

EPIMORPHISMS AND ACYCLIC TYPES IN UNIVALENT FOUNDATIONS

ULRIK BUCHHOLTZ , TOM DE JONG , AND EGBERT RIJKE 

Abstract. We characterize the epimorphisms in homotopy type theory (HoTT) as the fiberwise acyclic maps and develop a type-theoretic treatment of acyclic maps and types in the context of synthetic homotopy theory as developed in univalent foundations. We present examples and applications in group theory, such as the acyclicity of the Higman group, through the identification of groups with 0-connected, pointed 1-types. Many of our results are formalized as part of the `agda-unimath` library.

§1. Introduction. Univalent foundations relies on a homotopical refinement of the *propositions as types* approach to logical reasoning in dependent type theory, thence also known as homotopy type theory (HoTT) [54, Chapter 3]. One virtue of HoTT is that many advanced concepts from homotopy theory can be expressed in simple logical terms, sidestepping encodings in terms of combinatorial or point-set topological notions of spaces. The corresponding program is known as *synthetic homotopy theory* [4, 15, 52]. Additional benefits are that most results can be developed in a basic system of very modest proof-theoretic strength [42], way below that of classical second-order arithmetic, and that the results apply more generally than classical homotopy theory, namely in any higher topos [51]. Here, we consider the notion of epimorphism of types in HoTT—and its deep connections to synthetic homotopy theory—paying close attention to the logical principles needed throughout.

A map $f : A \rightarrow B$ is an *epimorphism* if it has the desirable property that for any map $f' : A \rightarrow X$, there is at most one extension (dashed in the diagram below) of f' along f .

$$\begin{array}{ccc}
 A & \xrightarrow{f'} & X \\
 f \downarrow & \nearrow & \\
 B & &
 \end{array}
 \tag{1}$$

In (1-)category theory, this property is often equivalently phrased as: for any two maps $g, h : B \rightarrow X$, if $g \circ f = h \circ f$, then $g = h$. It is well known that a map between sets is an epimorphism precisely when it is surjective. In HoTT one also considers *higher types* that don't necessarily behave as sets, because in general,

Received August 28, 2024.

2020 *Mathematics Subject Classification.* Primary 03B38, 55P35, 55P40, 68V20.

Key words and phrases. univalent foundations, homotopy type theory, synthetic homotopy theory, acyclic space, epimorphism, suspension, Higman Group.

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0022-4812/00/0000-0000

DOI:10.1017/jsl.2024.76



equality types can have non-trivial structure. As a consequence, the notion of epimorphism in HoTT becomes more involved and rather interesting. We shall illustrate this with an example.

Epimorphisms and the circle. To see that something unusual is going on in the presence of higher types, we will show that, while the terminal map $\mathbf{2} \rightarrow \mathbf{1}$ is an epimorphism of sets, it is *not* an epimorphism of (higher) types. In fact, we claim that the type of extensions of $\mathbf{2} \rightarrow S^1$ along $\mathbf{2} \rightarrow \mathbf{1}$ is equivalent to \mathbb{Z} .

Recall that the circle S^1 is the higher inductive type generated by a base point $\text{pt} : S^1$ and an identification loop $\text{loop} : \text{pt} = \text{pt}$. A standard result [54, Section 8.1] is that the loop space $\text{pt} = \text{pt}$ is equivalent to the type of integers \mathbb{Z} .

Now consider diag. (1) where f is the map $\mathbf{2} \rightarrow \mathbf{1}$ and f' is the constant map $\mathbf{2} \rightarrow S^1$ pointing at the base point. The type of extensions of f' along f is then equivalent to $\sum_{x:S^1} (x = \text{pt}) \times (x = \text{pt})$. Since $\sum_{x:S^1} (x = \text{pt})$ is contractible it follows that the type of extensions of f' along f is equivalent to the loop space $\text{pt} = \text{pt}$, which is in turn equivalent to \mathbb{Z} . Therefore we see that the type of extensions of g along f has infinitely many elements. Thus, in a suitable, higher sense, the map $\mathbf{2} \rightarrow \mathbf{1}$ is *not* an epimorphism of (higher) types.

Related work. The first main result of our paper is the characterization of epimorphisms in homotopy type theory (HoTT) as those maps whose fibers are all *acyclic*. This result is expected from classical results in algebraic topology [2, 30] and higher topos theory [33, 41], but new in HoTT. Traditionally, acyclic spaces are important in K-theory [58] and a space is defined to be acyclic if its reduced integral homology vanishes. We instead define a type to be acyclic if its *suspension* is contractible and we relate these two definitions in Section 6.

One can understand our results purely type-theoretically, but, at the same time, our results apply to all Grothendieck $(\infty, 1)$ -topoi—and not just the ∞ -topos of spaces—since HoTT can be seen as an internal language of higher topoi [51]. This highlights an important difference between our work and that of Raptis [41] and Hoyois [33]. The former applies to the ∞ -topos of spaces only and sometimes relies on tools only available there such as Whitehead’s Principle [54, Section 8.8]. Hoyois establishes the acyclic maps as the left class of a modality on an arbitrary ∞ -topos, but uses site presentations to do so. In contrast, the arguments of our work are fully internal. The closure results of this paper (Section 2.2) along with the Blakers–Massey theorem and its dual (Section 4.1) would follow if we could construct the modality of acyclic maps in HoTT, but since we don’t (yet) know how to do this, we offer direct proofs instead. In the case of Blakers–Massey, Raptis gives a non-constructive argument, while we offer a more constructive account, but still relying on an axiom. In algebraic topology, such direct proofs were given for acyclic maps between CW-spaces (spaces having the homotopy type of a CW-complex) by Hausmann and Husemoller [30, Section 2]; and Alonso [2, Section 4] further studied acyclic maps between (path-connected) CW-spaces. In a few places (i.e., Corollary 2.27, Proposition 4.5, and Theorem 2.26) we give direct references to analogous results in [2, 30, 41] for comparison.

We emphasize that the proofs of the nontriviality and acyclicity of the examples in Section 7 are new. For the first example, Hatcher [29, Example 2.38] proves acyclicity of the complex using a calculation in homology, whereas we derive acyclicity fairly

directly from Eckmann–Hilton [54, Theorem 2.1.6]. The second example is the classifying type of the Higman group [31] and the originality of our proof that this group is nontrivial is further commented on below and in Section 7.2.

Finally, we mention that all our proofs are fully constructive and do not rely on classical principles such as the axiom of choice and excluded middle. We also do not make use of impredicativity in the form of propositional resizing [54, Section 3.5].

1.1. Outline.

- Our first main result is Theorem 2.9 which characterizes the epimorphisms in homotopy type theory as the fiberwise acyclic maps. In Section 2 we further prove closure properties and show that the epimorphisms are also exactly the balanced maps of Raptis [41].
- Section 3 introduces relativized notions of acyclicity and epimorphisms to k -types (Theorem 3.6) with applications in group theory via the delooping of a group as a pointed connected 1-type.
- Although we leave establishing the acyclic maps as the left class of an orthogonal factorization system to future work, we do prove Blakers–Massey for acyclic maps (Theorem 4.12), and we study what the corresponding right class should be: the hypoabelian maps (Section 5). These results depend on an additional axiom that we call the *plus principle* (Section 4).
- In Section 6 we relate our definition of an acyclic type to the classical definition which requires its reduced integral homology to be zero. We also discuss the situation for maps, which is a bit more subtle.
- Finally, we exhibit interesting examples of acyclic types (Section 7).
 - The first example is a type-theoretic incarnation of Hatcher’s two-dimensional complex [29, Example 2.38] and positively answer a question raised by Rezk [43, p. 11].
 - The second example is the classifying type of the Higman group [31]. In proving its nontriviality we make use of recent results by David Wörn [57]. It is noteworthy that our proof is constructive and avoids combinatorial group theory [38], relying on higher categorical tools instead.

1.2. Foundations and preliminaries. We work in homotopy type theory (HoTT), also known as univalent foundations, and employ the conventions and notations of the HoTT book [54]. We make use of the univalence axiom and its consequences (such as function extensionality) without mentioning this explicitly.

We also assume familiarity with *pullbacks* [54, Exercise 2.11], *higher inductive types* [54, Section 6] and specifically, *pushouts* [54, Section 6.8]. We also assume familiarity with k -types [54, Section 7.1] and recall that a type A is *k -connected* [54, Section 7.5] if its *k -truncation* [54, Section 6.9] is contractible, i.e., $\|A\|_k \simeq \mathbf{1}$, and that this notion extends to maps by considering *fibers* [54, Definition 4.2.4].

A recurring idea, developed in [10, 17], is to regard a group G via its *classifying type* BG : this is a 0-connected, pointed 1-type such that taking loops at the point recovers the group G , i.e., we have an isomorphism of groups $G \cong (\text{pt} =_{BG} \text{pt})$. (The latter is indeed a group as we can compose and invert loops and these operations are neutral on the trivial constant loop.)

1.3. Formalization. We used the Agda proof assistant [40] to formalize substantial parts of this paper in the agda-unimath library [46]. Where appropriate, definitions, lemmas, theorems, etc. are marked with the symbol \star that is a link to (the HTML rendering of) the relevant file in the agda-unimath library.

§2. Epimorphisms and acyclic maps. In the theory of (1-)categories, a map $f : A \rightarrow B$ is an epimorphism if any two maps $g, h : B \rightarrow X$ are equal as soon as $g \circ f = h \circ f$. In other words, f is an epimorphism if precomposition with f is injective. To get a homotopically well behaved notion, i.e., to ensure that being epic is a *property* of a map, we replace the notion of injection by the notion of embedding in our definition of epimorphism. Recall [54, Definition 4.6.1] that a map is an embedding if its action on identity types is an equivalence. For sets the notions of embedding and injection coincide, so that we recover the original notion of epimorphism in such cases.

DEFINITION 2.1 (\star Epimorphism). A map $f : A \rightarrow B$ is an *epimorphism* if for every type X , the precomposition map

$$(B \rightarrow X) \xrightarrow{f^*} (A \rightarrow X)$$

is an embedding. We also say that f is an *epi* or that it is *epic*.

Thus, f is epic precisely when the canonical function $g = h \rightarrow g \circ f = h \circ f$ is an equivalence for all maps $g, h : B \rightarrow X$.

REMARK 2.2. Notice that the fiber of f^* at a map $g : B \rightarrow X$ is exactly the type of extensions of g along f , as considered at the start of the introduction to this paper. Since embeddings can be characterized as those maps whose fibers are all propositions [45, Theorem 12.2.3], we see that f is an epimorphism if and only if for all $g : B \rightarrow X$ the type of extensions of g along f is a proposition, i.e., every such g has at most one extension along f .

The universal quantification over types X in Definition 2.1 should be understood as ranging over types X in a type universe, so a priori this definition only makes sense relative to a universe. However, it is consequence of the characterization of epimorphisms (Theorem 2.9) that the notion is actually independent of the type universe.

LEMMA 2.3 (\star). A map $f : A \rightarrow B$ is epic if and only if the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f \downarrow & & \downarrow \text{id} \\ B & \xrightarrow{\text{id}} & B \end{array}$$

is a pushout.

PROOF. The square is a pushout if and only if the map

$$(B \rightarrow X) \rightarrow \sum_{g : B \rightarrow X} \sum_{h : B \rightarrow X} (g \circ f = h \circ f)$$

$$k \mapsto (k, k, \text{refl})$$

is an equivalence for every type X . Note that the right hand side is equivalent to $\sum_{g:B \rightarrow X} \text{fib}_{(-) \circ f}(g \circ f)$, so that we have this equivalence exactly when $\text{fib}_{(-) \circ f}(g \circ f)$ is contractible for all types X and maps $g : B \rightarrow X$. But this happens if and only if $(-) \circ f$ is an embedding, i.e., when f is epic. \dashv

Since we work in dependent type theory, it is natural to also consider the following, seemingly stronger notion

DEFINITION 2.4 (⚙️ **Dependent epimorphism**). A *dependent epimorphism* is a map $f : A \rightarrow B$ such that for every type family P over B , the precomposition map

$$\prod_{b:B} P(b) \xrightarrow{f^*} \prod_{a:A} P(f(a))$$

is an embedding.

Note that every equivalence is a (dependent) epimorphism. Obviously, every dependent epimorphism is an epimorphism, but the converse holds as well. In fact, this will be a consequence of our characterization of the epimorphisms as the acyclic maps, to which we now turn.

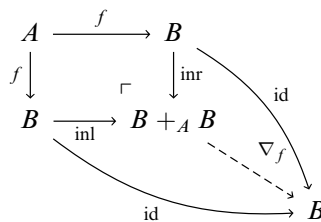
2.1. Acyclic maps. The notion of an acyclic map is defined fiberwise using the suspension of a type which we recall from [54, Section 6.5]:

DEFINITION 2.5 (⚙️ **Suspension ΣA**). The *suspension* of a type A is the pushout of the terminal maps $\mathbf{1} \leftarrow A \rightarrow \mathbf{1}$ and we denote it by ΣA . Equivalently, it is the higher inductive type generated by two points $N, S : \Sigma A$ (north and south) and for every $a : A$, an identification $m_a : N = S$ (the meridians).

DEFINITION 2.6 (⚙️ **Acyclicity**). A type is *acyclic* if its suspension is contractible; a map is acyclic if all of its fibers are acyclic types. Note that a type A is acyclic precisely when the unique map $A \rightarrow \mathbf{1}$ is acyclic.

Section 6 explains the relation to the traditional formulation of acyclicity (using reduced homology), while Section 7 presents examples of acyclic types. It is natural to consider variations on the notion of acyclicity where one instead requires the n -fold suspension (for $n \geq 2$) to become contractible. Proposition 4.7 shows that these notions reduce to the above notion of an acyclic type, at least in the presence of an additional principle (Section 4). For now, we work towards characterizing the epimorphisms as the acyclic maps.

DEFINITION 2.7 (⚙️ **Codiagonal ∇_f**). The *codiagonal* ∇_f of a map $f : A \rightarrow B$ is the dashed map in the pushout diagram:



The reason for introducing the codiagonal is that it is the “fiberwise suspension” as made precise by the following

LEMMA 2.8 (♣). *For every $f : A \rightarrow B$ and $b : B$ we have an equivalence $\text{fib}_{\nabla_f}(b) \simeq \Sigma \text{fib}_f(b)$ between the fiber of ∇_f at b and the suspension of the fiber of f at b .*

PROOF. By the flattening lemma [46, The flattening lemma for pushouts] (cf. [54, Lemma 6.12.2]), we can pull back the pushout square in Definition 2.7 along a point inclusion $b : \mathbf{1} \rightarrow B$ to obtain the pushout square

$$\begin{array}{ccc} \text{fib}_f(b) & \longrightarrow & \text{fib}_{\text{id}}(b) \\ \downarrow & \lrcorner & \downarrow \\ \text{fib}_{\text{id}}(b) & \longrightarrow & \text{fib}_{\nabla_f}(b). \end{array}$$

But the spans $\text{fib}_{\text{id}}(b) \leftarrow \text{fib}_f(b) \rightarrow \text{fib}_{\text{id}}(b)$ and $\mathbf{1} \leftarrow \text{fib}_f(b) \rightarrow \mathbf{1}$ are equivalent, so that $\text{fib}_{\nabla_f}(b)$ is also the pushout of the second span, i.e., it is the suspension of $\text{fib}_f(b)$, as desired. \dashv

THEOREM 2.9 (♣ Characterization of epimorphisms). *The following are equivalent for a map $f : A \rightarrow B$:*

- (i) f is an epi,
- (ii) f is a dependent epi,
- (iii) f is acyclic,
- (iv) its codiagonal ∇_f is an equivalence.

PROOF. The equivalence of (iii) and (iv) follows from Lemma 2.8. Moreover, (i) and (iv) are seen to be equivalent by Lemma 2.3. Finally, suppose that f is an epi; we show that it is a dependent epi as well. We need to prove that precomposition by f is an embedding $\prod_{b:B} P(b) \hookrightarrow \prod_{a:A} P(f(a))$ for an arbitrary type family P over B . The equivalence $A \simeq \sum_{b:B} \text{fib}_f(b)$ induces a commutative square

$$\begin{array}{ccc} \prod_{b:B} P(b) & \xrightarrow{f^*} & \prod_{a:A} P(f(a)) \\ \downarrow & & \simeq \uparrow \\ \prod_{b:B} (\text{fib}_f(b) \rightarrow P(b)) & \xrightarrow{\cong} & \prod_{w:\sum_{b:B} \text{fib}_f(b)} P(\pi_1(w)) \end{array}$$

so it suffices to prove that the leftmost map, which is the functorial action of \prod at the constants map $P(b) \rightarrow (\text{fib}_f(b) \rightarrow P(b))$, is an embedding. By a special case of [45, Exercise 13.12(a)], which is formalized in [46, Functoriality of dependent function types], it suffices for this constants map to be an embedding for all $b : B$. But this is indeed the case, because $\text{fib}_f(b)$ is acyclic, so that precomposition by the terminal map $\text{fib}_f(b) \rightarrow \mathbf{1}$ is an embedding. \dashv

We will state further results in terms of acyclicity, often tacitly using that the acyclic maps are exactly the epis.

For acyclic *types*, we arrive at the following.

COROLLARY 2.10 (♣). *The following are equivalent:*

- (i) *the type A is acyclic,*
- (ii) *for all types B , the constants map $B \rightarrow (A \rightarrow B)$ is an embedding,*
- (iii) *for all types B and elements $x, y : B$, the constants map $x = y \rightarrow (A \rightarrow x = y)$ is an equivalence.*

PROOF. Writing $!_A : A \rightarrow \mathbf{1}$ for the unique map from A to the unit type, the commutative diagram

$$\begin{array}{ccc}
 (\mathbf{1} \rightarrow B) & \xrightarrow{(!_A)^*} & (A \rightarrow B) \\
 \simeq \uparrow & \nearrow \text{const}_{B,A} & \\
 B & &
 \end{array}$$

informs us that $(!_A)^*$ is an embedding if and only if $\text{const}_{B,A}$ is, which proves the equivalence of (i) and (ii) via Theorem 2.9. For the equivalence of (ii) and (iii) we let $x, y : B$ and use the commutative diagram

$$\begin{array}{ccc}
 (x = y) & \xrightarrow{\text{ap}_{x,y}^{\text{const}_{B,A}}} & (\text{const}_{B,A}(x) = \text{const}_{B,A}(y)) \\
 & \searrow \text{const}_{x=y,A} & \downarrow \simeq \\
 & & \prod_{a:A} \text{const}_{B,A}(x)(a) = \text{const}_{B,A}(y)(a) \\
 & & \downarrow \simeq \\
 & & (A \rightarrow x = y)
 \end{array}$$

to see the map $\text{ap}_{x,y}^{\text{const}_{B,A}}$ is an equivalence if and only if $\text{const}_{x=y,A}$ is. But the former condition for all $x, y : B$ says precisely that $\text{const}_{B,A}$ is an embedding. \dashv

Examples of acyclic types, other than the unit type, are presented in Section 7.

2.2. Closure properties. We show that acyclic types and maps enjoy several closure properties. By Theorem 2.9 these properties also hold for epimorphisms of course.

LEMMA 2.11 (♣). *The class of acyclic maps is closed under composition and has the right cancellation property: if $g \circ f$ and f are acyclic, then so is g .*

PROOF. Suppose that f is acyclic, i.e., that precomposition with f is an embedding. Then, for any type X , the composite

$$(C \rightarrow X) \xrightarrow{g^*} (B \rightarrow X) \xrightarrow{f^*} (A \rightarrow X),$$

which is $(g \circ f)^*$, is an embedding if and only if g^* is, by the composition and *left* cancellation properties of the class of embeddings. \dashv

It follows from Lemma 2.11 that the acyclic types are closed under dependent sums:

PROPOSITION 2.12 (♣). *If A is an acyclic type and B is a family of acyclic types over A , then $\sum_{a:A} B(a)$ is acyclic.*

PROOF. The composite $\sum_{a:A} B(a) \xrightarrow{\pi_1} A \rightarrow \mathbf{1}$ is acyclic, because A and $\text{fib}_{\pi_1}(a) \simeq B(a)$ are acyclic types. ⊣

COROLLARY 2.13 (♣). *Finite products of acyclic types are acyclic.*

PROOF. This is the non-dependent version of the previous result. ⊣

Another consequence is that (inhabited) locally acyclic types are acyclic

LEMMA 2.14 (♣). *If all identity types of an inhabited type A are acyclic, then so is A .*

PROOF. Since being acyclic is a property, we may take a point $a : A$. The composite $\mathbf{1} \xrightarrow{\ulcorner a \urcorner} A \rightarrow \mathbf{1}$ is acyclic and so is $\text{fib}_{\ulcorner a \urcorner}(x) \simeq (a = x)$ for every $x : A$ by assumption. But then $A \rightarrow \mathbf{1}$ is acyclic too by Lemma 2.11. ⊣

The converse of Lemma 2.14 fails, as we will see later when we have produced an example of an acyclic type.

PROPOSITION 2.15 (♣). *The acyclic maps are stable under pullbacks and pushouts.*

PROOF. Given a pullback square

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_2} & B \\ \pi_1 \downarrow & \lrcorner & \downarrow f \\ A & \xrightarrow{g} & C \end{array}$$

we have, for every $a : A$, an equivalence $\text{fib}_{\pi_1}(a) \simeq \text{fib}_f(g(a))$ by [54, Lemma 7.6.8], which proves that π_1 is acyclic if f is.

Assume f is acyclic and consider the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ f \downarrow & \lrcorner & \downarrow \text{inr} \\ B & \xrightarrow{\text{inl}} & B +_A C. \end{array}$$

The type of extensions of a map $h : C \rightarrow X$ along inr is equivalent to the type of extensions of $h \circ g$ along f by the universal property of pushout. By acyclicity of f , the latter extension problem has at most one solution (recall Remark 2.2). But then so does the former, so inr is acyclic. ⊣

REMARK 2.16. Note that the acyclic types are not closed under coproducts: while $\mathbf{1}$ is acyclic, the coproduct $\mathbf{1} + \mathbf{1}$ is not, since $\Sigma(\mathbf{1} + \mathbf{1}) \simeq S^1$ which is, of course, not contractible.

LEMMA 2.17 (♣). *Acyclic types are closed under retracts.*

PROOF. If A is a retract of B , then ΣA is a retract of ΣB by functoriality of the suspension. If B is acyclic, then $\Sigma B \simeq \mathbf{1}$ and retracts of contractible types are contractible. ⊣

COROLLARY 2.18. *The acyclic maps are also closed under retracts (in the sense of [54, Definition 4.7.2]).*

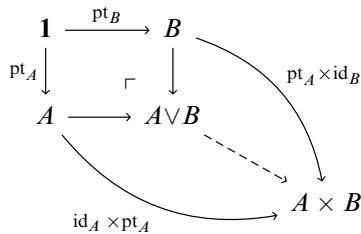
PROOF. Since fibers of a retract are retracts of fibers [54, Lemma 4.7.3], this follows from Lemma 2.17. \dashv

REMARK 2.19. The truncation of an acyclic type need not be acyclic. For a counterexample, we turn to classical K-theory and we don't spell out the details. The Volodin space $X(R)$ of a ring R is the fiber at the base point of the acyclic map $\text{BGL}(R) \rightarrow \text{BGL}(R)^+$, and hence acyclic [58, Example IV.1.3.2]. Its fundamental group $\text{St}(R)$ is the Steinberg group of R , and its higher homotopy groups contain the higher K-theory of R (by the long exact sequence of a fibration). The Steinberg group is not acyclic, since $H_3(\text{St}(R)) \cong K_3(R)$ [58, Example IV.1.9], and, e.g., $K_3(\mathbb{F}_q) \cong \mathbb{Z}/(q^2 - 1)$ by Quillen's computation of the K-theory of a finite field [58, Corollary IV.1.13]. Since the 1-truncation of $X(R)$ is the classifying space of the Steinberg group of R [58, Corollary IV.1.7.2], this shows that the truncation of an acyclic type need not be acyclic.

2.3. Balanced maps. We connect the acyclic maps to the notion of *balanced maps* due to Raptis [41]. To do so, we first recall the construction of joins and smash products [13, p. 33], relate them in Lemma 2.22 and prove a general lemma about joins with an acyclic type (Lemma 2.23).

DEFINITION 2.20 (\star Join $A * B$). The *join* of two types A and B is the pushout of the projections $A \leftarrow A \times B \rightarrow B$ and we denote it by $A * B$.

DEFINITION 2.21 (\star Smash product $A \wedge B$). The *smash product* of two pointed types A and B is the pushout of the span $\mathbf{1} \leftarrow A \vee B \rightarrow A \times B$, where right leg is the induced (dashed) map in the following diagram:



We denote the smash product by $A \wedge B$.

LEMMA 2.22. *For pointed types A and B , their join is equivalent to the suspension of their smash product, i.e., $A * B \simeq \Sigma(A \wedge B)$.*

PROOF. For an arbitrary pointed type X , we have a sequence of natural equivalences

$$\begin{aligned}
 & (\Sigma(A \wedge B) \rightarrow_{\text{pt}} X) \\
 & \simeq (A \wedge B \rightarrow_{\text{pt}} \Omega X) && \text{(loop-suspension adjunction [54, Lemma 6.5.4])} \\
 & \simeq (A \rightarrow_{\text{pt}} (B \rightarrow_{\text{pt}} \Omega X)) && \text{(by [24, Theorem 4.3.28])} \\
 & \simeq (A * B \rightarrow_{\text{pt}} X) && \text{(by [18, Lemma 6.1])}
 \end{aligned}$$

so $A * B$ and $\Sigma(A \wedge B)$ must be equivalent. \dashv

LEMMA 2.23. *The join of an acyclic type with an inhabited type is contractible.*

PROOF. Suppose that A is acyclic and B is inhabited. In particular, A is also inhabited. Since we are proving a proposition, we may assume that both types are actually pointed so that $A * B \simeq \Sigma(A \wedge B)$ by Lemma 2.22. Now we just calculate:

$$\begin{aligned} \Sigma(A \wedge B) &\simeq S^1 \wedge (A \wedge B) && \text{(by [13, Proposition 4.2.1])} \\ &\simeq (S^1 \wedge A) \wedge B && \text{(by [36, Proposition 9])} \\ &\simeq \Sigma A \wedge B && \text{(by [13, Proposition 4.2.1])} \\ &\simeq \mathbf{1} \wedge B && \text{(by acyclicity of } A) \\ &\simeq \mathbf{1}, && \text{(straightforward)} \end{aligned}$$

so that $A * B$ is contractible, as desired. ◻

DEFINITION 2.24 (Balanced map). A map $f : A \rightarrow B$ is *balanced* if for every surjection $g : X \rightarrow B$, the pullback square

$$\begin{array}{ccc} F & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow f \\ X & \xrightarrow{g} & B \end{array}$$

is also a pushout square.

LEMMA 2.25 (Pushouts are fiberwise). *A square*

$$\begin{array}{ccc} F & \xrightarrow{g'} & A \\ f' \downarrow & & \downarrow f \\ X & \xrightarrow{g} & B \end{array}$$

is a pushout square if and only if the induced squares of fibers

$$\begin{array}{ccc} \text{fib}_{g \circ f'}(b) & \longrightarrow & \text{fib}_f(b) \\ \downarrow & & \downarrow \\ \text{fib}_g(b) & \longrightarrow & \mathbf{1} \end{array}$$

are pushouts for every $b : \mathbf{1} \rightarrow B$.

PROOF. If the first square is a pushout square, then by the flattening lemma [46, The flattening lemma for pushouts] (cf. [54, Lemma 6.12.2]), we can pull it back along any point $b : \mathbf{1} \rightarrow B$ to get that the induced square of fibers is a pushout. Conversely, suppose that each fiber square is a pushout and consider the cogap map h in the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{g'} & A \\
 f' \downarrow & \lrcorner & \downarrow \\
 X & \longrightarrow & P \\
 & \searrow & \downarrow h \\
 & & B
 \end{array}$$

g (curved arrow from X to B)

We once again use the flattening lemma to obtain pushout squares

$$\begin{array}{ccc}
 \text{fib}_{g \circ f'}(b) & \longrightarrow & \text{fib}_f(b) \\
 \downarrow & \lrcorner & \downarrow \\
 \text{fib}_g(b) & \longrightarrow & \text{fib}_h(b)
 \end{array}$$

for every $b : B$. By assumption, $\text{fib}_h(b) \simeq \mathbf{1}$, so h is an equivalence, as desired. \dashv

The following result for the ∞ -category of spaces is due to Raptis [41, Theorem 2.1] where it is established using different techniques.

THEOREM 2.26. *A map is acyclic if and only if it is balanced.*

PROOF. In the forward direction, we note that Lemma 2.25 implies that it suffices to prove the proposition for acyclic maps into the unit type, i.e., when $B \equiv \mathbf{1}$ in Definition 2.24. In this case, $F \simeq A \times X$ and $P \simeq A * X$ (recall Definition 2.20) and we have to show that the latter is contractible. But this follows from Lemma 2.23 because A is acyclic and X is inhabited (as $g : X \rightarrow \mathbf{1}$ is assumed to be surjective).

Conversely, take $X = \mathbf{2} \times B$ and let $g : \mathbf{2} \times B \rightarrow B$ be the projection. Let $b : B$ be arbitrary and let F be the fiber of f at b . By pullback-stability of pushout squares, the pushout square on the left below pulls back to the pushout square on the right:

$$\begin{array}{ccc}
 \mathbf{2} \times A & \longrightarrow & A \\
 \mathbf{2} \times f \downarrow & \lrcorner & \downarrow f \\
 \mathbf{2} \times B & \longrightarrow & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbf{2} \times F & \longrightarrow & F \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbf{2} & \longrightarrow & \mathbf{1}.
 \end{array}$$

The square on the right being a pushout means $\mathbf{1} \simeq \mathbf{2} * F \simeq \Sigma F$, so F is acyclic. \dashv

The following appears for CW-spaces as [30, Theorem 2.5] and [2, Corollary 2.10(b)], and is discussed in the context of the ∞ -category of spaces on [41, p. 774].

COROLLARY 2.27. *If $f : A \rightarrow B$ is an acyclic map of connected types, then the fiber sequence for any $b : B$ is also a cofiber sequence. That is, the pullback square*

$$\begin{array}{ccc}
 \text{fib}_f(b) & \longrightarrow & A \\
 \downarrow & & \downarrow f \\
 \mathbf{1} & \xrightarrow{b} & B
 \end{array}$$

is also a pushout square.

The dual Blakers–Massey theorem holds for all left classes of modalities [3, Theorem 3.27], but the cited proof only uses stability under base change, so

it also holds for acyclic maps, as also observed by Raptis [41, Section 3.3], so we record that as well.

PROPOSITION 2.28 (dual Blakers–Massey for acyclic maps). *If in a pullback square*

$$\begin{array}{ccc} Q & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

the pushout product $f \square g$ is acyclic, then so is the cogap map $A +_Q B \rightarrow C$.

§3. Acyclic maps and epimorphisms of k -types. It turns out that we can get nice characterizations and examples if we consider a notion of epimorphism with respect to k -types only.

DEFINITION 3.1 (⚙️ (Dependent) k -epimorphism). We define a map $f : A \rightarrow B$ to be a k -epimorphism if for all k -types X , the precomposition map $(B \rightarrow X) \xrightarrow{f^*} (A \rightarrow X)$ is an embedding. We say that f is a *dependent k -epimorphism* if for every family P of k -types over B , the precomposition map

$$\prod_{b:B} P(b) \xrightarrow{f^*} \prod_{a:A} P(f(a))$$

is an embedding.

Note that we did not require f to be a map between k -types in the above definition. However, as the following result shows, being k -epic is stable under k -truncation:

LEMMA 3.2 (⚙️). *A map $f : A \rightarrow B$ is k -epic if and only if its k -truncation $\|f\|_k : \|A\|_k \rightarrow \|B\|_k$ is.*

PROOF. For every k -type X , we have a commutative square

$$\begin{array}{ccc} (\| * B \|_k \rightarrow X) & \xrightarrow{\|f\|_k^*} & (\| * A \|_k \rightarrow X) \\ \downarrow \lrcorner_k^* & & \downarrow \lrcorner_k^* \\ (B \rightarrow X) & \xrightarrow{f^*} & (A \rightarrow X). \end{array}$$

Since the vertical maps are equivalences by [54, Lemma 7.5.7 and Corollary 7.5.8], it follows that the top map is an embedding if and only if the bottom map is an embedding. ⊣

We work towards characterizing the k -epimorphisms.

DEFINITION 3.3 (⚙️ k -acyclicity). A type is k -acyclic if its suspension is k -connected (i.e., the k -truncation of its suspension is contractible). A map is k -acyclic if all of its fibers are.

Note that every $(k + 1)$ -acyclic type is k -acyclic. Since suspensions are inhabited, every type is $(- 1)$ -acyclic.

As with acyclic types, it is natural to consider variations on the notion of k -acyclicity where one instead requires the n -fold suspension (for $n \geq 2$) to become k -connected. In Proposition 3.29 we show that these notions suitably reduce to the above notion.

We recall the notion of a k -equivalence from [20] (where the notion was introduced for an arbitrary modality)

DEFINITION 3.4 (\star k -equivalence). A map is a k -equivalence if its k -truncation is an equivalence.

In general, not every k -equivalence is k -connected. For example, the “degree 2” map $d_2 : S^1 \rightarrow S^1$ that sends loop to loop \bullet loop is a 0-equivalence, but is not 0-connected. Alternatively, any map $\mathbf{1} \rightarrow \mathbf{2}$ is a (-1) -equivalence but no such map is (-1) -connected (= surjective). However, for retractions the notions do coincide and this observation proves very useful in the characterization of k -epis.

LEMMA 3.5 (\star). A retraction is a k -equivalence if and only if it is k -connected.

PROOF. For $k \geq 0$, this follows from [54, Corollary 8.8.5] together with the fact that if $r \circ s = \text{id}$, then $\pi_n(r) \circ \pi_n(s) = \text{id}$, so $\pi_n(r)$ must be surjective.

For $k = -1$, it follows from the fact that retractions are (-1) -connected (= surjective), and for $k = -2$, the claim is trivial. \dashv

THEOREM 3.6 (\star Characterization of k -epimorphisms). For a map $f : A \rightarrow B$, the following are equivalent:

- (i) f is a k -epi,
- (ii) its codiagonal ∇_f is a k -equivalence,
- (iii) its codiagonal ∇_f is k -connected,
- (iv) f is k -acyclic,
- (v) f is a dependent k -epi.

PROOF. (i) \iff (ii): By replaying the proof of Lemma 2.3, we see that f is a k -epi if and only if the square

$$\begin{array}{ccc} \|A\|_k & \xrightarrow{\|f\|_k} & \|B\|_k \\ \|f\|_k \downarrow & & \downarrow \text{id} \\ \|B\|_k & \xrightarrow{\text{id}} & \|B\|_k \end{array}$$

is a pushout of k -types (recall [54, Theorem 7.4.12]). But this just means that $\|\nabla_f\|_k$ is an equivalence, i.e., that ∇_f is a k -equivalence.

(ii) \implies (iii): This follows at once from Lemma 3.5, because the codiagonal is, by definition, a retraction.

(iii) \implies (iv): This holds because the codiagonal is the fiberwise suspension (Lemma 2.8).

(iv) \implies (v): Suppose that f is k -acyclic; we show that f is a dependent k -epi. Let P be a family of k -types over B . As in the proof of Theorem 2.9, it suffices to

¹More generally, using [20, Proposition 2.31] one can prove that a section is an L' -equivalence if and only if it is L' -connected for any localization L .

show that we have embeddings $P(b) \hookrightarrow (\text{fib}_f(b) \rightarrow P(b))$ for every $b : B$. For every $b : B$, the type $\text{fib}_f(b)$ is k -acyclic by assumption, so the codiagonal of $\text{fib}_f(b) \rightarrow \mathbf{1}$ is a k -equivalence and hence the map $\text{fib}_f(b) \rightarrow \mathbf{1}$ is a k -epi by the equivalence of (i) and (ii) proved above. Thus, since every $P(b)$ is a k -type, we have the desired embeddings $P(b) \hookrightarrow (\text{fib}_f(b) \rightarrow P(b))$.

(v) \implies (i): By specializing to non-dependent functions. ⊖

As with Corollary 2.10, the characterization theorem implies

COROLLARY 3.7 (⚙️). *The following are equivalent:*

- (i) *the type A is k -acyclic,*
- (ii) *for all k -types B , the constants map $B \rightarrow (A \rightarrow B)$ is an embedding,*
- (iii) *for all k -types B and $x, y : B$, the constants map $x = y \rightarrow (A \rightarrow x = y)$ is an equivalence.*

COROLLARY 3.8 (⚙️). *A type/map is 0-acyclic if and only if it is (-1) -connected.*

PROOF. The epimorphisms of sets are precisely the surjective maps, i.e., those maps with (-1) -connected fibers, so this follows from Theorem 3.6. ⊖

The following result implies that the k -sphere S^k is an example of a $(k + 1)$ -acyclic type.

PROPOSITION 3.9 (⚙️). *Every k -connected map is $(k + 1)$ -acyclic and k -equivalences are k -acyclic.*

PROOF. The first claim follows because taking the suspension increases the connectedness by one [54, Theorem 8.2.1]. Notice that the other claim is trivial for $k = -2$. For $k = k' + 1$ with $k' \geq -2$, we have that a k -equivalence is k' -connected by [20, Proposition 2.30], and hence k -acyclic by the above. ⊖

These implications are strict as shown in Remark 3.24. However, for simply-connected (= 1-connected) maps/types we do have the following result

PROPOSITION 3.10. *A simply connected type is $(k + 1)$ -acyclic if and only if it is k -connected.*

PROOF. The right-to-left implication is Proposition 3.9. For the converse, we may assume $k \geq 1$ and we use the Freudenthal suspension theorem [54, Theorem 8.6.4]: The unit map of the loop-suspension adjunction, $\sigma : A \rightarrow \Omega\Sigma A$, is $2n$ -connected whenever A is n -connected (for $n \geq 0$). Since A is assumed to be $(k + 1)$ -acyclic, the map $\Omega\Sigma A \rightarrow \mathbf{1}$ is k -connected. Hence, starting with the assumption that A is 1-connected, we in turn conclude that A is $\min(2, k)$ -connected, then $\min(4, k)$ -connected, etc., and hence k -connected. ⊖

A consequence of the above result is that joins of k -acyclic types become $2k$ -connected, as we show after this lemma.

LEMMA 3.11. *The suspension of the join of two pointed types A and B is the smash products of their suspensions, i.e., $\Sigma(A * B) \simeq \Sigma A \wedge \Sigma B$.*

PROOF. We have the chain of equivalences:

$$\begin{aligned}
 \Sigma(A * B) &\simeq \Sigma^2(A \wedge B) && \text{(by Lemma 2.22)} \\
 &\simeq S^1 \wedge S^1 \wedge (A \wedge B) && \text{(by [13, Proposition 4.2.1])} \\
 &\simeq S^1 \wedge (A \wedge B) \wedge S^1 && \text{(the smash product is clearly commutative)} \\
 &\simeq (S^1 \wedge A) \wedge (B \wedge S^1) && \text{(by [36, Proposition 9])} \\
 &\simeq \Sigma A \wedge \Sigma B && \text{(by [13, Proposition 4.2.1]).}
 \end{aligned}$$

⊣

PROPOSITION 3.12. *If A is a k -acyclic type and B is an l -acyclic type with $k, l \geq 0$, then their join $A * B$ is $(k + l)$ -connected. In particular, the join of two k -acyclic types is $2k$ -connected.*

PROOF. Since $k, l \geq 0$, the types A and B are inhabited, and since we are proving a proposition we may assume them to be pointed so that Lemma 3.11 applies and $\Sigma(A * B) \simeq \Sigma A \wedge \Sigma B$. Now, ΣA is k -connected and ΣB is l -connected by assumption so that their smash product is $(k + l + 1)$ -connected by [13, Proposition 4.3.1], indeed showing that $A * B$ is $(k + l)$ -connected. ⊣

3.1. Closure properties. Before characterizing 1- and 2-acyclic types, we record a few general closure properties of k -acyclic maps which are proved analogously to the ones for acyclic maps.

LEMMA 3.13 (♣). *The class of k -acyclic maps is closed under composition and has the right cancellation property: if $g \circ f$ and f are k -acyclic, then so is g .*

Moreover, the k -acyclic maps are closed under retracts, pullbacks and pushouts.

Finally, k -acyclic types are closed under Σ -types, and if all identity types of an inhabited type A are k -acyclic, then so is A itself.

COROLLARY 3.14 (♣). *A type is $(k + 1)$ -acyclic if and only if its k -truncation is.*

PROOF. The first map in the composite $A \xrightarrow{|-|_k} \|A\|_k \rightarrow \mathbf{1}$ is k -connected [54, Corollary 7.5.8] and so $(k + 1)$ -acyclic by Proposition 3.9. The result now follows from Lemma 3.13. ⊣

REMARK 3.15. The k -acyclic types are not closed under taking exponentials: the circle S^1 is 1-acyclic (as it is connected), but $(S^1 \rightarrow S^1) \simeq S^1 \times \mathbb{Z}$ is not by the upcoming Theorem 3.17.

3.2. Characterizing 1-acyclic types and applications in group theory. The following theorem says that there are no interesting 1-acyclic sets and is used to characterize the 1-acyclic types.

THEOREM 3.16 (♣). *For a set A , we have*

$$A \text{ is 1-acyclic} \iff A \text{ is acyclic} \iff A \text{ is contractible.}$$

PROOF. The right-to-left implications are trivial, so it suffices to show that every 1-acyclic set is contractible. If A is 1-acyclic, then by Corollary 3.7, the constants map

$$\Omega(\mathbf{BG}, \text{pt}) \rightarrow (A \rightarrow \Omega(\mathbf{BG}, \text{pt}))$$

must be an equivalence, where \mathbf{BG} is the classifying type [10, 17] of the free group G on the set A . Now, $\Omega(\mathbf{BG}, \text{pt}) \simeq G$, so in particular, the unit $\eta : A \rightarrow G$ of the free group G must be constant. Hence, $\eta(x) = \eta(y)$ for every $x, y : A$. But the unit is left-cancellable [39, Chapter X] (see [11] or [57, Example 11] for a proof in homotopy type theory), so A is a proposition. Finally, if A is 1-acyclic, then it is 0-acyclic, i.e., inhabited. Thus, A is contractible, as we wished to show. \dashv

THEOREM 3.17 (\spadesuit Characterization of 1-acyclic types). *A type is 1-acyclic if and only if it is connected.*

PROOF. All connected types are 1-acyclic by Proposition 3.9. Conversely, if A is 1-acyclic, then the composite $A \xrightarrow{\text{!}_0} \|A\|_0 \rightarrow \mathbf{1}$ is 1-acyclic. But the first map is 1-acyclic by Corollary 3.14, so that $\|A\|_0$ is 1-acyclic by the right-cancellation property of the class of k -acyclic maps. But $\|A\|_0$ is a set by definition, so A is connected by Theorem 3.16. \dashv

It follows directly from Theorem 3.17 that a map is 1-acyclic if and only if it is connected. Combined with Theorem 3.6 this can be used to give a constructive proof of the following fact about group homomorphisms.

THEOREM 3.18. *The following are equivalent for a group homomorphism $f : G \rightarrow H$:*

- (i) *f is an epi of groups;*
- (ii) *$\mathbf{B}f : \mathbf{BG} \rightarrow_{\text{pt}} \mathbf{BH}$ is an epi of pointed connected 1-types;*
- (iii) *$\mathbf{B}f : \mathbf{BG} \rightarrow \mathbf{BH}$ is connected;*
- (iv) *f is surjective as a map of sets.*

PROOF. The implication (i) \Rightarrow (ii) follows immediately from the equivalence between the category of groups and the category of pointed connected 1-types [17, Theorem 5.1].

To see that (ii) implies (iii), we suppose that $\mathbf{B}f$ is an epi in the category of pointed connected 1-types and pointed maps, and show that is also an epi of 1-types, which by Theorems 3.6 and 3.17 is equivalent to being connected. Assume we are given a map $g : \mathbf{BG} \rightarrow X$ whose codomain X is a 1-type. We are to show that g has at most one extension along $\mathbf{B}f$ (recall Remark 2.2). Now we can make g a pointed map by pointing X at $\text{pt}_X \equiv g(\text{pt}_{\mathbf{BG}})$. By connectedness of \mathbf{BG} , the map g factors through the connected component $X_0 \equiv \sum_{x:X} \|x = \text{pt}_X\|$ of pt_X as a map $g_0 : \mathbf{BG} \rightarrow X_0$. Because \mathbf{BH} is connected and the maps $\mathbf{B}f$ and g_0 are pointed, the type of bare extensions of g along $\mathbf{B}f$ is equivalent to $\sum_{e:\mathbf{B}H \rightarrow X} (e \circ \mathbf{B}f \sim g) \times \prod_{z:\mathbf{B}H} \|e(z) = \text{pt}_X\|$, which is in turn equivalent to $\sum_{e_0:\mathbf{B}H \rightarrow X_0} (e_0 \circ \mathbf{B}f \sim g_0)$, i.e., the type of extensions of g_0 along $\mathbf{B}f$. Now the type of pointed extensions of g_0 along $\mathbf{B}f$ is equivalent to the type

$$\sum_{e_0:\mathbf{B}H \rightarrow X_0} \sum_{r : e_0(\text{pt}_{\mathbf{B}H}) = \text{pt}_{X_0}} \sum_{H : e_0 \circ \mathbf{B}f \sim g_0} (q \bullet H(\text{pt}_{\mathbf{BG}}) \bullet \text{ap}_{e_0}(p)^{-1} = r),$$

which, by contracting r with its identification, is equivalent to the type of bare extensions of g_0 along $\mathbf{B}f$ as ordinary maps between (connected) types. Thus, if

$Bf : BG \rightarrow_{\text{pt}} BH$ is epic in the category of pointed connected 1-types, then Bf is 1-epic and hence connected (by Theorems 3.6 and 3.17).

If Bf is connected, then f is surjective as a map of sets by [10, Lemma 4.11.4], so (iii) \Rightarrow (iv). Finally, (iv) straightforwardly implies (i). \dashv

REMARK 3.19. Many traditional proofs of the fact that the epimorphisms of groups are precisely the surjections rely on excluded middle. For instance, the suggested proof in Mac Lane’s [35, Exercise I.5.5] relies heavily on a case analysis that requires the law of excluded middle. A notable exception is Todd Trimble’s proof [55] which is constructive and uses group actions. The above gives a different constructive proof relying instead on deloopings of groups and flattening (via the characterization of k -epis in Theorem 3.6).

We now give an application of Theorem 3.18 by presenting several structural characterizations of a group being generated by a subset.

COROLLARY 3.20. *Given an injection $\iota : S \hookrightarrow G$ from a set S to a group G , the following are equivalent:*

- (i) S generates G ;
- (ii) for every group H , the map

$$\begin{aligned} \iota^* : \text{Grp}(G, H) &\rightarrow (S \rightarrow H) \\ \varphi &\mapsto (s \mapsto \varphi(\iota(s))) \end{aligned}$$

which evaluates a group homomorphism φ is an embedding;

- (iii) the map $\hat{\iota} : FS \rightarrow G$ from the free group generated by S to G , induced by ι , is surjective;
- (iv) the map $B\hat{\iota} : BFS \rightarrow BG$ is connected.

PROOF. For (i) \Rightarrow (ii), note that if every element of G is a finite combination of elements in S and their inverses, then a group homomorphism from G to another group H is completely determined by its effect on S . Items (iii) and (iv) are equivalent by Theorem 3.18. Moreover, (i) and (iii) is clearly equivalent.

Finally, to see that (ii) and (iv) are equivalent, we consider the commutative diagram

$$\begin{array}{ccc} \text{Grp}(G, H) & \xrightarrow{\iota^*} & (S \rightarrow H) \\ \downarrow \simeq & & \uparrow \simeq \\ (BG \rightarrow_{\text{pt}} BH) & \xrightarrow{(B\hat{\iota})^*} & (BFS \rightarrow_{\text{pt}} BH) \end{array}$$

where the vertical maps are equivalences by [17, Theorem 5.1]. This diagram tells us that ι^* is an embedding if and only if $(B\hat{\iota})^*$ is. But the latter happens exactly when $B\hat{\iota}$ is an epi in the category of pointed connected 1-types and pointed maps, which, as we saw in Theorem 3.18, is equivalent to $B\hat{\iota}$ being connected. \dashv

3.3. Characterizing 2-acyclic types. The notion of 2-acyclicity turns out to be closely related to perfect groups. Most textbooks, e.g., [25, Exercise 19, Section 5.4], define a group G to be perfect if it equals its own commutator subgroup G' . Since the

abelianization [25, Proposition 7, Section 5.4] of a group G is given by the quotient G/G' , we can reformulate perfectness as follows

DEFINITION 3.21 (Perfectness). A group is *perfect* if its abelianization is trivial.

An example of a perfect group is the alternating group A_5 on 5 generators. Given a group G , we recall from [17, Section 6] that 2-truncating the suspension of the classifying type BG of G gives the classifying type of the abelianization of G as an abelian group, i.e., $\|\Sigma BG\|_2$ is the second delooping of its set of elements. This immediately yields the following result

PROPOSITION 3.22. *The classifying type of a group G is 2-acyclic if and only if G is perfect.*

REMARK 3.23. The classifying type of a group is always 1-acyclic by Theorem 3.17.

REMARK 3.24. While Proposition 3.9 tells us that every k -equivalence is k -acyclic, the converse fails. In fact, even a $(k + 1)$ -acyclic map need not be a k -equivalence as $BG \rightarrow \mathbf{1}$ is a 1-equivalence if and only if G is trivial, but it is a 2-acyclic map if and only if G is perfect by Proposition 3.22.

THEOREM 3.25 (Characterization of 2-acyclic types). *A type A is 2-acyclic if and only if A is connected and $\pi_1(A, a)$ is perfect for every $a : A$.*

PROOF. Note that connectedness is necessary, because 1-acyclic types are connected. Moreover, if A is connected, then $B\pi_1(A, a) = \|A\|_1$ for every $a : A$. By Corollary 3.14, the type A is 2-acyclic if and only if $\|A\|_1$ is. So by Proposition 3.22, this happens exactly when $\pi_1(A, a)$ is perfect. \dashv

Connected maps preserve 2-acyclicity

COROLLARY 3.26. *If $f : A \rightarrow B$ is connected and A is 2-acyclic, then so is B (which is equivalent to the image of f by connectedness).*

PROOF. Note that B is connected, because A and f are. So by Theorem 3.25 it suffices to prove that $\pi_1(B, b)$ is perfect for every $b : B$. Given $b : B$, there exists $a : A$ with $f(a) = b$ as f is connected. By connectedness of f and [54, Corollary 8.4.8(ii)], the map $\pi_1(f, a) : \pi_1(A, a) \rightarrow \pi_1(B, b)$ is a surjection for every $a : A$. But $\pi_1(A, a)$ is perfect and any quotient of a perfect group is perfect, so $\pi_1(B, b)$ is also perfect. \dashv

The following gives a necessary condition on the fundamental group functor for 2-acyclicity:

PROPOSITION 3.27. *If $f : A \rightarrow B$ is 2-acyclic, then it is connected, and for every $a : A$, we have that $\ker(\pi_1(f, a))$ is perfect and the abelianization of $\pi_1(f, a)$ is an isomorphism.*

PROOF. If $f : A \rightarrow B$ is 2-acyclic, then it is certainly connected by the characterization of 2-acyclic types. Let $a : A$ be arbitrary and write $F := \text{fib}_f(f(a))$. We have an exact sequence [54, Section 8.4]

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(A) \xrightarrow{\pi_1(f,a)} \pi_1(B) \rightarrow \pi_0(F) \cong \mathbf{1},$$

where the equivalence at the end holds because f is connected. Now $\ker(\pi_1(f, a)) = \text{im}(\pi_1(i))$ and $\pi_1(F)$ is perfect, because F is 2-acyclic. But any quotient of a perfect group is perfect, and hence, $\text{im}(\pi_1(i)) = \ker(\pi_1(f))$ is perfect. Finally, because abelianization is right exact (being a left adjoint, abelianization preserves all colimits), the exact sequence induces another exact sequence

$$\pi_1(F)^{\text{ab}} \xrightarrow{\pi_1(i)^{\text{ab}}} \pi_1(A)^{\text{ab}} \xrightarrow{\pi_1(f,a)^{\text{ab}}} \pi_1(B)^{\text{ab}} \rightarrow \mathbf{1}.$$

But $\pi_1(F)^{\text{ab}}$ is trivial since $\pi_1(F)$ is perfect, so the middle map $\pi_1(f, a)^{\text{ab}}$ is an isomorphism. –

We remark that the conditions in Proposition 3.27 are not *sufficient* for deriving 2-acyclicity: for example, the base point map $\mathbf{1} \rightarrow S^2$ satisfies the conditions of the proposition but is not 2-acyclic, as its fiber at the base point is ΩS^2 whose fundamental group \mathbb{Z} is not perfect. At present, we do not know of a necessary and sufficient characterization.

3.4. Iterated suspensions and stabilization. As mentioned before, it is natural to consider variations on the notion of k -acyclicity where one instead requires the n -fold suspension (for $n \geq 2$) to become k -connected. We show that these notions suitably reduce to the notion of k -acyclicity. More precisely, we have a stabilization result which says that the n -fold suspension of a type X is k -connected if and only if X is $(k - n + 1)$ -acyclic, for $n \geq 1$ and $k \geq 2$.

LEMMA 3.28. *The suspension of a type X is 2-acyclic if and only if X is connected.*

PROOF. If X is connected, then ΣX is 2-acyclic since suspensions increase connectedness by one [54, Theorem 8.2.1]. For the converse, suppose that ΣX is 2-acyclic. In particular, $\pi_2(\Sigma^2 X)$ is trivial. The unit of the set truncation $X \rightarrow \|X\|_0$ induces a map $\Sigma^2 X \rightarrow \Sigma^2 \|X\|_0$ whose fibers are 1-connected because its domain is 1-connected (as X is inhabited) and its codomain is 2-connected. The long exact sequence [54, Section 8.4] of this map then tells us that $\pi_2(\Sigma^2 \|X\|_0)$ is also trivial. By [17, Section 6], the abelian group $\pi_2(\Sigma^2 \|X\|_0)$ is in fact the free abelian group on the *pointed* set $\|X\|_0$. Below we describe an adaptation (to pointed sets) of an argument due to David W arn [56] to prove that the unit of the free-forgetful adjunction between abelian groups and pointed sets is injective. This implies that $\|X\|_0$ injects into the trivial group $\pi_2(\Sigma^2 \|X\|_0)$, showing that X is indeed connected.

The central idea, going back to Roswitha Harting [28] and recently also used in homotopy type theory by Jarl Taxer as Flaten in [53], is to regard a (pointed) set as a filtered colimit. For a pointed set $\text{pt}_X : X$ we consider the following category I_X : its objects are pairs (n, f) with n a natural number and $f : [n] \rightarrow_{\text{pt}} X$, where $[n]$ is the standard $(n + 1)$ -element set pointed at 0. A morphism between such objects (n, f) and (m, g) is a pointed map $p : [n] \rightarrow_{\text{pt}} [m]$ such that $f = g \circ p$. One can show that I_X is a filtered category and that $X \cong \text{colim}_{(n,f):I_X} [n]$.

The functor F that produces the free abelian group on a pointed set preserves colimits, as it is a left adjoint, so $FX \cong \text{colim}_{(n,f):I_X} [n]$. The forgetful functor sending an abelian group to its underlying set preserves filtered colimits [12, Proposition 2.13.5] and one can check that the forgetful functor from pointed sets to sets creates limits (we adopt [44, Definition 3.3.1]), so that the forgetful functor U sending an

abelian group to its underlying set pointed at the neutral element also preserves filtered colimits. Hence, $UFX \cong \text{colim}_{(n,f):IX} UF[n]$.

Because each $[n]$ has decidable equality, we can directly check that the unit maps $[n] \rightarrow UF[n]$ are all injective. But filtered colimits commute with finite limits in sets (see [12, Theorem 2.13.4] or [44, Theorem 3.8.9]) and the forgetful functor from pointed sets to sets creates such colimits and limits, so they also commute in pointed sets. Thus, since monos can be characterized using pullbacks, the unit map $X \rightarrow UFX$ must also be injective, as desired. \dashv

PROPOSITION 3.29. *For natural numbers $n \geq 1$ and $k \geq 2$, the n -fold suspension $\Sigma^n X$ of a type X is k -connected if and only if X is $(k - n + 1)$ -acyclic.*

PROOF. If X is $(k - n + 1)$ -acyclic, then ΣX is $(k - n + 1)$ -connected, so that $\Sigma^n X$ is $k = (k - n + 1 + (n - 1))$ -connected by [54, Theorem 8.2.1].

For the converse, note that the $n = 1$ case holds by definition. For $n = 2$, suppose that $\Sigma^2 X$ is k -connected. Then ΣX is k -acyclic, so by Lemma 3.28 and the fact that $k \geq 2$, we see that X is connected. But then ΣX is simply connected and by Proposition 3.10 even $(k - 1)$ -connected. Hence X is $k - 1 = (k - 2 + 1)$ -acyclic, as we wished to show.

Now suppose that $n > 2$ and that $\Sigma^n X$ is k -connected. Since $\Sigma^n X \simeq \Sigma^2 \Sigma^{n-2} X$, we see that $\Sigma^{n-2} X$ is $k - 2 + 1 = (k - 1)$ -acyclic, i.e., $\Sigma^{n-1} X$ is $(k - 1)$ -connected. So by induction hypothesis, X is $k - 1 - (n - 1) + 1 = (k - n + 1)$ -acyclic, as desired. \dashv

§4. The plus principle. From the definition of epimorphisms, we know that the type of extensions of a map $f' : A \rightarrow X$ along an epimorphism $f : A \rightarrow B$,

$$\begin{array}{ccc} A & \xrightarrow{f'} & X \\ f \downarrow & \nearrow h & \\ B & & \end{array}$$

is a proposition: indeed, it is the fiber at f' of the precomposition embedding $f^* : (B \rightarrow X) \hookrightarrow (A \rightarrow X)$. It is then a natural question to ask for equivalent reformulations of this proposition that might be easier to check. First we observe that a necessary condition is $\ker(\pi_1(f)) \subseteq \ker(\pi_1(f'))$, either in the sense of inclusion among congruence relations $\|A\|_1 \times \|A\|_1 \rightarrow \text{Set}$, or inclusions of subgroups for each $a : A$.

We don't know whether this is sufficient in general. However, there is a seemingly quite innocuous assumption under which it is, which we dub the *plus principle* (PP)

PRINCIPLE 4.1 (♣ PP). *Every acyclic and simply connected type is contractible.*

Hoyois highlighted this in the context of Grothendieck $(\infty, 1)$ -topoi [33, Remark 4], and it seems to be open whether it's true in general in that context. It follows from *Whitehead's principle* [54, Section 8.8] (every infinitely connected type is contractible, a.k.a. *hypercompleteness*) by the following

PROPOSITION 4.2. *Any acyclic and simply connected type is infinitely connected.*

PROOF. This follows directly from Proposition 3.10. \dashv

REMARK 4.3 (Anel). While Whitehead’s principle does *not* hold in the ∞ -topos of parametrized spectra (an object is hypercomplete if and only if the spectrum part is trivial), the plus principle *does* hold there, as observed by Mathieu Anel (private communication). The outline of his argument is as follows: we write S , Sp and PSp for the ∞ -categories of spaces, spectra, and parametrized spectra, respectively. The canonical functors $S \rightarrow PSp \rightarrow S$ are both left and right adjoint to each other and hence both preserve suspensions (as well as n -connected/truncated objects). The inclusion functor $Sp \rightarrow PSp$ preserves weakly contractible colimits and hence suspensions. Moreover, the suspension functor in Sp is an equivalence. Now, if E is an object of PSp and B is its image in S by $PSp \rightarrow S$ (its base), then B is respectively acyclic and simply-connected if E is. Thus, if E is acyclic and simply-connected, then $B \simeq \mathbf{1}$, and thus E is a spectrum. But then $\Sigma E \simeq \mathbf{1}$ implies that $E \simeq \mathbf{0}$ as a spectrum (i.e., E is terminal in PSp). So all acyclic simply connected objects in PSp are terminal.

From the plus principle itself we can deduce an analogous result for maps. We add (PP) to indicate that the result assumes the plus principle.

LEMMA 4.4 (PP). *Any acyclic 1-equivalence is an equivalence.*

PROOF. Consider an acyclic 1-equivalence $f : A \rightarrow B$. We show that each fiber is contractible, so let $b : B$ be given, and let F be the fiber of f at b :

$$\begin{array}{ccc}
 F & \xrightarrow{g} & A \\
 \downarrow & \lrcorner & \downarrow f \\
 \mathbf{1} & \xrightarrow{b} & B
 \end{array}
 \quad
 \begin{array}{c}
 \dots \longrightarrow \pi_2(B) \\
 \delta \curvearrowright \\
 \pi_1(F) \xrightarrow{\pi_1(g)} \pi_1(A) \xrightarrow{\pi_1(f)} \pi_1(B) \\
 \curvearrowright \\
 \pi_0(F) = \mathbf{1}.
 \end{array}$$

From the displayed fragment of the long exact sequence [54, Section 8.4] (relative to any base point of F) we have $\text{im}(\delta) = \ker(\pi_1(g)) = \pi_1(F)$, since $\pi_1(f)$ is an isomorphism. Thus, $\pi_1(F)$ is abelian as well as perfect (by Theorem 3.25) and hence trivial. By (PP) it follows that F is contractible. \dashv

We show that the necessary condition identified above is sufficient under (PP)

PROPOSITION 4.5 (PP). *Let $f : A \rightarrow B$ be acyclic and $f' : A \rightarrow X$ any map. Then f' extends along f if and only if we have the inclusion $\ker(\pi_1(f)) \subseteq \ker(\pi_1(f'))$.*

PROOF. The condition is necessary by functoriality of π_1 , since any extension h satisfies $f' = h \circ f$.

To establish sufficiency, we note that it suffices to consider the case where f' is surjective, since otherwise we just extend the corestriction to the image of f' . Now form the pushout:

$$\begin{array}{ccc}
 A & \xrightarrow{f'} & X \\
 f \downarrow & \lrcorner & \downarrow g \\
 B & \xrightarrow{g'} & P.
 \end{array}$$

Then g is acyclic and surjective on π_1 . By the previous lemma, it suffices to show that g is also injective on π_1 . Picking a base point in A , making the whole square pointed, we get a pushout square in groups by the Seifert–van Kampen theorem [54, Theorem 8.7.12]:

$$\begin{array}{ccc}
 \pi_1(A) & \xrightarrow{\pi_1(f')} & \pi_1(X) \\
 \pi_1(f) \downarrow & \lrcorner & \downarrow \pi_1(g) \\
 \pi_1(B) & \xrightarrow{\pi_1(g')} & \pi_1(P) \\
 & \searrow \varphi & \dashrightarrow \psi \\
 & & \pi_1(X)
 \end{array}$$

$\swarrow \text{id}$

The inclusion $\ker(\pi_1(f)) \subseteq \ker(\pi_1(f'))$ yields the dotted map φ as

$$\pi_1(B) \simeq \pi_1(A) / \ker(\pi_1(f)) \rightarrow \pi_1(A) / \ker(\pi_1(f')) \rightarrow \pi_1(X).$$

This induces the dashed map ψ , a retraction of $\pi_1(g)$. –

The above result was established for CW-spaces in [30, Proposition 3.1] and by different means for path connected CW-spaces in [2, Corollary 4.4]

COROLLARY 4.6 (PP). *Let A be a pointed, connected type with a given perfect normal subgroup $P \trianglelefteq \pi_1(A)$. Then the type of acyclic maps $f : A \rightarrow X$ with $\ker(\pi_1(f)) = P$ is a proposition.*

Another application of the plus principle is the following stabilization result.

PROPOSITION 4.7 (PP). *For any natural number $n \geq 1$, the n -fold suspension $\Sigma^n X$ of a type X is contractible if and only if X is acyclic.*

PROOF. The right-to-left implication is immediate. We prove the converse by induction. For $n = 1$ it follows by definition. For $n = 2$, we assume that $\Sigma^2 X$ is contractible so that ΣX is acyclic. Then X must be connected by Lemma 3.28 so that ΣX is simply connected and hence contractible by the plus principle. For $n > 2$, we assume that $\Sigma^n X$ is contractible. By the above, $\Sigma^{n-2} X$ must be acyclic and hence $\Sigma^{n-1} X$ is contractible, so that X is acyclic by induction hypothesis. –

Although it seems plausible, we do not know whether, in the absence of Whitehead’s Principle, a type X is acyclic as soon as its suspension spectrum $\Sigma^\infty X$ is contractible.

4.1. The Blakers–Massey theorem for acyclic maps. The Blakers–Massey theorem holds in HoTT for the left class of any modality [3, Theorem 4.2].

Here we show directly that it holds for acyclic maps, assuming the plus principle. First we need the following lemma, also observed by Raptis [41, Lemma 3.6] with a different proof.

LEMMA 4.8 (PP). *If A and B are inhabited, then the type $A * B$ is contractible if and only if it is acyclic.*

PROOF. For the nontrivial direction, assume $A * B$ is acyclic. Since A and B are inhabited and we are proving a proposition, we may assume they are pointed. Then

$A * B$ is equivalent to $\Sigma(A \wedge B)$ by Lemma 2.22. Since A and B are (-1) -connected, [13, Proposition 4.3.1] tells us that $A \wedge B$ is 0-connected, so that its suspension is 1-connected by [54, Theorem 8.2.1]. But now $A * B$ is acyclic and 1-connected, hence contractible by the plus principle. \dashv

In addition, we shall need the following observations, giving a constructive treatment of [41, Proposition 3.7].

LEMMA 4.9. *If the join $A * B$ is inhabited, then either A or B is inhabited.*

PROOF. Since we are proving a proposition, we may assume we have an element of the join $A * B$. By join induction, we get two (point constructor) cases, so the conclusion follows. \dashv

LEMMA 4.10 (PP). *Let A be a pointed type and B be any type. If the type $\Omega A * B$ is acyclic, then it is contractible.*

PROOF. The type $\Omega A * B$, pointed at $\text{inl}(\text{refl})$, is 0-connected with perfect fundamental group by Theorem 3.25, so it suffices to show that $\pi_1(\Omega A * B)$ is abelian. By the naive van Kampen theorem [54, Theorem 8.7.4], we can express the fundamental group as a set quotient of the type of sequences

$$(\text{refl}, \alpha_0, p_1, b_1, r_1, b'_1, p'_1, \alpha_1, p_2, \dots, b'_n, p'_n, \alpha_n, \text{refl})$$

where

- $n : \mathbb{N}$,
- $p_k, p'_k : \Omega A, b_k, b'_k : B$, for $0 < k \leq n$,
- $\alpha_k : p'_k = p_{k+1}$ for $0 \leq k \leq n$ with $p'_0 \equiv p_{n+1} \equiv \text{refl}$,
- $r_k : b_k = b'_k$ for $0 < k \leq n$.

To prove the proposition that two such codes give commuting elements, we look whether any of them has $n > 0$. If so, we know B is inhabited, and then $\Omega A * B$ is contractible by Lemma 4.8. Otherwise, the two codes represent 2-loops $\alpha, \beta : \Omega^2 A$, which commute by the Eckmann–Hilton argument. \dashv

LEMMA 4.11 (PP). *For any types A and B with elements $a, a' : A$ and $b, b' : B$ we have that the join $(a =_A a') * (b =_B b')$ is contractible if and only if it is acyclic.*

PROOF. Suppose the join is acyclic. By Lemma 4.9, one join summand is inhabited, so without loss of generality, we may assume we have $p : a =_A a'$. Concatenating with the inverse of p gives an equivalence $(a =_A a') \simeq \Omega(A, a)$. Now apply Lemma 4.10. \dashv

Now we can prove the Blakers–Massey theorem for acyclic maps. As observed by Raptis [41, Section 3.3], the conclusion is slightly stronger than naively expected.

THEOREM 4.12 (PP, Blakers–Massey for acyclic maps). *Consider a pushout square:*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ f \downarrow & & \downarrow \\ B & \longrightarrow & P. \end{array}$$

Then the relative pushout product $\Delta f \square_A \Delta g$ is acyclic if and only if it is contractible, and in that case the square is cartesian. This holds also if the absolute pushout product $\Delta f \square \Delta g$ is acyclic.

PROOF. Assume that the relative pushout product is acyclic. The fibers of $\Delta f \square_A \Delta g$ are joins of the form

$$((a, \text{refl}) =_F (a', p)) * ((a, \text{refl}) =_G (a'', q))$$

for $a, a', a'' : A$, $p : f(a) = f(a')$, and $q : g(a) = g(a'')$, where $F \equiv \text{fib}_f(f(a))$ and $G \equiv \text{fib}_g(g(a))$. Now apply Lemma 4.11 to get that the fibers are contractible. Then the little Blaker–Massey theorem gives that the square is cartesian. The final remark follows from the fact that $\Delta f \square_A \Delta g$ is a pullback of $\Delta f \square \Delta g$. \dashv

§5. Hypoabelian types and orthogonality. In the context of higher topos theory, Hoyois showed that the acyclic maps are part of an orthogonal factorization system [33]. While we leave a type-theoretic construction of this factorization system to future work (see Section 8), we consider what the corresponding right class should be, namely that of *hypoabelian* maps.

DEFINITION 5.1 (Hypoabelianness). A type X is *hypoabelian* if every perfect subgroup of $\pi_1(X, x)$ is trivial, for every $x : X$. A map $f : X \rightarrow Y$ is hypoabelian if all its fibers are.

We note that a type X is hypoabelian if and only if its 1-truncation is. An equivalent definition says that the perfect core (i.e., the largest perfect subgroup) of each $\pi_1(X, x)$ is trivial. We also remark that this definition, at least in the absence of propositional resizing [54, Section 3.5], should be understood as being relative to a type universe.

Recall that a map $f : A \rightarrow B$ is *left orthogonal* to a map $g : X \rightarrow Y$, denoted $f \perp g$, if we have a contractible type of lifts for all squares as below left, or equivalently, if the square below right is a pullback square.

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & \nearrow & \downarrow g \\ B & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} X^B & \xrightarrow{f^*} & X^A \\ g_* \downarrow & & \downarrow g_* \\ Y^B & \xrightarrow{f_*} & Y^A. \end{array}$$

In case of maps to the terminal type $\mathbf{1}$, we write $A \perp X$, and say that A is left orthogonal to X . This is equivalent to the constants map $\text{const} : X \rightarrow (A \rightarrow X)$ being an equivalence.

LEMMA 5.2 (PP). For all acyclic types A and hypoabelian types X , we have $A \perp X$.

PROOF. We need to show that the type of extensions of a map $g : A \rightarrow X$ along the terminal map $A \rightarrow \mathbf{1}$ is contractible. This is a proposition since A is acyclic, so it suffices to check that there exists an extension. Picking a base point of A , it suffices by Proposition 4.5 to check that $\pi_1(A) = \ker(\pi_1(g))$, or equivalently, that $\text{im}(\pi_1(g))$ is trivial. This follows since $\pi_1(A)$ is perfect, and the fact that the image of a perfect group is perfect. \dashv

COROLLARY 5.3 (PP). *For all acyclic maps $f : A \rightarrow B$ and hypoabelian maps $g : X \rightarrow Y$, we have $f \perp g$.*

PROOF. This is a general fact about two classes \mathcal{L} and \mathcal{R} of maps defined in terms of fibers, i.e., a map is in \mathcal{L}/\mathcal{R} if and only if all its fibers are. Suppose we have orthogonality of terminal maps in \mathcal{L} against terminal maps in \mathcal{R} . Then we get $f \perp g$ for all $f \in \mathcal{L}$ and $g \in \mathcal{R}$. Indeed, expressing a lifting problem in terms of a map $\varphi : B \rightarrow Y$ and a fiberwise map $\psi : \prod_{b:B} A(b) \rightarrow X(\varphi(b))$,

$$\begin{array}{ccc} \sum_{b:B} A(b) & \xrightarrow{\psi} & \sum_{y:Y} X(y) \\ \downarrow & \nearrow \text{---} & \downarrow \\ B & \xrightarrow{\varphi} & Y \end{array}$$

the type of lifts is

$$\begin{aligned} \sum_{h:\prod_{b:B} X(\varphi(b))} \prod_{b:B} \prod_{a:A(b)} \psi_b(a) = h(b) &\simeq \prod_{b:B} \sum_{x:X(\varphi(b))} \prod_{a:A(b)} \psi_b(a) = x \\ &\simeq \prod_{b:B} \text{fib}_{\text{const}}(\psi_b). \end{aligned}$$

So if all the constants maps, $X(\varphi(b)) \rightarrow (A(b) \rightarrow X(\varphi(b)))$, are equivalences, then all the fibers, $\text{fib}_{\text{const}}(\psi_b)$, are contractible, so this type of lifts is contractible too. \dashv

Nilpotent types [49] are a special case of hypoabelian types. We will show (Corollary 5.6) that every nilpotent type that is the limit of its Postnikov tower is right orthogonal to acyclic types *without* assuming the plus principle. (In the classical model, every type is the limit of its Postnikov tower.)

Recall that in homotopy type theory, the Postnikov tower of a type X is given by the truncation maps

$$X \rightarrow \dots \rightarrow \|X\|_n \rightarrow \dots \rightarrow \|X\|_1 \rightarrow \|X\|_0.$$

Following [37, Definition 7.2.2.20], we define a type Y to be an *EM n -gerbe* if it is $(n - 1)$ -connected and n -truncated. If $n \geq 2$, then this determines an abelian group $H := \pi_n(Y, y)$, which doesn't depend essentially on $y : Y$. If $n = 1$, we additionally require that $\pi_1(Y, y)$ is abelian for any/all $y : Y$. Now, any map $P : A \rightarrow K(H, n + 1)$ determines a family of n -gerbes over A via the equivalence $K(H, n + 1) \simeq \sum_{Z:\mathcal{U}} \|Z = K(H, n)\|_0$. We call such a P a *principal EM fibration*. (See also [16], the resulting gerbes are *banded* by H .)

LEMMA 5.4. *Given a pointed acyclic type A , and a pointed EM n -gerbe Y , we have that $(A \rightarrow_{\text{pt}} Y)$ is contractible.*

PROOF. Since Y is a pointed n -gerbe, we may assume $Y \simeq K(H, n) \simeq \Omega K(H, n + 1)$, where H is the associated abelian group. Now, using [54, Lemma 6.5.4], we have

$$\begin{aligned} (A \rightarrow_{\text{pt}} Y) &\simeq (A \rightarrow_{\text{pt}} \Omega K(H, n + 1)) \\ &\simeq (\Sigma A \rightarrow_{\text{pt}} K(H, n + 1)) \\ &\simeq (\mathbf{1} \rightarrow_{\text{pt}} K(H, n + 1)) \simeq \mathbf{1}. \end{aligned} \quad \dashv$$

PROPOSITION 5.5. *Given a pointed acyclic type A and a pointed nilpotent type X , we have that $(A \rightarrow_{\text{pt}} X)$ is contractible if in addition X is the limit of its Postnikov tower.*

To prove Proposition 5.5, we recall from [49, Theorem 2.58] that X is nilpotent if and only if each map $\|X\|_{n+1} \rightarrow \|X\|_n$ in the Postnikov tower of X factors as a finite composition of principal EM fibrations, i.e., maps classified by $K(A, n + 1)$ for abelian groups A .

PROOF. Since $X \simeq \varprojlim_n \|X\|_n$, we get an equivalence between $(A \rightarrow_{\text{pt}} X)$ and $\varprojlim_n (A \rightarrow_{\text{pt}} \|X\|_n)$, so it suffices to show that the type $(A \rightarrow_{\text{pt}} \|X\|_n)$ is contractible for all n , which we do by induction on n .

In the step case, since we’re proving a proposition, we may assume that the map $\|X\|_{n+1} \rightarrow \|X\|_n$ is factored as

$$\|X\|_{n+1} = Y_k \rightarrow Y_{k-1} \rightarrow \dots \rightarrow Y_0 = \|X\|_n,$$

with each map a principal EM fibration. The result now follows from Lemma 5.4. \dashv

COROLLARY 5.6. *For all acyclic types A and nilpotent types X that are limits of their Postnikov towers, we have $A \perp X$.*

PROOF. We’re proving a proposition, so fix a base point $\text{pt} : A$. The evaluation map at pt fits in the diagram below:

$$\begin{array}{ccc} X & \xrightarrow{\text{const}} & (A \rightarrow X) \\ & \searrow \text{id} & \downarrow \text{ev}_{\text{pt}} \\ & & X. \end{array}$$

By 3-for-2 for equivalences, it suffices to show that ev_{pt} is an equivalence. For each $x_0 : X$, the fiber is the type of pointed maps $(A \rightarrow_{\text{pt}} X)$, where X is pointed at x_0 . And this is contractible by the previous proposition. \dashv

§6. Acyclicity via (co)homology. Classically, the acyclic types are characterized as types A whose reduced integral homology vanishes, i.e., $\tilde{H}_i(A) = 0$ for all i . This is in fact the origin of the name *acyclic*, meaning that every cycle is a boundary, using the chain complex model of homology. For a definition of reduced homology in homotopy type theory, see [22, Definition 3.10]. Up to equivalence, $\tilde{H}_i(X)$ is defined for pointed types X as $\pi_i(\mathbb{H}\mathbb{Z} \wedge \Sigma^\infty X)$. For an unpointed type A , we define *unreduced homology* as $H_i(A) \equiv \tilde{H}_i(A_+)$, where A_+ is the free pointed type on A , viz., A with a disjoint base point.

DEFINITION 6.1. A type A is *homologically k -acyclic* if A is inhabited, and any one of the following equivalent conditions hold:

- (i) We have $\tilde{H}_i(A) = 0$ for $i \leq k$ and any choice of base point.
- (ii) The map $A \rightarrow \mathbf{1}$ induces isomorphisms $H_i(A) \rightarrow H_i(\mathbf{1})$ for $i \leq k$.

A type is *homologically acyclic* if it is homologically k -acyclic for all k .

Note that the augmentation map $H_0(A) \rightarrow \mathbb{Z}$ is an equivalence if and only if A is connected, and if A is pointed, then the inclusion $A \rightarrow A_+$ gives equivalences $\tilde{H}_i(A) \simeq H_i(A)$ for $i > 0$. This establishes the equivalence of the two conditions.

PROPOSITION 6.2. *A type is k -acyclic if and only if it is homologically k -acyclic.*

PROOF. Fix a type A . For $k = 0$ both conditions amount to A being connected. For $k > 0$ we get from the suspension property of homology that $\tilde{H}_i(\Sigma A) \simeq \tilde{H}_{i-1}(A)$ for all i . Since ΣA is simply connected, the truncated Whitehead’s theorem [54, Theorem 8.8.3] implies that $\|\Sigma A\|_{k+1}$ is contractible if and only if $\pi_i(\Sigma A)$ vanishes for $i \leq k + 1$. By Hurewicz’ theorem [22, Proposition 3.17], this happens if and only if $\tilde{H}_i(\Sigma A)$ vanishes in the same range. \dashv

COROLLARY 6.3. *Any acyclic type is homologically acyclic.*

The converse holds assuming Whitehead’s principle (WP).

COROLLARY 6.4 (WP). *Any homologically acyclic type is acyclic.*

We define the notion of being *cohomologically acyclic* in analogy with being homologically acyclic, just using integral cohomology [14, 19] instead of integral homology. Then we have the following.

LEMMA 6.5. *Any homologically acyclic type is cohomologically acyclic.*

PROOF. Fix a type A . By the suspension property of cohomology, we may assume that A is pointed and 1-connected. (Otherwise, consider the suspension ΣA .) Thus, by Propositions 3.10 and 6.2, A is in fact k -connected for any k , but then

$$\tilde{H}^i(A) \equiv \|A \rightarrow_{\text{pt}} \mathbf{K}(\mathbb{Z}, i)\|_0 \simeq \|\mathbf{1} \rightarrow_{\text{pt}} \mathbf{K}(\mathbb{Z}, i)\|_0 \simeq 0,$$

as desired. \dashv

Note that it is more subtle to characterize k -acyclicity cohomologically, since there are types A with $\tilde{H}^i(A) = 0$ for $i \leq k$ that are not k -acyclic. Consider for example $A = \mathbf{K}(\mathbb{Z}/2\mathbb{Z}, 2)$, which has

$$\tilde{H}^2(A) = \|A \rightarrow_{\text{pt}} \mathbf{K}(\mathbb{Z}, 2)\|_0 = \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0,$$

but $\tilde{H}_2(A) = \pi_2(A) = \mathbb{Z}/2\mathbb{Z}$. This can classically be fixed by requiring the induced map $\tilde{H}^{k+1}(A) \rightarrow \text{Hom}(\tilde{H}_{k+1}(A), \mathbb{Z})$ to be an isomorphism. However, this characterization relies on the Universal Coefficient Theorem which is not expected to hold in HoTT due to the presence of higher Ext groups [23]. Rather, we expect there is a Universal Coefficient Spectral Sequence as in Adams [1, (UCT2)]. Even assuming this, we would still need some argument to infer that cohomological acyclicity implies that the homology groups are finitely presented. However, the traditional proofs of this are very classical [29, Proposition 3F.12].

Let us now move on to the homological characterization of acyclicity of maps. Here it is not sufficient to just consider integer coefficients. But we can always move to a universal cover, by virtue of the following observation.

LEMMA 6.6. *A map $f : A \rightarrow B$ is acyclic if and only if, for all $b : B$, the pullback of f to the 1-connected cover of B at b , is acyclic,*

$$\begin{array}{ccc}
 \tilde{A} & \xrightarrow{f'} & \tilde{B}_b \\
 \downarrow & \lrcorner & \downarrow \\
 A & \xrightarrow{f} & B,
 \end{array} \tag{1}$$

where $\tilde{B}_b := \sum_{y:B} \|b = y\|_0$.

PROOF. The fibers of f and f' are identified. ⊖

We also need the following result, which would follow from a Universal Coefficient Spectral Sequence for homology [1, (UCT1)]. However, it also has a direct proof.

LEMMA 6.7. *If a type A is homologically k -acyclic, then it is so for any abelian coefficient group L : The map $A \rightarrow \mathbf{1}$ induces isomorphisms $H_i(A; L) \rightarrow H_i(\mathbf{1}; L)$ for $i \leq k$.*

PROOF. It suffices to consider the case where A is connected, and then we assume it is pointed and consider reduced homology. Again, by suspending and shifting if necessary, we may assume A is simply connected, so being (homologically) k -acyclic amounts to being k -connected. Now conclude by Christensen and Scoccola [22, Proposition 3.19]. ⊖

With these preliminaries, we are ready to present the following definition, which refines Hausmann and Husemöller [30, Definition 1.2] by considering k -acyclicity instead of acyclicity *simpliciter*

DEFINITION 6.8. A map $f : A \rightarrow B$ is *homologically k -acyclic* if any one of the following equivalent conditions hold:

- (i) All fibers of f are homologically k -acyclic.
- (ii) For any local coefficient system $L : B \rightarrow \mathbf{AbGroup}$, the induced maps

$$f_* : H_i(A; f^*L) \rightarrow H_i(B; L)$$

are isomorphisms for $i \leq k$ and surjective for $i = k + 1$.

- (iii) The induced maps

$$f_* : H_i(A; f^*\mathbb{Z}\pi_1 B) \rightarrow H_i(B; \mathbb{Z}\pi_1 B)$$

are isomorphisms for $i \leq k$ and surjective for $i = k + 1$.

- (iv) For each $b : B$, the map f' as in (1) induces isomorphisms $H_i(\tilde{A}) \rightarrow H_i(\tilde{B}_b)$ for $i \leq k$ and a surjection for $i = k + 1$.

PROOF OF THE EQUIVALENCE. For (i) implies (ii): We use the Serre Spectral Sequence for homology, as developed by van Doorn [24, Section 5.5]:

$$E_{p,q}^2 = H_p(B; \lambda b.H_q(F(b); L(b))) \Rightarrow H_{p+q}(A; f^*L),$$

where $F(b)$ is the fiber of f at b . By (i) and Lemma 6.7, we have $H_q(F(b); L(b)) = 0$ for $0 < q \leq k$, so the first possibly nontrivial differential (by total degree) is the transgression

$$d_{k+2,0}^{k+2} : H_{k+2}(B; L) \rightarrow H_0(B; \lambda b.H_{k+1}(F(b); L(b))).$$

Thus, convergence immediately gives isomorphisms $H_i(A; f^*L) \rightarrow H_i(B; L)$ for $i \leq k$ and a short exact sequence

$$0 \rightarrow E_{0,k+1}^\infty \rightarrow H_{k+1}(A; f^*L) \rightarrow H_{k+1}(B; L) \rightarrow 0,$$

yielding (ii).

It is clear that (ii) implies (iii).

For (iii) implies (iv), we look at the map of fibrations induced by a horizontal reading of (1), with fibers $\pi_1(B, b)$. The implications follow from naturality of the Serre Spectral Sequences and the Five Lemma.

For (iv) implies (i), we fix $b : B$ and use Lemma 6.6 to get a fiber sequence $F(b) \rightarrow \tilde{A} \rightarrow \tilde{B}_b$. For notational simplicity, we may as well assume that B is already simply connected, as well as pointed at $b : B$. Then we can use the Serre Spectral Sequence for homology with constant integral coefficients, in particular, we look at naturality with respect to the map of fiber sequences:

$$\begin{array}{ccccc} F(b) & \longrightarrow & A & \xrightarrow{f} & B \\ \downarrow & & f \downarrow & & \downarrow \text{id} \\ \mathbf{1} & \longrightarrow & B & \xrightarrow{\text{id}} & B. \end{array}$$

The comparison maps the left edge of the domain, $E_{0,i}^2 = H_0(B; H_i(F(b))) = H_i(F(b))$, to the left edge of the codomain, $E_{0,i}^2 = H_0(B; H_i(\mathbf{1})) = H_i(\mathbf{1})$, yielding (i), as desired. ⊖

COROLLARY 6.9. *A map is k -acyclic if and only if it is homologically k -acyclic. Any acyclic map is homologically acyclic, and the converse follows from (WP).*

We can also make the analogous definition for being a cohomologically acyclic map using the Serre Spectral Sequence for cohomology, which has even been formalized [24, Section 2.3]. But again we only know that homologically acyclic maps are cohomologically acyclic, as the converse would again require a Universal Coefficient Spectral Sequence and an argument to ensure finitely presented homology groups.

§7. Examples of acyclic types. We finally give some nontrivial examples of acyclic types.

7.1. A 2-dimensional acyclic type. Our first example is Hatcher’s 2-dimensional complex [29, Example 2.38]. We import this as the higher inductive type (HIT) X with constructors:

$$\text{pt} : X, \quad a, b : \Omega X, \quad r : a^5 = b^3, \quad s : b^3 = (ab)^2.$$

DEFINITION 7.1 (⚙ Hatcher structure and algebra). A *Hatcher structure* on a pointed type A is given by identifications

$$a, b : \Omega A, \quad r : a^5 = b^3, \quad s : b^3 = (ab)^2.$$

A *Hatcher algebra* is a pointed type equipped with Hatcher structure.

The HIT X is precisely the *initial* Hatcher algebra.

LEMMA 7.2 (♣). *Every loop space, pointed at refl, has a unique Hatcher structure.*

PROOF. The type of Hatcher structures on a loop space ΩA is

$$\sum_{a,b:\Omega^2 A} (a^5 = b^3) \times (b^3 = (ab)^2).$$

By Eckmann–Hilton [54, Theorem 2.1.6], we have $ab = ba$, so the last component is equivalent to $b = a^2$, and can be contracted away to obtain: $\sum_{a:\Omega^2 A} (a^5 = a^6)$. But, cancelling a^5 , this is equivalent to the contractible type $\sum_{a:\Omega^2 A} (a = \text{refl})$. \dashv

PROPOSITION 7.3 (♣). *The type X is acyclic.*

PROOF. For all pointed types Y , we have:

$$\begin{aligned} (\Sigma X \rightarrow_{\text{pt}} Y) &\simeq (X \rightarrow_{\text{pt}} \Omega Y) \\ &\simeq \text{Hatcher-structure}(\Omega Y) \\ &\simeq \mathbf{1}, \end{aligned}$$

where the first equivalence is [54, Lemma 6.5.4], the second is the universal property of X , and the third is Lemma 7.2.

Thus, ΣX has the universal property of the unit type and hence must be contractible. \dashv

In his lecture notes on higher topos theory, Charles Rezk asked [43, p. 11] whether it is possible to give a purely type-theoretic proof of the fact that $X \rightarrow \mathbf{1}$ is an epimorphism. Together with our characterization of the epimorphisms as the acyclic maps, Proposition 7.3 positively answers Rezk’s question.

The nontriviality of the type X follows from the following result since any 0-connected map $X \rightarrow \text{BA}_5$ gives a surjection $\pi_1(X) \rightarrow \text{A}_5$ by [54, Corollary 8.4.8].

PROPOSITION 7.4. *The type X has a 0-connected map to BA_5 , the classifying type of the alternating group A_5 .*

PROOF. Let $a = (1\ 2\ 3\ 4\ 5)$ and $b = (2\ 5\ 4)$ in A_5 . We have $ab = (1\ 2)(3\ 4)$, so these satisfy $a^5 = b^3$ and $b^3 = (ab)^2$, and thus induce a group homomorphism $\pi_1(X) \rightarrow \text{A}_5$, corresponding to a (pointed) map $X \rightarrow \|X\|_1 \rightarrow \text{BA}_5$. Since a and b generate A_5 , the group homomorphism is surjective, so the map $X \rightarrow \text{BA}_5$ is 0-connected by [54, Corollary 8.8.5]. \dashv

7.2. Higman’s type. Another interesting example of an acyclic type is the classifying type of Higman’s group H [31] which is given by the presentation

$$\text{H} = \langle a, b, c, d \mid a = [d, a], b = [a, b], c = [b, c], d = [c, d] \rangle.$$

(Here, $[x, y]$ denotes the commutator $[x, y] = xyx^{-1}y^{-1}$.) We will show that BH is acyclic, and moreover, that this presentation is *aspherical*, meaning that the presentation complex is already a 1-type, see also [26]. The presentation complex is easily imported into HoTT as the HIT BH with a point constructor $\text{pt} : \text{BH}$, four path constructors $a, b, c, d : \Omega \text{BH}$, and four 2-cell constructors corresponding to the relations.

PROPOSITION 7.5. *The type BH is acyclic.*

PROOF. By Eckmann–Hilton, similarly to Lemma 7.2 and Proposition 7.3. \dashv

To show that BH is not contractible, we make use of the following result due to David W arn [57, Lemma 8, Theorem 9].

THEOREM 7.6 (W arn). *Given a span $A \leftarrow R \rightarrow B$ of 0-truncated maps of 1-types, its pushout $A +_R B$ is a 1-type, the inclusion maps are 0-truncated, and the gap map is an embedding.*

THEOREM 7.7. *The type BH is a 1-type, and the generators a, b, c, d have infinite order.*

PROOF. Indeed, BH can be re-expressed as an iterated pushout as follows:

$$\begin{array}{ccc}
 \mathbf{B}\langle b \rangle & \longrightarrow & \mathbf{B}\langle b, c \rangle & & \mathbf{B}\langle a, c \rangle & \longrightarrow & \mathbf{B}\langle a, b, c \rangle \\
 \downarrow & & \lrcorner & \downarrow & \downarrow & & \lrcorner & \downarrow \\
 \mathbf{B}\langle a, b \rangle & \longrightarrow & \mathbf{B}\langle a, b, c \rangle & & \mathbf{B}\langle c, d, a \rangle & \longrightarrow & \mathbf{BH}.
 \end{array} \tag{3}$$

Here, each type is the HIT that uses only the constructors of BH that involve the mentioned generators. In particular, $\mathbf{B}\langle b \rangle$ is the circle (the classifying type of the free group on one generator, \mathbb{Z}) and $\mathbf{B}\langle a, c \rangle$ is the classifying type of the free group on two generators. We need to show that all maps in the span parts are 0-truncated maps of 1-types, because then the above theorem kicks in, showing in the end that BH is a 1-type, with all four elements a, b, c, d generating infinite cyclic subgroups of $H = \pi_1(\mathbf{BH})$. Indeed, if we can show that we have a span of 0-truncated maps between 1-types in the right square in (3), then Theorem 7.6 tells us that BH is a 1-type and that, e.g., the map $\mathbf{B}\langle a, b, c \rangle \rightarrow \mathbf{BH}$ is 0-truncated. Moreover, if we can then show that we have a span of 0-truncated maps between 1-types in the left square in (3), then Theorem 7.6 implies that the map $\mathbf{B}\langle b \rangle \rightarrow \mathbf{B}\langle b, c \rangle \rightarrow \mathbf{B}\langle a, b, c \rangle$ is also 0-truncated. Combining this with the above, we see that the composite map $\mathbf{B}\langle b \rangle \rightarrow \mathbf{BH}$ is 0-truncated, i.e., that the generator b has infinite order in H . One can similarly derive this for the other generators.

We start by looking at the types of the form $\mathbf{B}\langle a, b \rangle$. These are the classifying types of the Baumslag–Solitar [7] groups

$$\mathbf{BS}(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle,$$

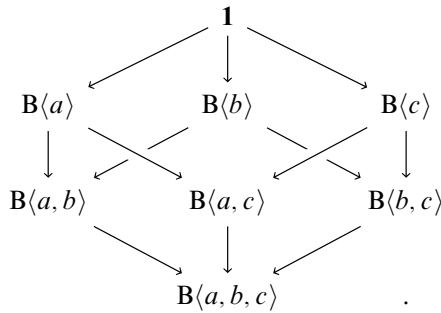
and are so-called HNN-extensions [32, 38], so we have coequalizer diagrams,² as below left, or equivalently pushouts, as below right:

$$\begin{array}{ccc}
 S^1 & \xrightarrow[b^2]{b} & \mathbf{B}\langle b \rangle & \longrightarrow & \mathbf{B}\langle a, b \rangle \\
 & & & & \downarrow [1,2] \\
 & & & & \mathbf{B}\langle b \rangle & \longrightarrow & \mathbf{B}\langle a, b \rangle.
 \end{array}$$

²In ordinary 1-category theory, every coequalizer is epic. The usual proof shows that every (homotopy) coequalizer is 0-epic. But coequalizers need not be 1-epic (even coequalizers of 1-types): the coequalizer $\mathbf{B}\langle b \rangle \rightarrow \mathbf{B}\langle a, b \rangle$ is 0-truncated but not 1-epic as this would make it 0-connected (by Theorem 3.17) and hence an equivalence, which it is not.

The maps in the span of the pushout square are 0-truncated maps of 1-types, so Theorem 7.6 applies. The (identical) inclusion maps can be identified with the map $B\langle b \rangle \rightarrow B\langle a, b \rangle$. The other inclusion, $B\langle a \rangle \rightarrow B\langle a, b \rangle$, is also 0-truncated, as it has a retraction, $B\langle a, b \rangle \rightarrow B\langle a \rangle$, defined by sending b to the neutral element; this is well defined, since the relation becomes $aa^{-1} = 1$.

This takes care of the input span to the left pushout square in (3). It remains to see that the maps of the form $B\langle a, c \rangle \rightarrow B\langle a, b, c \rangle$ are 0-truncated. The follows by descent [45, Section 25] from looking at the commutative cube:



The back faces are pullbacks: The subgroup $\langle b \rangle$ is normal in $\langle a, b \rangle$, and has trivial intersection with the subgroup $\langle a \rangle$, so the pullback of $B\langle a \rangle \rightarrow B\langle a, b \rangle \leftarrow B\langle b \rangle$ is the double coset $\langle a \rangle \backslash \langle a, b \rangle / \langle b \rangle$. Since the product of the two subgroups is the whole group, this is contractible, and similarly for the back right face. In addition, the top and bottom faces are pushouts, so the front faces are pullbacks as well by descent. Since the front bottom maps are (individually and jointly) surjective, and the maps on the sides are 0-truncated, the map in front is as well, as desired. \dashv

We conclude that BH is a nontrivial acyclic 1-type.

There are also analogs of the Higman group with any number n of generators, and the same argument shows that these classify infinite acyclic groups for $n \geq 4$. For $n < 4$, these groups are trivial, see e.g., [48] for the case $n = 3$.

The usual proofs that the Higman group is not the trivial group, e.g., [50, Proposition 6(b), Section 1.4], rely on nontrivial results from combinatorial group theory, specifically that we have embeddings into HNN extensions and amalgamations [38, Theorems 2.1(I) and 2.6, Chapter IV]. A noteworthy aspect of our proof is that it completely avoids combinatorial group theory and associated case distinctions, instead using tools from higher topos theory such as flattening (cf. Remark 3.19).

Finally, we note that nullification at any nontrivial acyclic type, such as BH or Hatcher’s example X of Section 7.1, provides a nontrivial modality for which all types are separated, as conjectured in [21, Example 6.6].

§8. Concluding remarks. In this paper we characterized the epimorphisms in univalent mathematics as the acyclic maps. The ensuing study of acyclic types, and the relativized versions to k -types, led to a further development of synthetic homotopy theory with applications in group theory.

There are numerous directions for future work. Our primary objective is to establish the acyclic maps as a modality [47], as Raptis and Hoyois did in the context of higher topos theory [33, 41]. Moreover, we would hope to show that this modality is accessible, perhaps under a mild extra assumption. In spaces, it can be explicitly described as nullification at a small collection of spaces [9]. An essential ingredient in constructing the modality is Quillen's plus-construction [58, Definition 1.4.1].

We left open the question of giving a cohomological characterization of k -acyclicity, as well as whether we can prove in HoTT that cohomologically acyclic types are homologically acyclic.

Another thread for future research is to construct acyclic types of a different nature than the examples presented in this paper, by considering automorphism groups [27], e.g., $\text{Aut}(\mathbb{N})$, or binate groups [8].

Additionally, we could work towards type-theoretic developments of the Barratt–Priddy(–Quillen) theorem [5] and the Kan–Thurston theorem [34]. The latter says that an ∞ -group can be presented by a pair (G, P) of a group G and perfect normal subgroup $P \triangleleft G$ [6].

Finally, we could try to generalize our results on epimorphisms to arbitrary wild categories with pullbacks and universal pushouts that satisfy descent. We might also have to impose the requirement that these categories are locally cartesian closed for the notion of a dependent epimorphism to make sense.

Acknowledgments. We are grateful to Mathieu Anel for Remark 4.3 and to Fredrik Nordvall Forsberg for detailed comments that helped to improve the paper. Further, we thank the anonymous reviewers for their thorough reports and insightful comments. In particular they drew our attention to an omission in the proof of Theorem 3.18 and suggested the current proof of Theorem 2.26 (our previous proof relied on the plus principle in one direction).

Funding. The second author was supported by The Royal Society (grant reference URF\R1\ 191055). The third author was supported by the TydiForm project and the MURI grant, US Air Force Office of Scientific Research, award numbers FA9550-21-1-0024 and FA9550-21-1-0009 respectively.

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