Date-stamping Multiple Bubble Regimes[*]

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January 10, 2020

Abstract

Identifying the start and end dates of explosive bubble regimes has become a prominent issue in the econometric literature. Recent research has demonstrated the advantage of a model-based minimum sum of squared residuals estimator, combined with Bayesian Information Criterion model selection, over recursive unit root testing methods in providing accurate date estimates for a single explosive regime. However, in the context of multiple bubbles, a large number of models are possible, making such a model-based method unappealing. In this paper, we propose a two-step procedure for dating multiple explosive regimes. First, recursive unit root tests are used to identify a 'date window' in which an explosive episode starts and ends. Second, a model-based BIC approach is used to precisely estimate the regime change points within each date window. In addition, our method allows us to distinguish between different types of explosive episode, such as whether or not each explosive regime crashes before reverting back to a unit root process, and date any crash regimes. Monte Carlo simulations highlight the effectiveness of our procedure when compared to existing methods of dating. The value of the new methodology is also demonstrated through an empirical application to housing markets.

Keywords: Explosive autoregression; Break date estimation; Multiple bubbles

JEL Classification: C13; C22; G14

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1 Introduction

The role that asset price bubbles play in financial instability has become increasingly apparent in recent years as a result of events such as the US housing price bubble of the mid-2000s which triggered the financial crisis that followed. Indeed, as Bernanke (2013) notes, whilst it is unavoidable that there will periodically be bubbles in the financial system, now that the damage these bubbles can cause has been realised, it is crucial that central banks and other regulators take their emergence seriously. This increasing interest in asset price bubbles has corresponded with a renewed focus within the econometric literature, and there is now a substantial body of work regarding econometric detection and dating of asset price bubbles. Much of this literature concerns rational bubbles, where investing in an asset can be a rational choice even when the price is not justified by the underlying fundamental value of the asset, due to a belief that its price will continue to increase beyond the price paid. In such models, a rational bubble is present if explosive characteristics are manifest in the time path of prices, but not for the corresponding fundamentals. Explosive dynamics in prices can be modelled by an autoregression with parameter greater than one, therefore much attention has recently been paid to detecting explosive autoregressive behaviour in price series.

In a now seminal paper, Phillips et al. (2011) [PWY] propose a test for explosive autoregressive behaviour through taking the supremum of forward recursive right-tailed unit root tests applied to prices. The PWY test is designed for circumstances in which a maximum of one explosive regime is present in a series. Of course, in practice, it is possible that over a given period of time a series may contain more than one explosive regime, and Phillips et al. (2015a) [PSY] propose an extension of the PWY methodology capable of detecting multiple explosive regimes based on forward and backward recursions of right-tailed unit root tests.[1] The PSY approach offers valuable improvements over the original PWY methodology and has now become the standard tool for detecting explosive autoregressive regimes in applied work.

An equally important issue to the detection of explosive behaviour is determining the timing of such regimes. Accurate identification of the start and end points of historical explosive regimes is vital for a number of reasons. First, by understanding the timing of explosive dynamics in a given price series, we can reconcile this with other macroeconomic events. Next, by analysing the start and end dates of explosive regimes, we are able to construct timelines of explosive behaviour across countries or across different classes of assets. These timelines can provide useful information about the evolution of explosive behaviour throughout the economy. Finally, if we know the dates over which a price series is and is not explosive, we can create an indicator for that asset which can then be used as a variable in regression models, to analyse the determinants of explosive behaviour, or the effect that explosiveness in a particular asset has on other variables of interest. Clearly the validity of any of the above analyses requires precise date estimation of the explosive regimes.

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[1]Limit theory for the PSY procedure is developed in the companion paper Phillips et al. (2015b).

PWY first proposed a method for dating the start and end point of a single explosive regime, based on sequences of recursive unit root test statistics compared to threshold critical values. Using this approach, PWY find evidence of a stock market bubble in the NASDAQ index from February 1973 - June 2005. In line with their extension to the PWY detection procedure, PSY also extend the PWY dating methodology to allow dating the start and end points of multiple explosive regimes. PSY apply their dating procedure to the S&P 500 stock market index and identify multiple periods of bubble behaviour corresponding to macroeconomic events such as Black Monday, the dot-com bubble, and the sub-prime mortgage crisis. The dating procedure of PSY has now superseded that of PWY, and has been widely used to estimate the timing of bubbles in a variety of asset markets, for example housing (Anundsen et al., 2016; Pavlidis et al., 2016), commodities (Etienne et al., 2014, 2015; Figuerola-Ferretti and McCrorie, 2016), oil and energy markets (Tsvetanov et al., 2016; Pavlidis et al., 2018; Sharma and Escobari, 2018; Figuerola-Ferretti et al., 2019), exchange rates (Hu and Oxley, 2017), and cryptocurrencies (Corbet et al., 2018).

In the context of a single explosive regime, in a paper in this journal, Harvey et al. (2017) [HLS] propose an alternative method of date-stamping to that proposed in PWY and PSY, where the regime change points are estimated using minimum sum of squared residual estimators and a Bayesian Information Criterion (BIC) model selection procedure. Specifically, the HLS procedure allows for selection between a number of different types of explosive episodes, where the explosive regime may be followed by a stationary collapse regime, may revert back to pre-explosive conditions without a stationary collapse, or may still be ongoing at the end of the sample. The estimated dates of regime change are found to be consistent, and it is demonstrated that this BIC-based procedure delivers much improved dating accuracy in comparison to the recursive unit root based method of PSY. When a collapse regime is present, it also provides a consistent estimate of the end of collapse regime date (an issue not considered by PWY and PSY).

Whilst the BIC procedure of HLS offers superior dating accuracy, a limitation of the procedure is that it is designed to date only a single explosive regime in a time series. In this paper we consider extension of the HLS-type approach to the multiple explosive regime context. When the possibility of multiple explosive regimes is entertained, where each regime may or may not terminate in a collapse, the model-based nature of the BIC procedure becomes challenging, with the number of potential models to be considered growing exponentially in the number of explosive regimes. Selecting between such a large number of models is unappealing in practice and would not be expected to work well in finite samples without the imposition of some restrictions. Instead, we propose a dating methodology based on minimum sum of squared residual estimators and BIC model selection, but using prior information gleaned from the PSY dating procedure as a means of reducing the dimensionality of the model selection and date estimation problem. Specifically, we propose a two-step procedure, where, assuming that pre-testing for the presence of explosive behaviour has occurred, in the first stage we apply

the PSY dating procedure, and use these preliminary date estimates to split our sample into a number of sub-sample 'date windows', each of which is assumed to contain a single explosive regime. In the second stage, we apply the HLS procedure to each date window, in order to obtain improved estimates of the regime change points. Monte Carlo simulations demonstrate the superior accuracy delivered by our two-step procedure in comparison to the dating methodology of PSY. In particular, we observe that whereas PSY has a tendency to date the regime change points later than they occur, our methodology is often able to identify the exact start and end points of each explosive regime, especially when considering end dates.

To demonstrate the effectiveness of our proposed date-stamping procedure, we undertake an analysis of explosive behaviour in the house prices of 22 countries from 1975:Q1-2018:Q2. We find that 20 of these countries contain at least one explosive episode within the time period, with a maximum of four explosive regimes detected in some countries. We show that our BIC date-stamping procedure typically finds earlier start dates and end dates for the explosive regimes than the PSY procedure. Our procedure also finds that many of the detected explosive regimes subsequently collapse, and is able to estimate the end point of the collapse. We next consider the extent to which episodes of explosive behaviour in the house price series can be interpreted as speculative bubbles. Following PWY, our interpretation of a bubble is the departure of prices away from an asset's fundamental value, and therefore the finding of explosive behaviour in house prices is not sufficient evidence of the presence of a bubble, without an analysis of house prices relative to fundamentals. Consequently, we repeat our analysis using price-to-fundamental ratios, with real personal disposable income employed as a proxy of the fundamental of housing. We note several key phases in the behaviour of house prices across countries. The first phase corresponds to the house price bubble seen in many countries in the late 1980s, the second phase describes the house price bubble in the mid-2000s whose collapse preceded the global financial crisis. Finally, when considering the price series, we observe the emergence of explosive behaviour in a number of countries in the later part of the sample, with this phase still ongoing at the end of the sample in some cases. However, when considering the price-to-fundamental ratios instead, there is less evidence of end-of-sample explosiveness.

The next section outlines the multiple explosive regime framework that we consider. Section 3 presents our proposed date-stamping procedure. In Section 4 we present a finite sample Monte Carlo simulation comparison of the accuracy of our proposed methodology with the recursive unit root based method of PSY. An empirical application of our date-stamping procedure to housing markets is considered in Section 5. Section 6 concludes.

The following notation is used throughout the paper: $\lfloor . \rfloor$ denotes the integer part and $\mathbb{I}(.)$ denotes the indicator function.

2   Multiple Explosive Regime Model

To fix ideas, we first consider the following model for a time series $\{y_t\}$, $t = 1, ..., T$, that permits a single explosive autoregressive regime in an otherwise unit root process:

$$
\begin{aligned}
y_t &= \mu + u_t \\
u_t &= (1 + \rho_t)u_{t-1} + v_t \\
\rho_t &= \rho^* \mathbb{I}(\lfloor \tau_1^* T \rfloor < t \leqslant \lfloor \tau_2^* T \rfloor)
\end{aligned}
\tag{1}
$$

with $u_0 = 0$ and $\{v_t\}$ a zero mean weakly dependent process satisfying Assumption 1 of HLS, $\rho^* > 0$ and $0 < \tau_1^* < \tau_2^* < 1$. Such a specification implies that y_t follows an underlying unit root process up to time $t = \lfloor \tau_1^* T \rfloor$, after which it follows an explosive autoregression up to time $t = \lfloor \tau_2^* T \rfloor$, before reverting back to unit root behaviour for the remainder of the sample. This form of temporary explosive autoregression was first considered by PWY as a foundation for testing and dating a single period of explosivity. PWY augment the specification above to permit an instantaneous level adjustment, or collapse, at time $t = \lfloor \tau_2^* T \rfloor + 1$, while HLS allow for a more general dynamic model for the collapse adjustment by incorporating a post-explosive stationary autoregressive regime after time $t = \lfloor \tau_2^* T \rfloor$, i.e. (1) is replaced with

$$
\rho_t = \rho_1^* \mathbb{I}(\lfloor \tau_1^* T \rfloor < t \leqslant \lfloor \tau_2^* T \rfloor) + \rho_2^* \mathbb{I}(\lfloor \tau_2^* T \rfloor < t \leqslant \lfloor \tau_3^* T \rfloor)
$$

where $\rho_1^* > 0$, $\rho_2^* < 0$ and $0 < \tau_1^* < \tau_2^* < \tau_3^* < 1$. The mean reversion implicit in the stationary regime generates a model of a post-explosive collapse period, as the underlying autoregressive process for y_t creates an exponential decay from the final explosive observation towards its mean μ, thereby offsetting the explosive period to an extent dependent on the magnitude of ρ_2^* and the collapse duration $\lfloor \tau_3^* T \rfloor - \lfloor \tau_2^* T \rfloor$. Such a model allows for a wide spectrum of collapse regimes, including short-lived sharp collapses (large in magnitude negative ρ_2^* and small $\lfloor \tau_3^* T \rfloor - \lfloor \tau_2^* T \rfloor$) and long, slower downward readjustments (small in magnitude negative ρ_2^* and large $\lfloor \tau_3^* T \rfloor - \lfloor \tau_2^* T \rfloor$). Modelling the collapse in this manner (as opposed to an instantaneous level adjustment) is appealing empirically, as it allows flexibility in the speeds at which different financial agents react at the end of an explosive regime. When the collapse regime finishes, the y_t series no longer reverts towards μ, and unit root behaviour resumes from whatever level was reached at the end of the collapse adjustment period; that is, typical market conditions resume thereafter.

While PSY consider a multiple explosive regime variant of the PWY model (including instantaneous collapses), in this paper we build on the HLS foundation due to its more flexible collapse specification. We therefore consider the following model with multiple explosive and

collapse regimes:

$$y_t = \mu + x_t + u_t \tag{2}$$

$$u_t = (1 + \rho_t)u_{t-1} + v_t \tag{3}$$

$$\rho_t = \sum_{j=1}^{N} \{\rho_{j1}^* \mathbb{I}(\lfloor \tau_{j1}^* T \rfloor < t \leqslant \lfloor \tau_{j2}^* T \rfloor) + \rho_{j2}^* \mathbb{I}(\lfloor \tau_{j2}^* T \rfloor < t \leqslant \lfloor \tau_{j3}^* T \rfloor) - \mathbb{I}(t = \lfloor \tau_{j3}^* T \rfloor + 1)\} \tag{4}$$

$$x_t = \sum_{j=1}^{N} u_{\lfloor \tau_{j3}^* T \rfloor} \mathbb{I}(t > \lfloor \tau_{j3}^* T \rfloor) \tag{5}$$

where the inclusion of x_t in (2) and the additional indicator function in (4) are introduced to prevent the magnitude of one explosive regime from entering the dynamics of subsequent explosive regimes (see discussion below). Our specification (2)-(5) implies that y_t follows an underlying unit root process but is subject to N explosive regimes, each of which can terminate with a collapse regime. We assume that, for a given j, $\rho_{j1}^* \geq 0$ and $\rho_{j2}^* \leq 0$ allowing for an explosive regime immediately followed by a stationary (collapse) regime, cf. Harvey et al. (2016) and Phillips and Shi (2018). We also impose $\tau_{11}^* > 0$, $\tau_{N3}^* \leq 1$, $\tau_{(j+1)1}^* > \tau_{j3}^*$ to ensure an initial unit root regime prior to the onset of explosive behaviour, together with an ordering of explosive regimes within the sample period. Note that we implicitly assume that stationary regimes do not exist without an immediately prior explosive regime; also, if the jth explosive regime does not end in collapse (i.e. $\rho_{j2}^* = 0$ such that unit root conditions resume directly after the explosive regime), we set $\tau_{j3}^* = \tau_{j2}^*$. The model also permits the Nth explosive (or collapse) regime to run to the end of the sample period, by letting $\tau_{N2}^* = 1$ (or $\tau_{N3}^* = 1$). We do not assume knowledge of N or any of the corresponding $\tau_{j1}^*, \tau_{j2}^*, \tau_{j3}^*$, hence we allow for an unknown number of explosive regimes at unknown times, with or without collapse.

For illustration, the model for x_t and u_t in the example case of $N = 2$ (with μ set to zero for simplicity) expands to the following:

$$
\begin{array}{lll}
x_t = 0 & u_t = v_t & t = 1 \\
x_t = 0 & u_t = u_{t-1} + v_t & t = 2, ..., \lfloor \tau_{11}^* T \rfloor \\
x_t = 0 & u_t = (1 + \rho_{11}^*)u_{t-1} + v_t & t = \lfloor \tau_{11}^* T \rfloor + 1, ..., \lfloor \tau_{12}^* T \rfloor \\
x_t = 0 & u_t = (1 + \rho_{12}^*)u_{t-1} + v_t & t = \lfloor \tau_{12}^* T \rfloor + 1, ..., \lfloor \tau_{13}^* T \rfloor \\
x_t = u_{\lfloor \tau_{13}^* T \rfloor} & u_t = v_t & t = \lfloor \tau_{13}^* T \rfloor + 1 \\
x_t = u_{\lfloor \tau_{13}^* T \rfloor} & u_t = u_{t-1} + v_t & t = \lfloor \tau_{13}^* T \rfloor + 2, ..., \lfloor \tau_{21}^* T \rfloor \\
x_t = u_{\lfloor \tau_{13}^* T \rfloor} & u_t = (1 + \rho_{21}^*)u_{t-1} + v_t & t = \lfloor \tau_{21}^* T \rfloor + 1, ..., \lfloor \tau_{22}^* T \rfloor \\
x_t = u_{\lfloor \tau_{13}^* T \rfloor} & u_t = (1 + \rho_{22}^*)u_{t-1} + v_t & t = \lfloor \tau_{22}^* T \rfloor + 1, ..., \lfloor \tau_{23}^* T \rfloor \\
x_t = u_{\lfloor \tau_{13}^* T \rfloor} + u_{\lfloor \tau_{23}^* T \rfloor} & u_t = v_t & t = \lfloor \tau_{23}^* T \rfloor + 1 \\
x_t = u_{\lfloor \tau_{13}^* T \rfloor} + u_{\lfloor \tau_{23}^* T \rfloor} & u_t = u_{t-1} + v_t & t = \lfloor \tau_{23}^* T \rfloor + 2, ..., T
\end{array}
$$

Specifically, y_t starts as a unit root process, then if $\rho_{11}^* > 0$, an explosive regime exists over $\lfloor \tau_{11}^* T \rfloor + 1 \leqslant t \leqslant \lfloor \tau_{12}^* T \rfloor$, with AR(1) parameter $1 + \rho_{11}^*$. If $\rho_{12}^* < 0$, this is then followed

by a stationary collapse over $\lfloor \tau_{12}^* T \rfloor + 1 \leqslant t \leqslant \lfloor \tau_{13}^* T \rfloor$, with AR(1) parameter $1 + \rho_{12}^*$; after the collapse regime, y_t returns to a unit root process. If, instead, $\rho_{12}^* = 0$, then no collapse regime occurs and y_t returns to a unit root process after $\lfloor \tau_{12}^* T \rfloor$ $(= \lfloor \tau_{13}^* T \rfloor)$. Then if $\rho_{21}^* > 0$, a second explosive regime exists over $\lfloor \tau_{21}^* T \rfloor + 1 \leqslant t \leqslant \lfloor \tau_{22}^* T \rfloor$, followed by potential collapse over $\lfloor \tau_{22}^* T \rfloor + 1 \leqslant t \leqslant \lfloor \tau_{23}^* T \rfloor$, before unit root dynamics return until the end of sample. Note that at the two points $t = \lfloor \tau_{13}^* T \rfloor + 1$ and $t = \lfloor \tau_{23}^* T \rfloor + 1$, the level of the process is recalibrated to account for the process level at the end of each explosive/collapse regime, while the u_t recursion is recalibrated to v_t. This has the effect of preventing the magnitude of the first explosive regime from entering the dynamics of the second explosive regime (and so on in a model with more than two explosive periods), thereby ensuring that the stochastic orders of magnitude (in T) of multiple explosive regimes remain the same. Without such recalibrations, equal magnitudes of ρ_{11}^* and ρ_{21}^* will result in much more pronounced explosivity for the second regime compared to the first.

Our setup assumes that each explosive regime is preceded by a unit root regime, ensuring that, following one explosive period (with possible collapse), unit root behaviour is restored before the emergence of the next. Given this, we can define the jth "explosive episode" as the sub-sample containing the following behaviour: unit root followed by explosive followed by possible stationary collapse followed by unit root. Adapting the notation of HLS, we can then specify two possible DGPs for such explosive episodes for $j = 1, ..., N - 1$, where we denote by τ_{js} and τ_{je} the start and end points of the jth episode (with $\tau_{1s} = 0$):

DGP 2: $\tau_{js} < \tau_{j1}^* < \tau_{j2}^* < \tau_{je}$, $\tau_{j3}^* = \tau_{j2}^*$, $\rho_{j1}^* > 0$

                (unit root, then explosive, then unit root)

DGP 4: $\tau_{js} < \tau_{j1}^* < \tau_{j2}^* < \tau_{j3}^* < \tau_{je}$, $\rho_{j1}^* > 0$, $\rho_{j2}^* < 0$

                (unit root, then explosive, then stationary collapse, then unit root)

where the numbering of the DGPs mirrors that of HLS. For the final explosive episode, $j = N$ with $\tau_{je} = 1$, it is possible that the explosive or collapse regimes run to the end of the sample, hence in this case the possibilities are:

DGP 1: $\tau_{js} < \tau_{j1}^* < 1$, $\tau_{j2}^* = 1$, $\rho_{j1}^* > 0$

                (unit root, then explosive until sample end)

DGP 2: $\tau_{js} < \tau_{j1}^* < \tau_{j2}^* < 1$, $\tau_{j3}^* = \tau_{j2}^*$, $\rho_{j1}^* > 0$

                (unit root, then explosive, then unit root)

DGP 3: $\tau_{js} < \tau_{j1}^* < \tau_{j2}^* < 1$, $\tau_{j3}^* = 1$, $\rho_{j1}^* > 0$, $\rho_{j2}^* < 0$

                (unit root, then explosive, then stationary collapse until sample end)

DGP 4: $\tau_{js} < \tau_{j1}^* < \tau_{j2}^* < \tau_{j3}^* < 1$, $\rho_{j1}^* > 0$, $\rho_{j2}^* < 0$

                (unit root, then explosive, then stationary collapse, then unit root)

We emphasise that DGP 1 and DGP 3 do not imply that the explosive or collapse regime continues in perpetuity, but that any subsequent regime change occurs outside our observed sample period.

3   Date-Stamping Procedure

Before we begin to estimate the regime change points of the multiple explosive regime process y_t, we must first confirm that y_t does contain explosive behaviour. PSY propose a method for detecting temporary explosiveness based on forward and backward recursions of right-tailed Dickey-Fuller unit root tests. Let r_1 and r_2 denote fractions of the sample, such that $r_2 > r_1$, then $ADF_{r_1}^{r_2}$ denotes the OLS augmented DF test statistic, computed over $t = \lfloor r_1 T \rfloor, ..., \lfloor r_2 T \rfloor$. The GSADF test of PSY considers the supremum of a series of $ADF_{r_1}^{r_2}$ statistics where the end point over which the sub-sample statistics are computed varies over $r_2 \in [r_0, 1]$ where r_0 is the minimum window width, and the start point varies over $r_1 \in [0, r_2 - r_0]$. The GSADF statistic is given by

$$GSADF(r_0) = \sup_{r_2 \in [r_0,1], r_1 \in [0, r_2 - r_0]} ADF_{r_1}^{r_2}$$

If the GSADF test statistic exceeds the relevant critical value then we infer that y_t exhibits at least one explosive regime.

Once the presence of explosive behaviour has been confirmed, our concern is the precise estimation of τ_{j1}^*, τ_{j2}^* and τ_{j3}^* for each explosive regime j. We propose a two-step dating procedure, which capitalises on the accuracy advantages offered by HLS in date-stamping a single explosive regime, and the capabilities of PSY to detect multiple explosive episodes.

The first step of our procedure is to apply the BSADF dating procedure of PSY, to obtain a preliminary estimate of explosive regime start and end dates. The BSADF statistic is given by:

$$BSADF_{r_2}(r_0) = \sup_{r_1 \in [0, r_2 - r_0]} ADF_{r_1}^{r_2}$$

The date fraction estimates are then given by:

$$\begin{aligned}
\widehat{\tau}_{j1}^{PSY} &= \inf_{r_2 \in [r_0, 1]} \left\{ r_2 : \ BSADF_{r_2}(r_0) > scv_{r_2} \right\} \\
\widehat{\tau}_{j2}^{PSY} &= \inf_{r_2 \in [\widehat{\tau}_{j1}^{PSY} + \ln(T)/T, 1]} \left\{ r_2 : \ BSADF_{r_2}(r_0) < scv_{r_2} \right\}
\end{aligned}$$

where scv_{r_2} is the critical value of the sup ADF statistic based on $\lfloor r_2 T \rfloor$ observations, and where $\widehat{\tau}_{j1}^{PSY}$ and $\widehat{\tau}_{j2}^{PSY}$ are estimates of τ_{j1}^* and τ_{j2}^* with the necessary restriction that $\widehat{\tau}_{(j+1)1}^{PSY} > \widehat{\tau}_{j2}^{PSY}$. That is, the start date fraction for the first $(j = 1)$ explosive episode is the first value of r_2 where $BSADF_{r_2}(r_0)$ exceeds its critical value. The end date fraction for the explosive regime is then given by the first value of r_2 (after a minimum regime length) where $BSADF_{r_2}(r_0)$ drops below the threshold critical value. The start date fraction for the second $(j = 2)$ explosive regime is then given by the first value of r_2, after τ_{12}^*, where $BSADF_{r_2}(r_0)$ exceeds the critical

value, and so on. In practice, the date estimation is obtained using integer dates, with $\left\lfloor \widehat{\tau}_{j1}^{PSY} T \right\rfloor$ and $\left\lfloor \widehat{\tau}_{j2}^{PSY} T \right\rfloor$ corresponding to estimates of the first observations of the explosive and post-explosive regimes respectively, i.e. estimates of $\left\lfloor \tau_{j1}^{*} T \right\rfloor + 1$ and $\left\lfloor \tau_{j2}^{*} T \right\rfloor + 1$ in the context of (2)-(5). For a valid explosive regime to be identified, we also require that $BSADF_{r_2}(r_0)$ exceed scv_{r_2} for a minimum of $\ln(T)$ consecutive dates. When implementing the GSADF detection and BSADF dating procedures, we adopt PSY's recommended minimum window width of $r_0 = 0.01 + 1.8/\sqrt{T}$. One feature we observed when implementing the dating procedure was that a long series of rejections may be interrupted by a very small number of non-rejections, giving the appearance of multiple explosive regimes, when in fact, the true DGP contained a single explosive episode. In empirical work, this phenomenon may be obvious from visual inspection of a price series, but it is difficult to apply such judgmental criteria in Monte Carlo simulations. In this paper we mitigate this issue by assuming that if up to 3 non-rejections are surrounded on either side by an explosive regime of length $\ln(T)$, then they can be treated as a single explosive episode, in essence joining these two runs of rejections together.

Once the estimates $\widehat{\tau}_{j1}^{PSY}$ and $\widehat{\tau}_{j2}^{PSY}$ have been obtained for $j = 1, ..., \hat{N}$, where $\hat{N}$ is the number of explosive regimes detected by PSY, we can use these estimates to split our full sample into $\hat{N}$ sub-sample date windows, so that each date window contains one explosive episode. The start and end points of these date windows (s_j and e_j) are given by

$$s_j = \begin{cases} 1 & j = 1 \\ e_{j-1} + 1 & j > 1 \end{cases}$$

and

$$e_j = \begin{cases} \lfloor \widehat{\tau}_{j2}^{PSY} T \rfloor + (\lfloor \widehat{\tau}_{(j+1)1}^{PSY} T \rfloor - \lfloor \widehat{\tau}_{j2}^{PSY} T \rfloor)/2 & j < \hat{N} \\ T & j = \hat{N} \end{cases}$$

such that we use the mid-point between the end of one explosive regime and the start of the next as the sample splitting point. In fact, once we have more information about the change points for regime j, we can adjust s_{j+1} in order to provide more accurate estimation of the change points of regime $j + 1$, as we will discuss later.

By splitting y_t into sub-samples on the basis of PSY date estimation, we can now assume that each date window contains a single explosive regime only, and therefore, in the second step of our date-stamping procedure, HLS-type date estimation can be applied to each date window in turn. As the model specifications that we consider (DGP 1 to DGP 4) all contain a period of explosiveness, this two-step procedure will always find the same number of explosive regimes, $\hat{N}$, as implementation of the PSY procedure alone. As a result, our procedure does not concern itself with detection of explosive processes, but focuses solely on improving the accuracy of date estimation and allowing for different model specifications for each explosive episode.

Following HLS, we first suppose that it is known which of DGP 1 to DGP 4 is the true DGP for each of the $j = 1, ..., \hat{N}$ explosive episodes in the full sample, and consider estimating the relevant change-points for each explosive episode by minimising the sum of squared residuals

across all candidate dates. We do this using the fitted OLS regression models below for the relevant DGP, recalling that Models 1 and 3 are only relevant for the final explosive regime $j = \hat{N}$:

$$\text{Model 1:} \quad \Delta y_t = \widehat{\mu}_1 D_t(\tau_{j1}, 1) + \widehat{\rho}_{j1} D_t(\tau_{j1}, 1) y_{t-1} + \widehat{v}_{1t}$$

$$\text{Model 2:} \quad \Delta y_t = \widehat{\mu}_1 D_t(\tau_{j1}, \tau_{j2}) + \widehat{\rho}_{j1} D_t(\tau_{j1}, \tau_{j2}) y_{t-1} + \widehat{v}_{2t}$$

$$\text{Model 3:} \quad \Delta y_t = \widehat{\mu}_1 D_t(\tau_{j1}, \tau_{j2}) + \widehat{\mu}_2 D_t(\tau_{j2}, 1) + \widehat{\rho}_{j1} D_t(\tau_{j1}, \tau_{j2}) y_{t-1}$$
$$+ \widehat{\rho}_{j2} D_t(\tau_{j2}, 1) y_{t-1} + \widehat{v}_{3t}$$

$$\text{Model 4:} \quad \Delta y_t = \widehat{\mu}_1 D_t(\tau_{j1}, \tau_{j2}) + \widehat{\mu}_2 D_t(\tau_{j2}, \tau_{j3}) + \widehat{\rho}_{j1} D_t(\tau_{j1}, \tau_{j2}) y_{t-1}$$
$$+ \widehat{\rho}_{j2} D_t(\tau_{j2}, \tau_{j3}) y_{t-1} + \widehat{v}_{4t}$$

where $t = s_j, ..., e_j$. The dummy variables are defined as $D_t(a, b) = 1(\lfloor aT \rfloor < t \leq \lfloor bT \rfloor)$, with the constant dummy variables associated with μ_1 and μ_2 ensuring invariance of the residuals of each model to the series mean μ. The regime change-point estimators obtained from these four models are then given by:

Model 1: $\widehat{\tau}_{j1} = \arg\min_{\tau_{j1}} SSR_{j1}(\tau_{j1})$

subject to $(s_j/T) < \tau_{j1} < 1$, $y_T > y_{\lfloor \tau_{j1} T \rfloor}$

Model 2: $(\widehat{\tau}_{j1}, \widehat{\tau}_{j2}) = \arg\min_{\tau_{j1}, \tau_{j2}} SSR_{j2}(\tau_{j1}, \tau_{j2})$

subject to $(s_j/T) < \tau_{j1} < \tau_{j2} < (e_j/T)$, $y_{\lfloor \tau_{j2} T \rfloor} > y_{\lfloor \tau_{j1} T \rfloor}$

Model 3: $(\widehat{\tau}_{j1}, \widehat{\tau}_{j2}) = \arg\min_{\tau_{j1}, \tau_{j2}} SSR_{j3}(\tau_{j1}, \tau_{j2})$

subject to $(s_j/T) < \tau_{j1} < \tau_{j2} < 1$, $y_{\lfloor \tau_{j2} T \rfloor} > y_{\lfloor \tau_{j1} T \rfloor}$, $y_{\lfloor \tau_{j2} T \rfloor} > y_T$

Model 4: $(\widehat{\tau}_{j1}, \widehat{\tau}_{j2}, \widehat{\tau}_{j3}) = \arg\min_{\tau_{j1}, \tau_{j2}, \tau_{j3}} SSR_{j4}(\tau_{j1}, \tau_{j2}, \tau_{j3})$

subject to $(s_j/T) < \tau_{j1} < \tau_{j2} < \tau_{j3} < (e_j/T)$, $y_{\lfloor \tau_{j2} T \rfloor} > y_{\lfloor \tau_{j1} T \rfloor}$, $y_{\lfloor \tau_{j2} T \rfloor} > y_{\lfloor \tau_{j3} T \rfloor}$

where $SSR_{ji}(.) = \Sigma_{t=s_j+1}^{e_j} \widehat{v}_{it}^2$ for $i = 1, ..., 4$ for Models 1-4 respectively. The constraint that $y_T > y_{\lfloor \tau_{j1} T \rfloor}$ for Model 1, and corresponding constraints on y_t for Models 2-4, ensure that we detect upwards, positive explosive regimes (followed by downwards collapse regimes in Models 3 and 4), given our interest in bubble detection and dating. We note, however, that these restrictions could be modified in an obvious way if negative episodes of explosivity were of interest to a practitioner (e.g. $y_T < y_{\lfloor \tau_{j1} T \rfloor}$ in Model 1). In a single explosive regime environment, HLS show that for correct pairings of the true DGP and the estimated model, the estimated regime start dates, end dates, and end of collapse dates are consistently estimated, and, under equivalent conditions to those in HLS, this same result applies in the current multiple explosive regime context.

Of course, accurate date estimation requires the selection of the correct model for each explosive regime j. Adapting HLS, we propose using a BIC-based model selection procedure in

which the chosen model is given by

$$m_j^{opt} = \begin{cases} \arg\min_{m \in \{2,4\}} BIC_{jm} & j = 1, ..., \hat{N} - 1 \\ \arg\min_{m \in \{1,2,3,4\}} BIC_{jm} & j = \hat{N} \end{cases}$$

where

$$\begin{aligned} BIC_{j1} &= T_j \ln\{T_j^{-1} SSR_{j1}(\hat{\tau}_{j1}, 1)\} + 3\ln(T_j) \\ BIC_{j2} &= T_j \ln\{T_j^{-1} SSR_{j2}(\hat{\tau}_{j1}, \hat{\tau}_{j2})\} + 4\ln(T_j) \\ BIC_{j3} &= T_j \ln\{T_j^{-1} SSR_{j3}(\hat{\tau}_{j1}, \hat{\tau}_{j2}, 1)\} + 6\ln(T_j) \\ BIC_{j4} &= T_j \ln\{T_j^{-1} SSR_{j4}(\hat{\tau}_{j1}, \hat{\tau}_{j2}, \hat{\tau}_{j3})\} + 7\ln(T_j) \end{aligned}$$

with T_j denoting the number of observations in the jth date window. As in HLS, the scalar multiplying the penalty $\ln(T_j)$ represents the number of coefficients being estimated plus the number of estimated regime change points in that model. Asymptotic results in HLS establish that model selection based on BIC will lead to the selection of the true model in the limit.

In a multiple explosive episode dating context, one challenge that presents itself is ensuring that our method of partitioning the sample separates the explosive episodes into different date windows. Due to the recursive nature of the PSY dating method, HLS demonstrate that the PSY date estimates tend to occur later than the true regime change points. When two explosive episodes (j and $j+1$) occur in quick succession, taking the mid-point of $\hat{\tau}_{j2}^{PSY}$ and $\hat{\tau}_{(j+1)1}^{PSY}$ may be problematic. This is because, if $\hat{\tau}_{j2}^{PSY} > \tau_{j2}^*$ and/or $\hat{\tau}_{(j+1)1}^{PSY} > \tau_{(j+1)1}^*$, then it is possible that $s_{j+1} > \lfloor \tau_{(j+1)1}^* T \rfloor$, in which case the $(j+1)$th date window begins with an explosive regime, making accurate dating of the start of the $(j+1)$th explosive regime impossible. To limit this potential problem, we require each date window to start with at least one observation of a unit root regime. Now for $j = 1, ..., \hat{N} - 1$, we only fit Models 2 and 4, hence the jth date window always ends with a fitted post-explosive/collapse unit root regime. Consequently, working sequentially, we can then adjust the starting point of the $(j+1)$th date window, i.e. s_{j+1}, to be equal to the first observation of the post-explosive/collapse unit root regime of the previous (jth) date window. Hence, we set $s_{j+1} = \lfloor \hat{\tau}_{j2} T \rfloor + 1$ or $s_{j+1} = \lfloor \hat{\tau}_{j3} T \rfloor + 1$ when the fitted model for the jth date window is Model 2 or Model 4, respectively.

Following HLS, we require that $\hat{\tau}_{j1} \geq \pi$, that $\hat{\tau}_{j2} - \hat{\tau}_{j1} \geq \pi$ and that $\hat{\tau}_{j3} - \hat{\tau}_{j2} \geq \pi/2$ imposing minimum length requirements on the initial unit root regime, explosive regime length, and length of collapse respectively; in this paper we set $\pi = 0.1$. However, as we split our sample into $\hat{N}$ date-windows, it is possible that the number of observations in a given date window, T_j, is small enough that $\lfloor \pi T_j \rfloor < 2$ and/or $\lfloor \pi T_j/2 \rfloor < 2$. In this multiple explosive episode context, we therefore further impose a lower bound of 2 for these minimum length requirements. Additionally, in line with HLS, we allow the final regime of any fitted model to be a minimum of 1 observation in length. Although we are primarily concerned with historical detection of explosivity, allowing the final regime to be only 1 observation, and fitting Model 1 or Model 3 (where either an explosive process or stationary collapse is ongoing at the end of

the sample), means that the procedure could be implemented in a real-time context to detect emerging explosive (or stationary collapse) behaviour. HLS illustrate this potential use of BIC date estimation through a pseudo-real-time monitoring application to the NASDAQ composite index.

4 Finite Sample Simulations

To assess the performance of our two-step BIC procedure in accurately date-stamping multiple explosive regimes, we conduct a finite sample Monte Carlo simulation exercise. For a single explosive episode, $N = 1$, our two-step procedure becomes identical to the BIC procedure of HLS, so we do not examine this case, instead focusing on $N = \{2, 3\}$. For the case of $N = 2$, we set the sample size to $T = 200$, and for $N = 3$ we use $T = 300$. We consider six DGPs which together demonstrate the performance of our two-step procedure over all model specifications, across different lengths of explosive and stationary regimes, across different locations of the regimes within the sample, and for different magnitudes of explosive behaviour and stationary collapse. Specifically, we generate simulated data from (2)-(5) with $v_t \sim iidN(0, 1)$, $\mu = 0$, and the following settings for each of the six DGPs considered:

A : $N = 2$: First explosive regime with collapse, Second explosive regime with collapse

$\quad\quad\quad j = 1 : \ (\tau_{11}^*, \tau_{12}^*, \tau_{13}^*) = (0.2, 0.3, 0.4), \ (\rho_{11}^*, \rho_{12}^*) = (0.1, -0.05)$

$\quad\quad\quad j = 2 : \ (\tau_{21}^*, \tau_{22}^*, \tau_{23}^*) = (0.6, 0.7, 0.8), \ (\rho_{21}^*, \rho_{22}^*) = (0.1, -0.05)$

B : $N = 2$: First explosive regime with collapse,

$\quad\quad\quad$ Second explosive regime with collapse running to sample end

$\quad\quad\quad j = 1 : \ (\tau_{11}^*, \tau_{12}^*, \tau_{13}^*) = (0.3, 0.5, 0.55), \ (\rho_{11}^*, \rho_{12}^*) = (0.05, -0.05)$

$\quad\quad\quad j = 2 : \ (\tau_{21}^*, \tau_{22}^*, \tau_{23}^*) = (0.75, 0.85, 1), \ (\rho_{21}^*, \rho_{22}^*) = (0.075, -0.05)$

C : $N = 2$: First explosive regime without collapse, Second explosive regime with collapse

$\quad\quad\quad j = 1 : \ (\tau_{11}^*, \tau_{12}^*) = (0.2, 0.3), \ \rho_{11}^* = 0.075$

$\quad\quad\quad j = 2 : \ (\tau_{21}^*, \tau_{22}^*, \tau_{23}^*) = (0.6, 0.7, 0.75), \ (\rho_{21}^*, \rho_{22}^*) = (0.075, -0.075)$

D : $N = 2$: First explosive regime without collapse, Second explosive regime running to sample end

$\quad\quad\quad j = 1 : \ (\tau_{11}^*, \tau_{12}^*) = (0.4, 0.5), \ \rho_{11}^* = 0.05$

$\quad\quad\quad j = 2 : \ (\tau_{21}^*, \tau_{22}^*) = (0.95, 1), \ \rho_{21}^* = 0.05$

$E:$ $N = 3:$ First explosive regime with collapse, Second explosive regime with collapse, Third explosive regime running to sample end

$j = 1:$ $(\tau_{11}^*, \tau_{12}^*, \tau_{13}^*) = (0.2, 0.35, 0.4),$ $(\rho_{11}^*, \rho_{12}^*) = (0.075, -0.075)$

$j = 2:$ $(\tau_{21}^*, \tau_{22}^*, \tau_{23}^*) = (0.6, 0.7, 0.75),$ $(\rho_{21}^*, \rho_{22}^*) = (0.075, -0.075)$

$j = 3:$ $(\tau_{31}^*, \tau_{32}^*) = (0.9, 1),$ $\rho_{31}^* = 0.075$

$F:$ $N = 3:$ First explosive regime with collapse, Second explosive regime with collapse, Third explosive regime with collapse running to sample end

$j = 1:$ $(\tau_{11}^*, \tau_{12}^*, \tau_{13}^*) = (0.3, 0.4, 0.5),$ $(\rho_{11}^*, \rho_{12}^*) = (0.075, -0.05)$

$j = 2:$ $(\tau_{21}^*, \tau_{22}^*, \tau_{23}^*) = (0.6, 0.7, 0.75),$ $(\rho_{21}^*, \rho_{22}^*) = (0.075, -0.05)$

$j = 3:$ $(\tau_{31}^*, \tau_{32}^*, \tau_{33}^*) = (0.85, 0.95, 1),$ $(\rho_{31}^*, \rho_{32}^*) = (0.075, -0.05)$

Figures 1-6 display histograms of the estimates of the start and end dates $\lfloor \tau_{j1}^* T \rfloor$ and $\lfloor \tau_{j2}^* T \rfloor$ from the PSY dating procedure and our two-step BIC procedure. We report results only for the replications where at least one explosive regime was detected according to the GSADF test performed at the 5% significance level. Here and throughout the paper, in the ADF statistics underlying the GSADF and BSADF statistics we use a fixed lag length of one (cf. HLS, PSY). When the test procedures found more explosive regimes than the true number of regimes, N, we report results for the estimated dates that are closest to the true DGP dates. To ensure that the generated explosive regimes are upward in direction, we only retain simulated processes for which $y_{\lfloor \tau_{j2}^* T \rfloor} > y_{\lfloor \tau_{j1}^* T \rfloor}$ $\forall j$. Monte Carlo simulations were carried out in Gauss 18 using 10,000 retained replications.

Consider first DGP A in Figure 1. Here, both explosive regimes are followed by a stationary collapse period, before unit root behaviour is restored. For both explosive regimes, the explosive and collapse phases last for 0.1 of the sample, although the magnitude of the collapse is smaller in absolute value than the magnitude of the explosive process. Examining the estimated start dates of the first explosive regime, it is clear that our BIC procedure correctly identifies the true start date more often than it estimates any other date, with a symmetric distribution around this date. In contrast, PSY estimates of the start date typically fall somewhat later than the true date. Turning our attention to the estimated end date of the first explosive regime, the BIC procedure shows excellent accuracy, with the estimated end date equalling the true end date in almost every replication. The variance of PSY end date estimates is also shown to be lower compared to its start date estimates, but as with the start dates, the estimated end point is late in most replications. Considering now the second explosive regime in the DGP, we note that the performance of the two procedures is near identical to that exhibited for the first explosive regime, with BIC offering superior date estimation of both start and end dates, thereby demonstrating the procedure's ability to accurately date multiple explosive episodes.

DGP B considers two explosive regimes, the first of which collapses before unit root be-

haviour resumes, whilst the second is followed by a collapse which continues until the end of the sample. The first explosive phase lasts for 0.2 of the sample, but is half the magnitude of the explosive phases considered in DGP A. This smaller magnitude results in a more dispersed distribution of estimated start dates for both the BIC and PSY procedures, although BIC still produces estimates equal to the true end date more frequently than any other date, whilst PSY start date estimates are generally later than the true start date. The explosive phases are followed by relatively short collapses of length 0.05. As in DGP A, the BIC estimated end dates equal the true end date in almost every replication, whilst PSY end date estimates are less accurate. The second explosive regime, although shorter in length, lasting for 0.1 of the sample, is of a higher magnitude than the first explosive regime; correspondingly the start date estimates of BIC are more tightly distributed around the true start date.

Consider now DGP C, where the first explosive regime does not end in a collapse, but instead reverts back to a unit root process once the explosive phase ends. In Figure 3, the same general pattern of behaviour is observed for the start date estimates, but it is apparent that when considering end dates, the absence of a collapse phase results in substantially less accurate estimation for the PSY procedure, whilst the BIC procedure still estimates the true end date more frequently than any other date, by some margin. DGP D features no stationary collapses for either explosive regime, with the first reverting back to a unit root process, and the second continuing to the end of the sample. Examining Figure 4, for the first regime, despite there being no stationary collapse following the explosive behaviour, BIC still estimates the true end date in many replications, whereas PSY end dates are noticeably more inaccurate, as in DGP C. Considering the second explosive regime, setting $\tau_{21}^* = 0.95$ such that the regime does not begin until very close to the end of the sample does not seem to have unduly affected the accuracy of the BIC procedure, with the distribution of estimated start dates still centred around the true start date. The second explosive regime continues to the end of the sample, and here the BIC and PSY procedures behave more similarly, as might be expected given that the tendency for the PSY estimated end date to be placed later than the true end date is not an issue here.

Considering now the DGPs containing three explosive episodes, Figure 5 displays the estimated dates for DGP E where the first two explosive regimes terminate in stationary collapses, whilst the third continues until the end of the sample. The first regime is placed early in the sample, with $\tau_{11}^* = 0.2$, and the third regime originates towards the end of the sample, with $\tau_{31}^* = 0.9$. A similar pattern of behaviour is observed as in the two explosive regime DGPs, with the distribution of BIC start date estimates being centred around the true values, and with BIC end date estimates being almost exclusively equal to the true end dates. Meanwhile, PSY date estimates are most often later than the true start and end values. Finally, we consider DGP F in Figure 6 where the first and second explosive regimes are followed by stationary collapses before reverting back to unit root processes, and the third is followed by a collapse that continues until the end of the sample. The three explosive regimes are placed closer together

in the sample, with a gap of 0.1 between the end of collapse of one and the start of the next. This placing within the sample has a small detrimental effect on the start date estimation of the second and third explosive regimes, relative to the first, but nevertheless, the comparative performance of the PSY and BIC estimates is found to be qualitatively similar to that of DGP E.

In addition to examining the accuracy of the date estimates, we also consider the extent to which the BIC procedure is able to select the correct model for each explosive regime. Of course, as HLS note, identification of the correct model is not essential for accurate date estimation. For example, if BIC were to mistakenly select Model 2 instead of Model 4, this would still provide estimates of τ_{j1}^* and τ_{j2}^*. Table 1 displays the frequencies with which the BIC procedure selects each model across the Monte Carlo replications for which at least one explosive regime was detected.[2] For all of the explosive regimes considered, BIC selects the correct model more frequently than it selects any other model. We note only two instances where the probability of selecting the correct model is less than a half: the second regime of DGP A, where in some replications the reversion back to a unit root after the collapse is not detected, most likely due to this occurring towards the end of the sample; and the second regime of DGP C, where a similar phenomenon occurs. In both cases, an incorrect selection of Model 3 over Model 4 still yields accurate identification of the start and end dates of the explosive regimes, as exhibited in Figures 1 and 3. Overall our results demonstrate a clear accuracy advantage of our proposed BIC date-stamping procedure over recursive unit root based methods of date estimation.

5 Empirical Application

To demonstrate the practical value of our proposed date-stamping procedure, we examine the historical bubble behaviour of house prices for a set of 22 countries. The sub-prime crisis and subsequent financial distress of the late 2000s has led to increased scrutiny of the dynamics of the housing market. Several recent studies have implemented the PSY detection and dating procedures to investigate the timing of explosive episodes in house prices. Anundsen et al. (2016) consider the effect of the interaction between credit and the housing market on financial stability. They apply the BSADF dating procedure of PSY to quarterly house price data for 20 OECD countries from 1975Q1-2014:Q2, and use the estimated dates to construct 'exuberance' indicators which are employed as explanatory variables in a logit regression of financial instability. Their findings suggest that the probability of a financial crisis increases substantially when bubbles in house prices coincide with similar exuberance in credit. Pavlidis et al. (2016) also consider the role of house price bubbles on financial stability. Applying the GSADF detection procedure and BSADF dating procedure of PSY to the International House

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[2]If an explosive regime $j + 1$ is detected, then BIC will only select between Models 2 and 4 for explosive episode j, as discussed in Section 3. In some replications, the PSY procedure detects fewer than N explosive regimes, therefore allowing Models 1 and 3 to be selected for $j < N$ in these replications.

Price Database of the Federal Reserve Bank of Dallas (Mack and Martinez-Garcia, 2011) for 22 countries, they find significant evidence of house price exuberance in 20 countries in the early 2000s, preceding the global financial crisis. The date estimates are used to construct a 'chronology of exuberance' tracking the evolution of house price bubbles across countries and time. Using these date estimates to construct an exuberance indicator, they estimate a probit model to assess the predictive ability of a number of financial and macroeconomic variables on house price exuberance. The Dallas Federal Bank has provided house price exuberance indicators from 2013:Q2 onwards based on the methodology of Pavlidis et al. (2016). In related work, Pavlidis et al. (2019) use exuberance indicators constructed from BSADF date-stamping to examine the role of interest rates and policy uncertainty in explaining explosive behaviour. Shi (2017) investigates regional house price behaviour in the US by decomposing a price-to-rent series into a fundamental and non-fundamental component and applying the BSADF test to the non-fundamental part. Whilst these papers all provide very interesting results, the validity of their analysis is clearly dependent on the accuracy of the estimated start and end dates of explosive behaviour.

As in Pavlidis et al. (2016, 2019), we examine real house price indices from the International House Price Database of the Federal Reserve Bank of Dallas (Mack and Martinez-Garcia, 2011) to detect and date house price bubbles in 22 countries.[3] Our sample period runs from 1975:Q1-2018:Q2. First, considering the GSADF detection procedure of PSY, we find evidence of explosive behaviour, at a 5% significance level, in all but two countries (Croatia and South Korea[4]), and therefore proceed to apply both the PSY BSADF dating procedure and our BIC procedure to date-stamp the detected explosive episodes. Whilst we restrict the BIC procedure to detect only upward explosive regimes, no such restriction applies to the PSY procedure and it is possible for PSY to detect rapid downward movements (see Phillips and Shi (2019)). In our simulation exercise, we generated only upward explosive processes, and as such would not expect this to be a major issue. However, we noted several occasions in our empirical application where PSY identifies such downward price movements. As these episodes are clearly not bubbles, we do not treat them as such, and discount these from our PSY analysis and the corresponding BIC procedure. As in our simulation exercise, when explosive regimes of length $ln(T)$ are separated by no more than 3 non-rejections of the BSADF test, we treat these as a single explosive episode. Of the 20 countries where at least one explosive regime is detected, PSY finds one period of explosiveness in nine countries (Denmark, France, Germany, Ireland, Italy, Japan, Netherlands, South Africa and Spain), two explosive regimes are detected in five countries (Australia, Belgium, Switzerland, UK and US), three explosive regimes are detected in four countries (Canada, New Zealand, Norway and Sweden), and four explosive regimes are

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[3]The database now contains information for an additional country, Israel, than in the analysis of Pavlidis et al. (2016) but, in unreported results, we find no evidence of explosive behaviour in this country.

[4]Our analysis focuses on a 5% significance level, but we note that evidence of explosive behaviour is found in Croatia when considering a 10% significance level.

detected in two countries (Finland and Luxembourg).

Figure 7 displays the house price index and estimates of the first and last explosive regime dates from the PSY and BIC dating procedures for the 20 countries where explosive behaviour is detected, along with the collapse regime dates where they are identified by the BIC procedure. We note the following key features from our analysis. First, when PSY and BIC are dating the same explosive episode, BIC tends to find start date estimates earlier than PSY in line with our simulation results. See, for example, the two explosive regimes detected in Australia, displayed in Figure 7a, where BIC estimates start dates of 1997:Q1 and 2012:Q4 compared to PSY start date estimates of 1998:Q4 and 2016:Q2. It is also evident that the BIC procedure typically provides earlier end dates than PSY. Considering Ireland, in Figure 7h, for example, BIC estimates the final date of the single explosive regime as 2007:Q1, corresponding to the maximum value of the house price index, whereas PSY finds an estimated end date of 2007:Q4, after the turning point of the price series. Across all countries, out of the approximately 35 occasions where BIC and PSY appear to date the same explosive regime, BIC finds an earlier (or identical) start date to PSY in 29 of these regimes, and an earlier (or identical) end date in 30 of these regimes. Another advantage of the BIC dating procedure is its ability to date a crash regime following an explosive episode. This is evident in our application; see, for example, the long collapse regime after the explosive episode in Japan (Figure 7j), and the collapse following the first explosive regime detected for Switzerland (Figure 7r).

One interesting feature of the analysis is that BIC and PSY do not always date the same regime. We have deliberately constructed our BIC procedure to provide the same *number* of detected upward explosive regimes as PSY, but within a given date window there is no guarantee that the two procedures will date the same regime. Consider Canada in Figure 7c. PSY dates the first explosive regime from 1988:Q3-1990:Q2 and the second explosive regime from 2002:Q4-2008:Q3. Therefore the first sub-sample date window that the BIC procedure uses to date runs from the start of the sample until 1996:Q3 (the mid-point of the estimated end date of the first explosive regime and the estimated start date of the second). However, within this date window, BIC fits an explosive regime much earlier in the sample from 1979:Q2-1981:Q3, with a collapse then lasting until 1982:Q4. From visual inspection of the house price series, this first explosive regime detected by BIC appears more pronounced than the later regime detected by PSY, but is not detected by the PSY procedure. Recalling from Section 3 that we use the recommended minimum window width of $r_0 = 0.01 + 1.8/\sqrt{T}$ for PSY dating, in our application this corresponds to 0.146, preventing the PSY procedure from detecting explosivity within the first 25 quarters. Conversely, our BIC procedure does not require such a large initial window, thus allowing us to date an explosive regime which originates within PSY's initial window. As a consequence of this, and due to the adjustment made in the BIC procedure in which the start date of the subsequent sub-sample date window is rolled back to the first unit root observation that follows the previous explosive episode, it is possible for BIC to detect the start of two explosive regimes before PSY has detected any, see Belgium, for example, in Figure 7b.

The application of the BIC procedure to house price data allows us to build up a chronology of events. We observe that many countries experienced explosive episodes in house prices during the 1980s, with the explosiveness originating in the late 1970s for some countries. This first phase of explosiveness typically continued until the late 1980s/early 1990s. Following this period, prices behaved differently across countries, with some displaying collapse behaviour of varying duration, while others returned to normal market behaviour immediately following the explosive period. The second key phase of activity is the house price exuberance that occurred in the majority of countries studied during the mid-2000s, typically terminating around the time of the global financial crisis. For example, explosive behaviour in the US emerged as early as 1997:Q3, before terminating in 2006:Q4 with a collapse that lasted until 2011:Q2. In the UK, house price exuberance emerged only two quarters later than in the US, before collapsing in 2007:Q4. The UK housing market suffered a shorter collapse phase than the US, ending in 2009:Q1. Finally, a third phase of explosiveness is observed in the later part of the sample for many countries, and which is still ongoing at the end of our sample for Luxembourg, Norway, Switzerland and the US.

As discussed in Section 1, if we define bubbles as deviations of prices away from the asset's fundamental value, then the detection of explosive behaviour in prices does not necessarily indicate the presence of a bubble. We therefore extend our analysis by examining the behaviour of a price-to-fundamental ratio for each country, where real personal disposable income (obtained from the same database) is used as a proxy of the housing fundamental value, following Pavlidis et al. (2016). Implementing the GSADF test of PSY, we find evidence of explosivity in the price-to-income ratios of 15 countries: Australia, Belgium, Canada, Denmark, France, Ireland, Italy, Japan, Luxembourg, Netherlands, South Africa, Spain, Switzerland, UK and US. Compared to the previous analysis of the price indices alone, we find that Finland, Germany, New Zealand, Norway and Sweden no longer reject at the 5% significance level.[5] As before, we estimate the start and end dates of detected explosive (and collapse) regimes using both the PSY and BIC date-stamping procedures. For Japan, the explosive episodes dated by PSY are negative, and this country is subsequently excluded from our analysis. PSY and BIC date estimates for the remaining 14 countries are displayed in Figure 8.

Examining the date-stamping results for the price-to-income ratios, we note several interesting features. First, for a number of countries (Australia, Canada, Luxembourg, Switzerland, US), fewer explosive episodes are detected in the price-to-income ratio than in the price index. Considering Canada, for example, we detected 3 explosive episodes in prices and 2 in the price-to-income ratio (see Figures 7c and 8c). This suggests that, in some cases, the finding of explosivity in prices may have been driven by the behaviour of income. We note only one country, Ireland, where more explosive regimes are detected in the price-to-income ratio than the price index. Comparing figures 7h and 8f, it is evident that the explosive period dated by BIC

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[5]At a 10% significance level, evidence of explosive behaviour is also found in the price-to-income ratios of Germany, New Zealand and Sweden.

in the price index (1995:Q2-2007:Q1) has been split into two sub-periods in the price-to-income ratio (1995:Q3-2000:Q1 and 2002:Q1-2007:Q2), with the collapse period ending in 2012:Q1 in both series. Calculating the year-on-year Q1 growth rate of income, we find that the growth rate declines from 6.2% to 5.3% to 1.4% in 1999, 2000 and 2001, and then increases to 2.1% and 3.7% in 2002 and 2003, and we suggest that fluctuations in the growth rate of income are likely responsible for the division of the long explosive regime into two sub-periods when considering the price-to-income ratio.

Now turning our attention to the date estimates themselves, we observe that in some instances the explosive regimes detected in the price-to-income ratio are shorter than those in the price index. In the United States, for example, BIC estimates a start date of 2000:Q3 in the price-to-income ratio (compared to 2001:Q3 for PSY), which is somewhat later than the 1997:Q3 start date we find for the price index. Pavlidis et al. (2016) observe a similar phenomenon, noting that income growth in the United States was strong during the 1990s, in part due to the growth of new information technologies, and therefore the behaviour of house prices in the late 1990s may be partly driven by this income growth. We observe similar results in the UK and France, where the explosive behaviour exhibited during the mid-2000s originates in 2001:Q2 and 2001:Q3 respectively according to BIC estimation (compared to 1997:Q4 and 1998:Q1 in the price index).

Considering the full set of results from the price-to-income ratios across all countries, we observe phases of explosive behaviour in housing during the 1980s and during the mid-2000s for a number of countries, consistent with the overall results from the price indices. It is interesting to note that, when considering price indices, BIC finds evidence of ongoing explosive behaviour at the end of the sample in 4 countries (Luxembourg, Norway, Switzerland and the US), but that when examining the price-to-income ratio instead, this result no longer holds for Norway, Switzerland and the US. This suggests that the end-of-sample explosiveness in prices that we observe in these countries is not indicative of an asset bubble, but instead driven in part by fundamentals. We do, however, note that when considering these price-to-income ratios, end-of-sample explosive behaviour now emerges in Belgium, in addition to Luxembourg.

Thus far we have primarily focused on the upward explosive regimes detected in the price-to-income ratios; however, for bubbles that have terminated in-sample, it is also of interest to examine the post-bubble behaviour observed in each case, analysis of which is afforded by the new methodology in contrast to PSY. Of the 18 in-sample bubbles detected, we find 11 terminate with a stationary collapse regime, and the remaining 7 return directly to unit root dynamics. It appears, therefore, that in the majority of cases, a period of readjustment occurs before the resumption of normal market conditions. The length and magnitude of this collapse period can vary substantially for different countries and different bubbles. For example, a very lengthy collapse regime is observed for the 1980s bubble in Switzerland, while the corresponding 1980s bubble in the UK terminated with a much shorter collapse duration. It is also interesting that the pre-financial crisis house price bubble of the mid-2000s observed in many countries

terminated in differing ways across countries. For the US and Ireland, we find relatively long crash regimes, with protracted periods of downward adjustment in house prices, while for Denmark, France, Luxembourg and the UK, the collapse is completed more quickly. We find that the mid-2000s bubbles in Australia, Canada and Spain do not terminate in a collapse regime at all, with normal market conditions resuming immediately upon termination of the bubble. For Spain, this is contrary to prevailing opinion regarding the Spanish house price bubble (see Akin et al., 2014, for example) and visual inspection of Figure 8k suggests downward movements did arise during the financial crisis period. We note that following the end of bubble date, the price-to-income ratio appears to level off for approximately four quarters, before beginning to decline; we speculate that this levelling off of prices is responsible for BIC fitting a unit root regime rather than a stationary collapse immediately after the explosive episode. Belgium is an interesting case in that whilst it exhibits the bubble behaviour of the mid-2000s, unlike other countries this regime is ongoing at the end of the sample. Ball (2010) observes that in countries such as Ireland and the UK, the house price boom of the mid-2000s corresponded to increasing mortgage-to-GDP ratios, but that this was not the case in Belgium whose mortgage-to-GDP ratio was comparatively low throughout the 2000s. This would have limited Belgium's reliance on mortgage backed securities, and therefore its exposure to the global market. As a consequence, house prices in Belgium have not (by the sample end date) been subject to a collapse like that identified in the majority of other countries.

6   Conclusion

In this paper we have proposed a new methodology for estimating the start and end dates of multiple asset price bubbles in a series. Our procedure capitalises on the ability of the multiple bubble detection procedure proposed by PSY, based on recursive right-tailed Dickey-Fuller tests, combined with a parametric model-based procedure for bubble date estimation. Specifically, our approach first uses the PSY approach to determine the number of explosive regimes and preliminary date estimates, then uses these estimated dates to split the sample into sub-periods each containing a single explosive episode, before subsequently employing a BIC-based date estimation method applied to each sub-sample to obtain more accurate estimates of the explosive regime start and end dates, and additionally estimating the end date of any post-explosive collapse period. Our Monte Carlo simulations showed that the new methodology offers substantial improvements in dating accuracy relative to PSY, which has a tendency to date the regime change points later than they actually occur. An additional advantage of our procedure is its ability to identify whether each explosive regime ends in collapse, or whether normal (unit root) market behaviour is restored when the explosivity terminates. We therefore anticipate the new procedures to be very valuable to researchers and practitioners engaged in identifying the timeline of bubble episodes in economic and financial time series. An empirical application of our methodology to house price data for 22 countries detected episodes of explosive behaviour

in 20 of these, with the number of explosive regimes ranging between one and four. In line with the simulation results, we find the start and end dates from our new methodology to be typically placed earlier than the corresponding PSY date estimates. Summarising across countries, we identified three key periods of house price explosivity: the 1980s, the mid-2000s, and ongoing exuberance at the end of our sample.

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Table 1: BIC model selection frequencies for DGPs $A - F$

	Model 1	Model 2	Model 3	Model 4
A				
$j = 1$	0.007	0.133	0.128	**0.732**
$j = 2$	0.033	0.062	0.451	**0.454**
B				
$j = 1$	0.013	0.127	0.134	**0.726**
$j = 2$	0.026	0.060	**0.633**	0.281
C				
$j = 1$	0.049	**0.765**	0.080	0.106
$j = 2$	0.089	0.176	0.299	**0.436**
D				
$j = 1$	0.285	**0.684**	0.009	0.022
$j = 2$	**0.627**	0.343	0.016	0.014
E				
$j = 1$	0.006	0.027	0.001	**0.966**
$j = 2$	0.156	0.039	0.022	**0.782**
$j = 3$	**0.642**	0.015	0.002	0.342
F				
$j = 1$	0.002	0.088	0.027	**0.883**
$j = 2$	0.018	0.095	0.252	**0.635**
$j = 3$	0.014	0.061	**0.524**	0.401

bold indicates the true bubble model

Figure 1: Histogram of date estimates for DGP A
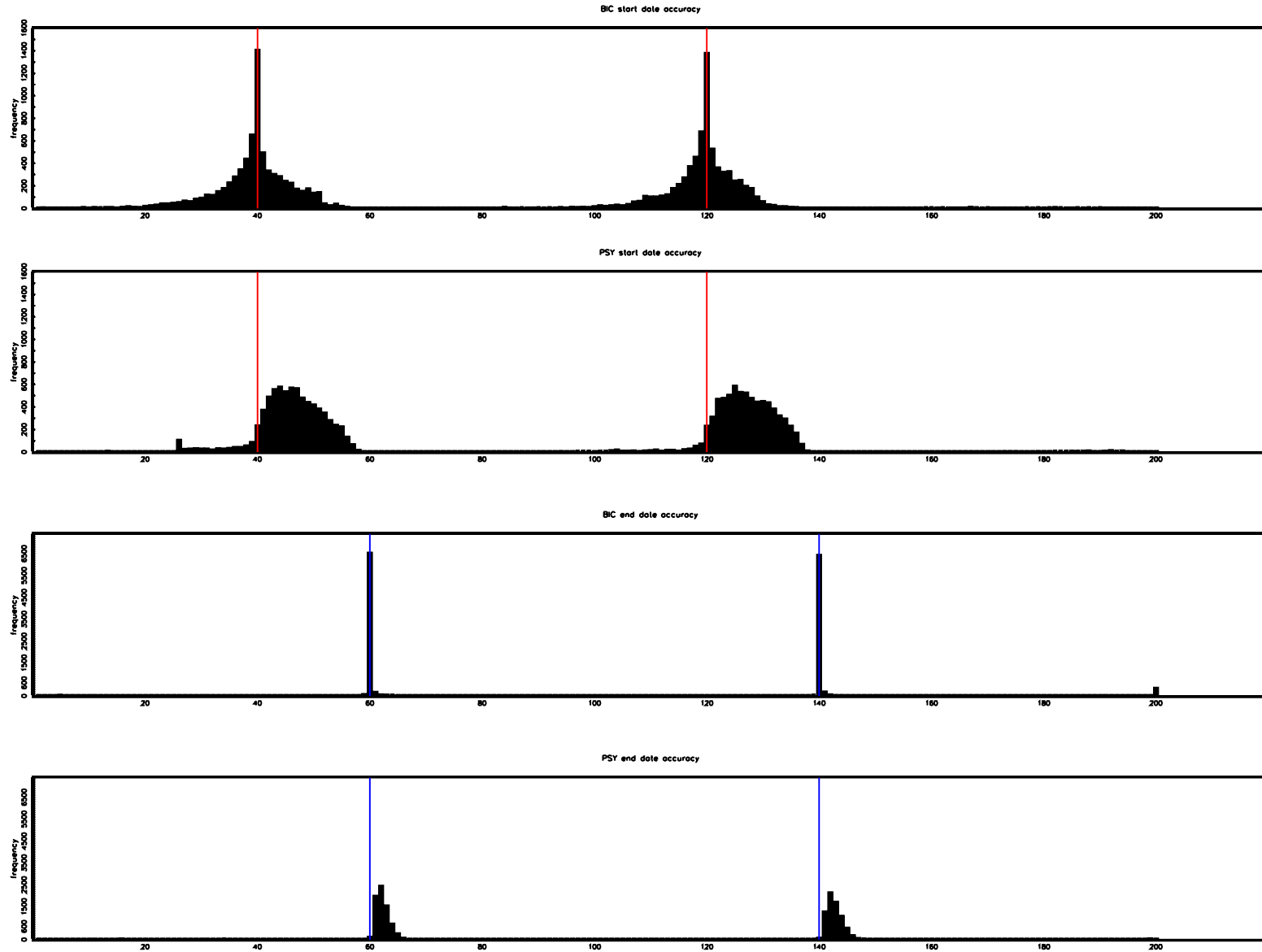
Figure 2: Histogram of date estimates for DGP B
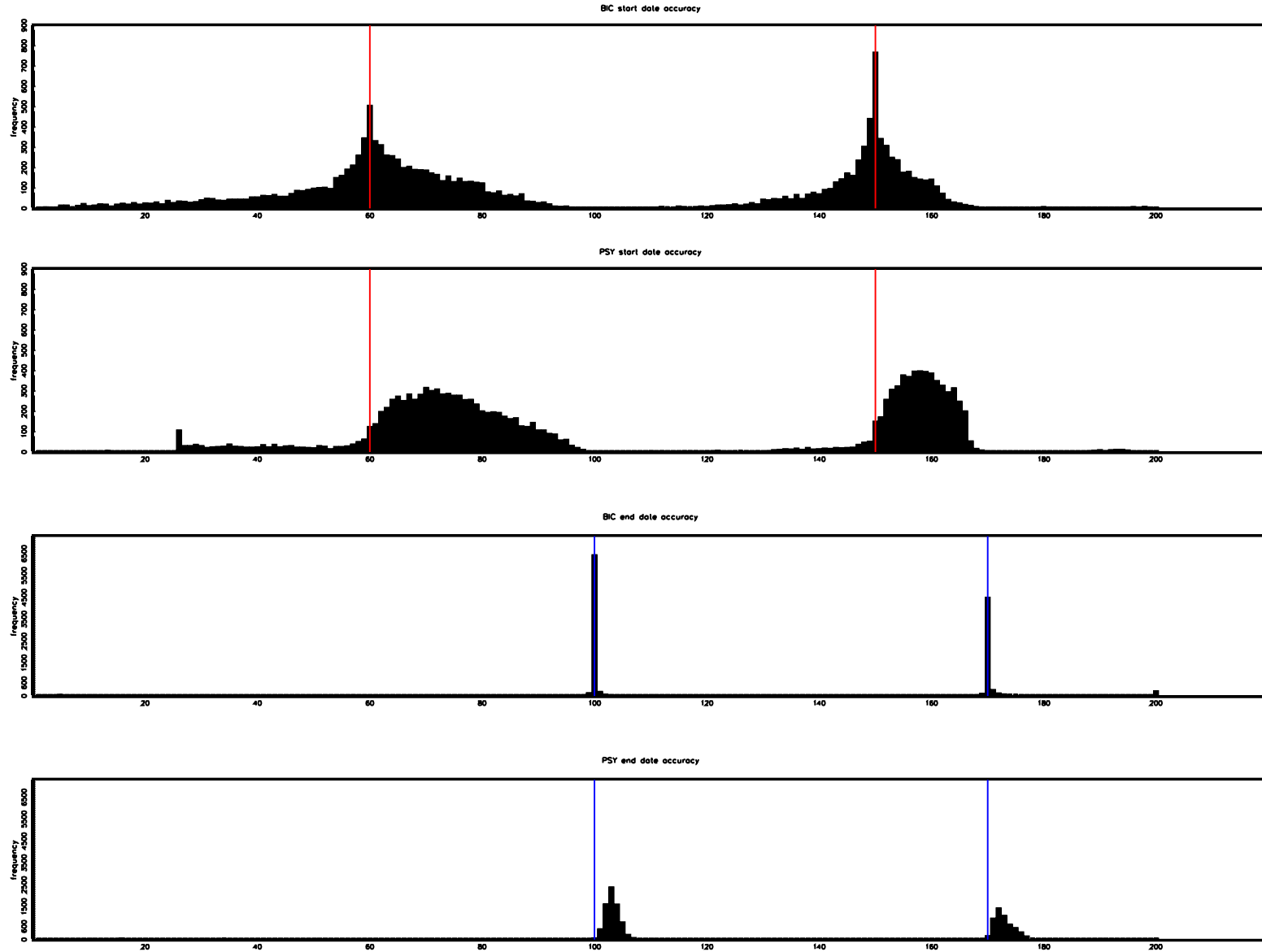
Figure 3: Histogram of date estimates for DGP C
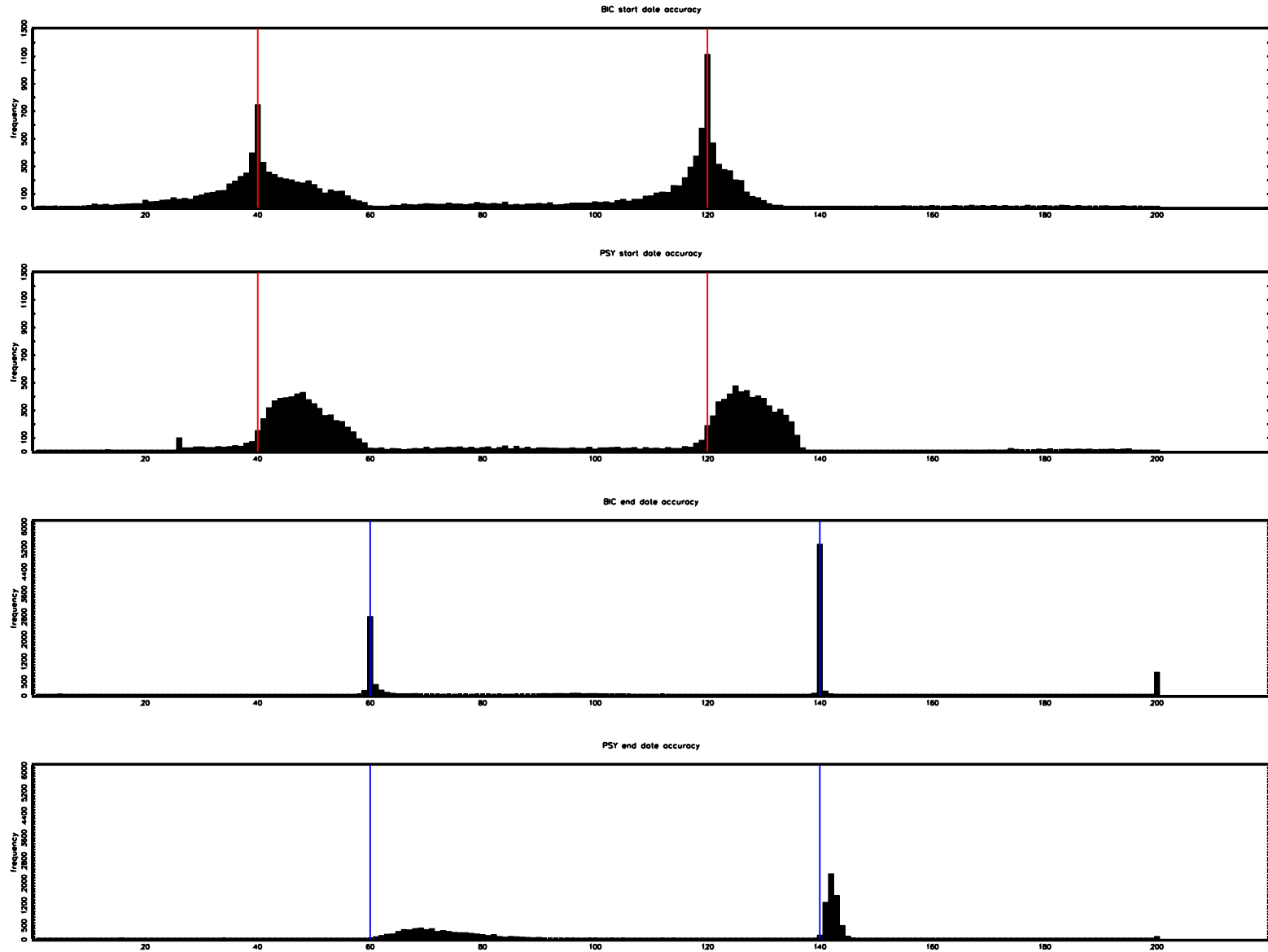
Figure 4: Histogram of date estimates for DGP D
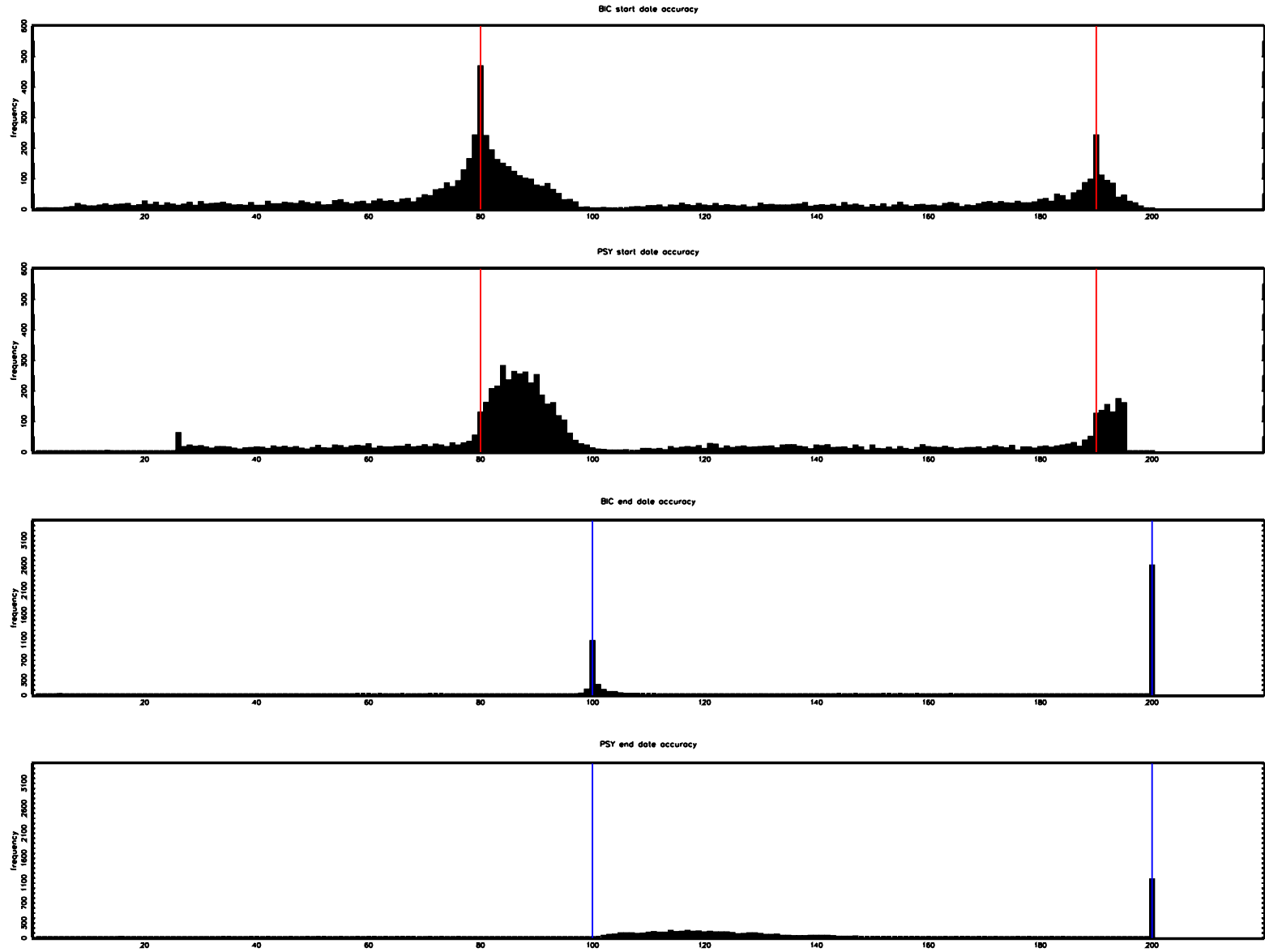
Figure 5: Histogram of date estimates for DGP E
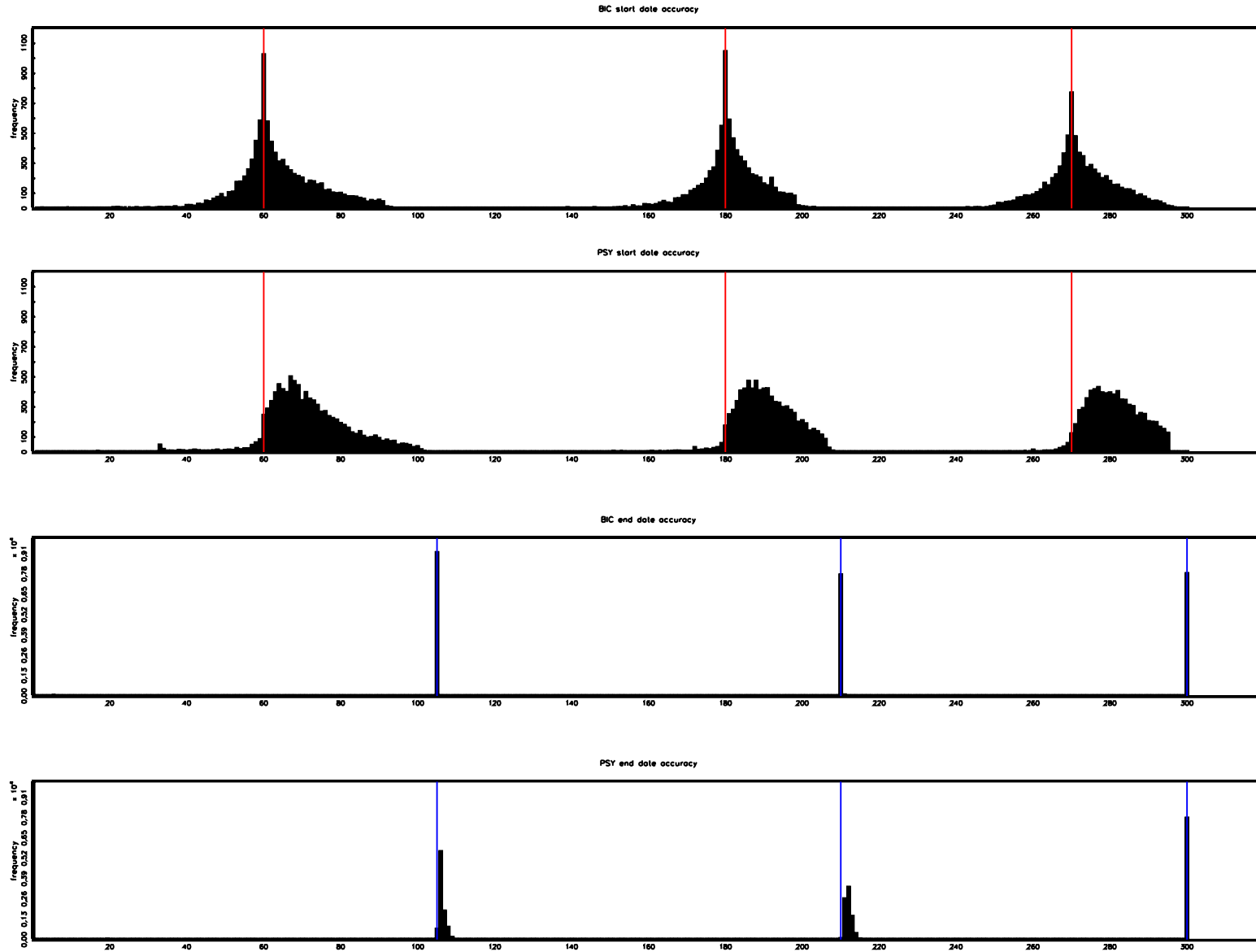
Figure 6: Histogram of date estimates for DGP F
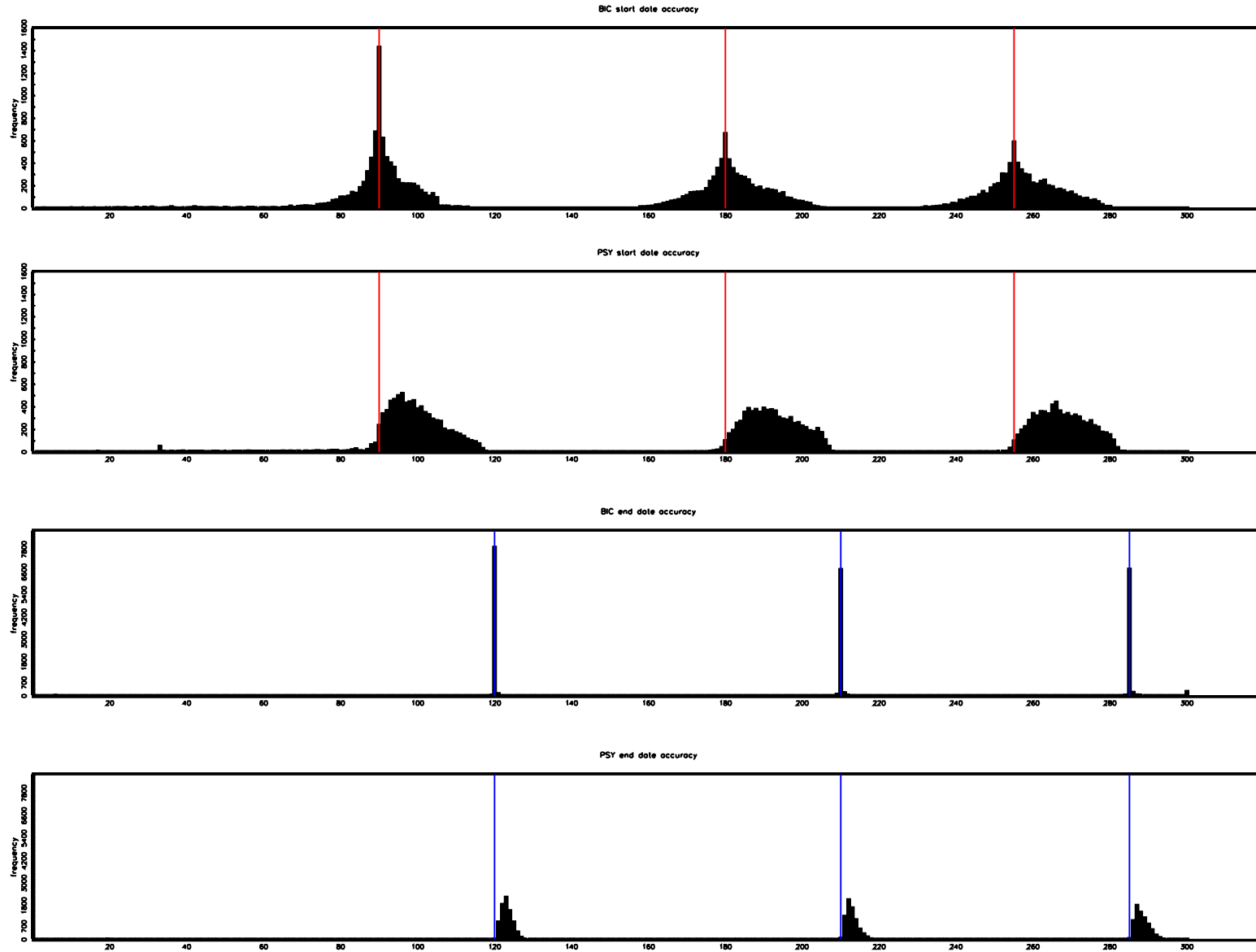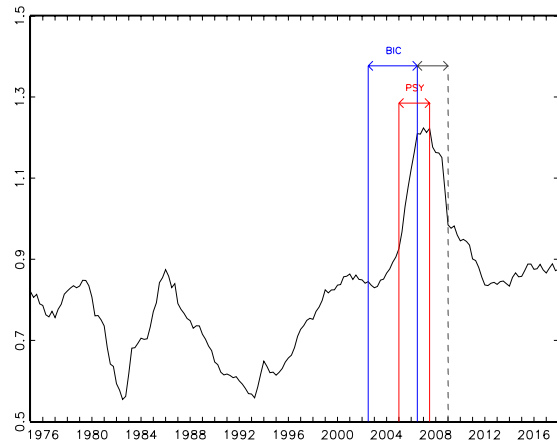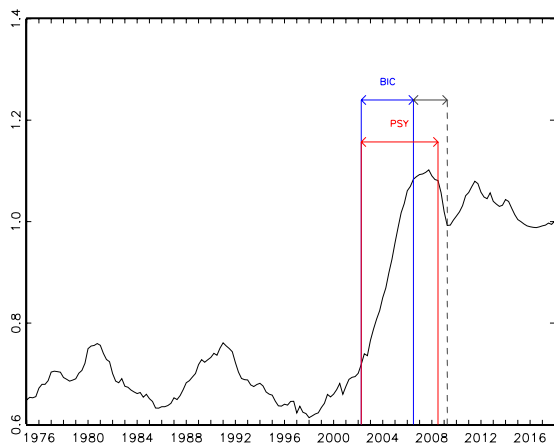
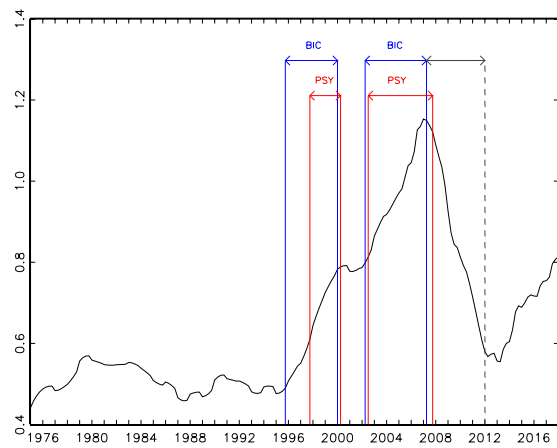Figure 7: House price index: BIC and PSY date estimates
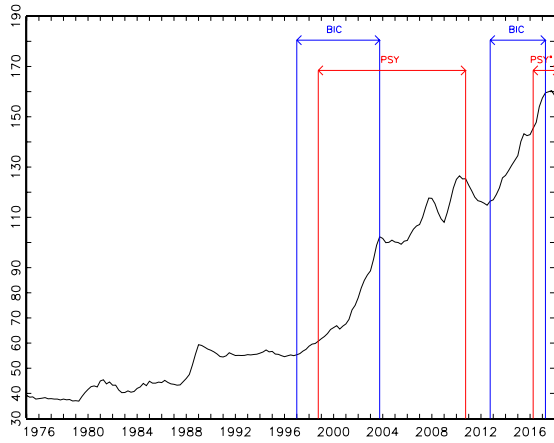
(a) Australia
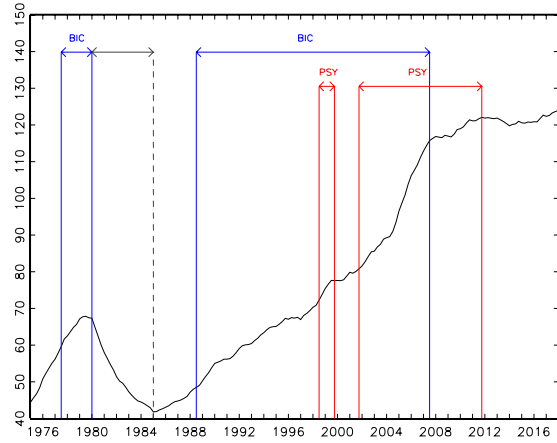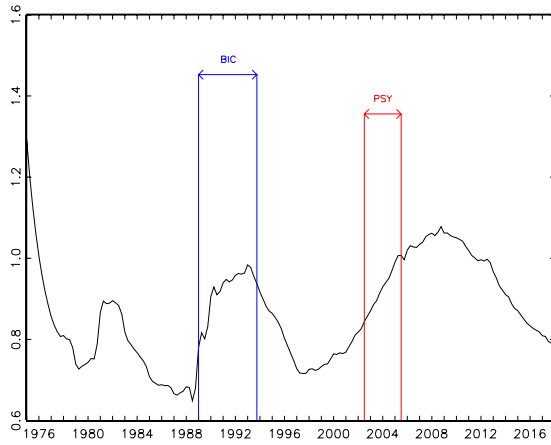
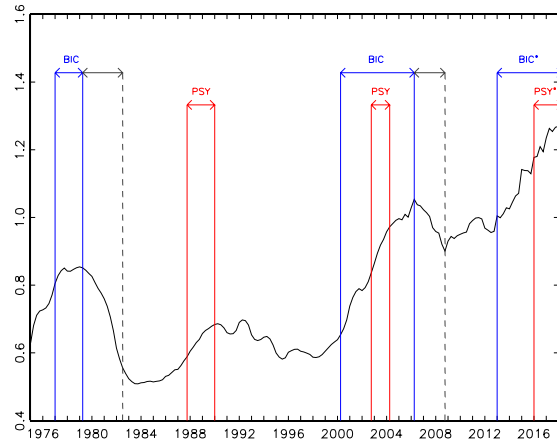(b) Belgium
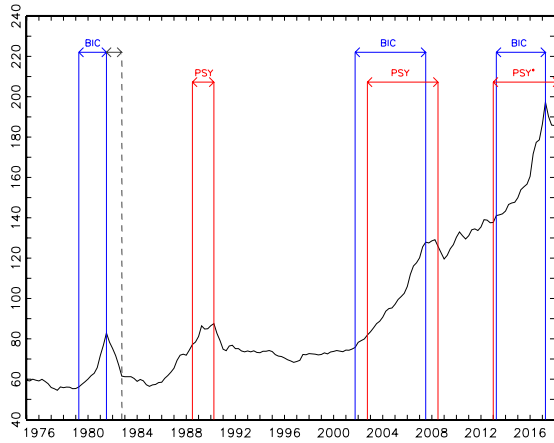
(c) Canada
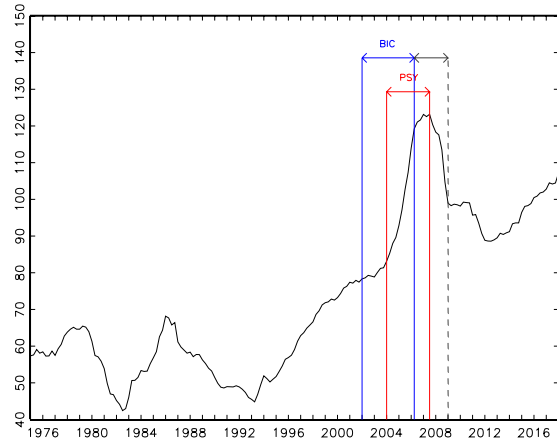
(d) Denmark
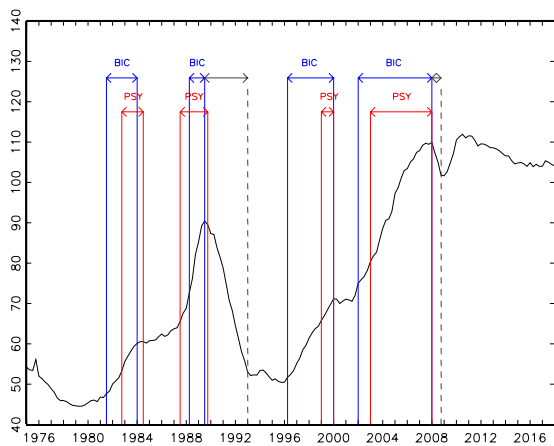
(e) Finland
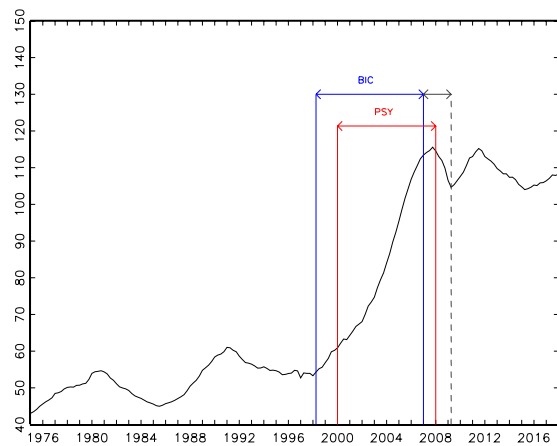
(f) France

⟷ BIC explosive regime, ⟷ BIC stationary regime, ⟷ PSY explosive regime
* indicates that the regime continues to the end of the sample

Figure 7: (continued) House price index: BIC and PSY date estimates
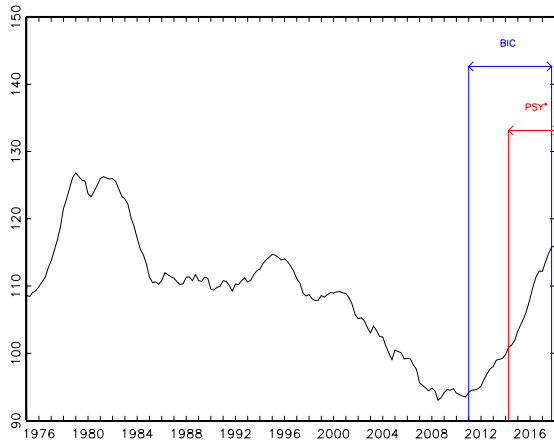
(g) Germany
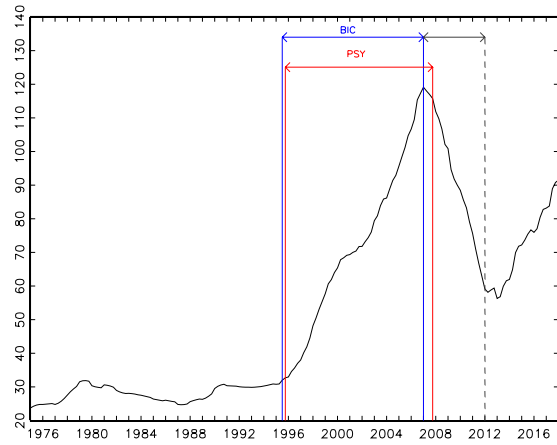
(h) Ireland
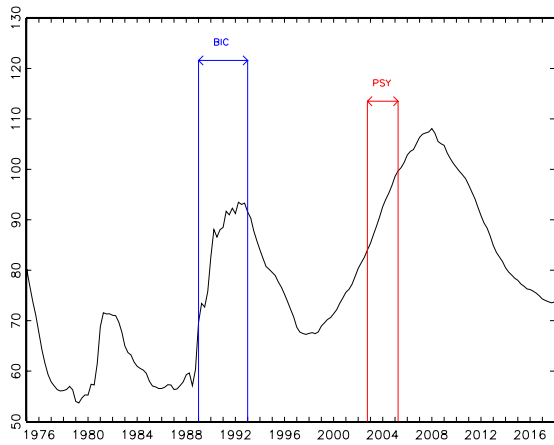
(i) Italy
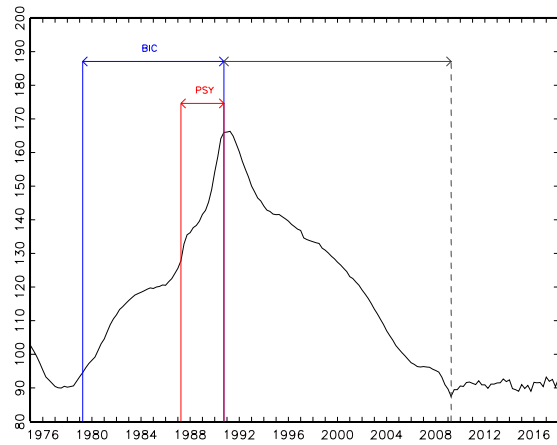
(j) Japan
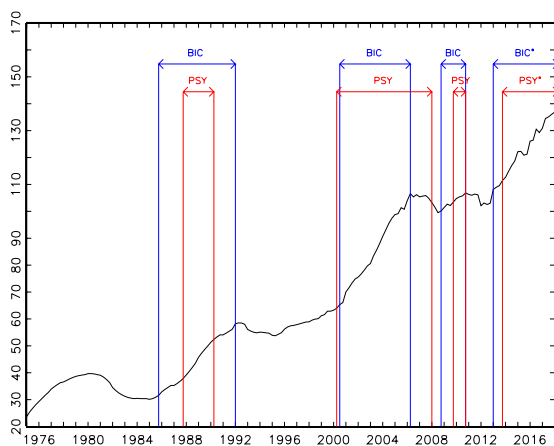
(k) Luxembourg
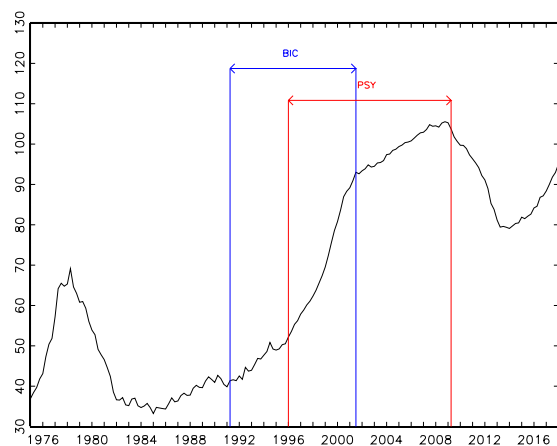
(l) Netherlands

⟷ BIC explosive regime, ⟷ BIC stationary regime, ⟷ PSY explosive regime

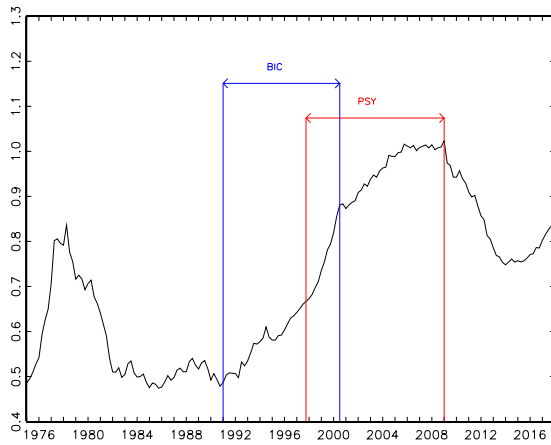* indicates that the regime continues to the end of the sample

Figure 7: (continued) House price index: BIC and PSY date estimates
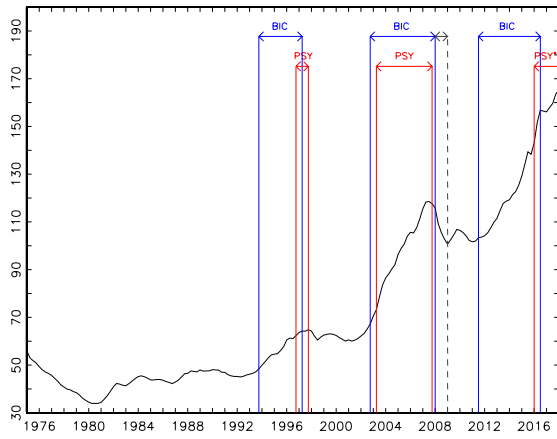
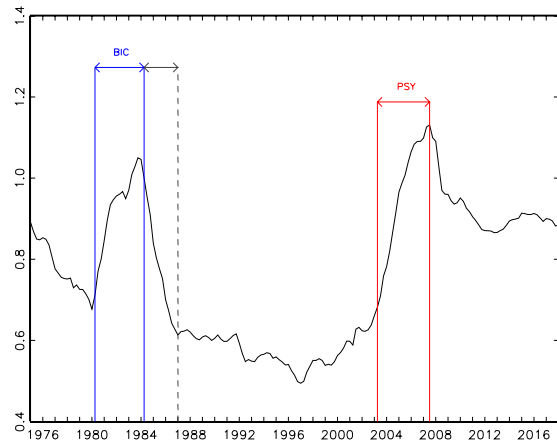(m) New Zealand
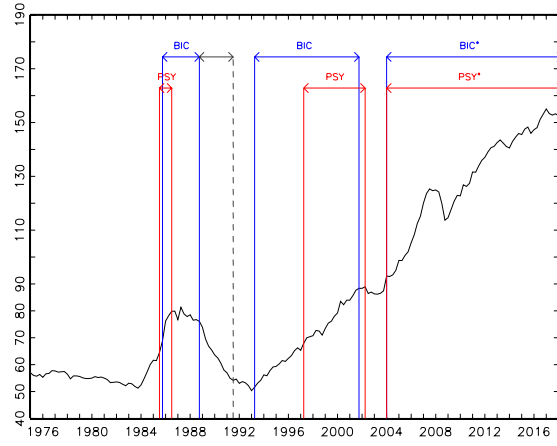
(n) Norway
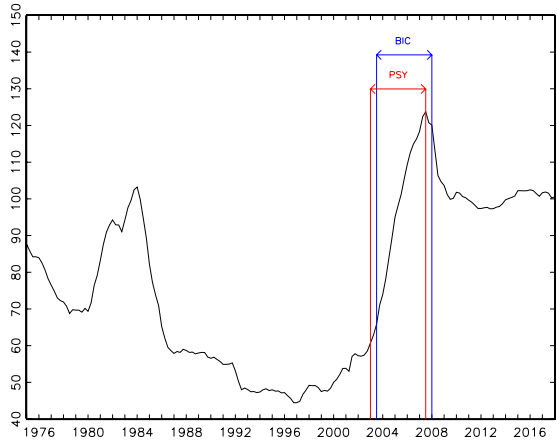
(o) South Africa
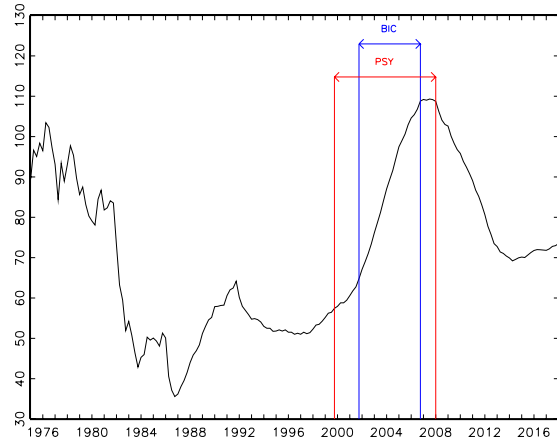
(p) Spain
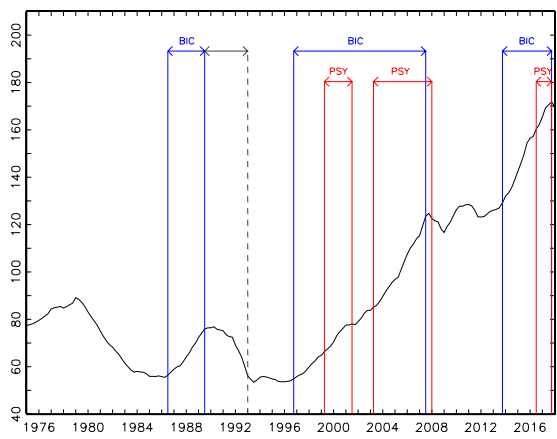
(q) Sweden
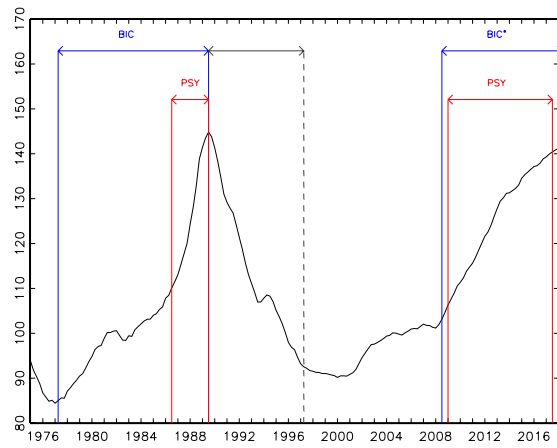
(r) Switzerland

⟷ BIC explosive regime, ⟷ BIC stationary regime, ⟷ PSY explosive regime
* indicates that the regime continues to the end of the sample

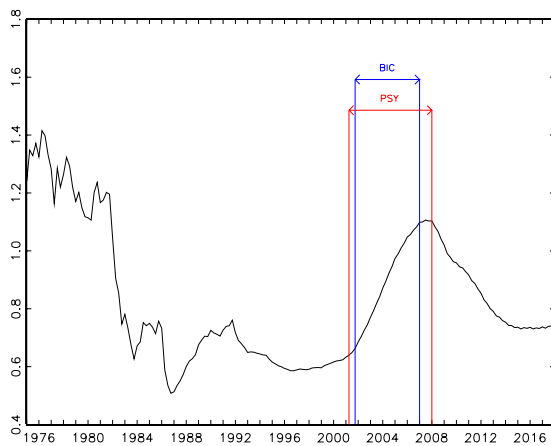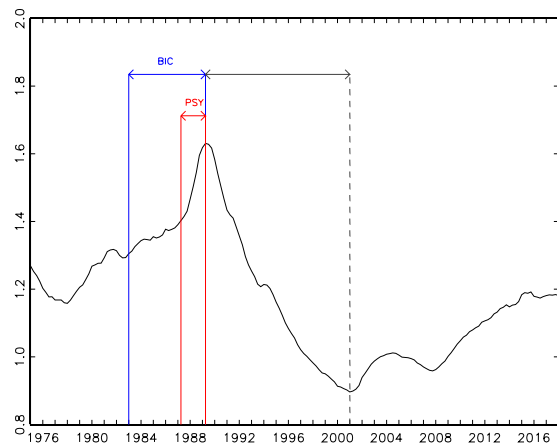Figure 7: (continued) House price index: BIC and PSY date estimates
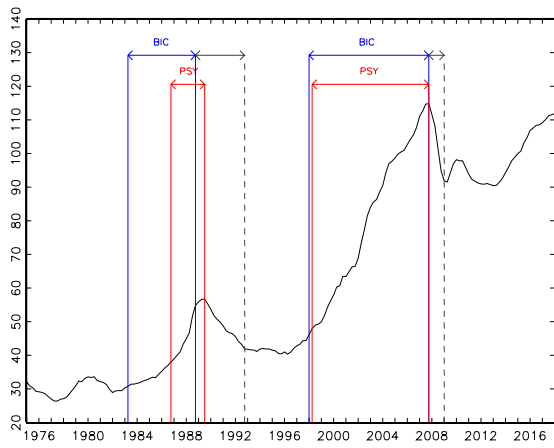
(s) United Kingdom
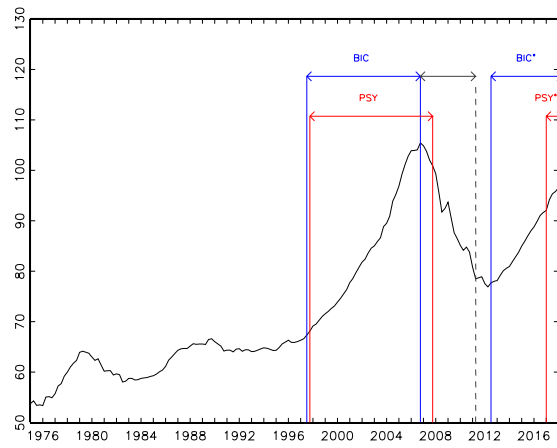
(t) United States

⟷ BIC explosive regime, ⟷ BIC stationary regime, ⟷ PSY explosive regime
* indicates that the regime continues to the end of the sample

Figure 8: House price-to-income ratio: BIC and PSY date estimates
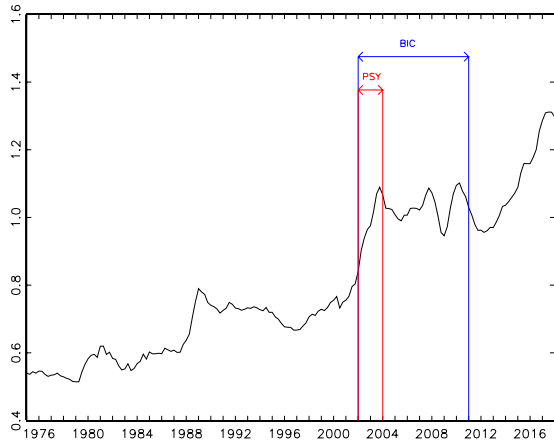
(a) Australia
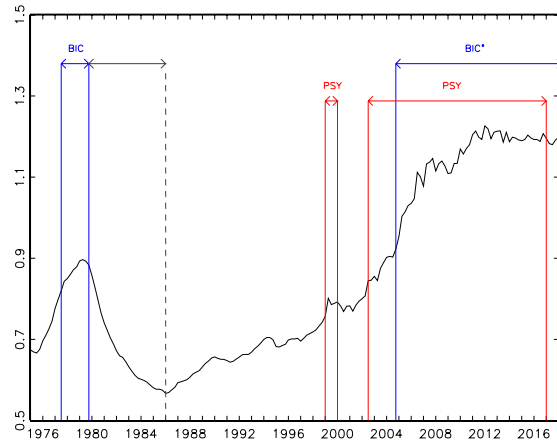
(b) Belgium
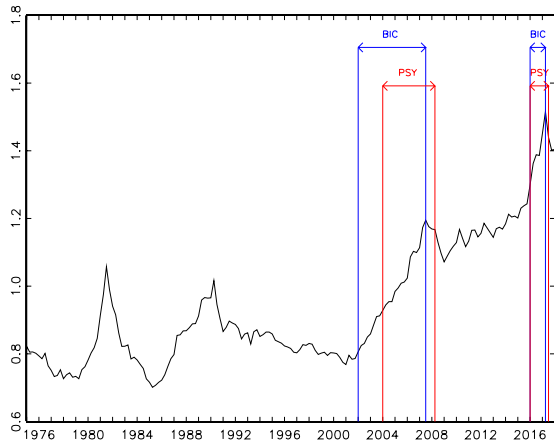
(c) Canada

(d) Denmark

(e) France

(f) Ireland

←→ BIC explosive regime, ←→ BIC stationary regime, ←→ PSY explosive regime

* indicates that the regime continues to the end of the sample

34

Figure 8: (continued) House price-to-income ratio: BIC and PSY date estimates

(g) Italy

(h) Luxembourg
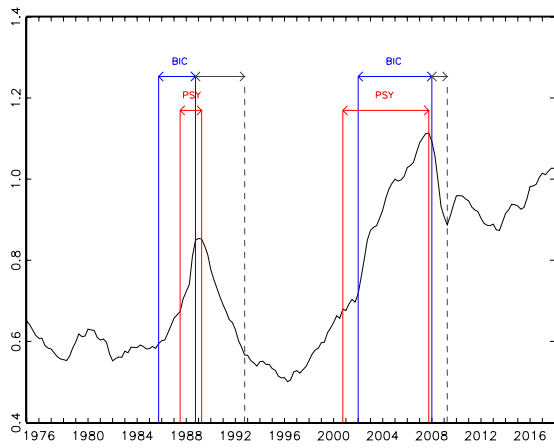
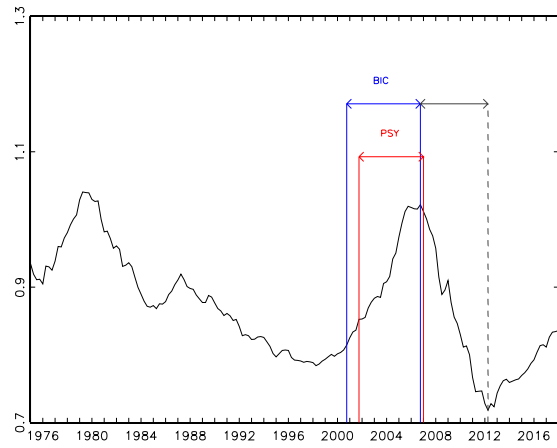(i) Netherlands

(j) South Africa

(k) Spain

(l) Switzerland

⟷ BIC explosive regime, ⟷ BIC stationary regime, ⟷ PSY explosive regime
* indicates that the regime continues to the end of the sample

Figure 8: (continued) House price-to-income ratio: BIC and PSY date estimates

(m) United Kingdom

(n) United States

⟷ BIC explosive regime, ⟷ BIC stationary regime, ⟷ PSY explosive regime
* indicates that the regime continues to the end of the sample