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Unemployment Risks and Optimal Retirement in an Incomplete Market

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We develop a new approach for solving the optimal retirement problem for an individual with an unhedgeable income risk. The income risk stems from a forced unemployment event, which occurs as an exponentially-distributed random shock. The optimal retirement problem is to determine an individual's optimal consumption and investment behaviors and optimal retirement time simultaneously. We introduce a new convex-duality approach for reformulating the original retirement problem and provide an iterative numerical method to solve it. Reasonably calibrated parameters say that our model can give an explanation for lower consumption and risky investment behaviors of individuals, and for relatively higher stock holdings of the poor. We also analyze the sensitivity of an individual's optimal behavior in changing her wealth level, investment opportunity, and the magnitude of preference for post-retirement leisure. Finally, we find that our model explains a counter-cyclical pattern of the number of unemployed job leavers.

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1. Introduction

Starting from pioneering works by Merton (1969, 1971), intertemporal models of optimal consumption and portfolio choice have evolved into some interesting generalizations. Among them, Bodie *et al.* (1992), Heaton and Lucas (1997), Koo (1998), Viceira (2001), Farhi and Panageas (2007) and others explored life-cycle models in which non-tradable labor income is incorporated into the problem of optimal consumption and portfolio choice.

Bodie *et al.* (1992) examine the impact of labor flexibility on an individual's optimal behavior including consumption and risky investment under the assumption that the labor supply is certain over the life cycle. Heaton and Lucas (1997) and Koo (1998) investigate how uncertain income stream can affect an individual's optimal consumption and investment behavior. Further, Viceira (2001) illustrates an individual's optimal behaviors when she faces an exogenous shock of retirement, assuming that exogenous retirement shocks and mortality risks arrive in a random way, but with constant probabilities. Farhi and Panageas (2007) investigate an individual's optimal consumption, investment and retirement behaviors simultaneously under the assumption that the individual's income rate is certain while working. All these papers permit only one of the following two assumptions, not both: risky labor income or endogenous (or voluntary) retirement opportunity.

Considering both income risks and endogenous retirement opportunity in the classical life-cycle models is a complicated job. Only a few researchers such as Liu and Neis (2003), Bodie *et al.* (2004), Dybvig and Liu (2010), and Jang *et al.* (2013) have successfully resolved the optimal voluntary retirement problems with income risks, but they consider a complete market in which income risks can be hedged away by purchasing and selling some financial instruments traded in that market. This paper deals with both income risks and endogenous retirement in an incomplete market. We assume individuals cannot eliminate income risks because they stem from exogenously forced unemployment events, and investigate the individual's optimal consumption, portfolio choice, and retirement behaviors in this serious situation.

Investigating the impact of income risks from unemployment events on an individual's optimal behaviors has been an important task for economists. Caroll (1992) shows that an unemployment event could have a major impact on an individual's current consumption and saving behaviors. Cocco *et al.* (2005) show that even a 0.05% probability of being unemployed in any given year has a large effect on an individual's portfolio choice, particularly early in life. Further, Wachter and Yogo (2010) stress the fact that unemployment risks could be significant for the poor in the sense that the optimal amount of risky investment increases at a sufficiently low wealth level. Also, Lynch and Tan (2011) show that the amount invested in the risky assets could be relatively lower in a persistent unemployment state and assert that the unemployment state pays only 10% of permanent labor income.[1] This paper might give a significant impetus to clarify the effect of unhedgeable unemployment risks on an individual's optimal behaviors in an incomplete market.

From a technical standpoint, solving financial problems constructed in an incomplete market, such as pricing derivatives and choosing an investor's optimal consumption and investment strategies, is extremely complex due to the non-uniqueness of the equivalent martingale measure, and obtaining a closed-form solution of the problems is hardly possible. Just a little literature, such as Kim and Omberg (1996), Chacko and Viceira (2005), Sangvinatsos and Wachter (2005), and Liu (2007), have provided closed-form solutions of financial problems defined in an incomplete market.

It is well-known that there are two approaches for solving financial problems derived from an incomplete market: the dynamic programming approach (DPA) and the martingale approach (MA). Koo (1998) and Henderson (2005) exploit the DPA to solve optimal consumption and investment problems incorporating labor income risks. They derive a non-linear partial differential equation and solve it. He and Pearson (1991), Svensson and Werner (1993), Teplá (2000), and Keppo *et al.* (2007) apply the MA to solve financial problems in the presence of income risks in an incomplete market.

Most existing literature concerning income risks, no matter they are hedgeable or unhedgeable, model the income stream as a standard geometric Brownian motion, which permits use of the MA. For instance, Karatzas *et al.* (1991) and He and Pearson (1991) suggest choosing the minimum local equivalent martingale measure with respect to unhedgable risks represented by a geometric Brownian motion (GBM). Further, Duffie *et al.* (1997) introduce a viscosity solution technique for solving portfolio choice problems in incomplete markets where a stochastic income evolved by a GBM cannot be hedged by utilizing a traded risky asset. However, we assume the income risks originally stem from forced unemployment events and such exogenous unemployment events occur following an exponential distribution with positive intensity. This kind of model is new in the optimal retirement literature, and a new approach is developed by using the DPA for solving the optimal retirement problem with the income risks. Specifically, we derive a differential equation and develop an iterative numerical method to solve it. As far as we know, this is the first paper to develop a method for solving the optimal retirement problem with a down-jump event of income which is modeled not using any Brownian motion.

With reasonably calibrated parameters we obtain some interesting features concerning an individual's optimal behaviors and income risks in the incomplete market. In our model, we find

- income risks stemming from forced unemployment events might significantly lower an individual's consumption, investment in a risky asset and voluntary retirement wealth level,
- income risks stemming from forced unemployment events might be an explanation for the findings of Cocco *et al.* (2005), Benzoni *et al.* (2007), and Lynch and Tan (2011), in that stock holdings in cash-on-hand can increase at a sufficiently low wealth level,
- certainty equivalent wealth gain, the maximum wealth that an individual is willing to give up in exchange for the market without unemployment risks, decreases as wealth and/or investment opportunity grow(s), and

- certainty equivalent wealth gain has a bigger value for an individual with a higher preference of leisure after retirement.

The first finding about voluntary retirement wealth level could be an explanation of a counter-cyclical pattern of the number of unemployed job leavers who have voluntarily left their current jobs; the proportion of job leavers increases during economic recessions and decreases during economic expansions.[2] Our result says that soaring income risks due to forced unemployment events during economic recessions induce myopic investors[3] who have slightly smaller wealth than the voluntary retirement wealth level planned during the past economic expansion to enter early retirement.

Our paper is organized as follows. In section 2 we establish a financial market with forced unemployment events and formulate our problem in the market. We also introduce a new dual approach and an iterative numerical method to solve the problem. In section 3 we display some analytical results including optimal consumption and investment strategies of an individual, and in section 4 we show numerical implications of our model in a normal market. In section 5 we analyze the relationship between the number of job leavers and business cycles and in Section 6 we conclude the paper.

2. The Model

2.1. The Financial Market

Following the conventional models, we assume an individual can trade two assets in the financial market: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price B_t evolves by the relationship

$$dB_t = rB_t dt,$$

where the positive constant value r is considered to be a risk-free interest rate. On the other hand, the stock price S_t follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ $(\mu > r)$ is the expected rate of the stock return, $\sigma > 0$ is the stock volatility, and W_t is a standard Brownian motion defined on a suitable probability space.

We assume that the individual is in the workforce at the beginning and wants to retire voluntarily someday in the future. The individual receives income at the rate of I_1 from labor. She is exposed to forced (or involuntary) unemployment risks such that she loses her job by compulsion whenever an exogenous unemployment shock arrives before the voluntary retirement date. Accordingly, she obtains income at the rate of I_2 $(I_2 < I_1)$ after the forced unemployment event. The forced unemployment event occurs following an exponential distribution with intensity δ, namely; for some time $t \geq 0$

$$\text{probability of } \{\tau_U \leq t\} = 1 - e^{-\delta t},$$

where τ_U is the *forced unemployment time*.[4]

We have two risk sources: the market risk (or the Brownian motion W_t) and the unemployment risk (or the Poisson arrival time τ_U). The market risk is hedgeable and can be partially diversified away by controlling dollar investment amount in the stock, but the unemployment risk cannot be hedgeable. Elmendorf and Kimball (2000) and Gormley *et al.* (2010) emphasize the important role of insurance against large and negative wealth shocks such as unemployment risks on an individual's optimal investment strategies. However, private insurance markets for hedging labor income risks are not sufficiently competitive compared to other insurance markets (Cocco *et al.* 2005). From the realistic point of view, we are assuming that there is no financial vehicle to eliminate or diminish the forced unemployment risks, thus, the financial market is considered to be *incomplete*. For the technical simplicity, we assume that the Brownian motion and the Poisson arrival event are independent.[5]

2.2. The Retirement Problem

The retirement problem explored in this paper can be thought of as a variation of the problem investigated by Farhi and Panageas (2007), but it allows an individual to be exposed to forced unemployment risks.

The individual has the following time-additive utility function of Cobb-Douglas type:

$$U(l(t), c(t)) \equiv \frac{1}{a}\ln(l(t)^{1-a}c(t)^a),$$

where $c(t)$ is per-period consumption and $l(t)$ is leisure at time t, and a is a weight for consumption satisfying $0 < a < 1$. We consider a binomial choice of leisure, in which the individual either works full time or she is retired permanently.[6] More specifically, the individual enjoys leisure $l(t) = l_1$ while she is working and $l(t) = l_2$ $(l_1 < l_2)$ when she retires. We assume that the wage rate w is constant and, then, the individual gets an income of $I_1 = w(l_2 - l_1) > 0$ per unit time during working status. We also assume she gets $I_2 > 0$ $(I_1 > I_2)$ per unit time after retirement.[7] The assumption of a positive income after retirement reflects the fact that most countries provide unemployment allowances and other public welfare services for retired people.

The wealth process $X(t)$ of the individual is given by

$$dX(t) = \begin{cases} \big(rX(t) - c(t) + I_1\big)dt + \pi(t)\sigma(dW(t) + \theta dt), & \text{for } \ 0 \le t < \tau \wedge \tau_U, \\ \big(rX(t) - c(t) + I_2\big)dt + \pi(t)\sigma(dW(t) + \theta dt), & \text{for } \ t \ge \tau \wedge \tau_U, \end{cases}$$

where π is the dollar amount invested in the stock and θ represents the Sharpe ratio $(\mu - r)/\sigma$. The individual accumulates wealth at the rates of $rX - c + I_1$ $(rX - c + I_2)$ before (after, respectively) voluntary or involuntary retirement. Note that the individual is exposed to forced unemployment risks and, thus, her labor income rate will decrease from I_1 to I_2 at the unemployment event. She is also exposed to the market risk stemming from stock investment and simultaneously compensated by the market premium, $\pi(\mu - r)$.

We assume that the individual can borrow money with her human capital. Following Friedman (1957) and Hall (1978) we define human capital h as the present value of future labor income discounted by the risk-free interest rate r:[8]

$$h = E\Big[\int_0^\infty e^{-rt} I_1 dt\Big] = \frac{I_1}{r}.$$

Then we impose a natural wealth constraint as the following:[9]

$$X(t) > -\frac{I_1}{r}, \quad \text{for} \quad t \geq 0. \tag{1}$$

This implies that the individual can consume and invest in the stock as long as her wealth level is above $-I_1/r$. If the wealth level approaches $-I_1/r$, she cannot consume and invest in the stock any more, i.e., consumption c and risky investment π should be zero. We exclude such trivial case and just consider the cases in the presence of the wealth constraint (1). We call the consumption and investment strategies satisfying the wealth constraint *admissible* strategies.

We normalize leisure prior to retirement as $l_1 = 1$, then the utility function during working should be

$$U_1(c) \equiv U(1, c) = \ln c.$$

We also let

$$K \equiv l_2^{\frac{1}{\alpha}-1} > 1,$$

which represents the preference for leisure after retirement. The retirement problem is to find the maximum of the individual's expected utility for consumption. The individual would like to maximize the utility by controlling her consumption c, risky portfolio π, and voluntary retirement time τ, i.e., the individual willingly obtains the following value function:

$$\Phi(x) \equiv \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t} U_1\big(c(t)\big) dt + e^{-\beta(\tau\wedge\tau_U)} \int_{\tau\wedge\tau_U}^\infty e^{-\beta(t-\tau\wedge\tau_U)} U_1\big(Kc(t)\big) dt\Big], \tag{2}$$

where $X(0) = x > -I_1/r$ is initial wealth of the individual and $\beta > 0$ is the individual's subjective discount rate. By utilizing the conditional expectation of τ_U, we can rewrite the individual's the value function $\Phi(x)$ given by (2) as the following:[10]

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^\tau e^{-(\beta+\delta)t} \Big\{U_1\big(c(t)\big) + \delta U_2\big(X(t)\big)\Big\} dt + e^{-(\beta+\delta)\tau} U_2\big(X(\tau)\big)\Big], \tag{3}$$

where

$$U_2(z) = \frac{1}{\beta}\Big[\ln\Big\{\beta\Big(z + \frac{I_2}{r}\Big)\Big\} + \frac{1}{\beta}\Big(r + \frac{\theta^2}{2} - \beta(1 - \ln K)\Big)\Big], \quad \text{for} \quad z > 0.$$

In fact, $U_2(z)$ is the value function of the classical Merton's problem with infinite investment horizon under the condition that the individual has a logarithmic utility and an income stream I_2 forever.

In our model, it is possible for an individual to involuntarily retire with negative wealth at the forced unemployment date. It makes our problem difficult to be well-defined because U_2 is not defined in the region of $(-I_1/r, -I_2/r]$. We extend our problem into the problem defined in the region of $(-I_1/r, \infty)$, which contains some negative values of wealth, by assuming that the post-retirement value function $U_2(z)$ is continuous at $z=0$ and the first derivative $U_2'(z)$ equals to zero for $z<0$.[11] Specifically, we assume that

$$U_2(z) = \begin{cases} \dfrac{1}{\beta}\Big[\ln\Big\{\beta\big(z+\dfrac{I_2}{r}\big)\Big\} + \dfrac{1}{\beta}\Big(r+\dfrac{\theta^2}{2}-\beta(1-\ln K)\Big)\Big], & \text{for } z>0, \\ \dfrac{1}{\beta}\Big[\ln\Big(\beta\dfrac{I_2}{r}\Big) + \dfrac{1}{\beta}\Big(r+\dfrac{\theta^2}{2}-\beta(1-\ln K)\Big)\Big], & \text{for } z\le 0. \end{cases}$$

2.3. Problem Reformulation

2.3.1. A new convex-duality approach We utilize the conventional dynamic programming approach to resolve our retirement problem in an incomplete financial market. For a fixed stopping time τ, we define

$$J_\tau(x) \equiv \max_{(c,\pi)} E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{U_1\big(c(t)\big)+\delta U_2\big(X(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big].$$

Then the value function $\Phi(x)$ given by (3) is rewritten as

$$\Phi(x) = \max_\tau J_\tau(x). \tag{4}$$

To solve the optimal stopping problem (4) we use the variational inequality approach. Variational Inequalities for the problem of this kind have been introduced by Bensoussan and Lions (1982) and Øksendal (2007). We will refer to Øksendal (2007) in the paper.[12] In fact, we can derive the following inequality:

$$\begin{aligned} (\beta+\delta)\phi(x)-(rx+I_1)\phi'(x)+\frac{\theta^2}{2}\frac{\phi'(x)^2}{\phi''(x)}+1+\ln\phi'(x) &\ge \delta U_2(x), \\ \phi(x) &\ge U_2(x), \\ \Big[(\beta+\delta)\phi(x)-(rx+I_1)\phi'(x)+\frac{\theta^2}{2}\frac{\phi'(x)^2}{\phi''(x)}+1+\ln\phi'(x)-\delta U_2(x)\Big]\Big(\phi(x)-U_2(x)\Big) &= 0, \end{aligned} \tag{5}$$

for any $x>-I_1/r$.

The retirement problem can be characterized by two regions: the continuation region where the individual works receiving labor income and the stopping region where she optimally enters voluntary retirement. The first inequality in (5) implies that the equality holds in the continuation region and the strict inequality holds in the stopping region. Moreover, if the strict inequality in the second inequality in (5) holds, i.e., if the value function prior to voluntary retirement is strictly larger than the value function after retirement, then the individual is in the continuation region and delay voluntary retirement. If the value function prior to retirement approaches the value function after retirement (i.e., the equality holds in the second inequality in (5)), then the individual is in the stopping region and, hence, optimally enters voluntary retirement. Note that for each value of $x > -I_1/r$ one of the first two equalities must hold, and, thus, we need the third equality in (5).

The continuation region and stopping region are determined by the so-called *critical wealth level*, over which it is optimal for an individual to enter voluntary retirement. Thus, we conjecture that the retirement problem can be solved by finding an optimal stopping boundary, or equivalently, a free boundary $\hat{x}$, which can be characterized by value matching and smooth pasting conditions. The problem can be formulated as follows:

$$\begin{cases} (\beta+\delta)\phi(x) - (rx+I_1)\phi'(x) + \dfrac{\theta^2}{2}\dfrac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) = \delta U_2(x), \quad -\dfrac{I_1}{r} < x < \hat{x}, \\ \phi(x) = U_2(x), \quad x \geq \hat{x}, \\ \phi(\hat{x}) = U_2(\hat{x}), \\ \phi'(\hat{x}) = \dfrac{1}{\beta}\dfrac{1}{\hat{x}+\frac{I_2}{r}}, \end{cases} \tag{6}$$

where $\hat{x}$ is the critical wealth level. If we find $\phi(x)$ satisfying C^1 and piecewise C^2, verifying the inequalities in the variational inequality (5) everywhere, then $\phi(x)$ in (6) is indeed a solution of the variational inequality (5).[13] Further, it is straightforward to verify that the solution $\phi(x)$ of (5) is equivalent to the solution $\Phi(x)$ of our optimal stopping problem (4) (see Theorem 10.4.1 in Øksendal, 2007).[14]

Now, we introduce a dual variable λ; we define the marginal value of the value function $\phi(x)$ as the variable λ. Then the critical wealth level $\hat{x}$ has an inverse relationship between the variable λ by the last equation in (6). Specifically,

$$\lambda(x) \equiv \phi'(x) \quad \text{and} \quad \hat{\lambda} \equiv \frac{1}{\beta}\frac{1}{\hat{x}+\frac{I_2}{r}}.$$

If we differentiate the first equation in (6) with respect to x and take the left derivative coefficient as the derivative coefficient of $U_2(x)$ at $x=0$, then we obtain

$$\lambda(x)(\theta^2+\beta+\delta-r) + \frac{\lambda'(x)}{\lambda(x)} - \lambda'(x)(rx+I_1) - \frac{1}{2}\theta^2\lambda(x)^2\frac{\lambda''(x)}{\lambda'(x)^2} = \begin{cases} \dfrac{\delta}{\beta}\dfrac{1}{x+\frac{I_2}{r}}, & \text{if } 0 < x \leq \hat{x}, \\ 0, & \text{if } -\dfrac{I_1}{r} < x \leq 0. \end{cases} \tag{7}$$

We introduce a modification of the conventional convex-duality approach by Karatzas and Shreve (1998) to solve our incomplete market problem. At first, we define a function $G(\cdot)$ which is the so-called convex-dual function[15] by

$$G\big(\lambda(x)\big) \equiv x + \frac{I_1}{r}. \tag{8}$$

Hence, we get the following relationships:

$$G'\big(\lambda(x)\big)\lambda'(x)=1 \quad\text{and}\quad G''\big(\lambda(x)\big)\lambda'(x)^2+G'\big(\lambda(x)\big)\lambda''(x)=0. \tag{9}$$

For the notational convenience, we let $G\big(\lambda(x)\big)=G$ and $\lambda(x)=\lambda$. Then by using the relationships given by (9), the equation (7) becomes

$$\lambda(\theta^2+\beta+\delta-r)+\frac{1}{\lambda}\frac{1}{G'}-\frac{rG}{G'}+\frac{1}{2}\theta^2\lambda^2\frac{G''}{G'}=\begin{cases}\dfrac{\delta}{\beta}\dfrac{1}{G-\frac{I_1}{r}+\frac{I_2}{r}}, & \text{if } G>\dfrac{I_1}{r}\\[2ex] 0, & \text{if } 0<G\le\dfrac{I_1}{r}.\end{cases} \tag{10}$$

If we rearrange the equation (10) and rewrite the last equation in (6) by using the definition (8) of the convex-dual function G, we obtain the following equations for $\lambda>\hat{\lambda}$:

$$\begin{aligned}&-\frac{1}{2}\theta^2\lambda^2G''(\lambda)-\lambda G'(\lambda)(\theta^2+\beta+\delta-r)+rG(\lambda)+\frac{\delta}{\beta}\frac{G'(\lambda)}{G(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\lambda)>\frac{I_1}{r}\}=\frac{1}{\lambda},\\ &G(\hat{\lambda})=\frac{1}{\beta\hat{\lambda}}+\frac{I_1-I_2}{r},\end{aligned} \tag{11}$$

where

$$\mathbf{1}\{G(\lambda)>\frac{I_1}{r}\}=\begin{cases}1, & \text{if } G(\lambda)>\dfrac{I_1}{r},\\[2ex] 0, & \text{if } 0<G(\lambda)\le\dfrac{I_1}{r}.\end{cases}$$

We add one more constraint,

$$G(\infty)=0, \tag{12}$$

which implies the individual's marginal utility λ goes to infinity as initial wealth x goes down to $-\frac{I_1}{r}$, which is the lower bound for initial wealth x.

Technically, for the case of $\delta=0$, without any forced unemployment risks, the problem formulated by equations in (11) and (12) has an analytic solution, whereas the problem for the case of $\delta>0$ is unlikely to have an explicit solution. To solve the problem for $\delta>0$, we first verify the existence of a solution satisfying equations in (11) and (12) in Appendix 7.1. For the next, we successfully obtain an analytic solution given by an implicit equation and develop an iterative numerical method to solve the implicit equation in the subsequent section.

2.3.2. The iterative method First, we define $\alpha_\delta>0$ and $\alpha_\delta^*<0$ as the two roots of

$$F(\alpha;\delta)\equiv-\frac{1}{2}\theta^2\alpha(\alpha-1)+\alpha(\beta+\delta-r)+r=0.$$

We conjecture the general solution of (11) as

$$G(\lambda)=\frac{1}{\lambda(\beta+\delta)}+A(\lambda)\lambda^{-\alpha_\delta}+A^*(\lambda)\lambda^{-\alpha_\delta^*} \tag{13}$$

subject to

$$A'(\lambda)\lambda^{-\alpha_\delta} + (A^*(\lambda))'\lambda^{-\alpha_\delta^*} = 0.$$

Putting the relationship in (13) into (11), we get

$$\begin{aligned} G(\lambda) = \frac{1}{\lambda(\beta+\delta)} + B(\hat{\lambda})\lambda^{-\alpha_\delta} + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda}\mu^{\alpha_\delta - 2}\ln\beta\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu \\ +(\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\infty}\mu^{\alpha_\delta^* - 2}\ln\beta\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big], \end{aligned} \tag{14}$$

for $\lambda > \hat{\lambda}$ and

$$\begin{aligned} &\frac{1}{\beta\hat{\lambda}} + \frac{I_1 - I_2}{r} \\ &= \frac{1}{\hat{\lambda}(\beta+\delta)} + B(\hat{\lambda})\hat{\lambda}^{-\alpha_\delta} + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[(\alpha_\delta^* - 1)\hat{\lambda}^{-\alpha_\delta^*}\int_{\hat{\lambda}}^{\infty}\mu^{\alpha_\delta^* - 2}\ln\beta\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}\Big]d\mu\Big], \end{aligned} \tag{15}$$

for

$$B(\hat{\lambda}) \equiv A(\hat{\lambda}) + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\hat{\lambda}^{\alpha_\delta - 1}\ln\frac{1}{\hat{\lambda}}.$$

Here,

$$\Big(G(\mu) - \frac{I_1}{r}\Big)^+ = \max\Big\{G(\mu) - \frac{I_1}{r}, 0\Big\}.$$

Moreover, the relationship of $\phi(\hat{x}) = U_2(\hat{x})$ in (6) implies

$$\begin{aligned} &\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K \\ &= \frac{\delta}{\beta}\ln\beta\hat{\lambda} + \hat{\lambda}(I_1 - I_2)\Big(1 + \frac{\alpha_\delta\theta^2}{2r}\Big) - \frac{\delta(\alpha_\delta^* - 1)}{\beta}\hat{\lambda}^{-\alpha_\delta^* + 1}\int_{\hat{\lambda}}^{\infty}\mu^{\alpha_\delta^* - 2}\ln\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu. \end{aligned} \tag{16}$$

The derivations of the relationships in (14) and (16) are in an online Appendix. Further, in Appendix 7.2 we prove the uniqueness of $G(\cdot)$ and the free boundary $\hat{\lambda}$, and the strictly decreasing property of $G(\cdot)$.

Now we state an iterative numerical method to find the solution $G(\cdot)$ of (11).[16]

The iterative procedure:

Step 0. Notice that, if $\delta = 0$, we can easily get $B(\hat{\lambda})$ from (15) and, thus, obtain $G(\lambda)$ from (14). Putting this $G(\lambda)$ into (16), we get $\hat{\lambda}$. Suppose $\delta \neq 0$, but has a sufficiently small value.[17] We exploit $G(\lambda)$ and $\hat{\lambda}$ for the case where $\delta = 0$ as the initial values of our iteration method.

Step 1. Since we have an initial $\hat{\lambda}$ and $G(\lambda)$, we can get $B(\hat{\lambda})$ from (15).

Step 2. Update $G(\lambda)$ by using equation (14).

Step 3. Putting the updated $G(\lambda)$ into (16), we can obtain a new $\hat{\lambda}$.

Step 4. Repeat steps 1, 2 and 3 until $\hat{\lambda}$ converges.

3. Analytical Results

3.1. Lower and Upper Bounds for Critical Wealth Level

Even though it is hard to have a closed form of the threshold level $\hat{x}$ in the optimal stopping problem (6), we can derive analytical lower and upper bounds.

To simplify notation, set L_δ to be the left hand side of (16) and $\psi_\delta(\hat{\lambda})$ to be its right hand side, that is, we let

$$
\begin{aligned}
L_\delta &\equiv \frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta)+\ln K,\\
\psi_\delta(\hat{\lambda}) &\equiv \frac{\delta}{\beta}\ln\beta\hat{\lambda}+\hat{\lambda}(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r})\\
&\quad -\frac{\delta(\alpha_\delta^*-1)}{\beta}\hat{\lambda}^{-\alpha_\delta^*+1}\int_{\hat{\lambda}}^{\infty}\mu^{\alpha_\delta^*-2}\ln\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\Big\}d\mu.
\end{aligned}
$$

Moreover, we define two functions

$$\underline{\phi}_\delta(\hat{\lambda}) \equiv \frac{\delta}{\beta}\ln\beta\hat{\lambda}+\hat{\lambda}(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r})$$

and

$$\overline{\phi}_\delta(\hat{\lambda}) \equiv \frac{\delta}{\beta}\ln\beta\hat{\lambda}+\hat{\lambda}(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r})+\frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{1}{\beta\hat{\lambda}},\frac{I_2}{r}\Big\}\Big),$$

which can be lower and upper bounds of $\psi_\delta(\hat{\lambda})$ respectively.

Proposition 3.1. If $I_2\ge r$,

$$\lambda_\delta^0\le\hat{\lambda}\le\lambda_\delta^1, \tag{17}$$

where λ_δ^0 and λ_δ^1 are found by

$$\overline{\phi}_\delta(\lambda_\delta^0)=L_\delta \quad\text{and}\quad \underline{\phi}_\delta(\lambda_\delta^1)=L_\delta.$$

Proof. It is obvious that

$$\psi_\delta(\hat{\lambda})\ge\underline{\phi}_\delta(\hat{\lambda}). \tag{18}$$

Since $\Big(G(\mu)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\le\max\Big\{\frac{1}{\beta\hat{\lambda}},\frac{I_2}{r}\Big\}$, under the assumption of $I_2\ge r$,

$$\psi_\delta(\hat{\lambda})\le\frac{\delta}{\beta}\ln\beta\hat{\lambda}+\hat{\lambda}(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r})+\frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{1}{\beta\hat{\lambda}},\frac{I_2}{r}\Big\}\Big)=\overline{\phi}_\delta(\hat{\lambda}). \tag{19}$$

Since both $\overline{\phi}_\delta(\hat{\lambda})$ and $\underline{\phi}_\delta(\hat{\lambda})$ are monotonically-increasing and continuous functions with

$$
\begin{aligned}
&\overline{\phi}_\delta(0)=\underline{\phi}_\delta(0)=-\infty,\ \text{and}\\
&\overline{\phi}_\delta(+\infty)=\underline{\phi}_\delta(+\infty)=+\infty,
\end{aligned} \tag{20}
$$

λ_δ^0 and λ_δ^1 are lower and upper bounds for $\hat{\lambda}$. **Q.E.D.**

The upper and lower bounds for the *critical wealth level* $\hat{x}$, at which the individual enters voluntary retirement, also can be written as

$$G(\lambda_\delta^1) - \frac{I_1}{r} \le \hat{x} \le G(\lambda_\delta^0) - \frac{I_1}{r},$$

if we use the result in the proposition and the definition of function $G(\cdot)$ in (8). Notice that $\underline{\phi}_\delta(\lambda_\delta^1)$ becomes $\overline{\phi}_\delta(\lambda_\delta^0)$, or equivalently, the upper and lower boundaries in the proposition are identical, where intensity δ is zero. For this case, the boundaries become the corresponding critical wealth level described in Farhi and Panageas (2007), in which individuals are not exposed to any forced unemployment risk.

3.2. Optimal Consumption and Investment Strategies

We find the optimal consumption and investment strategies by exploiting the terms of $\hat{\lambda}$, $B(\cdot)$ and $G(\cdot)$, which can be obtained by the iterative numerical method in the previous section.

THEOREM 1. *The optimal consumption c and risky portfolio π are given as*

$$\begin{aligned} c(t) &= (\beta+\delta)\Big(x+\frac{I_1}{r}\Big) - (\beta+\delta)B(\hat{\lambda})\lambda^*(x)^{-\alpha_\delta} \\ &\quad - \frac{2\delta(\beta+\delta)}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda^*(x)} \mu^{\alpha_\delta-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu \\ &\quad + (\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\infty} \mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big], \end{aligned} \tag{21}$$

$$\begin{aligned} \pi(t) &= \frac{\theta}{\sigma}\frac{1}{\lambda^*(x)(\beta+\delta)} + \frac{\theta}{\sigma}\alpha_\delta B(\hat{\lambda})\lambda^*(x)^{-\alpha_\delta} - \frac{2\delta}{\sigma\theta\beta\lambda^*(x)}\ln\beta(x+\frac{I_2}{r}) \\ &\quad + \frac{2\delta}{\sigma\theta(\alpha_\delta-\alpha_\delta^*)\beta}\Big[\alpha_\delta(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda^*(x)} \mu^{\alpha_\delta-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu \\ &\quad + \alpha_\delta^*(\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\infty} \mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big], \end{aligned} \tag{22}$$

where $\lambda^*(x)$ is a decreasing function with respect to wealth x and the solution of

$$\begin{aligned} x+\frac{I_1}{r} &= \frac{1}{\lambda^*(x)(\beta+\delta)} + B(\hat{\lambda})\lambda^*(x)^{-\alpha_\delta} \\ &\quad + \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda^*(x)} \mu^{\alpha_\delta-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu \\ &\quad + (\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\infty} \mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big] \end{aligned}$$

and

$$\begin{aligned} B(\hat{\lambda}) &= \hat{\lambda}^{\alpha_\delta}\Big[\frac{\delta}{\beta(\beta+\delta)}\frac{1}{\hat{\lambda}} + \frac{I_1-I_2}{r} \\ &\quad - \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big\{(\alpha_\delta^*-1)\hat{\lambda}^{-\alpha_\delta^*}\int_{\hat{\lambda}}^{\infty} \mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big\}\Big]. \end{aligned}$$

Proof. The relationship (8) and the first-order conditions with respect to $c(t)$ and $\pi(t)$, which were used in deriving the variational inequality (5), yield the optimal consumption $c(t)$ and risky portfolio proportion $\pi(t)$. **Q.E.D.**

The optimal consumption, c^M, of the classical Merton's (1969, 1971) problem in the presence of the income I_1 is represented as

$$c^M(t) = \beta\Big(x + \frac{I_1}{r}\Big),$$

so the marginal propensity to consume (MPC) out of wealth, $\dfrac{\partial c_t}{\partial x}$, is constant. However, in our model without any forced unemployment risk, namely $\delta = 0$, the first and second terms of the right hand side of (21), which spring up due to the voluntary retirement event, yield the optimal consumption

$$c^{VR}(t) = \beta\Big(x + \frac{I_1}{r}\Big) - \beta B(\hat{\lambda})\lambda^*(x)^{-\alpha_0}.$$

This implies the MPC out of wealth should be positive and the optimal consumption is a concave function with respect to wealth. Since $B(\hat{\lambda}) > 0$ for $\delta = 0$, the individual with voluntary retirement option consumes relatively less than the one in the classical Merton problem because she is likely to accumulate wealth by cutting down her consumption to enter retirement early. Moreover, since λ^* is a decreasing function with respect to wealth level the effect of the voluntary retirement option on her consumption behavior becomes significant as her wealth increases. This might be an explanation of the finding that individuals could dramatically decrease their consumption near the critical wealth level (Farhi and Panageas 2007, Dybvig and Liu 2010).

The third term of the right hand side of (21) consists of two parts which are closely associated with the impact of income risks stemming from forced unemployment events. Notice that the first (integral) part gives a negative effect to the individual's consumption and the second (integral) part affects it conversely. As an individual's wealth approaches the critical wealth level $\hat{x}$, $\lambda^*(x)$ gets closer to $\hat{\lambda}$, and subsequently, in the limit case the first part disappears and the second part has a fixed value. Hence, if an individual's wealth reaches near the critical wealth level, the third term of the right hand side in (21) could make a positive impact on the individual's consumption and offset the negative impact of the second term of the right hand side in (21), which stands for the voluntary retirement option value. So it might be possible for people facing forced unemployment risks to consume more than individuals not exposed to those, near retirement time.

When it comes to the optimal risky investment, the classical Merton's (1969, 1971) strategy, π^M, must be

$$\pi^M(t) = \frac{\theta}{\sigma}\Big(x + \frac{I_1}{r}\Big).$$

However, using the first and second terms of the right hand side of (22) we can get the optimal risky portfolio, π^{VR}, for the case where there exists no forced unemployment risk, i.e., $\delta=0$:

$$\pi^{VR}(t)=\frac{\theta}{\sigma}\left(x+\frac{I_1}{r}\right)+\frac{\theta}{\sigma}\alpha_0 B(\hat{\lambda})\lambda^*(x)^{-\alpha_0}. \tag{23}$$

Notice that the second term of (23) is associated with voluntary retirement and is a positive and increasing function with respect to initial wealth x. An individual permitted voluntary retirement is willing to take more risk than one considered in the classical Merton's set-up. Also it seems that the risky investment, π^{VR}, increases according to the growth of an individual's wealth level. Intuitively, people who have slightly smaller wealth than the critical wealth level give their attention to retire voluntarily as soon as possible, so they tend to invest more in risky assets even though they may end up losing relatively large amounts of money. Similar observations were reported by Farhi and Panageas (2007) and Dybvig and Liu (2010).

The third and fourth terms of the right hand side of (22) are closely associated with involuntary unemployment. Notice that the third term is negative, so it decreases an individual's stockholding, while the forth term consisting of two integral parts urges stock investment.

We can get an upper bound of π where individual's wealth goes up to a sufficiently close level to the critical wealth level and a lower bound where individual's wealth goes down to zero:

Corollary 3.1. For $I_2\le\dfrac{r}{\beta\hat{\lambda}}$

$$\lim_{x\uparrow\hat{x}}\pi(t)\le\frac{\theta}{\sigma}\frac{1}{\hat{\lambda}(\beta+\delta)}+\frac{\theta}{\sigma}\alpha_\delta B(\hat{\lambda})\hat{\lambda}^{-\alpha_\delta}-\frac{2\delta\alpha_\delta}{\sigma\theta(\alpha_\delta-\alpha_\delta^*)\beta\hat{\lambda}}\ln\frac{1}{\hat{\lambda}}.$$

Moreover,

$$\lim_{x\downarrow 0}\pi(t)\ge\frac{\theta}{\sigma}\frac{1}{\lambda^*(x)(\beta+\delta)}+\frac{\theta}{\sigma}\alpha_\delta B(\hat{\lambda})\lambda^*(x)^{-\alpha_\delta}-\frac{2\delta\alpha_\delta}{\sigma\theta(\alpha_\delta-\alpha_\delta^*)\beta}\ln\left(\frac{\beta I_2}{r}\right)\lambda^*(x)^{-\alpha_\delta}\hat{\lambda}^{\alpha_\delta-1}.$$

Proof. We first utilize inequality $\left(G(\mu)-\frac{I_1}{r}\right)^++\frac{I_2}{r}\le\max\left\{\frac{1}{\beta\hat{\lambda}},\frac{I_2}{r}\right\}$, and take the limit of $\lambda^*\downarrow\hat{\lambda}$ to derive the last term of the right hand side of the first inequality. On the other hand, the last term of the right hand side of the second inequality is derived if we use $G(\mu)\ge 0$ for all μ and take the limit of $x\downarrow 0$. **Q.E.D.**

4. Numerical Implications

In this section, we investigate the behaviors of critical wealth level, optimal consumption, and optimal risky investment in economically reasonable circumstances. We exploit the iterative procedure in Section 2.3.2.

	μ			σ			K		
δ	0.1023	0.1123	0.1223	0.1854	0.1954	0.2054	2	3	4
0	67.6714	79.6203	92.9475	84.8576	79.6203	75.0878	127.7719	79.6203	62.5382
0.01	63.5879	75.2544	88.3594	80.5051	75.2544	70.4416	118.6729	75.2544	59.6859
0.02	60.8109	71.9124	85.0592	77.3993	71.9124	66.9878	110.9534	71.9124	57.6477
0.03	58.8915	68.8722	82.6363	75.0959	68.8722	64.5996	105.1919	68.8722	56.1854

Table 1 Critical wealth levels $\hat{x}$ for various parameter values of δ, μ, σ, and K. Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$.

4.1. Baseline Parameters

The baseline parameters are fixed to $r = 3.71\%$, which is the annual rate of return from rolling over 1-month T-bills during the time period of 1926-2009[18] and we assume β has the same value as r. We utilize $\mu = 11.23\%$ and $\sigma = 19.54\%$, which are the return and standard deviation of the world's large stocks during the time period of 1926-2009.[19] We set $K = 3$ following the assumption used by Dybvig and Liu (2010), $I_1 = 1$, and $I_2 = 0.10$ following the results of Lynch and Tan (2011), who conclude the unemployment state pays only 10% of permanent labor income.

4.2. Critical Wealth Level

Table 1 shows critical wealth level $\hat{x}$'s for various parameter values, such as forced unemployment intensity δ, expected rate μ of stock return, stock volatility σ, and leisure K. It is obvious that an individual would be better off entering voluntary retirement wherever her wealth level is not less than $\hat{x}$, so that she can enjoy more leisure after retirement. Table 1 shows that such critical wealth level $\hat{x}$ decreases as forced unemployment intensity δ increases. Intuitively, individuals with a higher δ enter the voluntary retirement stage earlier even though the wealth level at retirement is not relatively high, because they willingly submit to such utility losses in exchange for avoiding utility losses stemming from forced unemployment risks.

On the other hand, the critical wealth level increases as μ increases or σ or K decreases, *ceteris paribus*. After retirement, an individual in our model faces a tradeoff between utility gains owing to the increase of leisure and utility losses from the significant reduction of income. In a financial market with a higher expected rate of stock return, an individual is willing to postpone her voluntary retirement because a better investment opportunity makes her worry less about forced unemployment risks. Similarly, a lower stock volatility also provides her with a better investment opportunity, so she worries less about forced unemployment risks. Evidently, a lower quantity of leisure after retirement hinders an individual from entering voluntary early retirement.

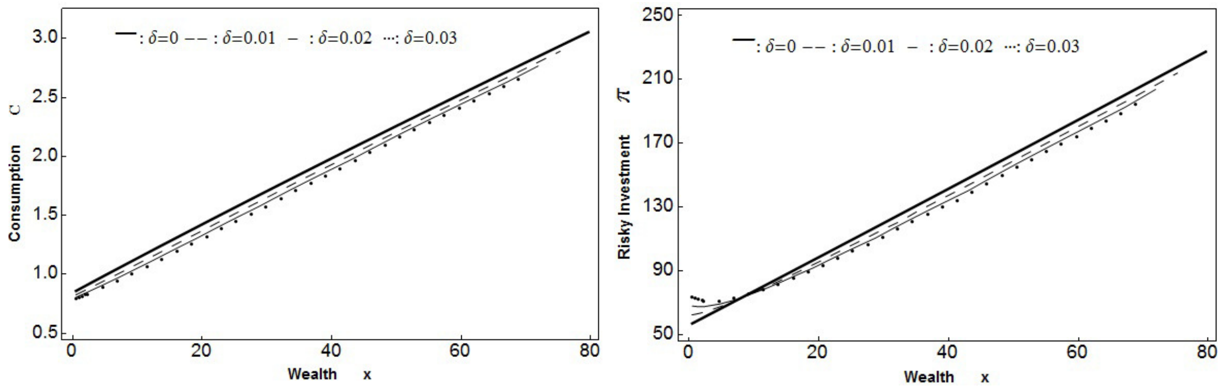

Figure 1 Optimal consumption and risky investment as a function of initial wealth. Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$.

4.3. Optimal Consumption and Risky Investment

Figure 1 shows the amounts of optimal consumption and investment in the risky stock as a function of initial wealth for several values of forced unemployment intensity δ. An individual's consumption c grows as initial wealth level x becomes higher, but it falls as her forced unemployment possibility δ grows. A higher demand on precautionary savings against a higher forced unemployment risk could be an explanation of the latter observation. (See Carroll, 1992; Malley and Moutos, 1996; Gruber, 1997).[20]

In terms of optimal risky investment, Figure 1 shows an interesting feature. Admittedly, if wealth is not small enough, the optimal risky investment increases as initial wealth increases. However, for poor people this might not be true; up to some wealth level, some of them with a higher forced unemployment possibility lessen their investment in the risky stock even though they have more wealth. This observation implies that forced unemployment risks could be an important explanation for the findings of Benzoni *et al.* (2007) and Lynch and Tan (2011), in that they find the stock holdings can increase at a sufficiently low wealth level. Moreover, the result seems to be consistent with that of Cocco *et al.* (2005), who address whether the amount of optimal risky investment can rise due to unemployment risks when wealth level is low enough.

Figure 2 says that optimal consumption to wealth ratio and optimal risky investment to wealth ratio decrease at decreasing rates as retirement time approaches, or equivalently, as x goes to $\hat{x}$.[21] The decreasing properties of optimal consumption and risky investment with respect to wealth ratio seem to result from the intense aspirations toward voluntary early retirement under our set-up. Notice also that the decreasing rate becomes bigger as forced unemployment possibility δ increases. This fact indicates that our model can reflect the higher intense aspirations toward voluntary early retirement of an individual facing a higher forced unemployment risk.

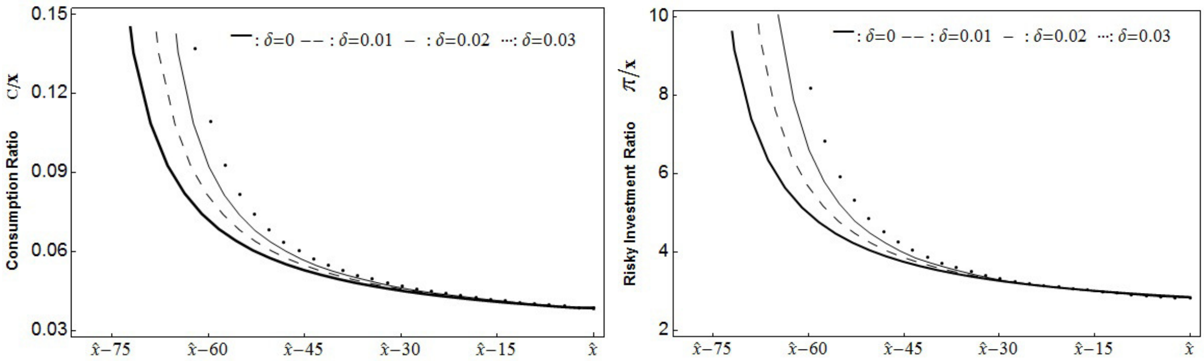

Figure 2 Optimal consumption to wealth ratio and optimal risky investment to wealth ratio as a function of initial wealth originated by the critical wealth level $\hat{x}$. Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$.

The second result in Figure 2 is compatible with the observation of Polkovnichenko (2007): an individual with a higher unemployment risks has to invest more in stock by using savings to finance consumption, because she cannot expand income much by choosing labor supply in the unemployment state. Economists have a consensus that the flexibility in labor supply causes higher investment in stock (e.g., Bodie *et al.*, 1992), and our model says that forced unemployment risks might lead to a much higher allocation to stock when the wealth of an individual is close to the critical wealth level.

4.4. Certainty Equivalent Wealth Gain

We define certainty equivalent wealth gain (henceforth, CEWG) to be the maximum wealth level that an individual in our model is willing to give up in exchange for the market without forced unemployment risks. We now investigate how much CEWG is in a normal economic situation and how sensitive CEWG is when μ, σ, and K change.

Definition 1. $\Delta(x)$ is the certainty equivalent wealth gain at initial wealth level x if it satisfies

$$\bar{\phi}(x - \Delta(x)) = \phi(x),$$

where $\bar{\phi}(x)$ is the value function $\phi(x)$ with $\delta = 0$.

Figure 3 represents CEWG as a function of initial wealth level. It decreases as wealth increases, implying that individuals become less sensitive to forced unemployment risks as they accumulate wealth. Intuitively, an individual confronting forced unemployment risks would be disentangled from those near the critical wealth level, because the possibility of unemployment for the small period is very low and she can retire soon. Thus, CEWG becomes smallest when wealth approaches the critical wealth level. Moreover, the value of CEWG becomes bigger as δ increases. A smaller

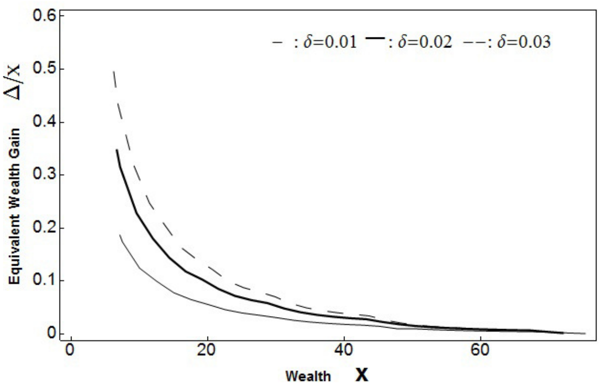

Figure 3 Certainty equivalent wealth gain to wealth ratio $\Delta(x)/x$ as a function of initial wealth x. Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$.

	$\delta = 0.01$		$\delta = 0.02$		$\delta = 0.03$	
$x \setminus \mu$	0.1023	0.1223	0.1023	0.1223	0.1023	0.1223
$\hat{x} - 50$	0.0889	0.0218	0.1986	0.0435	0.3328	0.0658
$\hat{x} - 40$	0.0398	0.0135	0.0885	0.0287	0.1321	0.0450
$\hat{x} - 30$	0.0208	0.0090	0.0460	0.0174	0.0699	0.0255
$\hat{x} - 20$	0.0121	0.0063	0.0231	0.0135	0.0340	0.0194
$\hat{x} - 15$	0.0076	0.0038	0.0178	0.0085	0.0282	0.0142
$\hat{x} - 10$	0.0040	0.0022	0.0087	0.0049	0.0148	0.0077
$\hat{x} - 5$	0.0019	0.0011	0.0043	0.0026	0.0072	0.0042
$\hat{x}$	0.0006	0.0003	0.002	0.001	0.0038	0.0018

Table 2 Certainty equivalent wealth gain to wealth ratios $\Delta(x)/x$ for various μ and δ. Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$. $\hat{x}$ is the critical wealth level which is different for each δ.

δ means a higher critical wealth level, so the individual has more time to prepare for forced unemployment risks; thus CEWG should be smaller.

Table 2 and 3 display the sensitivity of CEWG to the changes of expected rate μ of stock return and stock volatility σ, respectively. We find the values of CEWG for the extensive range of wealth, $[\hat{x} - 50,\ \hat{x}]$. It seems that an individual participating in a financial market with a higher expected stock return and/or a lower stock volatility has a lower CEWG. Intuitively, as investment opportunity in the financial market grows, individuals facing forced unemployment risks become less stressful because they are more likely to enter voluntary retirement.

The sensitivity of CEWG for various K, preference for leisure, is illustrated in Figure 4. It seems that CEWG is much bigger for a bigger preference for leisure if an individual's wealth is far less than her critical wealth level. This implies individuals with a higher preference of leisure have more stress if their wealth level is relatively low compared with their critical wealth level. This tendency is mitigated as their wealth gets closer to the critical wealth level.

	$\delta = 0.01$		$\delta = 0.02$		$\delta = 0.03$	
$x \setminus \sigma$	0.1854	0.2054	0.1854	0.2054	0.1854	0.2054
$\hat{x} - 50$	0.0310	0.0511	0.0649	0.1202	0.0953	0.1846
$\hat{x} - 40$	0.0192	0.0275	0.0373	0.0680	0.0548	0.0866
$\hat{x} - 30$	0.0109	0.0151	0.0224	0.0404	0.0350	0.0452
$\hat{x} - 20$	0.0069	0.0069	0.0135	0.0252	0.0195	0.0275
$\hat{x} - 15$	0.0060	0.0043	0.0115	0.0196	0.0164	0.0146
$\hat{x} - 10$	0.0040	0.0027	0.0097	0.0164	0.0146	0.0088
$\hat{x} - 5$	0.0018	0.0017	0.0043	0.0143	0.0074	0.0056
$\hat{x}$	0.0004	0.0006	0.0012	0.002	0.0021	0.0037

Table 3 Certainty equivalent wealth gain to wealth ratios $\Delta(x)/x$ for various σ and δ. Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$. $\hat{x}$ is the critical wealth level which is different for each δ.

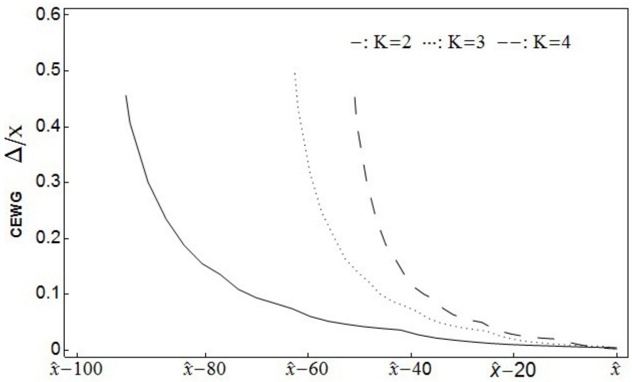

Figure 4 Certainty equivalent wealth gains to wealth ratios $\Delta(x)/x$ as a function of initial wealth level.Parameter values are set as follows: $\delta = 0.03$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $I_1 = 1$, and $I_2 = 0.10$.

5. Economic Downturns and Unemployment Risks

In 2000, "Issues in Labor Statistics" published by the US Bureau of Labor Statistics questions whether or not unemployed job leavers who have voluntarily quit their jobs is a meaningful gauge of confidence in the job market. Some analysts regard the increased number of job leavers as a meaningful gauge of rising confidence in the job market, and one possible explanation of their insistence is that it is good for workers to quit their job voluntarily if they have good prospects for a successful job search.

However, "Issues in Labor Statistics" asserts that the number of unemployed job leavers shows a counter-cyclical pattern; the proportion of job leavers increases during economic recessions and decreases during economic expansions and hence the use of unemployed job leavers as a gauge of confidence in the job market could be problematic. Figure 5 shows this phenomenon intimately, and our model might give a solid explanation for it. Notice that critical wealth level of an individual in

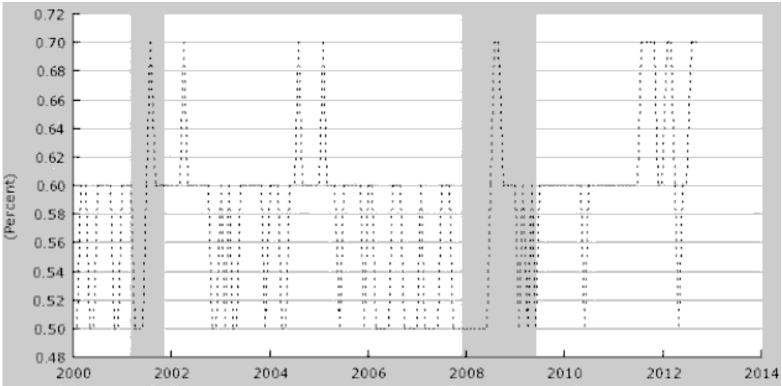

Figure 5 Unemployed job leavers as a percent of the labor force (Source: Bureau of Labor Statistics). The two shaded regions stands for US economic recessions quoted by NBER.

our model decreases as forced unemployment risk increases, implying a higher forced unemployment risk could compel individuals to retire at a lower wealth level. Generally, forced unemployment risks are relatively high during economic recessions and relatively low during economic expansions, so we might conclude that soaring forced unemployment risks during economic recessions induce myopic individuals who have slightly smaller wealth than their critical wealth level planned during the past economic expansion to enter early retirement.

More specifically, our model can show the behaviors of myopic individuals under different market conditions. Table 4 shows critical wealth levels for various δ and I_1 under two different market conditions, say, 'economic expansions' and 'economic recessions'. Ang and Bekaert (2002) show that the expected stock return and stock volatility in the US market was $\mu = 0.1394$ and $\sigma = 0.1313$ during the economic expansions and $\mu = 0.1394$ and $\sigma = 0.2600$ during the economic recessions, and we utilize the same parameters for Table 4.[22]

In the table, myopic individuals who draw up their retirement plan solely reflecting the current financial market conditions seem to retire at a much lower wealth level during the economic recessions than during the economic expansions, even though the force unemployment possibility and their income rate do not change. For instance, an individual with $I_1 = 1$ and $\delta = 0.02$ makes a plan to retire at the wealth level of 206.3668 during the economic expansions, however, the critical wealth level gets much smaller to 74.2596 during the economic recessions.

Admittedly, a smaller income rate I_1 or a bigger forced unemployment possibility δ during economic recession periods is most likely to consolidate the reduction of critical wealth level.[23] Since an individual with an asset portfolio and labor income is exposed to a bigger income risk stemming from forced unemployment events and worse financial market conditions during economic recessions, she inevitably lowers her voluntary retirement wealth level in order to get a relatively higher utility gain (mostly obtained from more leisure time) after retirement. For example, according to

	Economic Recessions				Economic Expansions			
$I_1 \setminus \delta$	0	0.01	0.02	0.03	0	0.01	0.02	0.03
1.6	130.8238	123.6475	118.3066	113.8545	351.8848	338.6856	327.7048	318.4957
1.4	114.4709	108.2522	103.6448	99.8449	307.9003	296.6821	287.3554	279.5401
1.2	98.1180	92.8475	88.9635	85.8013	263.9136	254.6262	246.9156	240.4587
1	81.7650	77.4323	74.2596	71.7168	219.9284	212.5128	206.3668	201.2310
0.8	65.4119	62.0041	59.5288	57.5844	175.9424	170.3294	165.6894	161.8225
0.6	49.0590	46.0203	44.7641	43.3963	131.9575	128.0568	124.8480	122.1871
0.4	32.7060	31.0927	29.9535	29.1254	87.9715	85.6640	83.7837	82.2405

Table 4 Critical wealth level $\hat{x}$ for various δ and I_1. Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $K = 3$, $\mu = 0.1394$ (0.1394), $\sigma = 0.2600$ (0.1313) are used for the parameter values under Economic Recessions (Economic Expansions, respectively). I_2 is fixed as 10% of I_1.

Table 4 a myopic individual with $\delta = 0.01$ and $I_1 = 1$ has the critical wealth level of 212.5128 during the economic expansions, but the critical wealth level plunges down to 77.4323 if an economic recession starts. When considering a smaller I_1 or higher δ, the reduction of critical wealth level becomes remarkable.

6. Conclusion

We developed a new approach for solving the optimal retirement problem for an individual with an unhedgeable income risk. The income risk stems from a forced unemployment event, which occurs as an exponentially-distributed random shock. The optimal retirement problem is to determine an individual's optimal consumption and investment behaviors and optimal retirement time simultaneously. Our approach for solving the problem originated from the combination of the DPA and the convex-duality approach, but we introduced a slightly different convex-dual function of the individual's value function from the conventional ones, and we also provided an efficient iterative numerical method.

Reasonably calibrated parameters show that our model can give an explanation for lower consumption and risky investment behaviors of individuals, and for relatively higher stock holdings of the poor. Exploiting the concept of CEWG, we glanced at an individual's optimal behaviors in changing her wealth level, investment opportunity, and the magnitude of preference of post-retirement leisure.

Finally, we find our model gives an explanation of a counter-cyclical pattern of the number of unemployed job leavers. It could provide an evidence that soaring forced unemployment risks during economic recessions induces people who have slightly smaller wealth than the critical wealth level planned during the past economic expansion to enter early retirement.

An interesting extension of our paper is to introduce a continuous-time Markov regime-switching model to discuss the effects of economic recessions and economic expansions on an individual's optimal strategies. We leave solving the retirement problem in a regime switching framework to the reader as an extension for future research.

7. Appendix

This version of the appendix contains the statements of important theorems without the proofs. An extended appendix concerning the details of the proofs is available online.

7.1. The Existence of a Solution to (11) and (12)

We assume that the subjective discount rate β equals to the risk-free interest rate r. This assumption is only used when verifying the existence of a solution to (11) and (12). Actually, if one wants to relax it, it is necessary to take a restriction on the free boundary $\hat{\lambda}$ to verify the existence. By taking the assumption of $\beta = r$, we can show the existence of a solution to (11) and (12) for *any* $\hat{\lambda} > 0$. Most importantly, without the verification for the existence of a solution we could find a numerical solution satisfying (11), (12) and (13) in a wide range of parameters.

We can provide a theorem concerning this issue.

Theorem 2. *We assume that $\beta = r$. Then for any $\hat{\lambda} > 0$, there exists a solution to the problem formulated by equations in (11) and (12), which has the boundedness such that*

$$\int_{\hat{\lambda}}^{\infty} \lambda^{1-n} (G'(\lambda))^2 d\lambda \leq C_n^*$$

and

$$\int_{\hat{\lambda}}^{\infty} \lambda^{3-n} (G''(\lambda))^2 d\lambda \leq C_n^{**},$$

where n is a natural number, C_n^* and C_n^{**} are constants.

7.2. Theorems Described in Section 2.3.2

Theorem 3. *(Uniqueness) If $\dfrac{2\delta}{\theta^2 \beta \hat{\lambda}} < 1$, the solution of (14) is unique.*

The following theorem permits us to take a monotonically-decreasing $G(\lambda)$ under suitable parameter conditions.

Theorem 4. *(Monotonicity) Suppose that*

$$I_2 \geq r, \quad \frac{1}{\beta + \delta} + \frac{2}{\theta^2 (\alpha_\delta - \alpha_\delta^*)} \ln (\beta \frac{I_2}{r}) > 0,$$

and

$$\frac{2\delta \alpha_\delta}{\theta^2 \beta} \Big(\ln \beta + \ln \max \Big\{ \frac{1}{\beta \hat{\lambda}}, \frac{I_2}{r} \Big\} \Big) < \frac{1}{\beta + \delta}. \tag{24}$$

Then, any solution of (11) satisfies $G'(\lambda) < 0$.

Theorem 5. *(Uniqueness of free boundary $\hat{\lambda}$) Suppose*

$$I_1 > \frac{\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta)+\ln K-\frac{\delta}{\beta}\ln\frac{r}{I_2}}{\frac{r}{\beta I_2}(1+\frac{\alpha_\delta\theta^2}{2r})}+I_2,$$
$$\frac{\delta}{\beta}\ln\frac{2\delta}{\theta^2}+\frac{2\delta}{\theta^2\beta}(I_1-I_2)\Big(1+\frac{\alpha_\delta\theta^2}{2r}\Big)+\frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{\theta^2}{2\delta},\frac{I_2}{r}\Big\}\Big)<\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta)+\ln K, \quad \textit{and}$$
$$\frac{\delta}{\beta}\ln\beta-\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}+e^{-\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}}(I_1-I_2)\Big(1+\frac{\alpha_\delta\theta^2}{2r}\Big)$$
$$+\frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{1}{\beta}e^{\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}},\frac{I_2}{r}\Big\}\Big)<\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta)+\ln K.$$

Then there exists a unique solution $\hat{\lambda}$ of (16) satisfying $0<\hat{\lambda}<\dfrac{r}{\beta I_2}$.

For the unique solution $\hat{\lambda}$ of (16), the corresponding $G(\lambda)$ is also the unique solution of (11).

7.3. Verification for the Optimal Stopping Problem

The verification for our optimal stopping problem (4) is executed in the following two steps: we first verify that the solution, $\phi(x)$, to the variational inequality (5) is the solution to the optimal stopping problem (4). Next, we verify that the solution to the free boundary problem (6) satisfies the variational inequality (5). We reuse some notations and definitions in Øksendal (2007).

First, we fix a domain G in $\mathbf{R}^k$ and let

$$dY_t = b(Y_t)dt+\sigma(Y_t)dB_t, \quad Y_0 = y,$$

be an Itô diffusion in $\mathbf{R}^k$. Define

$$\tau_G = \tau_G(y,\omega) = \inf\{t>0; Y_t(\omega)\notin G\}.$$

Let $f:\mathbf{R}^k\to\mathbf{R}$ and $g:\mathbf{R}^k\to\mathbf{R}$ be continuous functions satisfying

(a) $E^y\Big[\displaystyle\int_0^{\tau_G} f^-(Y_t)dt\Big]<\infty$ for all $y\in\mathbf{R}^k$, and

(b) the family $\{g^-(Y_\tau);\tau$ stopping time, $\tau\le\tau_G\}$ is uniformly integrable with respect to the probability law of Y_t for all $y\in\mathbf{R}^k$.

Let $\mathcal{T}$ denote the set of all stopping times $\tau\le\tau_G$. Consider the following problem: Find $\Phi(y)$ and $\tau^*\in\mathcal{T}$ such that

$$\Phi(y) = \sup_{\tau\in\mathcal{T}} J^\tau(y) = J^{\tau^*}(y),$$

where

$$J^\tau(y) = E^y\Big[\int_0^\tau f(Y_t)dt+g(Y_\tau)\Big] \quad \text{for} \quad \tau\in\mathcal{T}.$$

Note that since $J^0(y)=g(y)$ we have

$$\Phi(y)\ge g(y) \quad \text{for all} \quad y\in G.$$

We can now formulate the variational inequalities. As usual we let

$$L \equiv L_Y = \sum_{i=1}^{k} b_i(y)\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j}$$

be the partial differential operator.

Now, we can state Theorem 10.4.1 in Øksendal (2007).

THEOREM 6. *(Variational inequalities for optimal stopping)*

a) Suppose we can find a function $\phi : \overline{G} \to \boldsymbol{R}$ *such that*

(i) $\phi \in C^1(G) \cap C(\overline{G})$

(ii) $\phi \geq g$ *on* G $\lim_{t \to \tau_G^-} \phi(Y_t) = g(Y_{\tau_G})\chi_{\{\tau_G < \infty\}}$ *a.s.*

Define

$$D = \{x \in G; \phi(x) > g(x)\}.$$

Suppose Y_t *spends* 0 *time on* ∂D *a.s., i.e.*

(iii) $E^y\Big[\int_0^{\tau_G} \chi_{\partial D}(Y_t)dt\Big] = 0$ *for all* $y \in G$

and suppose that

(iv) ∂D *is a Lipschitz surface, i.e.* ∂D *is locally the graph of a function* $h : \boldsymbol{R}^{k-1} \to \boldsymbol{R}$ *such that there exists* $K < \infty$ *with*

$$|h(x) - h(y)| \leq K|x - y| \quad \textit{for all} \quad x, y.$$

Moreover, suppose the following:

(v) $\phi \in C^2(G \backslash \partial D)$ *and the second order derivatives of* ϕ *are locally bounded near* ∂D

(vi) $L\phi + f \leq 0$ *on* $G \backslash D$.

Then

$$\phi(y) \geq \Phi(y) \quad \textit{for all} \quad y \in G.$$

b) Suppose, in addition to the above, that

(vii) $L\phi + f = 0$ *on* D

(viii) $\tau_D \equiv \inf\{t > 0; Y_t \notin D\} < \infty$ *a.s. the probability law of* Y_t *for all* $y \in G$

and

(ix) the family $\{\phi(Y_\tau); \tau \leq \tau_D, \tau \in \mathcal{T}\}$ *is uniformly integrable with respect to the probability law of* Y_t *for all* $y \in G$.

Then

$$\phi(y) = \Phi(y) = \sup_{\tau \in \mathcal{T}} E^y\Big[\int_0^{\tau} f(Y_t)dt + g(Y_\tau)\Big]; \quad y \in G$$

and

$$\tau^* = \tau_D$$

is an optimal stopping time for this problem.

Now we recall the optimal stopping problem. For a fixed stopping time τ,

$$\Phi(x) = \max_{\tau} J_\tau(x),$$

where

$$J_\tau(x) = \max_{(c,\pi)} E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{U_1\big(c(t)\big) + \delta U_2\big(X(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big].$$

We let $c^*(t)$ and $\pi^*(t)$ denote optimal consumption and optimal risky portfolio, respectively. The partial differential operator

$$L = \frac{\partial}{\partial t} + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\frac{\partial}{\partial x} + \frac{1}{2}\pi^*(t)^2\sigma^2\frac{\partial^2}{\partial x^2}.$$

In our model, the domain G is given as

$$G = \{(x,t) \in \mathbf{R}\times\mathbf{R}; x > -I_1/r, t \geq 0\}.$$

Also, the domain D is

$$D = \{(x,t) \in G; \tilde{\phi}(x,t) > e^{-(\beta+\delta)t}U_2(x)\}$$

for a function $\tilde{\phi} : \overline{G} \to \mathbf{R}$. Then

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} =& \frac{\partial\tilde{\phi}}{\partial t} + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\frac{\partial\tilde{\phi}}{\partial x} \\ &+ \frac{1}{2}\pi^*(t)^2\sigma^2\frac{\partial^2\tilde{\phi}}{\partial x^2} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\}. \end{aligned}$$

Now we derive variational inequalities for our optimal stopping problem. We suppose that the function $\tilde{\phi}$ satisfies the following variational inequalities:

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} &= 0 \quad \text{on} \quad D, \\ L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} &\leq 0 \quad \text{on} \quad G\backslash D. \end{aligned}$$

Accordingly, they are equivalent to

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} &\leq 0, \\ \tilde{\phi}(x,t) &\geq e^{-(\beta+\delta)t}U_2(x), \\ \Big[L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\}\Big]\Big(\tilde{\phi}(x,t) - e^{-(\beta+\delta)t}U_2(x)\Big) &= 0. \end{aligned} \tag{25}$$

We conjecture the form of $\tilde{\phi}$ as

$$\tilde{\phi}(x,t) = e^{-(\beta+\delta)t}\phi(x).$$

Substituting it into the inequality (25) we get

$$\begin{aligned}
&\Big[-(\beta+\delta)\phi(x)+\Big(rx-c^*(t)+I_1+\pi^*(t)\sigma\theta\Big)\phi'(x)\\
&\qquad\qquad\qquad +\frac{1}{2}\pi^*(t)^2\sigma^2\phi''(x)+U_1\big(c^*(t)\big)+\delta U_2(x)\Big]\le 0,\\
&\qquad\qquad\qquad \phi(x)\ge U_2(x),\\
&\Big[-(\beta+\delta)\phi(x)+\Big(rx-c^*(t)+I_1+\pi^*(t)\sigma\theta\Big)\phi'(x)\\
&\qquad\qquad\qquad +\frac{1}{2}\pi^*(t)^2\sigma^2\phi''(x)+U_1\big(c^*(t)\big)+\delta U_2(x)\Big]\Big(\phi(x)-U_2(x)\Big)=0.
\end{aligned}$$

Optimality conditions for optimal consumption and risky portfolio are given by

$$c^*(t)=\frac{1}{\phi'(x)}\quad\text{and}\quad \pi^*(t)=-\frac{\theta}{\sigma}\frac{\phi'(x)}{\phi''(x)}.$$

Therefore, we obtain the variational inequality (5). Applying Theorem 6, the solution $\phi(x)$ to the variational inequality (5) is the solution $\Phi(x)$ to our optimal stopping problem (4).

Next, we provide a theorem verifying that the solution $\phi(x)$ to the free boundary problem (6) satisfies the variational inequality (5).

THEOREM 7. *If*

$$\begin{aligned}
&\frac{1}{\beta}\frac{r(I_1-I_2)}{I_2}\Big(1+\frac{\alpha_\delta\theta^2}{2r}\Big)<\frac{\theta^2}{2}\frac{\delta(1-\alpha_\delta)}{\beta(\beta+\delta)}+\ln K \quad \textit{and}\\
&\frac{1}{\beta}e^{\frac{\theta^2}{2}\frac{(1-\alpha_\delta)}{(\beta+\delta)}}(I_1-I_2)\Big(1+\frac{\alpha_\delta\theta^2}{2r}\Big)+\frac{\delta I_2}{\beta r}<\ln K
\end{aligned}$$

are true, the solution $\phi(x)$ to the free boundary problem (6) satisfies the variational inequality (5).

7.4. Convergence of the Iterative Procedure

7.4.1. Proof of the convergence We show that the approximation function $G(\cdot)$ obtained from the iterative procedure converges to the to the implicit equation (14) by using the Banach fixed-point theorem.

Consider a set $X=[\hat{\lambda},\infty)$ which is the domain of $\lambda(\cdot)$. Since the set $\mathbf{R}$ of real numbers is complete, the set $\mathcal{B}(X,\mathbf{R})$ of all bounded functions $f:X\to\mathbf{R}$ is a complete metric space with the supremum norm

$$d(f,g)\equiv\sup\{|f(x)-g(x)|:x\in X\}.$$

Note that the set $C_b(X,\mathbf{R})$ consisting of all continuous bounded functions $f:X\to\mathbf{R}$ is a closed subspace of $\mathcal{B}(X,\mathbf{R})$, so that, $C_b(X,\mathbf{R})$ is also a complete metric space. Hence, the continuous and decreasing function $G(\lambda)$, which is a solution to the differential equation (11) satisfying

$$0\le G(\lambda)\le G(\hat{\lambda}),$$

should be in $C_b(X,\mathbf{R})$.

Define for any $G(\lambda)\in C_b(X,\mathbf{R})$

$$T\big(G(\lambda)\big)\equiv\frac{1}{\lambda(\beta+\delta)}+B(\hat{\lambda})\lambda^{-\alpha_\delta}+\frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda}\mu^{\alpha_\delta-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\Big\}d\mu$$
$$+(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\infty}\mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\Big\}d\mu\Big],$$

then T is continuous and $T\big(G(\lambda)\big)$ is in $C_b(X,\mathbf{R})$ by the relationship of

$$|T\big(G(\lambda)\big)|\le\frac{2\delta}{\theta^2\beta\hat{\lambda}}\sup_{\lambda}|G(\lambda)|.$$

Moreover, if we take an assumption of

$$\frac{2\delta}{\theta^2\beta\hat{\lambda}}<1,$$

then the map $T:C_b(X,\mathbf{R})\to C_b(X,\mathbf{R})$ is a contraction mapping. This is because, for any $G_1(\lambda),\ G_2(\lambda)\in C_b(X,\mathbf{R})$, T satisfies the following:

$$\sup_{\lambda}|T\big(G_1(\lambda)\big)-T\big(G_2(\lambda)\big)|=\frac{2\delta}{\theta^2\beta\hat{\lambda}}\sup_{\lambda}|G_1(\lambda)-G_2(\lambda)|.$$

Let $G^i(\lambda)$ be a function and $B^i(\hat{\lambda}^i)$, $\hat{\lambda}^i$ be the two constants obtained from the i-th iteration. By the Banach fixed-point theorem, we have that $G^i(\lambda)$ converges uniformly to $G(\lambda)$ on the region of $[\hat{\lambda},\infty)$. Obviously, we know that $B^i(\hat{\lambda}^i)\to B(\hat{\lambda})$ and $\hat{\lambda}^i\to\hat{\lambda}$ as $i\to\infty$.

We display an example of the numerical experiments in the next subsection.

7.4.2. An example Let $G^i(\lambda)$ be the numerical solution derived by the i-th iteration. Then the corresponding value function $\Phi^i(x)$, $i=0,1,2,...$, can be calculated by the first relationship in (6): for any $-I_1/r<x<\hat{x}$,

$$\Phi^i(x)=\begin{cases}\dfrac{1}{\beta}\Big[rG^0\big(\lambda^*(x)\big)\lambda^*(x)-\dfrac{\theta^2}{2}\Big(G^0\big(\lambda^*(x)\big)\Big)'\big(\lambda^*(x)\big)^2-1-\ln\lambda^*(x)\Big], & \text{if}\quad i=0,\\ \dfrac{1}{\beta+\delta}\Big[rG^i\big(\lambda^*(x)\big)\lambda^*(x)-\dfrac{\theta^2}{2}\Big(G^i\big(\lambda^*(x)\big)\Big)'\big(\lambda^*(x)\big)^2-1-\ln\lambda^*(x) & \\ \qquad\qquad+\delta U_2\Big(G^i\big(\lambda^*(x)\big)-I_1/r\Big)\Big], & \text{if}\quad i=1,2,...,\end{cases}$$

where $G^i(\cdot)$, $B^i(\hat{\lambda}^i)$, $\hat{\lambda}^i$ are obtained from the iterative procedure in Section 2.3.2. Note that $\Phi^0(x)$ represents the value function without forced unemployment risks (i.e., $\delta=0$). Then $\Phi^i(x)$ ($i=1,2,...$) must satisfy the following inequality: for any $-I_1/r<x<\hat{x}$,

$$U_2(x)<\Phi^i(x)<\Phi^0(x). \tag{26}$$

The left inequality comes from the variational inequality of (5) and the right one must hold due to the fact that the value function without forced unemployment risks is clearly larger than that with the forced unemployment risks.

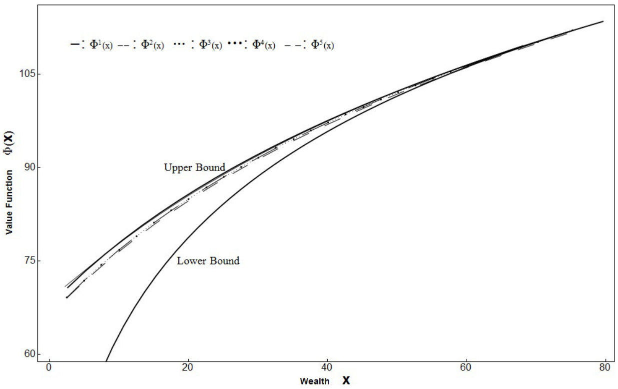

Figure 6 Numerical solutions as a function of initial wealth: $\delta = 0.01$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$ are used for parameter values.

Figure 6 shows the convergence of the numerical solution for a reasonable parameter set. The upper bound (lower bound) in the Figure 6 represents $\Phi^0(x)$ ($U_2(x)$, respectively). The figure shows that $\Phi^i(x)$ converges as the number of iteration increases and the numerical solutions satisfy the relationships in (26).

Table 5 shows the numerical solutions of $\Phi^i(x)$ $(i = 0, 1, ..., 5)$ for various initial wealth levels. It shows that the numerical results apparently seem to be convergent and are bounded by $U_2(x)$ and $\Phi^0(x)$. On the other hand, if we let $\hat{\lambda}^i$ $(i = 1, 2, ...)$ be the free boundary obtained from the i-th iteration, then we can observe in Table 6 that the numerical results for the free boundary seem to be convergent and stay between the lower bound of λ_δ^0 and the upper bound of λ_δ^1 (see the inequality (17) in Proposition 3.1) for various unemployment intensity δ's.

	lower bound	numerical solution					upper bound
x	$U_2(x)$	$\Phi^1(x)$	$\Phi^2(x)$	$\Phi^3(x)$	$\Phi^4(x)$	$\Phi^5(x)$	$\Phi^0(x)$
10	63.1217	77.8138	76.4976	76.7837	76.7839	76.7825	77.8959
20	78.7800	85.5165	84.6669	84.9565	84.9435	84.9428	85.732
30	88.6202	91.9625	91.3259	91.6085	91.5910	91.5908	92.1351
40	95.8131	97.4722	96.9468	97.2192	97.1998	97.2004	97.5626
50	101.4850	102.2700	101.8220	102.0880	102.0660	102.0660	102.2820
60	106.1690	106.5140	106.0870	106.3540	106.3310	106.3320	106.4620

Table 5 Numerical solutions for various initial wealth levels: $\delta = 0.01$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$ are used for parameter values.

Endnotes

δ	lower bound λ_δ^0	numerical result $\hat{\lambda}^1$	$\hat{\lambda}^2$	$\hat{\lambda}^3$	$\hat{\lambda}^4$	$\hat{\lambda}^5$	upper bound λ_δ^1
0.01	0.252839	0.380467	0.342191	0.345989	0.345784	0.345790	0.548594
0.02	0.183758	0.498807	0.335688	0.363804	0.361220	0.361278	0.707226
0.03	0.119902	0.641149	0.304488	0.387526	0.376163	0.376625	0.831393

Table 6 Numerical results of the free boundary $\hat{\lambda}$ for various unemployment intensity δ's: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$ are used for parameter values.

1. Cocco *et al.* (2005) define unemployment state as the state of zero income, however, Gakidis (1998) shows that even unemployed individuals may have other sources of income (e.g., income coming from unemployment benefits and social welfare).

2. See the article "Unemployed Job Leavers: A Meaningful Gauge of Confidence in the Job Market?" in Issues in Labor Statistics published by US Bureau of Labor Statistics on October 17, 2000.

3. Obviously, it might be reasonable to consider business cycles (therefore, non-myopic investors) in our model if we want to get some market-consistent results. Usually, researchers take a Markov regime-switching model to investigate the effect of business cycles on investors' optimal behaviors, but solving our problem in a regime-switching framework is much complex because regime risks are added in our model. In this paper we focus on optimal behaviors of myopic investors, and leave solving our problem in a regime-switching framework to the reader as an extension for future research. As a matter of fact, many existing literature analyze myopic investors' optimal behaviors (see, e.g., Gompers 1994, Marston and Craven 1998, Edmans 2009) and find out some economically meaningful results from their behaviors.

4. The forced unemployment time considered in this paper is not an optimal stopping time given by the set of information of stock price movements but a random time, resulting in market incompleteness.

5. We can allow a correlation between the Brownian motion and the Poisson arrival event as follows: assume that

$$\text{probability of } \{\tau_U \leq t\} = \int_0^t \delta_t dt,$$

where

$$d\delta_t = \delta_t(\mu_t^\delta dt + \sigma_t^\delta dW_t^*).$$

Here, μ_t^δ and σ_t^δ are functions with respect to time variable t and W_t^* is a standard Brownian motion such that

$$dW_t \cdot dW_t^* = \rho dt, \quad \text{for} \quad \rho \in [-1, 1].$$

In this case, however, the stochastic intensity δ can violate the condition of

$$\int_0^\infty \delta_t dt \leq 1,$$

because the intensity δ is a stochastic process with a random drift. We leave the problem in the presence of a correlation between the Brownian motion and the Poisson arrival event as an extension for future research.

6. Bodie *et al.* (1992) and Choi *et al.* (2008) allow a continum of choice between labor and leisure. However, Farhi and Panageas (2007) and Dybvig and Liu (2010) support empirical evidence that labor supply is largely indivisible. Concerning leisure choice (or equivalently, labor supply), we take the assumption of Farhi and Panageas (2007) and Dybvig and Liu (2010).

7. If we take the definition of unemployment in Lynch and Tan (2011), I_2 is about 10% of I_1.

8. It is possible to consider perceived unemployment risk in determining the discount rate. For instance, we can replace r by $r + \delta$, and show the discount rate of this type in the individual's modified objective function (3).

9. See Dybvig and Liu (2010), and Park and Jang (2014) to find the effects of various wealth constraints, e.g., a non-negative wealth constraint and a negative wealth constraint respectively, on an individual's optimal behaviors. Investigating the effect of a non-negative wealth constraint with unemployment risk would be an important research subject, and it was examined by Jang and Park (2015).

10. For the details of the derivation, see an online Appendix.

11. It guarantees the boundedness of the first derivative of the value function everywhere except zero. Roughly speaking, the investor with such preference is risk-averse for positive values of wealth and indifferent for how much they borrow up to the wealth constraint of $X(t) > -\dfrac{I_1}{r}$. Piecewise connected utility functions of this kind are seen in lots of articles in economics. For example, Venter (1983) introduces a utility function which is constant for negative values of wealth, and also says that it is designed to reflect bankruptcy laws.

12. Bensoussan and Lions (1982) showed the connection between optimal stopping and variational inequality. For the details of the derivation of the variational inequality for our problem, see Appendix 7.3.

13. For the details, see Theorem 7 in Appendix 7.3.

14. In Appendix 7.3, we provide the verification theorem for optimal stopping.

15. The reason for why we call the function G the convex-dual function is that it is a dual function of the marginal utility of objective function ϕ and a convex function. Actually, the convexity of G is a conjecture and should be verified. As we compared to the traditional convex-duality approach

given by Karatzas and Shreve (1998), the verification for the convexity is hardly possible. Instead, we numerically confirm the convexity of G with a wide range of parameters. Moreover, in Theorem 4 in Appendix 7.2, we have shown the monotonic decreasing property of G. Our objective function ϕ is increasing and concave, whereas the function G is decreasing and convex. In this sense, the function G is a convex-dual function of ϕ.

16. We can guarantee the uniqueness of $G(\cdot)$ for the case where the condition for Theorem 3 in Appendix 7.2 holds. However, it is not easy to check whether the condition holds or not because it contains $\hat{\lambda}$ which should be determined together with $G(\cdot)$ at the same time. Thus, we conducted numerical experiments for a wide range of parameters, and we assert that it works well for some reasonable parameters. Importantly, by utilizing the Banach fixed-point theorem we can verify that the iterative procedure converges. For the details, see Appendix 7.4. Furthermore, we also display an example of the numerical experiments in Appendix 7.4.2.

17. This condition of δ is necessary in order to guarantee the uniqueness and existence of the solution of (14) and to verify that the solution obtained from the new dual approach is a solution to the variational inequality of (5). We display the possible range of δ in Theorem 3, 5 and 7 in Appendix 7.2.

18. Source: Bureau of Labor Statistics.

19. Source: pp. 170 of Bodie, Kane, and Marcus (2011)

20. Much empirical literature alludes to precautionary savings against background risks, which cannot be controlled by an individual and are independent of endogenous risks. (See Kimball, 1992). Background risks consist of two main factors, income risk and housing effect, and Bodie *et al.* (1992), Koo (1998), and Heaton and Lucas (2000) emphasize the importance of income risk.

21. The former result is consistent with that of Carroll (1992). According to her, the property of relatively less increasing consumption at a higher wealth level is mostly due to an individual's impatience.

22. These parameters are also used in Jang *et al.* (2007).

23. Lynch and Tan (2011) find that the first and second moments of labor income growth depend on business cycles, and Emery *et al.* (2010) find that in the global crisis the default volume of corporate issuers skyrocketed. For example, the default rate of Moody's global speculative-graded bonds was 13.0% in 2009, which is about three times bigger than 4.4% in the previous year.

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References

Ang A, Bekaert G (2002) International asset allocation with regime shifts. *Rev. Financial Stud.* 15(4):1137–1187.

Bensoussan A, Lions JL (1982) *Application of Variational Inequalities in Stochastic Control.* North Holland.

Benzoni L, Collin-Dufresne P, Glodstein RS (2007) Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *J. Finance.* 62(5):2123–2167.

Bodie Z, Merton RC, Samuelson WF (1992) Labor supply flexibility and portfolio choice in a life cycle model. *J. Economic Dynamics and Control.* 16(3-4):427–449.

Bodie Z, Detemple JB, Otruba S, Walter S (2004) Optimal consumption-portfolio choices and retirement planning. *J. Economic Dynamics and Control.* 28(6):1115–1148.

Bodie Z, Kane A, Marcus AJ (2011) *Investments and Portfolio Management.* McGrawHill, Ninth Edition.

Carroll CD (1992) The buffer-stock theory of saving: some macroeconomic evidence. *Brookings Papers on Economic Activity.* 23(2):61–156.

Chacko G, Viceira LM (2005) Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Rev. Financial Stud.* 18(4):1369-1402.

Cocco JF, Gomes FJ, Maenhout PJ (2005) Consumption and portfolio choice over the life cycle. *Rev. Financial Stud.* 18(2):491–533.

Duffie D, Fleming W, Soner HM, Zariphopoulou T (1997) Hedging in incomplete markets with HARA utility. *Journal of Economic Dynamics and Control.* 21(4-5):753-782.

Dybvig PH, Liu H (2010) Lifetime consumption and investment: retirement and constrained borrowing. *J. Economic Theory.* 145(3):885–907.

Edmans A (2009) Blockholder trading, market efficiency, and managerial myopic. *J. Finance.* 64(6):2481–2513.

Elmendorf E, Kimball M (2000) Taxation of labor income and the demand for risky assets. *International Economic Review* 41(3):801–832.

Emery K, Ou S, Tennant J (2010) Corporate default and recovery rates, 1920-2009. *Moody's Investors Service.*

Friedman A (1982) *Variational Principles and Free-Boundary Problems.* Wiley, New York.

Friedman M (1957) *A Theory of the Consumption Function.* Princeton: Princeton University Press.

Farhi E, Panageas S (2007) Saving and investing for early retirement: a theoretical analysis. *J. Financial Econom.* 83(1):87–121.

Gakidis H (1998) Stocks for the old? earnings uncertainty and life-cycle portfolio choice. *Mimeo.*

Gompers PA (1994) The rise and fall of venture capital. *Business and Economic History.* 23(2):1–24.

Gormley T, Liu H, Zhou G (2010) Limited participation and consumption- saving puzzles: a simple explanation and the role of insurance. *J. Financial Econom.* 96:331–344.

Gruber J (1997) The consumption smoothing benefits of unemployment insurance. *American Economic Review.* 87(1):192–205.

Hall RE (1978) Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence. *J. Political Econom.* 91(6):249–265.

He H, Pearson N (1991) Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite dimensional case. *J. Economic Theory.* 54(2):259–305.

Heaton J, Lucas D (1997) Market frictions, savings behavior and portfolio choice. *Macroeconomic Dynamics.* 1(1):76–101.

Heaton J, Lucas D (2000) Portfolio choice and asset prices: the importance of entrepreneurial risk. *J. Finance.* 55(3):1163–1198.

Henderson V (2005) Explicit solutions to an optimal portfolio choice problem with stochastic income. *J. Economic Dynamics and Control.* 29(7):1237–1266.

Jang BG, Koo HK, Liu H, Loewenstein M (2007) Liquidity premia and transaction costs. *J. Finance.* 62(5):2329–2366.

Jang BG, Park S, Rhee Y (2013) Optimal retirement with unemployment risks. *J. Banking and Finance.* 37(9):3585-3604.

Jang BG, Park S (2015) Consumption, retirement, and asset allocation with unemployment risks and borrowing constraints. *Working paper.*

Karatzas I, Lehoczky J, Shreve SE, Xu GL (1991) Martingale and duality methods for utility maximization in incomplete markets. *SIAM J. Control Optim.* 29(3):702–730.

Karatzas I, Shreve SE (1998) *Methods of Mathematical Finance.* Springer, New York.

Keppo J, Meng X, Sullivan MG (2007) A computational scheme for the optimal strategy in an incomplete market. *J. Economic Dynamics and Control.* 31(11):3591-3613.

Kim TS, Omberg E (1996) Dynamic nonmyopic portfolio behavior. *Rev. Financial Stud.* 9(1):141–161.

Kimball MS (1993) Standard risk aversion. *Econometrica.* 61(3):589–611.

Koo HK (1998) Consumption and portfolio selection with labor income: a continuous time approach. *Math. Finance.* 8(1):49–65.

Liu J, Neis E (2003) Endogenous retirement and portfolio choice. *Working Paper.*

Liu J (2007) Portfolio selection in stochastic environments. *Rev. Financial Stud.* 20(1):1–39.

Lynch AW, Tan S (2011) Labor income dynamics at business-cycle frequencies: implications for portfolio choice. *J. Financial Econom.* 101(2):333–359.

Malley J, Moutos T (1996) Unemployment and consumption. *Oxford Economic Papers.* 48:584–600.

Marston CL, Craven BM (1998) A survey of corporate perceptions of short termism among analysts and fund managers. *European J. Finance.* 4(3):223–256.

Merton RC (1969) Lifetime portfolio selection under uncertainty: the continuous-time case. *Review of Economics and Statistics.* 51(3):247–257.

Merton RC (1971) Optimal consumption and portfolio rules in a continuous-time model. *J. Economic Theory.* 3(4):373–413.

Øksendal B (2007) *Stochastic Differential Equations: An Introduction with Applications.* Springer, Fifth Edition.

Park S, Jang BG (2014) Optimal retirement strategy with a negative wealth constraint. *Operations Research Letters.* 42(3):208–212.

Polkovnichenko V (2007) Life-cycle portfolio choice with additive habit formation preferences and uninsurable labor income risk. *Rev. Financial Stud.* 20(1):83–124.

Sangvinatsos A, Wachter J (2005) Does the failure of the expectations hypothesis matter for long-term investors?. *J. Finance.* 60(1):179–230.

Svensson LEO, Werner IM (1993) Nontraded assets in incomplete markets. *European Economic Review.* 37(5):1149–1168.

Teplá L (2000) Optimal hedging and valuation of nontraded assets. *European Finance Review.* 4:231–251.

Venter GG (1983) Utility with decreasing risk aversion. Proceedings of the Causality Actuarial Society. 133:144–155.

Viceira LM (2001) Optimal portfolio choice for long-horizon investors with nontradable labor income. *J. Finance.* 56(2):433–470.

Wachter JA, Yogo M (2010) Why do household portfolio shares rise in wealth?. *Rev. Financial Stud.* 23(11):3929–3965.