion). The equations of motion of this theory can be derived from an action that is *formally identical* to Eq. (7), but the æther is constrained to satisfy *only* the unit norm constraint of Eq. (2) and is not hypersurface orthogonal in general. Variation with respect to the metric and the æther yields

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[E^{æ}_{ab}\delta g^{ab} + 2\vec{\mathcal{Æ}}^a\delta u_a], \quad (14)$$

where the unit constraint of Eq. (2) has been imposed by constraining the æther's variation. $E^{æ}_{ab}$ is defined as

$$E^{æ}_{ab} = G_{ab} - T^{æ}_{ab}, \quad (15)$$

with $T^{æ}_{ab}$ being the stress energy tensor of the æther. Thus from Eq. (14), the equations of motion of Einstein-æther theory are

$$E^{æ}_{ab} = 0, \qquad \vec{\mathcal{Æ}}^a = 0. \quad (16)$$

$T^{æ}_{ab}$ and T^H_{ab} are formally identical as they come from formally identical actions under variation with respect to the metric. This means that if one imposes the hypersurface orthogonality condition (1) on the æther as an additional simplifying assumption at the level of the equations of motion in Einstein-æther theory, then the systems of equations (13) and (16) will have the same "Einstein equations." Moreover, any such hypersurface-orthogonal solution of Einstein-æther theory will also be a solution of Hořava gravity. The converse is not generically true. It has, however, been shown to hold for spherically symmetric, asymptotically flat solutions under the assumption that all leaves of the foliation reach the center and the center is regular [24]. It has also been shown to hold for static, spherically symmetric, asymptotically flat solutions without any further assumptions [12], as well as for static,

---

[1]We will not consider here versions of the theory where extra restrictions have been imposed, such as projectability or detailed balance [2,16–22].

spherically symmetric solutions with more general asymptotics but with a regular universal horizon [25].

III. EVOLUTION

As a genuine vector-tensor theory, Einstein-æther theory propagates not only the standard spin-2 mode of general relativity but also spin-1 and spin-0 modes. In Hořava gravity, on the other hand, u_a is hypersurface orthogonal and given in terms of T via (1), so there cannot be a dynamical spin-1 mode. This is certainly a significant difference in the dynamics of the two theories.

However, here we wish to focus on the more subtle differences that are not related to the existence of vector modes. To this end, we wish to compare the (nonperturbative) dynamics of Hořava gravity with that of Einstein-æther theory when the æther is constrained to be hypersurface orthogonal (at the level of the equations) throughout the evolution.[2] An important subcase that we will discuss later in more detail will be spherically symmetric collapse, as spherically symmetric vectors are hypersurface orthogonal. However, in this section we will opt to be as general as possible, and we will not assume any symmetries.

By virtue of Frobenius's theorem [26], a hypersurface orthogonal (hence twist-free) unit timelike æther satisfies

$$\nabla_a u_b - \nabla_b u_a = -u_a a_b + u_b a_a, \tag{17}$$

where a_a is the acceleration of the æther congruence defined as usual

$$a^a = \nabla_u u^a \Leftrightarrow a_a = N^{-1} \vec{\nabla}_a N. \tag{18}$$

Here $\vec{\nabla}_a$ is the projected covariant derivative on the preferred foliation, and the second expression in Eq. (18) above is a consequence of Eqs. (1) and (17). One may thus expand the covariant derivative of the æther as

$$\nabla_a u_b = -u_a a_b + K_{ab}, \tag{19}$$

where K_{ab} is the extrinsic curvature of the preferred leaves due to their embedding in spacetime,

$$K_{ab} = \frac{1}{2} \pounds_u \mathsf{p}_{ab}, \tag{20}$$

and is purely spatial (i.e. orthogonal to the æther) by definition. The mean curvature K, i.e. the trace of the extrinsic curvature tensor, is then given by

$$K = \mathsf{g}^{ab} K_{ab} = \mathsf{p}^{ab} K_{ab} = (\nabla \cdot u). \tag{21}$$

We may now use the above quantities and relations to adapt the equations of motion of both theories, Eqs. (13) and (16), to the foliation defined by the æther. For Hořava gravity this is imperative as mentioned earlier, for it is only in this foliation that the equations become second order in time derivatives. This is simply the preferred foliation determined by T, in which the theory is usually defined. For Einstein-æther theory, however, this is simply a choice which we make in order to facilitate the comparison with Hořava gravity. It is also worth noting that, even though we are adopting a foliation, we will refrain from adopting any coordinate system.

As already noted, for a hypersurface orthogonal æther Einstein's equations in both the theories are formally identical. One may furthermore show [6,27] that in that case, the covariantized Bianchi identities are formally identical as well. When adapted to the preferred foliation, the (generalized) Bianchi identities for both the theories read

$$\partial_T E^T{}_i = 0,$$
$$\partial_T E^T{}_T + (\sqrt{-\mathsf{g}})^{-1} \partial_i [\sqrt{-\mathsf{g}} N \vec{\mathcal{E}}^i] = 0, \tag{22}$$

where $i = \{1, 2, 3\}$ denote *coordinate indices* on the preferred leaves, E_{ab} is either of E_{ab}^{H} (10) or $E_{ab}^{æ}$ (15) (for a hypersurface orthogonal æther), and in writing Eq. (22) it was assumed that all Einstein's equations (but not the æther/T equations) are satisfied *on the given leaf*. Note that a $(1 + 3)$ decomposition of the equations of motion of Einstein-æther theory need not necessarily be performed with respect to the æther's foliation, and in general, the corresponding constraint equations are a combination of Einstein's equations and the æther's equation of motion. However, (only) when formulated as a theory of a one-form, the constraint equations of Einstein-æther theory adapted to the æther's foliation do not involve the æther's equations of motion but only Einstein's equations in the form of $(E^æ)^T{}_T = (E^æ)^T{}_i = 0$. Thus according to Eq. (22), these constraints are also preserved in time once the æther becomes "on shell" (16) as well. For Hořava gravity, on the other hand, the only allowed foliation to perform a $(1 + 3)$ decomposition of the equations of motion with respect to *is* the preferred foliation; only then can proper constraint equations be found in the form of $(E^{\mathrm{H}})^T{}_T = (E^{\mathrm{H}})^T{}_i = 0$ which are first order in the T-derivative and according to Eq. (22) are preserved in time once the khronon becomes on shell (13).

---

[2]Note that, in general, evolution could generate vorticity, so our condition is stronger that selecting vorticity-free initial data. It might well be that constraining the aether to be hypersurface orthogonal throughout the evolution could lead to an overconstrained system in the absence of extra symmetries, e.g. spherical symmetry. This is an interesting open question, but it will not concern us here, as we simply seek for the most general setting in which we can straightforwardly compare Einstein-æther theory and Hořava gravity.

In both theories, the constraint $E^T{}_T = 0$ constitutes the *energy constraint equation* explicitly given by

$$(1 - c_{13})K_{ab}K^{ab} - (1 + c_2)K^2$$
$$+ c_{14}[2(\vec{\nabla} \cdot a) + a^2] - \mathcal{R} = 0, \qquad (23)$$

where $\mathcal{R}$ is the intrinsic scalar curvature of the leaves of the foliation. The equations $E^T{}_i = 0$ become the *momentum constraint equations* and take the form

$$(1 - c_{13})\vec{\nabla}_c K^c{}_a = (1 + c_2)\vec{\nabla}_a K. \qquad (24)$$

The remaining Einstein's equations (i.e. those completely projected onto the preferred foliation), for both theories, reduce to an *evolution equation* for the mean curvature

$$c_\ell \nabla_u K = - (1 - c_{13})K_{ab}K^{ab} - \frac{c_{123}}{2}K^2$$
$$+ \left[1 - \frac{c_{14}}{2}\right](\vec{\nabla} \cdot a + a^2), \qquad (25)$$

where $c_{123} = c_2 + c_{13}$ and $c_\ell = 1 + (1/2)c_{13} + (3/2)c_2$, and an *evolution equation* for the traceless part $[K]_{ab} = K_{ab} - (K/3)\mathsf{p}_{ab}$,

$$\pounds_u[K]_{ab} = 2[K]_{ac}[K]_b{}^c - \frac{K}{3}[K]_{ab}$$
$$+ \frac{1 - c_{14}}{1 - c_{13}}\left[a_a a_b - \frac{a^2 \mathsf{p}_{ab}}{3}\right]$$
$$+ \frac{1}{1 - c_{13}}\left[\vec{\nabla}_a a_b - \frac{(\vec{\nabla} \cdot a)\mathsf{p}_{ab}}{3} - \mathcal{R}_{ab} + \frac{\mathcal{R}\mathsf{p}_{ab}}{3}\right], \qquad (26)$$

where $\mathcal{R}_{ab}$ is the Ricci curvature of the preferred hypersurfaces. Collectively, Eqs. (23), (24), (25), and (26) provide all Einstein's equations for both the theories.

As noted above, Eqs. (25) and (26) provide a set of *evolution equations for the extrinsic curvature* that are first order in time derivatives with respect to the preferred foliation in both the theories (for Einstein-æther theory, these equations were already obtained in [13]). Taken together with Eq. (20), these provide a set of first order evolution equations for the pair of conjugate variables consisting of the components of the induced metric and those of the extrinsic curvature. To turn these equations explicitly into a set of coupled partial differential equations, one needs to introduce a set of coordinates on the leaves of the hypersurfaces and perform a lapse-shift decomposition of the metric. However, this goes beyond our current goal; we merely wish to point out that even in the most general setting the "metric-extrinsic curvature

pair" can be evolved in the same manner with respect to the preferred foliation in both the theories. The difference between the dynamics of the theories—and the related issue of the existence of the instantaneous mode in Hořava theory—stems from the evolution of the æther/T. We will take this issue up next and study it in detail in spherical symmetry.

IV. THE INSTANTANEOUS MODE OF HOŘAVA GRAVITY

In terms of the kinematic variables introduced previously, the quantity $\vec{Æ}_a$ (11) is given by

$$\vec{Æ}_a = \frac{c_{123}}{(1 - c_{13})}\vec{\nabla}_a K - c_{14}[Ka_a + \pounds_u a_a - 2K_{ac}a^c]. \qquad (27)$$

This allows one to interpret the æther's equation of motion (16) in Einstein-æther theory as an evolution equation for the acceleration as follows [13]:

$$\pounds_u a_a = 2K_{ac}a^c - Ka_a + \frac{c_{123}}{c_{14}(1 - c_{13})}\vec{\nabla}_a K. \qquad (28)$$

Needless to say, the above is not satisfied, in general, in Hořava gravity; instead, the "khronon's equation of motion" in Hořava gravity is given by (13)

$$\nabla_a[N\vec{Æ}^a] = 0 \Leftrightarrow \vec{\nabla}_a[N^2\vec{Æ}^a] = 0, \qquad (29)$$

where the expression for $\vec{Æ}^a$ is identical with that given in Eq. (27). The difference in the dynamics in the two theories thus lies in the difference between the nature of Eqs. (28) and (29). To study them closely, we will henceforth restrict ourselves to spherical symmetry, which will allow us to integrate (29) very easily. Note that in spherical symmetry, the hypersurface orthogonality of the æther is guaranteed kinematically.

Toward setting up a suitable coordinate system that makes the spherical symmetry manifest (among other things), let us start with some basic observations: in any coordinate system adapted to the æther's foliation, the time coordinate is identical to T [and hence subject to the reparametrizations (4)]. Next, the unit spacelike vector s_a along the acceleration,

$$a_a = (a \cdot s)s_a, \qquad (s \cdot s) = 1, \qquad (u \cdot s) = 0, \qquad (30)$$

defines a natural spacelike direction in the spacetime which is orthogonal to the spherical directions by virtue of spherical symmetry. In order to be completely general (and in particular, to make our subsequent conclusions manifestly independent of any "gauge choices") we will now introduce a coordinate system adapted to the preferred foliation consisting of the "time coordinate" T and a "radial coordinate" R in which [along with Eq. (1)]

$$u^a = N^{-1}\partial_T - N^{-1}N^R\partial_R,$$

$$s^a = S^{-1}\partial_R,$$

$$s_a = SN^R\nabla_a T + S\nabla_a R, \qquad (31)$$

such that the functions $N = N(T,R)$, $S = S(T,R)$, and $N^R = N^R(T,R)$ describe the æther configuration completely in a manifestly spherically symmetric manner. Note that for the above choice of coordinates, the shift vector is $\vec{N}^a = N^R\partial_R$. Furthermore, the projector (12) can be written as

$$\mathsf{p}_{ab} = s_a s_b + \hat{\mathsf{g}}_{ab}, \qquad (32)$$

where $\hat{\mathsf{g}}_{ab}$ is the metric on a unit two-sphere up to a conformal factor which is the *areal radius r* squared

$$r = \sqrt{\frac{\text{Area of two-sphere}}{4\pi}}, \qquad \hat{\mathsf{g}}_{ab} = r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2),$$

and θ and ϕ are the usual polar coordinates on the unit two-sphere. In what follows, r will not be treated as a coordinate. Rather, the coordinate system that we have constructed consists of the coordinate functions $\{T, R, \theta, \phi\}$, and the areal radius is given as $r = r(T,R)$, in the same way as N, S, and N^R. Thus in the present coordinate system the full metric is

$$\mathsf{g}_{ab} = -N^2\mathrm{d}T^2 + S^2(N^R\mathrm{d}T + \mathrm{d}R)^2$$
$$+ r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \qquad (33)$$

and the æther and metric are completely specified by the four functions $N(T,R)$, $N^R(T,R)$, $S(T,R)$, and $r(T,R)$. Hence, the equations of motion of the two theories along with some suitable gauge choice will allow us to solve for these functions.

We may now integrate T's equation of motion in Hořava gravity (29) to obtain

$$\partial_R[r^2 SN^2\vec{Æ}^R] = 0 \Leftrightarrow (s\cdot\vec{Æ}) = \frac{f_{\mathrm{IM}}(T)}{r^2N^2}, \qquad (34)$$

where $f_{\mathrm{IM}}(T)$ is a "constant" of integration. Plugging this into the expression (27) of $\vec{Æ}_a$ we then end up with a first order evolution equation for the acceleration very similar to (28)

$$\partial_T(a\cdot s) = N^R\partial_R(a\cdot s) - N\hat{K}(a\cdot s)$$
$$+ \frac{c_{123}N\partial_R K}{c_{14}(1-c_{13})S} - \frac{f_{\mathrm{IM}}(T)}{c_{14}r^2N}, \qquad (35)$$

where $\hat{K} = \hat{\mathsf{g}}^{ab}K_{ab}$. This equation contains, in the most explicit manner, the most crucial difference between the dynamics of Einstein-æther and Hořava theories. Indeed,

one obtains the æther's equation of motion in Einstein-æther theory (28) upon setting $f_{\mathrm{IM}}(T) = 0$ *for all T*, while $f_{\mathrm{IM}}(T) \neq 0$ characterizes those solutions of Hořava theory which are *not* solutions of Einstein-æther theory. Finally, as soon as one solves for $(a\cdot s)$ via (35), one may solve for the lapse by integrating (18) on a given T slice, i.e.

$$\partial_R\log N = (a\cdot s)S, \qquad (36)$$

which implies

$$\log N(T,R) = \log N(T,\infty) + \int_\infty^R \mathrm{d}R'(a\cdot s)S(T,R'). \qquad (37)$$

In this manner, all the relevant functions determining the spacetime-æther/T configuration in both the theories can be solved for.

In both theories there is still the reparametrization freedom of Eq. (4). In Hořava gravity, this is a symmetry of the theory itself, whereas in Einstein-æther theory it comes as a consequence of our restriction that the æther be hypersurface orthogonal. Equation (5) implies that $\log N$ picks up a function of T *additively* under the reparametrization (4). We can choose to reparametrize T such that

$$\log N(T,\infty) = 0 \qquad (38)$$

(which was also the choice in Ref. [13]). It then becomes apparent that generic Hořava gravity solutions are characterized by a nonzero $f_{\mathrm{IM}}(T)$ while $f_{\mathrm{IM}}(T) = 0$ in Einstein-æther theory.

The root of the difference between the two theories is the following: turning the T equation into an evolution equation for the lapse N in Hořava gravity involves integrating a divergence on each slice.[3] Hence there is an elliptic part in this system of equations that is absent in Einstein-æther theory. It should be stressed that this elliptic part is fundamentally different from the constraint equations, even though the latter are also elliptic. The main difference has to do with the fact that constraints are preserved by time evolution and hence need to be imposed only on an initial slice, while the divergence in Eq. (34) has to be integrated on *every slice* and $f_{\mathrm{IM}}(T)$ is to be determined by suitable boundary/asymptotic conditions. This will be discussed in more detail below. For instance, for generic functional forms of $f_{\mathrm{IM}}(T)$, Eq. (35) is singular if either $r = 0$ or $N = 0$. Thus, the physical requirement of regularity at the center or on a universal horizon where $N = 0$ for our choice of T can impose $f_{\mathrm{IM}}(T) = 0$.

This is simply the nonperturbative manifestation of the instantaneous mode discussed in Ref. [5] in a perturbative

---

[3]Recall that we are working in the preferred foliation, so the lapse N cannot be set to a constant by making a gauge choice, and hence it should be determined by the field equations.

setting. Indeed, when spherical perturbations around a black hole were considered in Ref. [5], it was the assumption of regularity on the universal horizon that forced the instantaneous mode to vanish, in agreement with what has been mentioned above.

The above conclusions can be generalized beyond spherical symmetry, albeit somewhat qualitatively. To that end, we may begin by recalling that in diffeomorphism invariant scalar-tensor theories the equation determining the scalar field is dynamically redundant, as it can be obtained by taking a divergence of the field equations for the metric (see the Appendix). Hence, one can in principle solve the latter only and neglect the scalar's equation altogether. Since Hořava gravity can be written as a diffeomorphism invariant scalar-tensor theory, one can apply this logic. This then implies that consistent solutions can be obtained by solving only Eqs. (23)–(26) (where we have conveniently neglected Eq. (13) only after forming the constraint equations). Equation (23) can then be turned into the following Poisson type elliptic equation for ϱ, defined through $N = \varrho^2$:

$$\vec{\nabla}^2 \varrho = \frac{\varrho}{4c_{14}} [\mathcal{R} - (1 - c_{13}) K_{ab} K^{ab} + (1 + c_2) K^2]. \quad (39)$$

As already pointed out in Ref. [28], this equation allows one to solve for the lapse N *on each slice of the preferred foliation*. One can then subsequently compute the acceleration from Eq. (18). Thus, Eqs. (39), (24), (25), and (26) provide a complete set of equations that can dynamically determine the spacetime and the foliation in Hořava gravity.

Since Eq. (39) is a second-order *elliptic* equation in ϱ that is not preserved by time evolution when T is not taken to be on shell [cf. Eqs. (22)], it is indeed expected that its solution should depend on two integration "constants"— actually functions of preferred time T. This matches precisely the result we obtained previously in spherical symmetry. While one of these functions of T can be set to a desired value by the yet-to-be-fixed reparametrization freedom of T, Eq. (4), the second one will be related to the instantaneous mode of the theory [analogous to the function $f_{\mathrm{IM}}(T)$ introduced above] and cannot be done away with even after fixing the said reparametrization freedom.

The above logic does not apply to Einstein-æther theory, simply because the æther equation is not dynamically redundant even when the æther is hypersurface orthogonal. Indeed, solutions of Eq. (39), which obviously also hold in Einstein-æther theory, do not always satisfy the æther's equation of motion (28).

Though slightly less rigorous than our spherically symmetric treatment, this last analysis has two advantages: it is more general, and it clearly demonstrates that in Hořava gravity and in the preferred foliation the equations can be thought of as a system of an elliptic equation that

needs to be imposed on every slice, elliptic equations that are preserved by time evolution and hence constitute constraints, and dynamical equations that generate the spacetime together with its foliation. Reference [28] has reached the same conclusion by means of a Hamiltonian analysis.

V. COMMENTS ON SPHERICAL COLLAPSE

The problem of spherically symmetric collapse provides one of the simplest settings to which the preceding analysis can be directly applied, thereby allowing us to compare the dynamics of the two theories explicitly. Indeed, spherically symmetric collapse in Einstein-æther theory have been previously studied in [13], while an analogous simulation in Hořava theory is yet to be performed.[4] In light of the relation between evolution in Einstein-æther theory and Hořava gravity as discussed in the previous sections, it is tempting to revisit the results of Ref. [13], potentially reinterpreting some of them, and to attempt to draw some general conclusions about spherically symmetric collapse in Hořava gravity.

To be more specific, in Ref. [13] spherically symmetric collapse in Einstein-æther theory with a minimally coupled scalar field ψ was studied, where ψ represented collapsing matter. The evolution of the system was performed by adapting Eqs. (16) to the foliation described by the æther that was hypersurface orthogonal due to spherical symmetry. This is the preferred foliation of Hořava gravity, as pointed out earlier. Equations (16) were supplemented with appropriate equations of motion for ψ.

Simulations were performed for two different values for the speed of the spin-0 mode s_0. In the first case the couplings c_3 and c_4 were set to zero, and the remaining two parameters of the theory, c_1 and c_2, were chosen such that s_0 was set to unity, i.e. equal to the speed of light. Two values of c_1 were considered. For $c_1 = 0.7$ a regular (Killing) horizon forms as a result of the collapse while for $c_1 = 0.8$ no such horizon seems to form and "… the evolution seems to become singular, thus indicating the formation of a naked singularity." The main reason for considering the specific values of the c_i parameters and s_0 is because no static solutions had been found for the same values and $c_1 \geq 0.8$ in Ref. [4]. Indeed, the result was interpreted as verifying the absence of black holes for these parameters. However, static black holes were later found for that very same choice of the couplings in Ref. [6], and it was argued there that the reason these solutions were not found in Ref. [4] was insufficient accuracy in the numerics performed there. This puzzling situation definitely deserves further investigation. However, these simulations are not presented in detail in Ref. [13], and so it is hard to interpret

---

[4]See, however, Ref. [14] where cuscuton theory is used as a proxy for Hořava gravity. We will discuss the relation between the two theories elsewhere.

them in light of the later results of Ref. [6] or our analysis in the previous sections. Hence, we will not consider them further.

The second set of parameters was chosen such that the speed of the spin-0 mode was set to $\sqrt{2}$. With suitably chosen initial conditions, evolution led to the formation of a *regular* spin-0 horizon *inside* the metric horizon. Furthermore, at sufficiently "late times," the geometry *outside the spin-0 horizon* settled down to the static solutions of [4] to high accuracy.[5] Moreover, the simulations of [13] also revealed that the preferred frame lapse function N "is driven to zero as the singularity is approached."

A vanishing of the lapse function at any given point of an evolution simulation in a gravity theory is strongly indicative of a breakdown of the corresponding foliation. A well known example of this is the study of spherically symmetric collapse in general relativity in Schwarzschild coordinates, where a similar situation is expected toward the formation of the Killing horizon. On the other hand, provided one can be certain about the horizon-crossing properties of a certain foliation, having the lapse vanish asymptotically in time and as the singularity is approached is clearly advantageous from a numerical perspective. Since studying evolution with respect to the foliation defined by the æther is merely a choice in Einstein-æther theory, determining whether this is the optimal choice is a point that deserves further discussion.

The æther's foliation in spherical symmetry will penetrate all Killing horizons, as the latter are null surfaces and the aether is always timelike. Considering also its privileged status in Einstein-aether theory, it was certainly a natural choice for Ref. [13]. One of the goals of Ref. [13] was indeed to verify whether regular spin-0 horizons emerge from spherical collapse in Einstein-æther theory. Nonetheless, this foliation is special, and there is a way in which using it in this setting resembles using Schwarzschild coordinates in spherically symmetric collapse in general relativity: it does not penetrate the universal horizon.

Indeed, the vanishing of the lapse function N in the preferred foliation can have an alternative interpretation as an asymptotic formation of a *universal horizon*. In a static and spherically symmetric geometry, a universal horizon [5,6] is a leaf of the preferred foliation that is also a constant r hypersurface (and hence a hypersurface generated by the Killing vector associated with staticity), turning it into an event horizon even for arbitrarily fast propagations [7]. In particular, the fact that a universal horizon is generated by a Killing vector implies [7] that the preferred frame lapse function, subjected to the boundary condition (38), will also vanish on the universal horizon. Moreover, a universal

horizon can only occur in the asymptotic future in the preferred time. These observations, along with the fact that the geometry "outside" settles down to the appropriate static (and essentially unique [5,25]) solution [4,6], thus strongly suggest that the simulations of Ref. [13] revealed the asymptotic formation of a universal horizon in the "late time" phase. Clearly, the notion of a universal horizon was introduced several years after Ref. [13] appeared, and it is natural that the above interpretation escaped its authors.

In situations where a universal horizon may form working in the preferred foliation is clearly not the optimum choice. The simulation will inevitably "stop" as the universal horizon is approached, and one may never cross it in this setup. If the simulations of Ref. [13] were to be performed again in a different foliation, it seems likely that one would be able to trace the formation and evolution of the universal horizon and verify whether the result leads to the static solutions of Ref. [6] all the way to the universal horizon and beyond.

We now turn our attention to what the simulation of Ref. [13] can teach us about spherical collapse in Hořava gravity. Taking into account the connection between Hořava gravity and Einstein-æther theory as discussed in detail in the previous sections, spherical collapse in the latter will be identical to spherical collapse in the former once boundary conditions that set $f_{\mathrm{IM}}(T) = 0$ in (29) have been chosen. The suitable boundary condition is simply regularity at the origin, $r = 0$ (up to the formation of the singularity and/or universal horizon). Note that using the preferred foliation is not a choice but a necessity in Hořava gravity. Hence, the fact that the evolution seemingly "ends" with an asymptotic formation of a universal horizon appears to be a confirmation of the claim that the universal horizon is also a Cauchy horizon in theories like Hořava gravity [5,7], where the preferred foliation actually determines the causal structure and boundary data are required to determine the evolution.

VI. CONCLUSIONS

Einstein-æther theory with the additional constraint that the æther be hypersurface orthogonal at the level of the field equations resembles the low-energy limit of Hořava gravity, but is a different theory. In both theories there is a special foliation but it has a different standing in each of them. This has been demonstrated clearly by comparing the initial value formulation in the two cases. In Hořava gravity the field equations are second order in time derivatives only in this foliation. Additionally, the system of evolution equations includes an elliptic equation that is not a constraint but instead needs to be imposed on each slice. The presence of such an elliptic equation implies that, in principle, evolution depends not only on initial data but also on future boundary/asymptotic data (for any type of boundary, including a conformal one). This is a key feature of the causal structure of the theory [7] and is intimately related

---

[5]Note that [13] predates [6], and thus the authors were only able to compare their results with [4].

with the presence of an instantaneous mode at the perturbative level [5]. In other words, this foliation is both dynamically and causally preferred. In Einstein-æther theory in contrast, the special foliation defined by the æther is not preferred in any of these two senses. It does not define causality, and one need not adopt it to set up an initial value problem.

On the contrary, we have argued that choosing this foliation is not ideal when performing spherical collapse simulations in Einstein-æther theory. This is because a universal horizon is actually a leaf of this foliation and the simulation cannot proceed past it in this slicing. This is reminiscent of spherical collapse in general relativity if performed in a foliation by constant Schwarzschild time surfaces. The collapse simulations of Ref. [13] have indeed been performed in the foliation defined by the hypersurface orthogonal æther. We have revisited them and argued that they might have indeed uncovered the dynamical formation of a universal horizon asymptotically in time. In stationary black holes the lapse of this foliation vanishes on the universal horizon (when appropriately normalized at spatial infinity) [7], and in some of the simulations of Ref. [13] the lapse indeed appears to vanish asymptotically in time.

Our results suggest that it would be particularly interesting to perform spherical collapse simulation in Einstein-æther theory in a different foliation than that used in Ref. [13]. Such simulations would also effectively describe collapse in (low energy) Hořava gravity with additional regularity conditions at the center/universal horizon that determine the purely elliptic part of the evolution problem. Hence, they could shed light in the dynamical formation and evolution of universal horizons.

ACKNOWLEDGMENTS

APPENDIX: DIFFEOMORPHISM INVARIANCE AND THE SCALAR EQUATION

Consider the Lagrangian density $\mathcal{L}_1[\mathsf{g}^{ab}, \phi]$ and $\mathcal{L}_2[\mathsf{g}^{ab}, u^d]$ that is a functional of the metric g^{ab} and the scalar field ϕ. The action of the diffeomorphism ξ^b on $\mathcal{L}_1$ can be written as

$$£_\xi \mathcal{L}_1[\mathsf{g}^{ab}, \phi] = \frac{\delta \mathcal{L}_1}{\delta \mathsf{g}^{ab}} £_\xi \mathsf{g}^{ab} + \frac{\delta \mathcal{L}_1}{\delta \phi} £_\xi \phi$$
$$\approx 2\left(\nabla^a \frac{\delta \mathcal{L}_1}{\delta \mathsf{g}^{ab}}\right) \xi^b + \frac{\delta \mathcal{L}_1}{\delta \phi} \xi^b \nabla_b \phi, \quad \text{(A1)}$$

where $\approx$ is used to denote equality up to total divergences, which we are willing to neglect here. Note that $\delta \mathcal{L}_1 / \delta \mathsf{g}^{ab} = 0$ and $\delta \mathcal{L}_1 / \delta \phi = 0$ are by definition the field equations of the metric and the scalar, respectively. Hence, when the Lagrangian densities are invariant under diffeomorphisms, the field equation for the scalar follows from the divergence of the field equation of the metric and vice versa, provided that ϕ is not constant.

The same calculation yields a different result when ϕ is replaced by a field with a higher tensorial rank. This is because the Lie derivative does not reduce to a directional derivative along the generator, as is the case for a scalar field. We will not repeat the calculation here, as it is essentially the same calculation that leads to Eqs. (22). For higher rank tensors, e.g. vectors, having the field on shell implies that $\delta \mathcal{L}_1 / \delta \mathsf{g}^{ab}$ is divergence-free but the converse is not true.

[1] T. Jacobson and D. Mattingly, Gravity with a dynamical preferred frame, Phys. Rev. D **64**, 024028 (2001).

[2] P. Horava, Quantum gravity at a Lifshitz point, Phys. Rev. D **79**, 084008 (2009).

[3] J. W. Elliott, G. D. Moore, and H. Stoica, Constraining the new æther: Gravitational Čerenkov radiation, J. High Energy Phys. 08 (2005) 066.

[4] C. Eling and T. Jacobson, Black holes in Einstein-æther theory, Classical Quantum Gravity **23**, 5643 (2006); **27**, 049802 (2010).

[5] D. Blas and S. Sibiryakov, Hořava gravity versus thermodynamics: The black hole case, Phys. Rev. D **84**, 124043 (2011).

[6] E. Barausse, T. Jacobson, and T. P. Sotiriou, Black holes in Einstein-æther and Hořava-Lifshitz gravity, Phys. Rev. D **83**, 124043 (2011).

[7] J. Bhattacharyya, M. Colombo, and T. P. Sotiriou, Causality and black holes in spacetimes with a preferred foliation, arXiv:1509.01558.

[8] P. Berglund, J. Bhattacharyya, and D. Mattingly, Mechanics of universal horizons, Phys. Rev. D **85**, 124019 (2012).

[9] E. Barausse and T. P. Sotiriou, Slowly rotating black holes in Hořava-Lifshitz gravity, Phys. Rev. D **87**, 087504 (2013).

[10] T. P. Sotiriou, I. Vega, and D. Vernieri, Rotating black holes in three-dimensional Hořava gravity, Phys. Rev. D **90**, 044046 (2014).

[11] T. Jacobson, Extended Hořava gravity and Einstein-æther theory, Phys. Rev. D **81,** 101502 (2010); **82,** 129901 (2010).

[12] E. Barausse and T. P. Sotiriou, A No-Go Theorem for Slowly Rotating Black Holes in Hořava-Lifshitz Gravity, Phys. Rev. Lett. **109,** 181101 (2012); **110,** 039902 (2013).

[13] D. Garfinkle, C. Eling, and T. Jacobson, Numerical simulations of gravitational collapse in Einstein-æther theory, Phys. Rev. D **76,** 024003 (2007).

[14] M. Saravani, N. Afshordi, and R. B. Mann, Dynamical emergence of universal horizons during the formation of black holes, Phys. Rev. D **89,** 084029 (2014).

[15] D. Blas, O. Pujolas, and S. Sibiryakov, Consistent Extension of Hořava Gravity, Phys. Rev. Lett. **104,** 181302 (2010).

[16] T. P. Sotiriou, M. Visser, and S. Weinfurtner, Phenomenologically Viable Lorentz-Violating Quantum Gravity, Phys. Rev. Lett. **102,** 251601 (2009).

[17] T. P. Sotiriou, M. Visser, and S. Weinfurtner, Quantum gravity without Lorentz invariance, J. High Energy Phys. 10 (2009) 033.

[18] S. Weinfurtner, T. P. Sotiriou, and M. Visser, Projectable Hořava-Lifshitz gravity in a nutshell, J. Phys. Conf. Ser. **222,** 012054 (2010).

[19] S. Mukohyama, Hořava-Lifshitz cosmology: A review, Classical Quantum Gravity **27,** 223101 (2010).

[20] D. Vernieri and T. P. Sotiriou, Hořava-Lifshitz gravity: Detailed balance revisited, Phys. Rev. D **85,** 064003 (2012).

[21] D. Vernieri and T. P. Sotiriou, Hořava-Lifshitz gravity with detailed balance, J. Phys. Conf. Ser. **453,** 012022 (2013).

[22] T. P. Sotiriou, Hořava-Lifshitz gravity: A status report, J. Phys. Conf. Ser. **283,** 012034 (2011).

[23] T. P. Sotiriou, M. Visser, and S. Weinfurtner, Lower-dimensional Hořava-Lifshitz gravity, Phys. Rev. D **83,** 124021 (2011).

[24] D. Blas, O. Pujolas, and S. Sibiryakov, Models of non-relativistic quantum gravity: The good, the bad and the healthy, J. High Energy Phys. 04 (2011) 018.

[25] J. Bhattacharyya and D. Mattingly, Universal horizons in maximally symmetric spaces, Int. J. Mod. Phys. D **23,** 1443005 (2014).

[26] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).

[27] T. Jacobson, Initial value constraints with tensor matter, Classical Quantum Gravity **28,** 245011 (2011).

[28] W. Donnelly and T. Jacobson, Hamiltonian structure of Hořava gravity, Phys. Rev. D **84,** 104019 (2011).