SUPPLEMENTARY MATERIALS: MULTISCALE ANALYSIS OF NUTRIENT UPTAKE BY PLANT ROOTS WITH SPARSE DISTRIBUTION OF ROOT HAIRS: NONSTANDARD SCALING *

JOHN KING†, JAKUB KÖRY‡, AND MARIYA PTASHNYK§

SM1. Parameter values. The scaling in the boundary conditions on Γ^ε should be interpreted in terms of the experimental values for nutrient uptake rates by root hairs for different plant types. Considering the nondimensionalization of dimensional Michaelis-Menten boundary condition

$$-D\nabla u \cdot \hat{\mathbf{n}} = \frac{F_h u}{K_h + u}$$

via $x = R\widetilde{x}$, $t = R^2\widetilde{t}/D$, $u = K_h\widetilde{u}$ gives

$$-\widetilde{\nabla}\widetilde{u} \cdot \mathbf{n} = \frac{F_h R}{K_h D}\frac{\widetilde{u}}{1+\widetilde{u}} = \frac{\varepsilon}{a_\varepsilon}\frac{r_h R^2 F_h}{K_h l^2 D}\frac{\widetilde{u}}{1+\widetilde{u}} = \frac{\varepsilon}{a_\varepsilon}\widetilde{\alpha}\frac{\widetilde{u}}{1+\widetilde{u}}, \tag{SM1.1}$$

where r_h denotes the dimensional hair radius, l denotes the dimensional inter-hair distance and $\widetilde{\alpha} = (r_h R^2 F_h)/(K_h l^2 D)$. Considering the range of phosphate uptake parameters F_h and K_h as reviewed in [SM1], and $D = 10^{-5}$ cm^2 s^{-1} [SM2], as well as $R = 1$ cm, $l = 0.01$ cm and $r_h \sim 10^{-4}$ cm, we conclude that $\widetilde{\alpha} = 10$ for wheat, while $\widetilde{\alpha} = 1$ arises when modelling sulphur and magnesium uptake by maize [SM3].

SM2. Derivation of macroscopic equations for nonlinear boundary conditions on root hair surfaces.

SM2.1. Case $\varepsilon \ln(1/a_\varepsilon) = O(1)$. Following the same procedure as in Section 3.2.1 of the main text, we obtain the same equations as in (3.29), but with different boundary conditions for u_2^I, u_3^I, and u_4^I, namely

$$\begin{aligned} &D_u\nabla_z u_2^I \cdot \hat{\mathbf{n}} = -\kappa g(u_0^I) \quad \text{on } \partial B_1, \quad D_u\nabla_z u_3^I \cdot \hat{\mathbf{n}} = -\kappa g'(u_0^I)u_1^I \quad \text{on } \partial B_1,\\ &D_u\nabla_z u_4^I \cdot \hat{\mathbf{n}} = -\kappa\big[g'(u_0^I)u_2^I + \frac{1}{2}g''(u_0^I)(u_1^I)^2\big] \quad \text{on } \partial B_1. \end{aligned} \tag{SM2.1}$$

Hence the corresponding solutions are

$$\begin{aligned} &u_j^I(t,x,z) = u_j^I(t,x), \quad j = 0,1, \quad u_2^I(t,x,z) = (\kappa/D_u)g(u_0^I)\ln(\|z\|) + U_2^I(t,x),\\ &u_3^I(t,x,z) = (\kappa/D_u)g'(u_0^I)u_1^I\ln(\|z\|) + U_3^I(t,x),\\ &u_4^I(t,x,z) = (\kappa/D_u)\big[g'(u_0^I)U_2^I(t,x) + \frac{1}{2}g''(u_0^I)(u_1^I)^2\big]\ln(\|z\|) + U_4^I(t,x). \end{aligned}$$

Then by matching inner approximation u_2^I and outer approximation u_2^O we obtain for u_2^O equation (3.32) with $g(u_0^I)$ instead of u_0^I and for u_0^O equation (3.33) with $g(u_0^I)$

*Submitted to the editors DATE.

Funding: Jakub Köry and John King acknowledge funding from FUTUREROOTS Project (project ID: 294729) between European Research Council and The University of Nottingham.

†School of Mathematical Sciences & Centre for Plant Integrative Biology, School of Biosciences, University of Nottingham, Nottingham NG7 2QL, United Kingdom (john.king@nottingham.ac.uk).

‡ School of Mathematics & Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, United Kingdom (jakub.koery@glasgow.ac.uk).

§School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AL, United Kingdom (m.ptashnyk@hw.ac.uk).

instead of u_0^I. We also obtain the same matching condition (3.45). Hence we obtain an effective equation

$$\partial_t u_0 = \nabla_x \cdot (D_u \nabla_x u_0) - 2\pi\kappa\, g(u_0)\, \chi_{\Omega_L} \quad \text{in } \Omega,\ t > 0, \tag{SM2.2}$$

Case $\varepsilon^2 \ln(1/a_\varepsilon) = O(1)$**.** Applying the formal asymptotic expansion ansatz (3.25) in multiscale problem (2.1)–(2.3), (2.6), (2.7) again yields (3.53), equipped here with the modified boundary condition

$$\begin{aligned}
&\Big(e^{\lambda/\varepsilon^2}\varepsilon^{-1} D_u \nabla_z + D_u \nabla_x\Big)(u_0 + \varepsilon u_1 + \cdots)\cdot \hat{\mathbf{n}} = -\varepsilon\,\kappa\, e^{\lambda/\varepsilon^2} g\big(u_0 + \varepsilon u_1 + \cdots\big)\\
&= -\varepsilon\,\kappa\, e^{\lambda/\varepsilon^2}\big[g(u_0) + \varepsilon g'(u_0)u_1 + \varepsilon^2 g'(u_0)u_2 + \varepsilon^2 \frac{1}{2} g''(u_0)u_1^2 + \cdots\big] \quad \text{on } \Omega_L \times \partial B_1.
\end{aligned}$$

In the case of inner solutions, for u_0^I and u_1^I we have the same equations and boundary conditions as in (3.29) and for u_2^I, u_3^I, and u_4^I we obtain the same equations as in (3.29) but with different boundary conditions

$$\begin{aligned}
&D_u \nabla_z u_2^I \cdot \hat{\mathbf{n}} = -\kappa g(u_0^I) && \text{on } \partial B_1,\\
&D_u \nabla_z u_3^I \cdot \hat{\mathbf{n}} = -\kappa g'(u_0^I)u_1^I && \text{on } \partial B_1,\\
&D_u \nabla_z u_4^I \cdot \hat{\mathbf{n}} = -\kappa\big[g'(u_0^I)u_2^I + \frac{1}{2} g''(u_0^I)(u_1^I)^2\big] && \text{on } \partial B_1.
\end{aligned} \tag{SM2.3}$$

Hence the inner approximation reads

$$\begin{aligned}
u_\varepsilon^I(t,x) &= u_0^I(t,x) + \varepsilon u_1^I(t,x) + \varepsilon^2 U_2^I(t,x) + \varepsilon^2(\kappa/D_u) g(u_0^I)\ln(\|z\|)\\
&\quad + \varepsilon^3\Big[(\kappa/D_u) g'(u_0^I)\, u_1^I \ln(\|z\|) + U_3^I(t,x)\Big]\\
&\quad + \varepsilon^4\Big[\frac{\kappa}{D_u}\big(g'(u_0^I)\, U_2^I + \frac{1}{2} g''(u_0^I)(u_1^I)^2\big)\ln(\|z\|) + U_4^I(t,x)\Big] + \cdots.
\end{aligned} \tag{SM2.4}$$

Then in terms of outer variables y the inner approximation u_ε^I has the form

$$\begin{aligned}
u_\varepsilon^I &= \Big(u_0^I + \lambda \frac{\kappa}{D_u} g(u_0^I)\Big) + \varepsilon\Big(u_1^I + \lambda\frac{\kappa}{D_u} g'(u_0^I)u_1^I\Big)\\
&\quad + \varepsilon^2\big[U_2^I + \frac{\kappa}{D_u} g(u_0^I)\ln(\|y\|) + \lambda\frac{\kappa}{D_u}\big(g'(u_0^I)U_2^I + \frac{1}{2} g''(u_0^I)(u_1^I)^2\big)\big] + \cdots.
\end{aligned}$$

In the same way as in Subsection 3.2.2, for the outer approximation we obtain

$$u_\varepsilon^O(t,x) = u_0^O(t,x) + \varepsilon u_1^O(t,x) + \varepsilon^2\Big(U_2^O(t,x) + 2\pi(\kappa/D_u) g(u_0^I(t,x))\psi(y)\Big) + \cdots.$$

Then the matching condition for inner and outer solutions for zero order terms implies

$$u_0^O(t,x) = u_0^I(t,x) + \lambda(\kappa/D_u) g(u_0^I(t,x)), \tag{SM2.5}$$

and the macroscopic equation for $u_0(t,x) = u_0^O(t,x)$ reads

$$\partial_t u_0 = \nabla_x \cdot (D_u \nabla_x u_0) - 2\pi\kappa\, g(h(u_0))\chi_{\Omega_L} \quad \text{in } \Omega,\ t > 0, \tag{SM2.6}$$

where $h = h(u_0)$ is the solution of $u_0 = h + \lambda\,(\kappa/D_u) g(h)$.

Adopting the Michaelis-Menten boundary condition (2.4), condition (SM2.5) can be rewritten as a quadratic equation

$$(u_0^I)^2 + u_0^I\big(\lambda(\kappa/D_u) + 1 - u_0^O\big) - u_0^O = 0, \tag{SM2.7}$$

with unique non-negative solution

$$u_0^I = \frac{1}{2}\Big[\sqrt{(u_0^O - \lambda(\kappa/D_u) - 1)^2 + 4u_0^O} + u_0^O - \lambda\frac{\kappa}{D_u} - 1\Big],$$

and the effective equation (SM2.6) thus becomes (3.64).

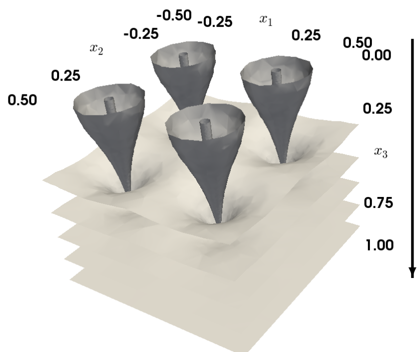

Fig. SM1: Isosurfaces of nutrient concentration support the intuition that with the chosen boundary conditions, the (steady-state) solution has the same behavior in every periodicity cell ($a_\varepsilon = 0.01$, $\varepsilon = 0.5$). The arrow points in the direction of increasing x_3 (i.e. away from the root surface located at $x_3 = 0$).

REFERENCES

[SM1] D. LEITNER, S. KLEPSCH, M. PTASHNYK, A. MARCHANT, G.J.D. KIRK, A. SCHNEPF, and T. ROOSE. A dynamic model of nutrient uptake by root hairs. *New Phytol.*, 185(3):792–802, 2010.

[SM2] P. S. NOBEL. *Physicochemical and Environmental Plant Physiology*. Academic Press, Oxford, 2009.

[SM3] T. ROOSE. *Mathematical model of plant nutrient uptake*. PhD thesis, Oxford University, Mathematical Institute, 2000.