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Optimal investment for a retirement plan with deferred annuities allowing for inflation and labour income risk

Iqbal Owadally, Chul Jang, Andrew Clare

Abstract

We construct an optimal investment portfolio model for an individual investor saving in a retirement plan. The investor earns stochastic labour income with both permanent and temporary shocks, and has access to equity, conventional bond, inflation-indexed bond and cash, as well as two types of deferred annuities: nominal and inflation-protected. The objective function consists of power utility in terms of real retirement income from the annuities as well as bequest from remaining wealth in tradable securities. Asset returns are represented by a vector autoregressive model underpinned by Nelson-Siegel real and nominal yield curves. The optimization problem is solved numerically using multi-stage stochastic programming with a hybrid scenario structure combining a scenario tree with scenario fans. Our numerical results show that deferred annuities are bought early and in increasing amounts during the working lifetime of the investor, with portfolio risk declining with age. Welfare is diminished by 40% if deferred annuities are not available. Inflation-protected deferred annuities are marginally more important in the presence of real labour income risk, but nominal deferred annuities are bought as a cheaper alternative if real yields are low or negative. Portfolio composition and annuity allocation vary depending on financial market expectations, but our central result about the importance of deferred annuities is robust to a variety of financial market conditions.

Keywords: Finance, Stochastic programming, Inflation-protected annuities, Interest rate model, Scenario fans

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1. Introduction

A life annuity is a product which is sold by life insurers to individuals and which makes a regular stream of payments to the annuity-holder while alive. An annuity is termed an "immediate annuity" if it starts paying out from the time that it is purchased. The insurer can also design an annuity which has a deferment period between the time that the annuity is purchased and its first payout date, and this is known as a "deferred annuity" (or deferred income annuity). Annuities are a key product in retirement planning since retirees face longevity risk, i.e. the risk that they will outlive their savings. By purchasing an annuity, individuals transfer longevity risk to life insurers, which can pool this risk by writing a large portfolio of annuities. Unsurprisingly, the annuity market is very large. In 2017 in the U.S., sales of annuities totaled \$203.5 billion, of which deferred annuities accounted for \$2.2 billion (Chen et al., 2019). In addition to the retail market, a type of deferred annuity is also provided by social security and defined benefit pension plans, in which case governments and employers, rather than insurers, back the retirement benefits.

All life annuities help individuals by removing longevity risk, but deferred annuities have a specific role in reducing longevity risk at very old ages for retirees (Ezra, 2016; Scott, 2008). Deferred annuities are also useful for younger working individuals as they can be used to secure retirement income well before retirement (Horneff et al., 2010; Maurer et al., 2013). Annuity payments may be fixed in nominal terms or may be linked to a specific index, for example an index of inflation. Inflation-indexed (or inflation-protected) deferred annuities can be particularly useful since payments may continue for a long time in the future, and retirees' purchasing power should be protected (Merton, 2014) .

Although deferred annuities are potentially very useful in retirement, there is little research on optimal retirement planning with these annuities. Target-date funds in the U.S. are authorised to include deferred annuities in 401(k) pension plans (U.S. Treasury Department, 2014), but in practice many pension plans implement simple so-called glide-path strategies (Donaldson et al., 2015) which disregard deferred annuities. Glide paths reduce equity allocation and increase bond allocation as retirement draws nearer, but such a strategy does not maximize utility in terms of retirement income in real terms (Merton, 2014).

Only a few studies look at optimal investment with deferred annuities prior to retirement. Horneff et al. (2010) and Maurer et al. (2013) find that it is optimal to start purchasing deferred annuities from about age 40 with continual purchase of up to about 80% of wealth at retirement. They assume one risk-free asset with constant interest rate, one risky asset, and stochastic labour income. A similar conclusion is reached by Konicz & Mulvey (2013) in a different setting with full consumption of annuity income during retirement. Huang et al. (2017) seek the conditions for purchase of deferred annuities when interest rates are mean-reverting but their analysis is limited because there is neither investment portfolio optimization nor labour income.

Unlike the sparse research on the use of deferred annuities in optimal investment prior to retirement, there is considerably more research on immediate annuities on and after retirement. Koijen et al. (2011) calculate the optimal allocation to three types of immediate annuities (nominal, inflation-protected and variable) when wealth is fully annuitized at retirement. They also find the optimal pre-retirement consumption and investment to hedge the optimal annuity portfolio at retirement. Another strand of annuity research concerns the timing of purchase of immediate annuities. Horneff et al. (2008, 2009) solve numerically for the optimal annuitization and investment decisions when nominal immediate annuities are available in retirement and variable immediate annuities, stocks and bonds are available before retirement. They find that annuity holdings increase over an individual's lifetime. They also obtain the typical life-cycle result that equity allocation falls while bond allocation increases over time.

An important aspect of retirement planning is labour income. Most occupational retirement plans require employees to make a pension contribution which is a fixed proportion of their income. Viceira (2001), Cocco et al. (2005) and Benzoni et al. (2007) examine how risks to labour income influence optimal investment choices over an individual's lifetime. An individual, whose labour income stream is deterministic and non-tradable during her working lifetime, can be regarded as holding a risk-free coupon bond. Therefore, the investor's optimal allocation to a risky asset is higher than if she had an uncertain labour income, and it decreases over time. On the other hand, if labour income is volatile and positively correlated with the risky asset return, the fraction of wealth invested in the risky asset decreases (Cocco et al., 2005). A similar result is found by Benzoni et al. (2007) who show that cointegration between labour income and stock dividends can lead the investor to take a short position in the risky assets. However, if labour income risk is idiosyncratic, risky asset allocation can be larger than in the absence of labour income (Viceira, 2001).

Our main contributions in this article are five-fold. First, we employ a rich and realistic model of financial markets which means that our results can be implemented by pension and financial planners. Previous studies on deferred annuities in individual portfolio optimization, such as by Horneff et al. (2009, 2010), Maurer et al. (2013), and Konicz & Mulvey (2013), use only a constant risk-free rate and a geometric Brownian motion-driven risky asset. Koijen et al. (2011) employ a model with time-varying equity return and a full term structure, which they argue is critical to long-term portfolio allocation. In this paper, we use a vector autoregressive model underpinned by a Nelson-Siegel model of the term structure of nominal and real interest rates, similar to Konicz et al. (2015). However, we go further than Koijen et al. (2011) and Konicz et al. (2015) in that we include deferred annuities whereas they do not.

Second, we model a richer set of instruments and features, again making our model implementable by retirement planners. In particular, we have inflation-indexed bonds as well as inflation-indexed annuities, both immediate and deferred. We assume that pension savers have stochastic labour income, with an inflation component, as would be the case in the real world. It is especially important to include inflation given the long-term nature of retirement planning. To our knowledge, no other study on deferred annuitization incorporates all these features. Further, short-selling is not allowed in our model. Koijen et al. (2011) find that large amounts of short-selling occur in their optimal solution (with immediate annuities only), but this cannot be implemented as individual investors do not have access to short positions within their long-only retirement funds.

Third, we use multi-stage stochastic programming (MSP) to contend with the numerous state variables that our more realistic setting imposes. Numerical dynamic programming is used by Koijen et al. (2011) but they fail to incorporate realistic features and constraints such as deferred annuities and no short sales. MSP is also used by Consigli et al. (2012), Dempster & Medova (2011), Konicz & Mulvey (2013, 2015) and Konicz et al. (2015) in individual retirement planning, but we extend these works either with a richer financial market model or with a richer set of instruments or both. MSP requires the use of scenarios, and we generate several scenario trees that are arbitrage-free. Furthermore, we introduce a new scenario structure which combines a scenario tree with scenario fans in order to track inflation scenarios which are used for projecting annuity income in retirement. We also devise an optimization model to construct scenarios of labour income shocks which are independent of the financial market scenarios.

Our fourth contribution in this article is to evaluate explicitly the welfare gains when individuals have immediate and deferred annuities at their disposal, in both nominally-fixed and inflation-indexed varieties. This is important because it enables individual investors as well as their financial advisers to observe in monetary terms the advantage of including these instruments in their pension planning.

The fifth and final contribution of our paper is that we carry out sensitivity analyses over different historical periods. Financial market conditions change, particularly after momentous events such as the 2008 financial crisis, and we generate new scenario trees and find the optimal investment over different periods. This should reinforce financial planners' confidence in the model, and provide them with a robust tool with which to advise investors saving for retirement.

2. Investment problem for a retirement plan

Assume that there is an individual, aged δ years old at time 0, who makes regular contributions to a personal retirement plan until his retirement at time T. He lives to a maximum age ω (the last age in an actuarial life table), so he cannot live beyond time $\tau = \omega - \delta$. The amount of the contributions is a fixed proportion ϕ of his uncertain nominal labour income L_t (at time t) during in the pre-retirement period $[0, T)$.

In the retirement fund, the individual can hold equity, nominal bond, inflation-linked bond, and cash. In the remainder of the paper, these financial assets are denoted using subscripts E, B, $\widetilde{B}$ and C respectively, with the set of financial assets being $\mathcal{F} = \{E, B, \widetilde{B}, C\}$.

Withdrawals from the retirement fund are allowed but only to buy deferred annuities (DAs) which will pay out, if he is alive at retirement time T, every year from time T until he dies. Two types of annuities can be purchased: nominal and inflation-protected annuities, denoted by the subscripts A and $\widetilde{A}$ respectively. The set of annuity products is $\mathcal{A} = \{A, \widetilde{A}\}$. Both types of annuities are irreversible contracts, so the individual can buy them from insurers, but not sell them back to insurers or on a secondary market.

There is no payout from the annuities if the individual dies before retirement. However, any remaining wealth in his retirement fund is bequeathed to his heirs. If the individual survives till retirement, then all of his accumulated wealth in the fund is used to purchase nominal and inflation-protected immediate annuities which pay out from the retirement date until death. An immediate annuity is merely a deferred annuity with a zero deferment period, so in the following we do not distinguish between immediate and deferred annuities.

We assume that the investor exhibits constant relative risk aversion (CRRA) with a power utility function $u(t,x) = e^{-\rho t}\left(x^{1-\gamma}\right)/(1-\gamma)$, at time t, with $\gamma > 0$ being a risk aversion coefficient, $0 \le \rho \le 1$ being a time preference coefficient, and x representing either annuity income or wealth bequest, both of the latter being in real terms.

If the individual survives to retirement at time T, then utility is gained by retirement income *in real terms* (i.e. net of inflation), following the policy prescription of Merton (2014). Real income in retirement is denoted by I_t for $t \in [T, \tau)$ and it is the nominal income from annuities deflated back to time 0 at the realized log-rates of price inflation $\{\Pi_t, t \in (0, \tau]\}$. We assume implicitly that all income from the retirement plan is used for consumption, so utility $u(t, I_t)$ is gained for annuity income, in real terms, every year in retirement.

If the individual dies before time T, then his wealth in the retirement plan (minus the value of deferred annuities purchased) is bequeathed to his estate, and utility is gained by the amount, in real terms, that is bequeathed. Real wealth bequeathed is W_t for $t \in (0, T]$, and it is the nominal value of wealth deflated back to time 0. A bequest parameter κ, to be introduced shortly, captures the relative importance of the bequest to retirement income.

During the pre-retirement period $[0, T)$, the investor can dynamically adjust his investment portfolio and purchase deferred annuities in order to maximize the expected utility of real annuity income in retirement and of real bequest before retirement. Let the total number of units of nominal deferred annuity purchased by time t be $X_{A,t}$ and, similarly, the total number of units of inflation-protected deferred annuity be $X_{\widetilde{A},t}$. (Recall that the subscript A stands for nominal annuity and $\widetilde{A}$ for inflation-protected annuity.)

One unit of nominal annuity guarantees a nominal income of £1 annually in retirement until death. One unit of inflation-protected annuity also guarantees a nominal income of £1 annually in retirement until death, but the number of units is increased in line with price inflation every year. Thus, income from the inflation-protected annuity is perfectly correlated with price inflation. Let the prices of one unit of nominal and inflation-protected annuities be $S_{A,t}$ and $S_{\widetilde{A},t}$ respectively at time $t \in [0, T]$. Then, at time t, the investor pays $S_{A,t}(X_{A,t} - X_{A,t-1})$ to buy the nominal annuity and $S_{\widetilde{A},t}(X_{\widetilde{A},t} - X_{\widetilde{A},t-1}e^{\Pi_t})$ to buy the inflation-protected annuity, where Π_t is the log-rate of price inflation in year $(t-1, t)$. (In both cases, $X_{A,-1} = X_{\widetilde{A},-1} \equiv 0$.)

The investor can also buy and sell units of a cash fund, equity fund, nominal bond fund and inflation-linked bond fund. (Recall that these are denoted by C, E, B and $\widetilde{B}$ respectively.) Let $X_{E,t}$ be the number of units of the equity fund held in the retirement plan at time t, and $S_{E,t}$ be the price of equity units at time t. A corresponding notation holds for the cash, nominal bond and inflation-linked bond funds. Note that we assume perfect divisibility of assets and that fractional units can be held.

At time $t \in [0, T)$, the individual can adjust asset allocations of the retirement plan by deciding how much to hold in cash, equity, and bonds, and how many annuity units to buy. At the retirement date T, all financial wealth is sold and the investor decides how many units of nominal and inflation-protected annuities to buy. The decision variable for the individual at time $t \in [0, T]$ is therefore $X_t = [X_{C,t},\ X_{B,t},\ X_{E,t},\ X_{\widetilde{B},t},\ X_{A,t},\ X_{\tilde{A},t}]'$.

3. Formulation for multi-stage stochastic programming

3.1. Preliminary notation and definitions

Stochastic programming is a mathematical framework for optimization problems with uncertain scenarios. The scenarios can be economic, financial, and demographic. Both the state space and time are discretized in multi-stage stochastic programming (MSP). The multiple discrete-time points are known as stages. An MSP model is constructed in a nodal form by using state variables generated in a scenario tree. The scenario tree starts at the initial stage from a unique root node and it ends at the terminal stage with multiple leaf nodes. The root node branches out to a number of children nodes at the second time stage. Each child node itself branches out to further nodes at the third time stage, and so on, until the leaf nodes are reached. Every node, except the root node, has a unique parent node. A scenario is the connected path through a series of parent nodes from a leaf node to the root node. In general, the scenario tree is non-recombining.

We set out here some notation pertaining to the scenario tree. For convenience, this notation is summarized in Table 1. The scenario tree is depicted in schematic form in Fig. 1. The root node is located at the first stage and is denoted by n_0. The set of all nodes in the scenario structure is $\mathcal{N}$, and $\mathcal{N}_t$ is the set of nodes at time t. Thus, $\mathcal{N}_0 = \{n_0\}$ contains the root node only, $\mathcal{N}_\tau$ is the set of leaf nodes, and $\mathcal{N} = \bigcup_{t\in[0,\tau]} \mathcal{N}_t$. A node $n \neq n_0$ branches off a parent node, denoted by n^-, which may itself have its own parent node n^{--}, etc. It is convenient to denote by s_n the set of all predecessor nodes of node n, i.e. $s_n = \{n, n^-, n^{--}, \ldots, n_0\}$. A node $n \notin \mathcal{N}_\tau$ forks into a set of children nodes, denoted by $\{n^+\}$ at the next stage, and these children nodes may themselves have their own children nodes $\{n^{++}\}$ at the following stage, etc. The time between each stage may

Table 1 – Variables and parameters for the MSP optimization problem.

Sets	
$\mathcal{N} = \bigcup_{t\in[0,\tau]} \mathcal{N}_t$	All nodes in the hybrid scenario structure
$\mathcal{N}_t$	All nodes at time t
$\mathcal{A} = \{A, \widetilde{A}\}$	Set of annuities guaranteeing fixed-nominal (A) and inflation-protected ($\widetilde{A}$) payments
$\mathcal{F} = \{C, B, \widetilde{B}, E\}$	Set of financial assets: cash (C), conventional bond (B), inflation-linked bond ($\widetilde{B}$), equity (E)
$s_n = \{n, n^-, n^{--}, \ldots, n_0\}$	All predecessor nodes of node n, including itself, up to root node n_0
$\{n, n^+, n^{++}, \ldots\}$	All successor nodes of node n, including itself
Decision variables	
$X^{buy}_{i,n}$, $i \in \mathcal{A} \cup \mathcal{F}$	The unit number of asset i to buy at node n
$X^{sell}_{i,n}$, $i \in \mathcal{F}$	The unit number of asset i to sell at node n
X_n	Vector collecting all the buy and sell decision variables at node n
$X_{i,n}$, $i \in \mathcal{A} \cup \mathcal{F}$	The unit number of asset i held at node n after rebalancing
Other main variables	
I_n	Real retirement income at node n
W_n	Real financial wealth at node n (excludes the value of annuities)
Parameters	
${}_tp_\delta$	Probability that a δ-year old person survives until age $\delta + t$
${}_{\Delta t}q_{\delta+t}$	Probability that a $(\delta + t)$-year old person dies over the following Δt years
$\mathbf{pr}_n$	Unconditional probability that a node n occurs
$S_{i,n}$, $i \in \mathcal{A} \cup \mathcal{F}$	Nominal price of asset i at node n
L_n	Nominal labour income per annum at node n
Π_n	Inflation log-rate over a Δt-long time interval ending at node n
$\Delta t, T, \tau$	Portfolio holding period, retirement date, maximum time respectively (all in years)
δ	Investor's starting age at $t = 0$
(γ, ρ, κ)	Investor's preference parameters; risk aversion, time preference, and bequest motive respectively
w_0	Current nominal wealth in cash at the root node n_0 before contribution and rebalancing
ϕ	Fixed contribution rate, as a proportion of nominal labour income L_n, to the retirement plan

vary, in general, but it is fixed in our model and is denoted by Δt. The unconditional probability that a node n occurs is $\mathbf{pr}_n$ and the conditional probability that a node n occurs given its parent node n^- is pr_n.

We make use of a hybrid structure of scenario tree followed by scenario fans, as illustrated in Fig. 1. The scenario tree consists of multiple branches from every node and it spans the decision period $[0, T]$, i.e. the period up to retirement during which investment and annuitization decisions are made. The scenario fans consist of only two branches from every node at retirement, and they span the projection period $(T, \tau]$,

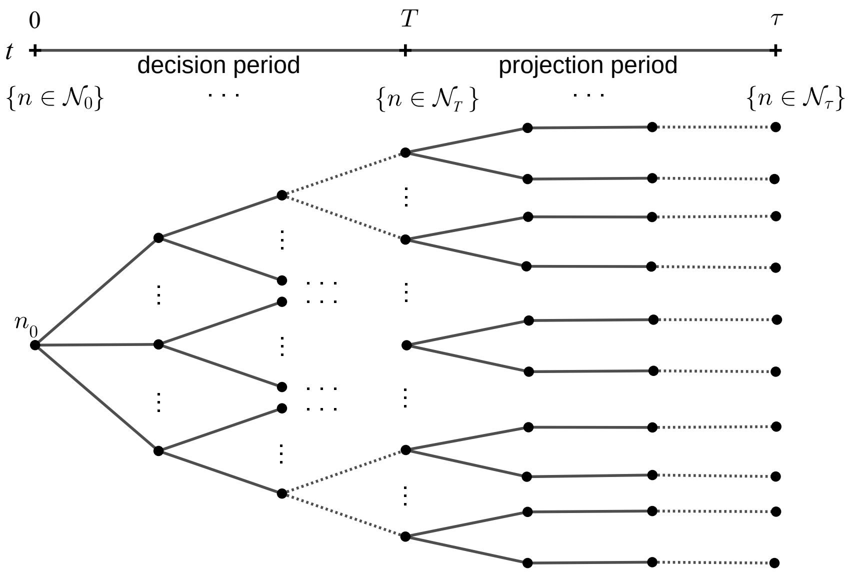


Fig. 1 – A hybrid structure of a scenario tree (from time 0 to T) and scenario fans (from $T+1$ to τ).

i.e. the post-retirement period. It is important to highlight that decisions are only made up to retirement, and the relevant scenarios are captured in the scenario tree. The scenario fans after retirement are only used to project inflation forward. They are required because income is received during retirement but it is the utility of retirement income in *real* terms (i.e. net of inflation) that must be evaluated. As explained by Dupačová et al. (2000), the scenario fan structure can be used in the special case in which the probability distributions of supporting variables are only affected by decisions made before the scenarios.[1]

In the following, survival and death probabilities are represented using standard actuarial notation. The probability that a δ-year old person survives until age $\delta + t$ is denoted by ${}_tp_\delta$. The probability that a $(\delta + t)$-year old person dies over the following Δt years is denoted by ${}_{\Delta t}q_{\delta+t}$ (typically abbreviated to $q_{\delta+t}$ when $\Delta t = 1$).

3.2. Optimization problem

The optimization problem for the investor in a retirement plan can be formulated on the scenario structure as a multi-stage stochastic programming (MSP) problem as follows. Let $X_{i,n}$ be the number of units of asset $i \in \mathcal{A} \cup \mathcal{F}$ held at node n in the scenario structure. We separate buy and sell decisions: $X^{buy}_{i,n}$ is the number of units of asset i to buy at node n, and $X^{sell}_{i,n}$ is the number of units of asset i to sell at node n. Annuities cannot be sold, so the decision variable at node n is $X_n = \left[X^{buy}_{C,n},\, X^{sell}_{C,n},\, X^{buy}_{B,n},\, X^{sell}_{B,n},\, X^{buy}_{E,n},\, X^{sell}_{E,n},\, X^{buy}_{\widetilde{B},n},\, X^{sell}_{\widetilde{B},n},\, X^{buy}_{A,n},\, X^{buy}_{\widetilde{A},n}\right]'$.

[1] The combined structure of the scenario tree and scenario fans, depicted in Fig. 1, still satisfies the non-anticipative condition. Here, the supporting variable is the inflation process to project annuity income, and decisions are all made before starting the scenario fan.

The objective function, budget constraints, and variable constraints for the retirement planning problem are given in nodal form suitable for MSP by the equations below (the equations in the usual time representation are reproduced in the online supplementary appendix, section S-1):

$$\max_{\{X_n, n\in\{\mathcal{N}_t, t\in[0,T]\}\}} \Bigg[\sum_{t\in[T,\tau)} \sum_{n\in\mathcal{N}_t} {}_t p_\delta \ u(t, I_n) \ \mathbf{pr}_n + \sum_{t\in[0,T)} \sum_{n\in\mathcal{N}_{t+\Delta t}} {}_t p_\delta \ {}_{\Delta t} q_{\delta+t} \kappa^\gamma u(t+\Delta t, W_n) \ \mathbf{pr}_n \Bigg], \tag{1a}$$

$$\text{s.t.} \quad I_n = \left(\sum_{i\in\mathcal{A}} X_{i,n}\right) \exp(-\textstyle\sum_{m\in s_n} \Pi_m) \qquad \text{for } n \in \{\mathcal{N}_t, t\in[T,\tau)\}, \tag{1b}$$

$$W_n = \sum_{i\in\mathcal{F}} X_{i,n^-} \cdot S_{i,n} \exp\left(-\textstyle\sum_{m\in s_n} \Pi_m\right) \text{ for } n \in \{\mathcal{N}_t, t\in(0,T]\}, \tag{1c}$$

$$\mathbb{1}_{\{n=n_0\}} w_0 + \mathbb{1}_{\{n\notin\mathcal{N}_T\}} \phi \ L_n \Delta t + \sum_{i\in\mathcal{F}} X_{i,n}^{sell} \ S_{i,n} = \sum_{i\in\mathcal{A}\cup\mathcal{F}} X_{i,n}^{buy} \ S_{i,n} \qquad \text{for } n \in \{\mathcal{N}_t, t\in[0,T]\}, \tag{1d}$$

$$X_{i,n} = \mathbb{1}_{\{n\neq n_0\}} X_{i,n^-} + X_{i,n}^{buy} - X_{i,n}^{sell} \qquad \text{for } i\in\mathcal{F} \text{ and } n \in \{\mathcal{N}_t, t\in[0,T]\}, \tag{1e}$$

$$X_{A,n} = \mathbb{1}_{\{n\neq n_0\}} X_{A,n^-} + \mathbb{1}_{\{n\in\{\mathcal{N}_t, t\in[0,T]\}\}} X_{A,n}^{buy} \qquad \text{for } n \in \{\mathcal{N}_t, t\in[0,\tau)\}, \tag{1f}$$

$$X_{\widetilde{A},n} = \mathbb{1}_{\{n\neq n_0\}} X_{\widetilde{A},n^-} \ e^{\Pi_n} + \mathbb{1}_{\{n\in\{\mathcal{N}_t, t\in[0,T]\}\}} X_{\widetilde{A},n}^{buy} \qquad \text{for } n \in \{\mathcal{N}_t, t\in[0,\tau)\}, \tag{1g}$$

$$X_{i,n} = X_{i,n}^{buy} = 0 \qquad \text{for } i\in\mathcal{F} \text{ and } n\in\mathcal{N}_T, \tag{1h}$$

$$X_{i,n}, X_{i,n}^{buy}, X_{i,n}^{sell} \geq 0 \qquad \text{for } i\in\mathcal{F} \text{ and } n \in \{\mathcal{N}_t, t\in[0,T]\}, \tag{1i}$$

$$X_{i,n}^{buy} \geq 0 \qquad \text{for } i\in\mathcal{A} \text{ and } n \in \{\mathcal{N}_t, t\in[0,T]\}. \tag{1j}$$

In Eq. (1a) above, the decision variables over which the expected utility is maximized are the portfolio and annuity purchase decisions over the planning horizon $[0, T]$. It is implicit in the objective function in Eq. (1a) that summations occur over the time stages in the scenario tree during the investor's lifetime $t \in [0, \tau)$. The decision variable must be chosen at every node n in the scenario tree component of the scenario structure, $\{\mathcal{N}_t, t \in [0, T]\}$. Eq. (1b) shows the real retirement income I_n at time $t \in [T, \tau)$ during retirement, this being the nominal income from annuities deflated back to time 0. Recall that the number $X_{\widetilde{A},n}$ of inflation-protected annuity units increases in line with price inflation, whereas the corresponding number for nominal units does not. Wealth W_n in Eq. (1c) is evaluated in real terms and does not include the value of annuities, so it is the inflation-adjusted total value of financial assets before rebalancing the portfolio at node n.

The cash balance constraint in Eq. (1d) sets off cash inflows against outflows. At the root node, wealth is initialized at the non-random amount w_0 specified on the l.h.s. of Eq. (1d). The incidence of cash flows in our model is such that contributions occur in advance every Δt years. A financial asset inventory constraint appears in Eq. (1e) and tracks the number of units of cash, equity, nominal and inflation-indexed bond funds held at node n. The annuity inventory constraints in Eq. (1f) and Eq. (1g) allow the investor to buy the two types of annuities during the pre-retirement planning period as well as at retirement, and they track the number of units of annuities for the whole lifetime $[0, \tau)$. Notice that the number of units of inflation-protected annuities in Eq. (1g) increases in line with inflation every year. At retirement time T, all assets are sold to purchase nominal and inflation-protected immediate annuities, and constraint Eq. (1h) guarantees this full annuitization. Eq. (1i) represents the no short-sales constraint, while Eq. (1j) ensures that annuities can only be bought and not sold.

3.3. Scenario generation for financial and annuity markets

There are two related features of the scenario tree which call for a modelling decision: the number of stages in the tree and the branching factor at every stage. (The branching factor is the number of children nodes that each node has.) A "curse of dimensionality" occurs in MSP as the number of stages increases and the scenario tree becomes 'denser' (Shapiro et al., 2009; Dupačová et al., 2000). This corresponds to a similar curse in numerical dynamic programming when the number of state variables increases. In MSP, there is a computational limit to the number of stages and branching factors in the scenario tree.

A common strategy is to use stages of increasing lengths with decreasing branching factors along the tree, as used for example by Consigli et al. (2012). We do not employ this strategy here for two reasons. First, branching factors cannot decrease beyond the minimum number of branches required to avoid arbitrage (Geyer et al., 2010) and to match the required moments of the conditional distributions of the variables stored at every node in our optimization problem (Høyland & Wallace, 2001). Second, we use a regular time interval between stages to replicate the regular financial reviews that an individual may have with a financial planner during their financial lifecycle, and the regular portfolio rebalancing that then occurs. The optimal investment solution that we obtain is then an approximation to optimal intertemporal investment without the distortion that would be caused by an uneven time-interval effect.[2]

To generate scenario trees for a portfolio optimization problem, scenario reduction and state aggregation are not suitable methods (Geyer et al., 2010). Scenario reduction methods do not admit no-arbitrage conditions explicitly, while state aggregation involves

[2] We thank a reviewer for suggesting that stages of increasing lengths, with a short period of 1 year between the first two stages, could be useful in a practical setting. For example, the model could be used in a receding horizon control fashion where it is fully solved but only the first step is implemented. It is then solved again the following year, when the individual is one year older, and again only the first step is implemented, etc. until retirement.

only the risk-neutral measure, not the real-world measure which should be used in a portfolio optimization problem. To generate the scenario tree, we combine the sequential approach of Høyland & Wallace (2001) with the moment matching method (Klaassen, 2002).

Details of the scenario generation procedure are given in section S-2 of the online supplementary appendix. In brief, we have 10 state variables at every node, we use a regular stage interval of $\Delta t = 5$ years and we have 6 stages (5 periods), so that the scenario tree spans 25 years in the decision period up to retirement (this is the period $[0, T]$ with $T = 25$ in Fig. 1). Given the values of the variables on any particular node n, the moments of the conditional distribution of these state variables after $\Delta t = 5$ years can be calculated using a suitable financial model fitted to market data. (The model will be described in section 4.) The values of the state variables on the children nodes $\{n^+\}$ are then determined by matching the first 4 moments of the conditional distribution. There are 85 moment specifications (10×4 central moments and $10(10-1)/2 = 45$ covariances). Based on this, Høyland & Wallace (2001) suggest that there should be a minimum of 8 branching factors. Starting from the root node, we can evaluate the state variables at each node sequentially throughout the scenario tree. This procedure gives a close fit to market data as is demonstrated in section S-2 of the online supplementary appendix.

Another consideration relevant to scenario generation is that the scenarios are arbitrage-free. The following procedure is used to preclude arbitrage:[3]

Step 1. Given a node $n \in \{\mathcal{N}_t, t \in [0, T)\}$, calibrate the first four conditional moments of the subtree branching from this node.

Step 2. Check if the generated scenarios preclude arbitrage opportunities among cash, nominal bond, inflation-linked bond, and equity funds: see Klaassen (2002). Return to Step 1 if any arbitrage opportunity is found.

Step 3. Check if each of the generated scenarios has conditional moments over the next stage that are within no-arbitrage bounds: see Geyer et al. (2014).

Step 4. Repeat Step 1 to Step 3, if Step 3 meets the always-arbitrage bound.

Step 2 can be subsumed within Step 1. We proceed in a sequential way and apply the above procedure from the first to the penultimate stage. To identify arbitrage opportunities among the four financial assets (cash, nominal bond, inflation-linked bond, and equity) in Steps 2 and 3, we use the two methods of Klaassen (2002) for two arbitrage types *ex-post* and the method of Geyer et al. (2014) for no-arbitrage bounds *ex-ante*.

3.4. Scenarios for pre-retirement labour income and post-retirement inflation

In the pre-retirement period (this is the period $[0, T)$ in Fig. 1), labour income is earned by the individual. In order to incorporate the effects of stochastic labour income

[3] We are grateful to a reviewer for suggesting the alternative method set out by Consiglio et al. (2016).

into the retirement planning problem, we generate labour income scenarios superposed upon the financial market scenarios described in the previous section. We can generate real labour income scenarios which are independent of the asset return scenarios during the individual investor's working lifetime $[0, T)$ because there are enough degrees of freedom in the eight branches of each node within the financial market scenarios, when the ten state variables are reduced to four financial assets returns for cash, equity, nominal bond and inflation-linked bond funds. Visiting each node in the working period of the scenario structure, we generate permanent real labour income shocks and temporary real labour income shocks on the eight children nodes. Details of the labour income model are given in section 4.4 below and the relevant scenario generation is discussed in section S-3 of the online supplementary appendix.

In the post-retirement period (this is the period $[T, \tau]$ in Fig. 1), there is no labour income, of course, but annuity income is earned. This income must be evaluated *in real terms*, i.e. deflated back to the root node n_0 at time 0. We use scenario fans to project inflation forward during retirement. We emphasize again that no decision is made in the post-retirement period. Log-rates of inflation are Normally distributed in our model (see section 4.3 below), consequently only two moments need to be matched to create scenario fans. For the scenario fans, we have an initial branching factor of 2 and we choose two points at one conditional standard deviation on either side of conditional mean inflation on each node in $\mathcal{N}_T$, i.e. at retirement (Dupačová et al., 2000).

In the scenario tree part of the hybrid scenario structure that we use, there are eight children nodes for every node over the first six stages (five periods), giving $8^5 = 32,768$ scenarios in the scenario tree. This is then doubled to $65,536$ in the overall hybrid scenario structure because of the scenario fans. Since post-retirement inflation is required only for the purpose of deflating retirement income back to the root node n_0 at time 0, $65,536$ scenarios for inflation is an ample number to capture Normally distributed log-rates of inflation, conditioned on the root node. Finally, intervals of one year are used along the fans, from time $T + 1 = 26$ to the terminal date $\tau = 80$ in Fig. 1. (As we indicate later, the terminal date will correspond to a maximum age in a mortality table.)

4. Financial assumptions

4.1. Term structure of interest rates

The Nelson-Siegel model is chosen to model the real and nominal yield curves along with a vector autoregressive (VAR) model for stochastic cash, bond, equity returns, and inflation rates. The Nelson-Siegel model is parsimonious and known to avoid the over-fitting problem and to return better out-of-samples predictions than affine term structure models (Diebold & Li, 2006). Ferstl & Weissensteiner (2011) combine the Nelson-Siegel formulation proposed by Boender et al. (2008) with the VAR model. Our model therefore incorporates asset return predictabilities and produces a seamless yield curve for pricing not only the cash and nominal and inflation-indexed bond funds, but also nominal and inflation-protected annuities.

The entire nominal yield curve is determined by a fitted Nelson-Siegel model with three time-varying parameters[4], $\boldsymbol{\beta}_t = [\beta_{1,t}, \beta_{2,t}, \beta_{3,t}]'$. For the real yield curve, we use the notation $\widetilde{\boldsymbol{\beta}}_t = [\widetilde{\beta}_{1,t}, \widetilde{\beta}_{2,t}, \widetilde{\beta}_{3,t}]'$. The Nelson-Siegel model for the s-year nominal spot rate at time t is as follows:

$$y(\boldsymbol{\beta}_t, s, \lambda) \;=\; \beta_{1,t} + (\beta_{2,t} + \beta_{3,t}) \left(\frac{1 - e^{-\lambda s}}{\lambda s} \right) - \beta_{3,t} e^{-\lambda s}, \tag{2}$$

where the scaling parameter λ is a constant. A corresponding equation holds for the real spot rate.

4.2. Time-varying investment opportunities

In order to incorporate predictabilities of asset returns and a pair of three Nelson-Siegel parameters ($\boldsymbol{\beta}_t$ and $\widetilde{\boldsymbol{\beta}}_t$), we use a VAR(1) model (for details, see Barberis, 2000; Campbell et al., 2003). Specifically, we use the combined approach of Konicz et al. (2015) to model interest rates, equity returns and inflation rates. Here, r_t is monthly log-returns on the equity fund. Monthly inflation log-rates are denoted by π_t. Our VAR model is given by

$$z_t \;=\; \Phi_0 \;+\; \Phi_1 z_{t-1} \;+\; v_t, \tag{3}$$

where $z_t = [r_t, \pi_t, \beta_{1,t}, \beta_{2,t}, \beta_{3,t}, \tilde{\beta}_{1,t}, \tilde{\beta}_{2,t}, \tilde{\beta}_{3,t}]'$. The accumulated return $R_{E,t}$ on the equity fund, defined near Eq. (6) as the log-return from time $t - \Delta t$ to t, is simply a sum of the monthly log-returns. The accumulated inflation Π_t, defined near Eq. (8) and Eq. (12), is a sum of the monthly inflation rates. In Eq. (3), the intercept term Φ_0 is a column vector. The slope term Φ_1 is a 8×8 coefficient matrix of the VAR model. The error term v_t is a column vector of i.i.d. innovations $\sim \mathrm{N}(0, \Sigma_z)$, where $\Sigma_z = \mathbb{E}[vv']$.

Given fixed values of λ and $\tilde{\lambda}$, if all eigenvalues of Φ_1 have moduli less than one, the stochastic process in Eq. (3) is stable with the unconditional expected mean μ_{zz} and covariance Γ_{zz} of z_t in the steady state:

$$\mu_{zz} \;=\; (I - \Phi_1)^{-1} c \tag{4}$$

$$vec(\Sigma_{zz}) \;=\; (I - \Phi_1 \otimes \Phi_1)^{-1} vec(\Sigma_z), \tag{5}$$

where I is an identity matrix and the operator $\otimes$ is the Kronecker product and vec is a vectorisation function, which transforms the 8×8 matrix Σ_z into a $8^2 \times 1$ vector.

Using historical nominal and real yield curves from the Bank of England from January 1985 to June 2017 with 0.5 to 25-year spot rates, monthly FTSE 100 data from Bloomberg, and retail price index as an inflation rate measure from the Office for Na-

[4] In the Nelson-Siegel model, the long interest rate is given by $\lim_{s\to\infty} y(\beta_t, s) = \beta_{1,t}$ and the short interest rate is $\lim_{s\to 0} y(\beta_t, s) = \beta_{1,t} + \beta_{2,t}$. The parameters $\beta_{1,t}$, $\beta_{2,t}$, and $\beta_{3,t}$ determine level, slope, and curvature of the yield curve respectively (Boender et al., 2008).

Table 2 – Estimated parameters and t-statistics for the VAR(1) model.

	Φ_0	Φ_1							
		r_{t-1}	π_{t-1}	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	$\tilde{\beta}_{1,t-1}$	$\tilde{\beta}_{2,t-1}$	$\tilde{\beta}_{3,t-1}$
r_t	-0.0034	-0.0013	-0.0584	0.1265	-0.0425	-0.0342	0.1064	-0.0831	-0.0658
	(-0.3415)	(-0.0251)	(-0.1051)	(0.4051)	(-0.2886)	(-0.3130)	(0.2384)	(-0.4921)	(-0.4000)
π_t	0.0021	0.0016	0.0474	0.0288	0.0727	0.0026	-0.0690	-0.0844	-0.0157
	(2.2691)	(0.3493)	(0.9413)	(1.0177)	(5.4389)	(0.2652)	(-1.7055)	(-5.5114)	(-1.0493)
$\beta_{1,t}$	0.0031	-0.0021	-0.0402	0.9246	0.0301	0.0046	0.0280	-0.0155	0.0242
	(2.9330)	(-0.3987)	(-0.6907)	(28.2607)	(1.9470)	(0.3996)	(0.5994)	(-0.8773)	(1.4043)
$\beta_{2,t}$	0.0047	0.0156	0.0264	-0.1462	0.9498	-0.0243	0.1527	-0.0422	-0.0367
	(3.2698)	(2.1473)	(0.3332)	(-3.2800)	(45.1243)	(-1.5571)	(2.3975)	(-1.7493)	(-1.5619)
$\beta_{3,t}$	0.0043	-0.0057	-0.0123	-0.1073	-0.0930	0.8791	0.2206	0.0446	0.0040
	(1.3922)	(-0.3662)	(-0.0719)	(-1.1206)	(-2.0568)	(26.2582)	(1.6133)	(0.8625)	(0.0800)
$\tilde{\beta}_{1,t}$	0.0007	0.0003	-0.0105	-0.0220	0.0028	-0.0047	1.0178	0.0014	0.0169
	(1.2741)	(0.1043)	(-0.3342)	(-1.2418)	(0.3383)	(-0.7583)	(40.2434)	(0.1499)	(1.8160)
$\tilde{\beta}_{2,t}$	0.0028	-0.0141	0.3517	-0.1208	-0.0159	-0.0385	0.1352	0.9486	0.0132
	(2.2349)	(-2.2155)	(5.0743)	(-3.1007)	(-0.8665)	(-2.8284)	(2.4294)	(45.0456)	(0.6422)
$\tilde{\beta}_{3,t}$	-0.0002	0.0082	-0.2255	0.0832	0.0234	0.0500	-0.1611	-0.0154	0.8587
	(-0.1062)	(0.8469)	(-2.1284)	(1.3969)	(0.8337)	(2.4032)	(-1.8940)	(-0.4785)	(27.3676)

Monthly data of FTSE 100 from Bloomberg, retail price index from Office for National Statistics and fitted yield curves from Bank of England respectively are used from January 1985 to June 2017 ($\lambda = 0.1519$ and $\tilde{\lambda} = 0.2508$ for the Nelson-Siegel nominal and real yield curves model); t-statistics in parentheses. R^2: 0.0114 (r_t), 0.1396 (π_t), 0.9484 ($\beta_{1,t}$), 0.9665 ($\beta_{2,t}$), 0.8431 ($\beta_{3,t}$), 0.9881 ($\tilde{\beta}_{1,t}$), 0.92626 ($\tilde{\beta}_{2,t}$), 0.8373 ($\tilde{\beta}_{3,t}$).

tional Statistics over the same period, the VAR model is stable. We choose the Nelson-Siegel scale parameters of $\lambda = 0.1519$ and $\tilde{\lambda} = 0.2508$ such that the sum of squared errors between the fitted Nelson-Siegel and historical yields is minimized. Our estimates for Φ_0 and Φ_1 in Eq. (3), along with t-statistics, are collected in Table 2. The level of R^2 for the equity return component is low, so it is difficult to confirm the existence of return predictability in the U.K. equity market. Table 3 exhibits the standard deviations and correlations of the residuals. Table 4 presents the unconditional expected mean μ_{zz} and standard deviation $\sigma_{zz} = \sqrt{diag(\Sigma_{zz})}$ of z_t at the steady state. The left hand panel of Fig. 2 displays the nominal term structure at the steady state, and it is clearly upward sloping and concave (as is the steady-state *real* term structure, not shown here). The right hand panel of Fig. 2 shows eight different stochastic realizations of the nominal term structure after 5 years.

4.3. Price dynamics of bonds and annuities

The investor rebalances his portfolio and buys deferred annuities at regular intervals of length Δt years during his retirement planning period $[0, T]$. There are $N \in \mathbb{N}$ such regular intervals, i.e. $T = N\Delta t$. Defining $R_{i,t}$ as the accumulated log-return of financial asset $i \in \mathcal{F}$ from time $t - \Delta t$ to t, the price $S_{i,t}$ of financial asset i is given by:

$$S_{i,t} = S_{i,t-\Delta t} \cdot \exp(R_{i,t}) \qquad \text{for } i \in \mathcal{F}, \tag{6}$$

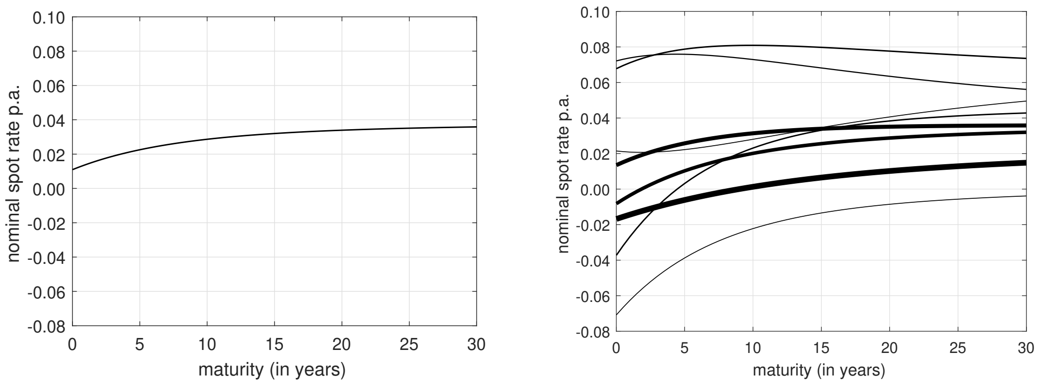
where $S_{i,0} = 1$ without loss of generality.

Table 3 – Standard deviations and cross correlations of residuals of the VAR(1) model.

	r_t	π_t	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$	$\tilde{\beta}_{1,t}$	$\tilde{\beta}_{2,t}$	$\tilde{\beta}_{3,t}$
$\sqrt{diag(\Sigma_z)}$	0.0442	0.0040	0.0046	0.0063	0.0135	0.0025	0.0055	0.0084
r_t	1.0000	-0.0192	0.1104	-0.0786	-0.1136	-0.0587	-0.0387	-0.0261
π_t		1.0000	0.0161	0.0191	0.0665	0.0545	-0.0086	-0.0378
$\beta_{1,t}$			1.0000	-0.3770	-0.3500	0.3960	-0.0562	-0.1534
$\beta_{2,t}$				1.0000	0.5465	0.2576	0.2980	0.0509
$\beta_{3,t}$					1.0000	0.3653	0.0040	0.2172
$\tilde{\beta}_{1,t}$						1.0000	-0.0936	-0.4195
$\tilde{\beta}_{2,t}$							1.0000	-0.3501
$\tilde{\beta}_{3,t}$								1.0000

Table 4 – Unconditional expected mean μ_{zz} and standard deviation σ_{zz} of the VAR(1) model.

	r_t	π_t	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$	$\tilde{\beta}_{1,t}$	$\tilde{\beta}_{2,t}$	$\tilde{\beta}_{3,t}$
μ_{zz}	0.0022	0.0027	0.0392	-0.0283	0.0135	-0.0012	-0.0191	0.0210
σ_{zz}	0.0445	0.0043	0.0219	0.0396	0.0341	0.0293	0.0206	0.0231



(a) Steady state at initial stage

(b) After 5 years

Fig. 2 – Left panel (a) shows the initial nominal term structure starting from the steady state. Right panel (b) shows different realizations of the nominal term structure after 5 years (the thicker the line, the likelier the occurrence).

The gross return of the nominal bond fund with a maturity of M years over a holding period of length Δt from time $t - \Delta t$ to t is approximated by

$$R_{B,t} \;=\; M \cdot y(\boldsymbol{\beta}_{t-\Delta t}, M, \lambda) \;-\; (M - \Delta t) \cdot y(\boldsymbol{\beta}_t, M - \Delta t, \lambda). \tag{7}$$

The function $y(\boldsymbol{\beta}_t, M, \lambda)$ denotes the M-year nominal spot rate at time t, determined by the Nelson-Siegel term structure model with a vector $\boldsymbol{\beta}_t$ of parameters and a scale parameter λ to be specified shortly.

In a similar fashion, the gross return of the inflation-linked bond fund with a maturity of M years over a holding period of length Δt from time $t-\Delta t$ to t is approximated by

$$R_{\tilde{B},t} \;=\; M \cdot y(\widetilde{\boldsymbol{\beta}}_{t-\Delta t}, M, \widetilde{\lambda}) \;-\; (M-\Delta t) \cdot y(\widetilde{\boldsymbol{\beta}}_t, M-\Delta t, \widetilde{\lambda}) \;+\; \Pi_t, \tag{8}$$

where Π_t denotes the gross rate of inflation between $t-\Delta t$ and t. The M-year real spot rate at time t is also denoted by the function y, but with parameters $\widetilde{\boldsymbol{\beta}}_t$ and $\widetilde{\lambda}$.

The gross interest rate on the cash fund from time $t-\Delta t$ to t is defined simply by changing bond maturity M in Eq. (7) to Δt. The cash fund return is given by

$$R_{C,t} \;=\; \Delta t \cdot y(\boldsymbol{\beta}_{t-\Delta t}, \Delta t, \lambda). \tag{9}$$

Naturally, the cash return at time t does not depend on the current nominal spot rate $y(\boldsymbol{\beta}_t, \Delta t, \lambda)$ at time t, but on the past spot rate $y(\boldsymbol{\beta}_{t-\Delta t}, \Delta t, \lambda)$. The price dynamics of the cash and bond funds are obtained by substituting $R_{i,t}$ from Eq. (7), Eq. (8) and Eq. (9) into Eq. (6).

The fair actuarial price of a nominal deferred annuity contract which pays £1 in nominal terms and in every year of retirement is

$$S_{A,t} \;=\; \sum_{s=T-t}^{\tau-t} {}_s p_{\delta+t} \cdot \exp\left(-s \cdot y(\boldsymbol{\beta}_t, s, \lambda)\right). \tag{10}$$

If $t=T$ in Eq. (10) above, then the annuity is of course an immediate annuity. Eq. (10) above holds verbatim for an inflation-protected deferred annuity, except that A is replaced by $\widetilde{A}$ and the yield is $y(\widetilde{\boldsymbol{\beta}}_t, s, \widetilde{\lambda})$. We assume static pricing mortality rates here, and we also ignore loading factors (expenses).

4.4. Labour income

We model exogenous labour income in accordance with Viceira (2001), Cocco et al. (2005) and Blake et al. (2007). An individual investor cannot change the amount of labour supply during his working lifetime $[0,T)$. Real labour income $\widetilde{L}_t$ at time t is determined by three components: a deterministic growth function f of age $\delta+t$, a permanent shock υ_t and a temporary shock ε_t. The deterministic function f is a polynomial, which can match the hump shape of the age profile of real wages. The permanent labour income shock υ_t can be either independent of, or correlated with, equity returns. Nominal labour income L_t at time t is the real labour income $\widetilde{L}_t$ inflated by two components: price inflation Π_t and real wage inflation G_t.

The real and nominal labour income processes of the investor are therefore given, respectively, by:

$$\widetilde{L}_t \;=\; \widetilde{L}_{t-\Delta t} \exp\left(f(\delta+t) - f(\delta+t-\Delta t) + \upsilon_t + \varepsilon_t - \varepsilon_{t-\Delta t}\right), \tag{11}$$

$$L_t \;=\; \widetilde{L}_t \exp\left(\textstyle\sum_1^t (\Pi_s + G_s)\right), \tag{12}$$

for $t \in (0,T)$. Here, $L_0 = \widetilde{L}_0$ w.p. 1, $\upsilon_t \sim$ i.i.d. $\mathrm{N}(0, \sigma_\upsilon^2 \Delta t)$ and $\varepsilon_t \sim$ i.i.d. $\mathrm{N}(0, \sigma_\varepsilon^2)$.

Table 5 – Percentiles of the nominal and real spot rates at retirement, for different maturities.

	Percentile	5y	10y	15y	20y	25y	30y
	0.05	-0.0563	-0.0406	-0.0302	-0.0233	-0.0182	-0.0148
nominal	0.50	0.0332	0.0381	0.0407	0.0420	0.0426	0.0430
	0.95	0.1212	0.1160	0.1109	0.1065	0.1030	0.1002
	0.05	-0.0550	-0.0473	-0.0458	-0.0460	-0.0464	-0.0468
real	0.50	0.0003	0.0042	0.0052	0.0054	0.0054	0.0054
	0.95	0.0557	0.0556	0.0560	0.0567	0.0573	0.0576

4.5. Prices and yield curves on the scenario tree

The financial model set out above is used to generate the hybrid scenario structure as described in sections 3.3 and 3.4. The ten state variables stored at each node are $[R_n, \Pi_n, r_n, \pi_n, \beta_{1,n}, \beta_{2,n}, \beta_{3,n}, \widetilde{\beta}_{1,n}, \widetilde{\beta}_{2,n}, \widetilde{\beta}_{3,n}]$, employing the same notation as before except that we index by node n. Asset prices can be evaluated at each node of the scenario tree. For example, the asset price in Eq. (6) is transformed into the nodal form simply by replacing t with n and $t - \Delta t$ with n^- as follows:

$$S_{i,n} \;=\; S_{i,n^-} \cdot \exp(R_{i,n}), \qquad \text{for } n \in \{\mathcal{N}_t, t \in (0,T]\} \text{ and } i \in \mathcal{F},$$

with $S_{i,n_0} = 1$. Pricing formulas for other assets are transformed similarly.

It is useful to consider the yield curves generated by the scenario tree. Percentiles of the nominal and real spot rates for different maturities at retirement date T, as calculated from the generated scenarios, are shown in Table 5. It is clear that the generated nominal yield curve is more volatile at the short end, than at the long end, consistent with empirical evidence. This is also evident from the right hand panel of Fig. 2, which shows eight different realizations of the nominal yield curve, with a greater spread at the short end than at the long end.

5. Numerical Results

In this section, we solve the investment and deferred annuitization problem for an individual who can invest in equities, cash, inflation-indexed and nominal bonds, and who can buy both inflation-protected and nominal annuities. We investigate particularly the welfare enhancement potentially conferred by inflation-protected deferred annuities in the presence of labour income risk.

5.1. Numerical example

Consider a 40-year-old individual ($\delta = 40$) who intends to retire at age 65 ($T = 25$).[5] His aim is to maximize and secure his retirement benefits *in real terms* and to set aside a portion of his portfolio as a bequest, also *in real terms*, if he dies before retirement. His personal retirement plan permits him to invest in an equity fund, a nominal bond fund (maturity $M = 20$ years), an inflation-linked bond fund (maturity $M = 20$ years), a cash fund (maturity $M = 5$ years), nominal deferred annuities, and inflation-protected deferred annuities as described in section 2.

In the base case, the VAR asset return model with Nelson-Siegel real and nominal yield curves is based on U.K. data from January 1985 to June 2017, as described in section 4.2. Mean, standard deviations and cross-correlations of key variables from the model are shown in the online supplementary section, Table S-1. A U.K. mortality table based on 2000–2006 experience[6] is used to price the deferred annuity.

The individual can rebalance his portfolio and buy deferred annuities every 5 years ($\Delta t = 5$) until retirement. There are therefore six stages (five periods) in the scenario tree part of the hybrid scenario structure. At time 0, the individual has wealth of $w_0 =$ £80,000 in his retirement plan. His annual wage starts at £40,000, so $L_0 = \widetilde{L}_0 = 40,000$ in Eq. (11) and Eq. (12), or equivalently $L_{n_0} = \widetilde{L}_{n_0} = 40,000$ in Eq. (S.5) and Eq. (S.6). He contributes ten per cent ($\phi = 10\%$) of his nominal labour income to his retirement plan.

Average real labour income with age typically exhibits a concave shape unless deflation occurs. National statistics show that the average real wage tends to decline after age 40–50 (Office for National Statistics, 2015). We employ a deterministic quadratic function for f in the real labour income mode of Eq. (11), and use the parameter estimates of Blake et al. (2007)[7].

5.2. Optimal annuitization and investment without risk to real labour income

First, we consider the case where there is no real labour income uncertainty: there are no stochastic shocks to real labour income, $\sigma_v = \sigma_\varepsilon \equiv 0$ in Eq. (11).

Our numerical results show that the optimal strategy to secure retirement income leads to fairly early and continual purchase of deferred annuities over the working lifetime of the investor. This is illustrated in Fig. 3, which shows average and percentile values of annual retirement income accumulated in real terms from holdings of nominal deferred

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[5] In the numerical results that follow, we assume that the individual has a CRRA (power) utility, as defined in section 2, with representative relative risk aversion coefficient γ of 3 or 5, and time impatience ρ of 0.02 or 0.04. This is in common with other long-term portfolio and annuity studies, e.g. Huang et al. (2017), Horneff et al. (2010), Koijen et al. (2011) and Viceira (2001).

[6] Institute and Faculty of Actuaries, S1PML/S1PFL—All pensioners (excluding dependants), male/female lives `www.actuaries.org.uk/research-and-resources/documents/s1pml-all-pensioners-excluding-dependants-male-lives`(www.actuaries.org.uk).

[7] The real labour income function scaled to 1 at the final age of 60 is $w(y) = 0.5963 + 2.3708y - 1.9671y^2$, where $y = (x - 20)/(60 - 20)$, $x = 20, 21, \ldots, 60$. The coefficients, estimated from the New Earnings Survey of the Office for National Statistics in 1998, are for a male all-occupation group. For details, refer to Blake et al. (2007)

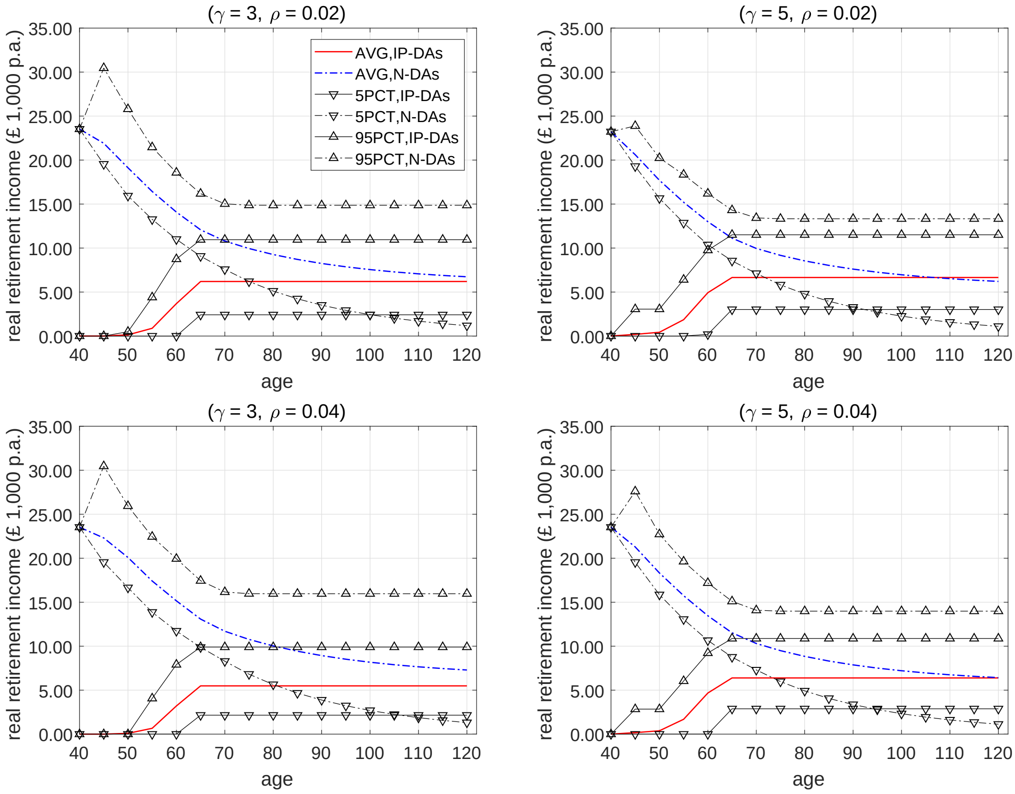


Fig. 3 – Average and percentiles of optimal secured retirement income ($\times$£1,000 p.a.) through either inflation-protected deferred annuities (IP-DAs) or nominal deferred annuities (N-DAs) for different coefficients of risk aversion γ and time preference ρ, in the absence of labour income risk and bequest motive. All values are presented in real terms.

annuities (N-DAs) and inflation-protected deferred annuities (IP-DAs), at various ages. From age 65 onwards, the annual annuity payouts, in real terms, are shown and these are constant for the infation-protected annuity but declining for the nominal annuity. The four panels represents four different individuals with different risk aversion and time preferences. There is no bequest motive ($\kappa = 0.$).

It is clear from Fig. 3 that nominal annuities are purchased at earlier ages than inflation-protected annuities. This is because a younger investor implicitly holds larger human capital than an older one *ceteris paribus*, labour income is perfectly correlated with inflation here, so the nominal annuity is riskier than the inflation-protected annuity, and at younger ages more risk can be taken. We also observe, by comparing the panels on the right to the ones on the left in Fig. 3, that a more risk-averse investor buys less nominal annuities and more inflation-protected annuities, the latter being less risky than the former in real terms. Comparing the bottom panels to the top ones in Fig. 3, we see that the more impatient an investor, the more he purchases nominal annuities. This is sensible since cash flows from nominal annuities decline in real terms, satisfying the more impatient investor.

Average optimal asset allocations over period till retirement are presented in Fig. 4. The bar graphs in the first and third rows show average asset allocations including

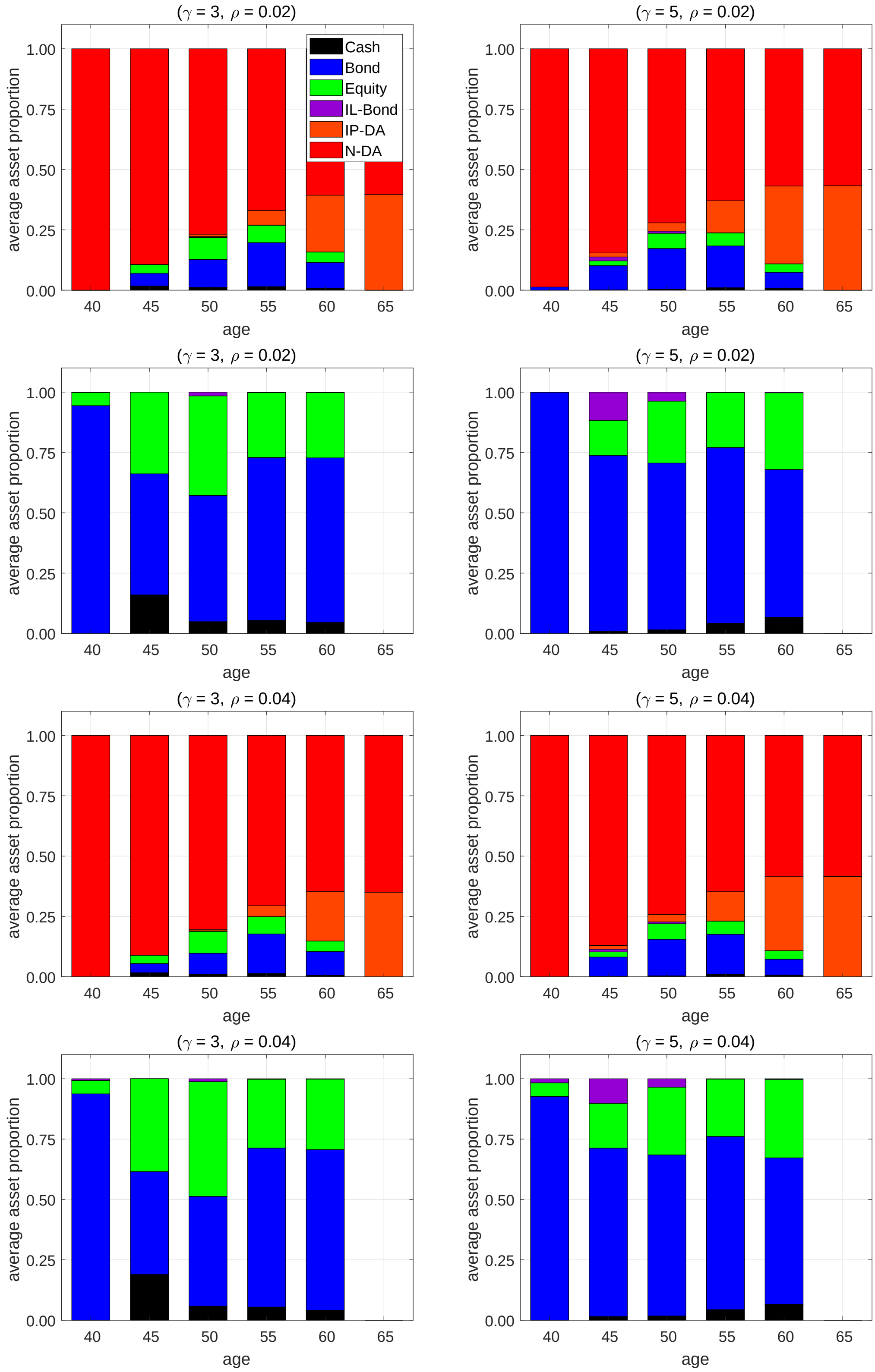
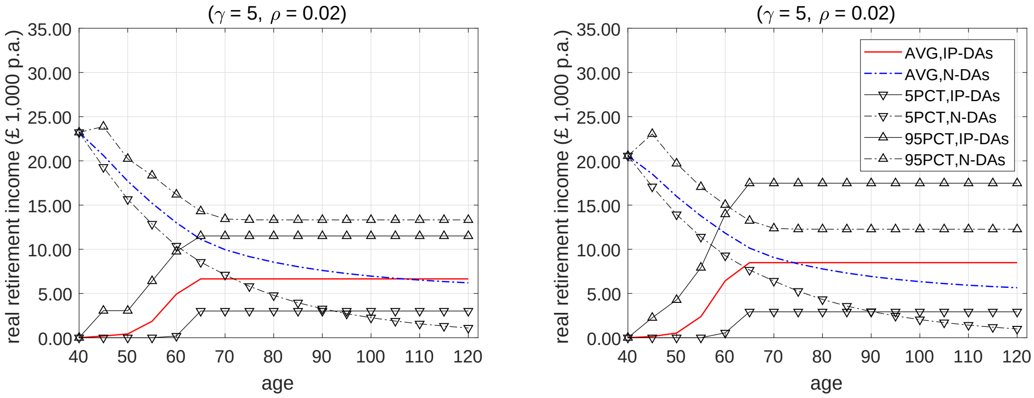


Fig. 4 – Optimal investment and deferred annuity proportions on average at different ages till retirement for different coefficients of risk aversion γ and time preference ρ, in the absence of labour income risk and bequest motive. The second and fourth rows show proportions of financial assets only. IL-Bond is inflation-linked bond, IP-DA is inflation-protected deferred annuity, N-DA is nominal deferred annuity.



(a) No labour income risk

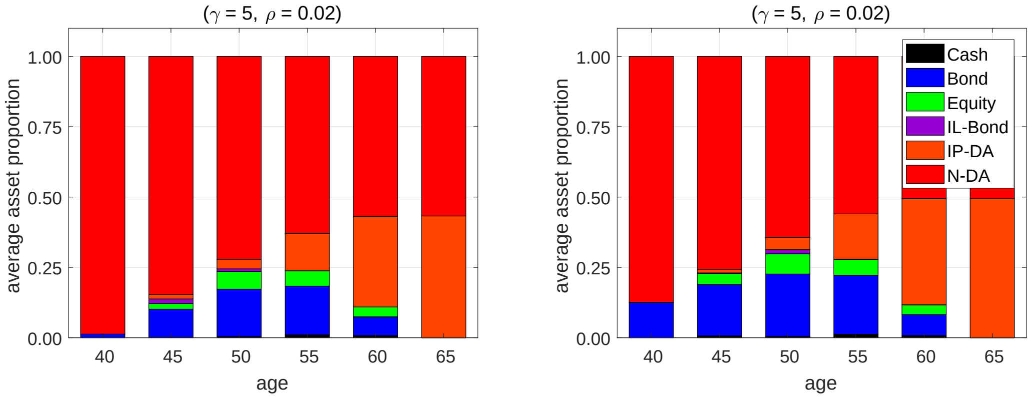
(b) With labour income risk

Fig. 5 – Average and percentiles of optimal secured retirement income ($\times$£1,000 p.a.) through either inflation-protected deferred annuities (IP-DAs) or nominal deferred annuities (N-DAs) in the absence of labour income risk (left panel) and with labour income risk (right panel). Risk aversion $\gamma = 5$ and time preference $\rho = 0.02$ are the same. All values are presented in real terms. There is no bequest motive.

deferred annuities, and those in the second and fourth rows concentrate on financial assets only, excluding deferred annuities. At retirement (age 65), only annuities are held. Comparing the right-hand panels to the left-hand panels, we find that equity allocation is lower, bond allocation higher, and annuity allocation marginally higher, the greater the investor's risk-aversion is.

A key result from Fig. 4 is the dominance of deferred annuity holdings over financial asset holdings. This is a function of the historical data, 1985–2017, that we used to parameterize our financial market model and price annuities. Long-term rates fell during this period, meaning that annuities were cheap at the start, and our model rightly favours annuities over financial assets, on average.

We also find that nominal deferred annuities dominate their inflation-protected cousins, and that nominal bonds dominate inflation-linked bonds. This initially surprising result follows from the fact that long-term real yields were on average negative over this historical period (the mean of $\widetilde{\beta}_{1,t}$ in online supplementary section Table S-1 is -0.12%) so inflation-indexed securities and products were expensive. Over this historical period, the long-term nominal yield is highly correlated with the long-term real yield (the correlation of $\beta_{1,t}$ with $\widetilde{\beta}_{1,t}$ in online supplementary section Table S-1 is 0.7319). Investing in the long-term nominal bond fund thus helps to hedge price changes in not only nominal but also inflation-protected annuities to secure real retirement income. The average long-term nominal rate is 3.92% (the mean of $\beta_{1,t}$ in online supplementary section Table S-1), which is higher than the average annual equity return (the average of monthly return r_t in online supplementary section Table S-1 is 0.22% accumulating to 2.63% p.a.), explaining the preference of nominal bonds over equities.



(a) No labour income risk (b) With labour income risk

Fig. 6 – Optimal investment and deferred annuity proportions on average at different ages till retirement in the absence of labour income risk (left panel) and with labour income risk (right panel). Risk aversion $\gamma = 5$ and time preference $\rho = 0.02$ are the same. There is no bequest motive. IL-Bond is inflation-linked bond, IP-DA is inflation-protected deferred annuity, N-DA is nominal deferred annuity.

5.3. Optimal annuitization and investment with risk to real labour income

Next, we consider the case where there are risks to real labour income. We use the parameterization of Cocco et al. (2005), based on income data for a college-educated worker, for the volatility of permanent and temporary shocks to real labour income: $\sigma_\upsilon^2 = 0.02$ and $\sigma_\varepsilon^2 = 0.05$ in Eq. (11).

Fig. 5 compares secured retirement income without and with labour income risk. More retirement income is received from the inflation-protected annuities than from the nominal annuities when labour income risk is present. Fig. 6 compares the average allocations in the two cases. Allocations to inflation-protected annuities and to bonds are marginally higher in the presence of labour income risk than in its absence. These results are as anticipated: implicit wealth, including human capital, is riskier if the investor has risky real labour income than if he does not, so in the first case he should choose less risky investment and annuity strategies in terms of achieving real retirement income.

5.4. Investigating the dynamic solution

The optimal investment and annuity strategies are, of course, dynamic. In this section, we investigate the dynamic strategy by considering the decisions made on specific scenario nodes. We demonstrate how the strategy is explained by realized asset prices on a node and expected asset returns on descendant nodes. The model parameters and assumptions are the same as in the case with labour income risk in subsection 5.3.

Table 6 shows optimal investment and deferred annuity decisions on the root node at age 40 and on selected nodes at age 45, so that we can compare different scenarios. Case (a) in Table 6 shows that, at age 40, the optimal decision is to buy 12.55 units of the bond fund and 20.57 units of nominal deferred annuity. The prices of the cash, bond, equity, and inflation-linked bond fund units are £1,000. The prices of the nominal and

Table 6 – Optimal investment and deferred annuity choices with labour income risk at ages 40 and 45 under different scenarios. Risk aversion and time preference are $\gamma = 5$ and $\rho = 0.02$. No bequest motive.

Age	Node	Conditional expected 5-year returns [risk premium in brackets]				Net purchases of units[a] (prices in parentheses)					
		Cash	Bond	Equity	IL-Bond	Cash	Bond	Equity	IL-Bond	IP-DA[b]	N-DA[b]
40	(*a*) root node	0.113	0.266	0.134	0.199	0.000	12.555	0.000	0.000	0.000	20.570
			[0.153]	[0.021]	[0.086]	(1,000)	(1,000)	(1,000)	(1,000)	(15,105)	(4,251)
45	(*b*) the lowest bond	0.394	0.556	0.369	0.351	0.000	31.653	0.000	0.000	0.000	9.379
	& N-DA prices		[0.163]	[-0.025]	[-0.043]	(1,113)	(596)	(2,485)	(889)	(6,703)	(1,816)
	(*c*) the largest	0.111	0.330	0.273	0.217	0.000	-8.360	0.000	0.000	0.000	10.796
	N-DA purchase		[0.219]	[0.162]	[0.106]	(1,113)	(1,172)	(800)	(1,310)	(12,644)	(3,740)
	(*d*) the largest cash	0.379	0.285	0.214	0.302	8.218	-12.555	0.000	0.000	2.659	0.000
	& IP-DA purchases		[-0.094]	[-0.165]	[-0.077]	(1,113)	(709)	(519)	(774)	(7,108)	(2,896)
	(*e*) the largest	0.129	0.239	0.125	0.205	0.000	34.029	0.000	0.000	0.000	0.000
	bond purchase		[0.111]	[-0.004]	[0.076]	(1,113)	(1,184)	(949)	(1,149)	(14,143)	(5,262)
	(*f*) the largest	-0.194	-0.203	-0.139	-0.140	0.000	-12.555	38.409	8.743	0.000	0.000
	equity purchase		[-0.009]	[0.055]	[0.054]	(1,113)	(2,412)	(800)	(1,959)	(44,020)	(16,742)

[a] Units in funds, or units of deferred annuities. A negative number means that there is a net sale.
[b] N-DA (IP-DA)= nominal (inflation-protected) deferred annuity. One unit of N-DA (IP-DA) is equivalent to £1,000 p.a. of secured nominal (real) retirement income.

inflation-protected deferred annuities are £4,251 and £15,105 for £1,000 p.a. of nominal and real retirement income, respectively.

Case (*b*) in Table 6 displays the scenario at age 45 with the lowest prices of nominal bond and nominal deferred annuity among all the scenarios at age 45. The lowest price of the nominal deferred annuity leads the risk-averse investor optimally to buy 9.379 units of this annuity. He also buys 31.653 units of the nominal bond fund because bond returns will mean-revert and this bond fund has an expected risk premium over cash of 16.3% over the next 5 years till age 50. All other financial assets are expected to have negative risk premiums, and are not purchased.

Case (*c*) in Table 6 presents the scenario at age 45 where the most units of nominal deferred annuity are bought. The price of this annuity has fallen (£3,740 compared to £4,251 in case (*a*)) whilst the price of the nominal bond fund fund has increased (£1,172, up from £1,000 in case (*a*)). The investor therefore sells about two thirds of his original holding of the bond fund at age 40 to buy nominal deferred annuities thereby securing £10,796 p.a. of nominal retirement income.

Case (*d*) refers to the scenario where the most units of the cash fund and inflation-protected deferred annuities are purchased. Cash is expected to return 37.9% over the next 5 years whereas the other assets are expected to return less, hence the large investment in the cash fund. The inflation-protected deferred annuity is cheap, having more than halved in price from case (*a*) from £15,105 to £7,108, encouraging the investor to secure real retirement income of £2,659 p.a. Similarly, the largest purchase of nominal bond in case (*e*) may be explained by its relatively cheap price and by its high expected return, with an expected premium of 11.1% over cash over the next 5 years. On the other hand, equity is the only asset whose price falls from case (*a*) to the scenario in case (*f*), and it also has the highest expected risk premium among the financial assets. Unsurprisingly, case (*f*) is the scenario at age 45 with the largest equity purchase.

Table 7 – Certainty equivalent values (£ p.a.) when the availability of different types of annuity is restricted, for different coefficients of risk aversion γ and time preference ρ.

Preferences		Availability					
γ	ρ	**IP+N DA+IA**	**IP DA+IA**	**N DA+IA**	**IP+N IA**	**IP IA**	**N IA**
3	0.02	14,396.83	9,316.08	13,801.32	12,567.78	8,911.84	8,424.12
		(0.00%)	(-35.29%)	(-4.14%)	(-12.70%)	(-38.10%)	(-41.49%)
3	0.04	20,551.62	13,025.18	20,003.43	17,751.67	12,459.99	12,206.63
		(0.00%)	(-36.62%)	(-2.67%)	(-13.62%)	(-39.37%)	(-40.60%)
5	0.02	13,227.98	10,180.07	11,990.06	11,250.87	9,458.91	8,250.79
		(0.00%)	(-23.04%)	(-9.36%)	(-14.95%)	(-28.49%)	(-37.63%)
5	0.04	15,976.37	12,037.21	14,785.08	13,391.44	11,184.49	10,167.68
		(0.00%)	(-24.66%)	(-7.46%)	(-16.18%)	(-29.99%)	(-36.36%)

IP+N/DA+IA: both inflation-protected and nominal annuities are available, in both deferred and immediate versions; **IP/DA+IA**: only inflation-protected annuities are available, both deferred and immediate; **N/DA+IA**: only nominal annuities are available, both deferred and immediate; **IP+N/IA**: both nominal and inflation-protected annuities are available, but only immediate annuities; **IP/IA**: only inflation-protected immediate annuities are available; **N/IA**: only nominal immediate annuities are available.

The dynamic optimal annuity and investment strategy is therefore explained by current asset prices and expected returns. The predictability of asset returns, in the VAR model fitted to data in section 4, is exploited by the multi-stage stochastic programming model, and the optimal strategy dynamically responds to market changes.

5.5. Availability of deferred annuities

We wish to compare the situations where the availability of different types of annuities is restricted. We consider six possible situations depending upon which feature is unavailable: nominal or inflation-protected payments, deferred or immediate payout. To compare these situations, we calculate a certainty equivalent (CE) value, which is the level of constant real retirement income which generates a utility equal to the maximized expected utility, in each of these situations. We use the numerical example in section 5.3, i.e. with labour income risk.

The results are tabulated in Table 7. The worst CE value is recorded when only nominal immediate annuities are available irrespective of the investor's risk and time preferences (last column of Table 7 labelled N/IA). Unsurprisingly, when there is utmost flexibility and all types of annuities are available, the highest CE value is recorded (third column of Table 7 labelled IP+N/DA+IA), but this is about 60% better in terms of equivalent real retirement income than the worst case. This suggests that individual investors' welfare is severely impaired in the current U.K. markets where the most common distribution strategies offered are nominal immediate annuities and income drawdown.

An interesting finding is that inflation-protected immediate annuities enhance welfare compared to their nominal cousins (compare the last two columns of Table 7) but that, when deferment is available, access to a nominal annuity is better than access to an inflation-protected annuity (compare the columns labelled IP/DA+IA and N/DA+IA).

As we saw in section 5.2, the average long-term real interest rate was slightly negative in the historical period (1985–2017) on which our model is calibrated, making inflation-protected deferred annuities very expensive.

These results show the importance of the availability of deferred annuities and also that, for long deferment periods, providing a nominal deferred annuity may be better than an expensive inflation-protected deferred annuity.

5.6. Model validation and results over different historical periods

In order to validate our results, we generate several scenario structures to verify that our optimal solutions do not vary significantly both qualitatively and quantitatively. For any given historical period over which our model is parameterized, we only show the results on one scenario structure so that comparisons do not involve sampling error.[8] However, we also calibrate the scenario structure over different historical periods in order to test that our central conclusions hold, and we describe this here.

In the earlier sections, our financial market model was parameterized on U.K. data in the period 1985–2017. Because of the negative average long-term real yield, and the steep upward sloping nominal yield curve, nominal bonds and nominal deferred annuities played a significant role in the average asset allocation. In this section, we calibrate our model on different historical sub-periods to examine if investment and annuity decisions change. Average allocations are shown in Fig. 7 for four different periods: the original Jan. 1985–Jun. 2017 period, Jan. 1985–Dec. 1997 (before the Asian financial crisis), Jan. 1997–Dec. 2007 (between the Asian and the global financial crisis), and Jan. 2008–Jun. 2017 (after the global financial crisis).

Average allocations are strikingly different over these different periods. First, this reminds us that our model produces *dynamic* asset allocation and annuitization decisions, but that only averages are shown in Fig. 7. Second, optimal portfolio decisions are sensitive to model calibration. Third, deferred annuities are bought early and in increasing amounts, on average, in all four periods in Fig. 7, and thus have a signficant role to play in retirement planning, irrespective of the economic cycle.

We note that our model produces sensible results in each sub-period considered. For example, in the 2008–17 sub-period (bottom right panel of Fig. 7), negative long-term real yields and near-zero nominal yields make deferred annuities and bonds expensive and unappealing, and the model takes advantage of the rising stock market with a large and falling equity allocation. In the 1985–97 sub-period (top right panel of Fig. 7), high short rates to combat inflation make cash attractive in the earlier years. This is discussed further in online supplementary section S-4.

[8] We thank a reviewer for suggesting that a number of scenario trees can be generated and used for formal validation purposes. For robustness, results can also be averaged over several scenario trees.

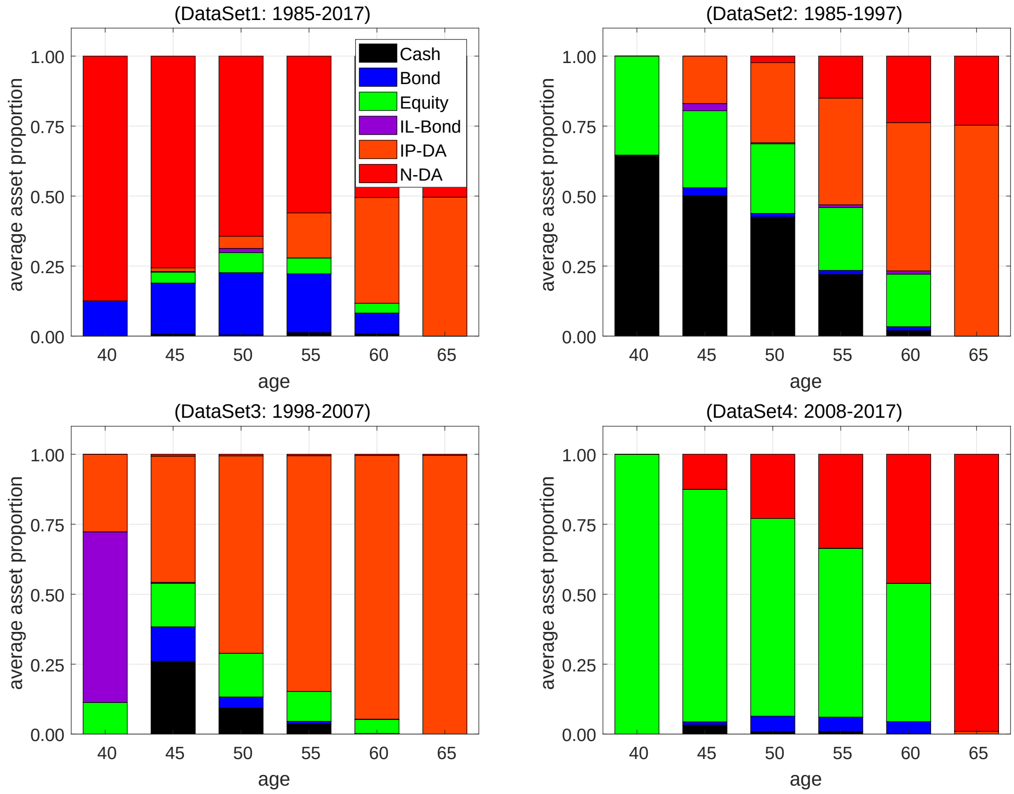


Fig. 7 – Optimal investment and deferred annuity allocations of overall wealth on average over the planning horizon and for different datasets. Constant relative risk aversion (γ), time preference (ρ), and bequest (κ) coefficients are $\gamma = 5$, $\rho = 0.02$, and $\kappa = 0.0$ respectively. Volatilities of the permanent and temporary shocks are 0.02 and 0.05 respectively.

6. Conclusion

We construct an optimal investment model with deferred annuities for an individual investor who is saving for retirement. The individual's labour income is uncertain and is subject to exogenous permanent and temporary shocks in addition to price inflation. Our results show that buying nominal and inflation-protected deferred annuities, continually and from an early age, is an optimal strategy when the objective is to maximize the expected utility of real retirement income and when the retirement income is secured by annuitization. The balance between inflation-protected and nominal deferred annuities depends on expectations of long-term real and nominal rates. In the presence of shocks to real labour income, inflation-protected deferred annuities are marginally preferred to nominal deferred annuities.

Optimal investment in financial assets (cash, nominal bond, inflation-linked bond and equity) also depends on financial market expectations, but portfolio riskiness typically declines with age, consistent with life-cycle models. If welfare is measured in terms of a certainty equivalent of real retirement income, we find that welfare falls by about 40% if deferred annuities are not available and only nominal immediate annuities are available at retirement. When we calibrate our model on different historical sub-periods, we find that our key result about the importance of early purchases of deferred annuities appears to be robust to different financial market expectations. The actual portfolio composition and annuity allocation vary, but the investment portfolio becomes less risky as retirement approaches.

Besides the results that we obtain in the context of retirement planning, we make some contributions in terms of modelling using multi-stage stochastic programming. We suggest a new hybrid scenario structure which combines a scenario tree in the pre-retirement phase with scenario fans in the post-retirement phase. We also implement an unconstrained non-linear program to generate shocks to real labour income which are independent of financial market scenarios. We also implement a scenario generation procedure that avoids arbitrage, using the methods of Klaassen (2002) and Geyer et al. (2014).

Future work will address some of the limitations of our model. Jurisdiction-specific taxes and transaction costs can be implemented in the model to enhance its practical usefulness. Retirement provision also takes place through employer-sponsored institutional vehicles, which deserve further investigation (Consiglio et al., 2015). Flexibility in the labour supply, i.e. hours worked, affects contributions to the retirement plan and is not considered here. The effects of housing and mortgage costs, and different mortality rates for impaired lives or different population groups, are also ignored. Power utility does not capture the elasticity of intertemporal substitution in consumption and an Epstein-Zin utility may be implemented. Other rational and behavioural factors such as habit formation and hyperbolic discounting can also be considered. Annuity products such as variable annuities can be added to the portfolio of annuities available. Life and health insurance can also affect optimal retirement planning. These extensions will be studied in subsequent work.

References

Barberis, N. (2000). Investing for the Long Run when Returns Are Predictable. *The Journal of Finance*, *55*, 225–264.

Benzoni, L., Collin-Dufresne, P., & Goldstein, R. S. (2007). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *The Journal of Finance*, *62*, 2123–2167.

Blake, D., Cairns, A., & Dowd, K. (2007). The impact of occupation and gender on pensions from defined contribution plans. *Geneva Papers on Risk and Insurance–Issues and Practice*, *32*, 458–482.

Boender, G., Dert, C., Heemskerk, F., & Hoek, H. (2008). Chapter 18 - A scenario approach of ALM. (pp. 829–860). San Diego: North-Holland.

Campbell, J. Y., Chan, Y. L., & Viceira, L. M. (2003). A multivariate model of strategic asset allocation. *Journal of Financial Economics*, *67*, 41–80.

Chen, A., Haberman, S., & Thomas, S. (2019). An overview of international deferred annuity markets. *SSRN Electronic Journal*, *62*, 2123–2167.

Cocco, J. F., Gomes, F. J., & Maenhout, P. J. (2005). Consumption and portfolio choice over the life cycle. *The Review of Financial Studies*, *18*, 491–533.

Consigli, G., Iaquinta, G., Moriggia, V., di Tria, M., & Musitelli, D. (2012). Retirement planning in individual asset-liability management. *IMA Journal of Management Mathematics*, *23*, 365–396.

Consiglio, A., Carollo, A., & Zenios, S. A. (2016). A parsimonious model for generating arbitrage-free scenario trees. *Quantitative Finance*, *16*, 201–212.

Consiglio, A., Tumminello, M., & Zenios, S. A. (2015). Designing and pricing guarantee options in defined contribution pension plans. *Insurance: Mathematics and Economics*, *65*, 267–279.

Dempster, M. A. H., & Medova, E. A. (2011). Asset liability management for individual households. *British Actuarial Journal*, *16*, 405–439.

Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields, . *130*, 337–364.

Donaldson, S., Kinniry, F., Maciulis, V., Patterson, A., & DiJoseph, M. (2015). *Vanguard's approach to target-date funds*. Technical Report Vanguard Valley Forge, PA. URL: `https://personal.vanguard.com/pdf/icrtdf.pdf`.

Dupačová, J., Consigli, G., & Wallace, S. W. (2000). Scenarios for Multistage Stochastic Programs. *Annals of Operations Research*, *100*, 25–53.

Ezra, D. (2016). Most People Need Longevity Insurance rather than an Immediate Annuity. *Financial Analysts Journal*, *72*, 23–29.

Ferstl, R., & Weissensteiner, A. (2011). Asset-liability management under time-varying investment opportunities. *Journal of Banking & Finance*, *35*, 182–192.

Geyer, A., Hanke, M., & Weissensteiner, A. (2010). No-arbitrage conditions, scenario trees, and multi-asset financial optimization. *European Journal of Operational Research*, *206*, 609–613.

Geyer, A., Hanke, M., & Weissensteiner, A. (2014). No-arbitrage bounds for financial scenarios. *European Journal of Operational Research*, *236*, 657–663.

Horneff, W. J., Maurer, R. H., Mitchell, O. S., & Dus, I. (2008). Following the rules: Integrating asset allocation and annuitization in retirement portfolios. *Insurance: Mathematics and Economics*, *42*, 396–408.

Horneff, W. J., Maurer, R. H., Mitchell, O. S., & Stamos, M. Z. (2009). Asset allocation and location over the life cycle with investment-linked survival-contingent payouts. *Journal of Banking & Finance*, *33*, 1688–1699.

Horneff, W. J., Maurer, R. H., & Rogalla, R. (2010). Dynamic portfolio choice with deferred annuities. *Journal of Banking & Finance*, *34*, 2652–2664.

Huang, H., Milevsky, M. A., & Young, V. R. (2017). Optimal Purchasing of Deferred Income Annuities When Payout Yields are Mean-Reverting. *Review of Finance*, *21*, 327–361.

Høyland, K., & Wallace, S. W. (2001). Generating scenario trees for multistage decision problems. *Management Science*, *47*, 295–307.

Klaassen, P. (2002). Comment on "Generating scenario trees for multistage decision problems". *Management Science*, *48*, 1512–1516.

Koijen, R. S. J., Nijman, T. E., & Werker, B. J. M. (2011). Optimal annuity risk management. *Review of Finance*, *15*, 799–833.

Konicz, A. K., & Mulvey, J. M. (2013). Applying a stochastic financial planning system to an individual: Immediate or deferred life annuities? *The Journal of Retirement*, *1*, 46–60.

Konicz, A. K., & Mulvey, J. M. (2015). Optimal savings management for individuals with defined contribution pension plans. *European Journal of Operational Research*, *243*, 233–247.

Konicz, A. K., Pisinger, D., & Weissensteiner, A. (2015). Optimal annuity portfolio under inflation risk. *Computational Management Science*, *12*, 461–488.

Konicz, A. K., Pisinger, D., & Weissensteiner, A. (2016). Optimal retirement planning with a focus on single and joint life annuities. *Quantitative Finance*, *16*, 275–295.

Maurer, R., Mitchell, O. S., Rogalla, R., & Kartashov, V. (2013). Lifecycle Portfolio Choice With Systematic Longevity Risk and Variable Investment-Linked Deferred Annuities. *Journal of Risk and Insurance*, *80*, 649–676.

Merton, R. C. (2014). The crisis in retirement planning. *Harvard Business Review*, *92*, 42–50.

Office for National Statistics (2015). Annual Survey of Hours and Earnings: 2015 Provisional Results. URL: `https://www.ons.gov.uk/employmentandlabourmarket/peopleinwork/earningsandworkinghours/bulletins/annualsurveyofhoursandearnings/2015provisionalresults`.

Scott, J. S. (2008). The longevity annuity: An annuity for everyone? *Financial Analysts Journal*, *64*, 40–48.

Shapiro, A., Dentcheva, D., & Ruszczynski, A. (2009). *Lectures on Stochastic Programming: Modeling and Theory*. Philadelphia: Society for Industrial and Applied Mathematics.

U.S. Treasury Department (2014). Treasury Issues Guidance to Encourage Annuities in 401(k) Plans. URL: `https://www.treasury.gov/press-center/press-releases/Pages/jl2673.aspx`.

Viceira, L. M. (2001). Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *The Journal of Finance*, *56*, 433–470.

Online Supplementary Material
"Optimal investment for a retirement plan with deferred annuities allowing for inflation and labour income risk"

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Abstract

This document provides online supplementary material for "Optimal investment for a retirement plan with deferred annuities allowing for inflation and labour income risk". A time representation of the optimization problem is given in section S-1 below. We provide further information on the parameterization of the scenario tree in section S-2. Scenario generation for labour income is discussed in section S-3. Further results, when our financial market model is parameterized using different historical periods, are stated in section S-4.

S-1. Time Representation of the Optimization Problem

The optimization problem presented in the paper in Eq. (1a)–Eq. (1j) is given in a nodal representation suitable for MSP. The equivalent equations in the more usual time representation are given here.

$$\max_{\{X_t, t\in[0,T]\}} \quad \mathbb{E}_0 \left[\sum_{t\in[T,\tau)} {}_tp_\delta \cdot u\left(t, I_t\right) + \sum_{t=[0,T)} {}_tp_\delta \cdot q_{\delta+t} \cdot \kappa^\gamma u\left(t+1, W_{t+1}\right) \right], \qquad \text{(S.1a)}$$

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$$
\begin{aligned}
\text{s.t.}\quad W_{t+1} &= W_t e^{-\Pi_{t+1}} + \exp\left(-\textstyle\sum_1^{t+1}\Pi_s\right)\Bigg[\phi\, L_t + \sum_{i\in\mathcal{F}}(S_{i,t+1}-S_{i,t})X_{i,t} \\
&\quad - S_{A,t}(X_{A,t}-X_{A,t-1}) - S_{\widetilde{A},t}\left(X_{\widetilde{A},t}-X_{\widetilde{A},t-1}e^{\Pi_t}\right)\Bigg] \text{ for } t\in[0,T), \qquad \text{(S.1b)}
\end{aligned}
$$

$$I_t = X_{\widetilde{A},T}\exp\left(-\textstyle\sum_1^T\Pi_u\right) + X_{A,T}\exp\left(-\textstyle\sum_1^t\Pi_u\right) \qquad \text{for } t\in[T,\tau), \qquad \text{(S.1c)}$$

$$X_{i,t} \geq 0 \qquad \text{for } i\in\mathcal{A}\cup\mathcal{F} \text{ and } t\in[0,T], \qquad \text{(S.1d)}$$

$$X_{A,t} \geq X_{A,t-1} \qquad \text{for } t\in(0,T], \qquad \text{(S.1e)}$$

$$X_{\widetilde{A},t} \geq X_{\widetilde{A},t-1}e^{\Pi_t} \qquad \text{for } t\in(0,T], \qquad \text{(S.1f)}$$

$$W_T = \left[S_{A,T}(X_{A,T}-X_{A,T-1}) + S_{\widetilde{A},T}\left(X_{\widetilde{A},T}-X_{\widetilde{A},T-1}e^{\Pi_T}\right)\right]\exp\left(-\textstyle\sum_1^T\Pi_t\right), \qquad \text{(S.1g)}$$

$$X_{i,T} = 0 \qquad \text{for } i\in\mathcal{F}, \qquad \text{(S.1h)}$$

$$W_t \geq 0 \qquad \text{for } t\in[0,T], \qquad \text{(S.1i)}$$

$$W_0 = w_0 \qquad \text{w.p. 1.} \qquad \text{(S.1j)}$$

In Eq. (S.1a) above, the decision variables over which the expected utility is maximized are the portfolio and annuity purchase decisions over the planning horizon $[0,T]$. The budget constraint, in Eq. (S.1b) above, displays the dynamics of real wealth W_t in the retirement plan, i.e. W_t is nominal wealth at time t deflated back to time 0 using the realized inflation rates $\{\Pi_s\}_1^t$. Nominal wealth is increased by contribution, which is a fixed proportion ϕ of nominal labour income L_t, as well as by price changes in financial assets (recall that $\mathcal{F}=\{E,B,\widetilde{B},C\}$ denotes the set of financial assets). Wealth is also reduced if there is are withdrawals to buy deferred annuities at time t.

Eq. (S.1c) shows the real retirement income I_t at time $t\in[T,\tau)$ during retirement, this being the nominal income from annuities deflated back to time 0. Recall that $X_{\widetilde{A},T}$ is the number of units of inflation-protected annuities held at retirement, and this number increases every year in line with price inflation. The constraint in Eq. (S.1d) means that short selling is not allowed. The constraint in Eq. (S.1e) means that units of nominal annuities cannot be sold, but new units can be bought. Similarly, the constraint in Eq. (S.1f) means that units of inflation-protected annuities cannot be sold, but new units can be bought, and in addition the number of units held increases in line with price inflation every year. The terminal conditions in Eq. (S.1g) and Eq. (S.1h) assert that, at retirement time T, all financial assets, except annuities, are sold off, and all wealth in the retirement plan is annuitized. Eq. (S.1i) ensures that wealth remains non-negative. The initial condition in Eq. (S.1j) states that the investor has a known initial wealth at time 0.

S-2. Scenario Generation: Moment Matching Method

Recall from section 3.3 that $\mathcal{N}$ is the set of all nodes in the structured scenarios; $\mathcal{N}_t$ is the set of nodes at time t. The scenario tree component of the structured scenarios consists of the nodes set $\{\mathcal{N}_t, t \in [0, T]\}$. The set of children nodes of node $\{\mathcal{N}_t, t \in [0, T)\}$ is denoted by $\{n^+\}$. Within the scenario tree component, the time interval between node n and its children nodes $\{n^+\}$ is Δt. At each node $n \in \mathcal{N}_t$, R_n is the equity log-return over a Δt-long time interval ending at time t (Eq. (6)); Π_n is the log inflation rate over a Δt-long time interval ending at time t (Eq. (8) and Eq. (1)); r_n is the equity log-return over a month ending at time t (Eq. (3)); π_n is the log inflation rate over a month ending at time t (Eq. (3)); and $\beta_{1,n}$, $\beta_{2,n}$, $\beta_{3,n}$, $\tilde{\beta}_{1,n}$, $\tilde{\beta}_{2,n}$ and $\tilde{\beta}_{3,n}$ are the Nelson-Siegel term structure parameters for nominal and real interest rates at time t (Eq. (7) to Eq. (10), and Eq. (3)).

The moment matching method (Høyland & Wallace, 2001; Klaassen, 2002) is used here for generating scenario trees of accumulated equity returns, accumulated inflation rates, three Nelson-Siegel parameters for real yield curves, and the other three Nelson-Siegel parameters for nominal yield curves. To be precise, we combine the sequential approach of Høyland & Wallace (2001) with the moment matching method.

A large multi-period tree consists of many small single-period sub-trees. The first-period sub-tree has a number of outcomes corresponding to each child node in the set $\{n_0^+\}$. The outcomes for the first-period sub-tree are obtained by matching the first four moments of the distributions of state variables from the financial model. For the second-period sub-trees, the conditional outcomes are obtained by matching the first four moments of the conditional distributions on outcomes of the first-period sub-tree. This procedure is executed sequentially for the third, fourth etc. sub-trees until the final-period sub-trees, whose outcomes are the nodes in $\mathcal{N}_T$. This ensures that all conditional distribution properties are matched fully throughout the whole scenario tree.

The scenario tree that we construct in our multi-stage stochastic programming problem has six stages. The time interval between the stages is Δt, so $T = 5\Delta t$. At each node n, we store the state variables $[R_n, \Pi_n, r_n, \pi_n, \beta_{1,n}, \beta_{2,n}, \beta_{3,n}, \widetilde{\beta}_{1,n}, \widetilde{\beta}_{2,n}, \widetilde{\beta}_{3,n}]$ employing the same notation as before except that we index by node n rather than by time. This means that, if node n occurs at time t, then R_n and Π_n denote equity log-return and inflation log-rate respectively over a Δt-long time interval ending at time t (Eq. (6) and Eq. (8)); r_n and π_n denote equity log-return and inflation log-rate respectively over a month ending at time t (Eq. (3)); and $\boldsymbol{\beta}_n = [\beta_{1,n}, \beta_{2,n}, \beta_{3,n}]$ and $\widetilde{\boldsymbol{\beta}}_n = [\widetilde{\beta}_{1,n}, \widetilde{\beta}_{2,n}, \widetilde{\beta}_{3,n}]$ denote the Nelson-Siegel term structure parameters for nominal and real spot rates respectively at time t (Eq. (2)). Initial state values at the root node n_0 are made equal to the estimated unconditional expected means laid out in Table 4. Every node in the scenario tree, except nodes at time T, branches off to eight children nodes. We require at least eight outcomes to perfectly match the first four moments of the ten state variables.

We derive the moments of the conditional distributions of the ten state variables $(R_{n^+}, \Pi_{n^+}, r_{n^+}, \pi_{n^+}, \beta_{1,n^+}, \beta_{2,n^+}, \beta_{3,n^+}, \tilde{\beta}_{1,n^+}, \tilde{\beta}_{2,n^+}, \tilde{\beta}_{3,n^+})$ on its unique parent node n. Let H be the number of monthly time steps of our vector autoregressive (VAR) model in Eq. (3) which matches the time interval Δt between the parent node and its children nodes in the scenario tree.

Eq. (S.2) and Eq. (S.3) show the first-two moments of the accumulated equity return, accumulated inflation rate, and six Nelson-Siegel parameters. Barberis (2000) and Pedersen et al. (2013) apply these equations to scenario generation for their asset-liability management models. Let $\zeta_{n^+} = \left[R_{n^+}, \Pi_{n^+}, \beta_{1,n^+}, \beta_{2,n^+}, \beta_{3,n^+}, \tilde{\beta}_{1,n^+}, \tilde{\beta}_{2,n^+}, \tilde{\beta}_{3,n^+}\right]'$ and $z_n = \left[r_n, \pi_n, \beta_{1,n}, \beta_{2,n}, \beta_{3,n}, \tilde{\beta}_{1,n}, \tilde{\beta}_{2,n}, \tilde{\beta}_{3,n}\right]'$, then we have the following equations:

$$\mathbb{E}(\zeta_{n^+}|z_n) = \left(\left(\sum_{h=1}^{H-1}(I + J(H-h))\,\Phi_1^{h-1}\right) + \Phi_1^{H-1}\right)\Phi_0 + \left(\Phi_1^H + \sum_{h=1}^{H-1} J\Phi_1^h\right) z_n \tag{S.2}$$

$$\begin{aligned}\mathbb{V}(\zeta_{n^+}|z_n) = {} & \Sigma_z + (J+\Phi_1)\,\Sigma_z\,(J+\Phi_1)' + \left(J + J\Phi_1 + \Phi_1^2\right)\Sigma_z\left(J + J\Phi_1 + \Phi_1^2\right)' \\ & + \cdots + \left(\Phi_1^{H-1} + \sum_{h=1}^{H-1} J\Phi_1^{h-1}\right)\Sigma_z\left(\Phi_1^{H-1} + \sum_{h=1}^{H-1} J\Phi_1^{h-1}\right)',\end{aligned} \tag{S.3}$$

where $J = diag([1,1,0,0,0,0,0,0])$. Note that Φ_0, Φ_1 and Σ_z are the coefficient and covariance matrices from the VAR model (Eq. (3)). If $J = 0$ (a 8×8 matrix of zeros), then Eq. (S.2) and Eq. (S.3) simplify to $\mathbb{E}(z_{n^+}|z_n)$ and $\mathbb{V}(z_{n^+}|z_n)$ respectively. In addition, we can evaluate the covariances: $\sigma_{R,r}$, $\sigma_{R,\pi}$, $\sigma_{\Pi,r}$, and $\sigma_{\Pi,\pi}$. For example, the covariance $\sigma_{R,r}$ between R_{n^+} and r_{n^+} follows:

$$\begin{aligned}\sigma_{R,r} = {} & J^{(1)}\Sigma_z J^{(1)'} + \Phi_1^{(1)}\Sigma_z\left(J^{(1)} + \Phi_1^{(1)}\right)' + \left(\Phi_1^{(1)}\Phi_1\right)\Sigma_z\left(J^{(1)} + \Phi_1^{(1)} + \Phi_1^{(1)}\Phi_1\right)' \\ & + \cdots + \left(\Phi_1^{(1)}\Phi_1^{H-2}\right)\Sigma_z\left(J^{(1)} + \sum_{h=2}^{H}\Phi_1^{(1)}\Phi_1^{H-h}\right)',\end{aligned} \tag{S.4}$$

where $\Phi_1^{(1)}$ is the first row of Φ_1 and $J^{(1)}$ is the first row of J.

Aggregating the moments information from Eq. (S.2) to Eq. (S.4), the conditional expectations and covariances of the ten state variables (R_{n^+}, Π_{n^+}, r_{n^+}, π_{n^+}, β_{1,n^+}, β_{2,n^+}, β_{3,n^+}, $\tilde{\beta}_{1,n^+}$, $\tilde{\beta}_{2,n^+}$, $\tilde{\beta}_{3,n^+}$) from the unique parent node n can be evaluated. These are used, as described in section 3.3, to generate the scenario tree component of the hybrid scenario structure. Table S-1 shows the conditional expectations, standard deviations and correlation coefficients of the ten state variables from the root node n_0, whose state values are given as the unconditional expected mean μ_{zz} in Table 4.

We verify how well our scenario tree captures the financial market model described in section 4 by comparing unconditional cumulative distributions of some of the variables generated on the scenario tree together with the actual cumulative distributions from the financial market model: see Fig. S-1. Although the scenario tree discretizes both the time set and the state space of variables, it replicates the distributions very closely, particularly in the later stages.

Table S-1 – Mean, standard deviations and cross correlations of financial variables Jan 1985–Jun 2017.

	R_t	Π_t	r_t	π_t	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$	$\tilde{\beta}_{1,t}$	$\tilde{\beta}_{2,t}$	$\tilde{\beta}_{3,t}$
$\mathbb{E}$	0.1317	0.1602	0.0022	0.0027	0.0392	-0.0283	0.0135	-0.0012	-0.0191	0.0210
$\sqrt{\mathbb{V}}$	0.3717	0.0586	0.0444	0.0043	0.0153	0.0296	0.0315	0.0163	0.0190	0.0204
R_t	1.0000	0.2332	0.1354	0.0324	0.1298	0.0095	-0.1362	0.0352	-0.1305	-0.0639
Π_t		1.0000	0.0388	0.2010	0.3571	0.4278	-0.0383	0.4091	0.1041	-0.1926
r_t			1.0000	-0.0120	0.1046	-0.0257	-0.0654	0.0407	-0.0484	-0.0398
π_t				1.0000	0.0993	0.2135	0.0554	0.1470	-0.1193	-0.0194
$\beta_{1,t}$					1.0000	0.1802	-0.0083	0.7319	-0.1772	-0.1432
$\beta_{2,t}$						1.0000	0.2242	0.6196	0.4923	-0.3369
$\beta_{3,t}$							1.0000	0.3791	-0.0812	0.2356
$\tilde{\beta}_{1,t}$								1.0000	0.0676	-0.3526
$\tilde{\beta}_{2,t}$									1.0000	-0.4440
$\tilde{\beta}_{3,t}$										1.0000

Monthly data of FTSE 100 from Bloomberg, retail price index from the Office for National Statistics, fitted nominal and real yield curves from the Bank of England are used from January 1985 to June 2017 ($\lambda = 0.1519$ and $\tilde{\lambda} = 0.2508$ for the Nelson-Siegel yield nominal and real curves model).

S-3. Scenario Generation for Labour Income

The labour income model given by Eq. (11) and Eq. (12) in Subsection 4.4 may be transformed into a nodal form with inflation Π_n already generated from the financial scenarios. For the sake of simplicity, we assume henceforth that real wage inflation $G_n = G$ at every node, i.e. it is constant. Then, real and nominal labour income scenarios follow

$$\widetilde{L}_n = \widetilde{L}_{n^-} \cdot \exp\left(f(\delta + t_n) - f(\delta + t_{n^-}) + \upsilon_n + \varepsilon_n - \varepsilon_{n^-}\right), \tag{S.5}$$

$$L_n = \widetilde{L}_n \cdot \exp\left(\textstyle\sum_{m \in s_n} (\Pi_m + G)\right), \tag{S.6}$$

for $n \in \{\mathcal{N}_t, t \in (0,T)\}$ and an initial value $\tilde{L}_{n_0} = L_{n_0}$ w.p. 1 at the root node. In equation Eq. (S.5) above, t_n is the time at which the node n belongs. And in equation Eq. (S.6) above, s_n is the set of all predecessor nodes of node n, i.e. $\{n, n^-, \ldots, n_0\}$. Evidently from Eq. (S.5), the temporary real labour income shock ε_{n^-} at the parent node n^- is removed from the real labour income at the current node n, while its permanent shock υ_{n^-} persists through $\widetilde{L}_{n^-}$.

Visiting each node $n \in \bigcup_{t\in[0,T)} \mathcal{N}_t$, we generate the permanent shocks υ_{n^+} and temporary shocks ε_{n^+}. Note that υ_{n^+} and ε_{n^+} are both column vectors with eight rows, since each node branches off to eight child nodes in the scenario fan part of the hybrid scenario structure. Scenarios of the independent shocks are calculated by solving an unconstrained non-linear optimization problem numerically with residuals e_{n^+} of financial asset returns:

$$\underset{\{\upsilon_{n^+}, \varepsilon_{n^+}\}}{\arg\min} \sum_{x \in \{\upsilon_{n^+}, \varepsilon_{n^+}\}} \left(o(x, e_{n^+}, pr_{n^+})^2 + \sum_{m=1,\ldots,4} g(x, m, pr_{n^+})^2 \right), \tag{S.7}$$

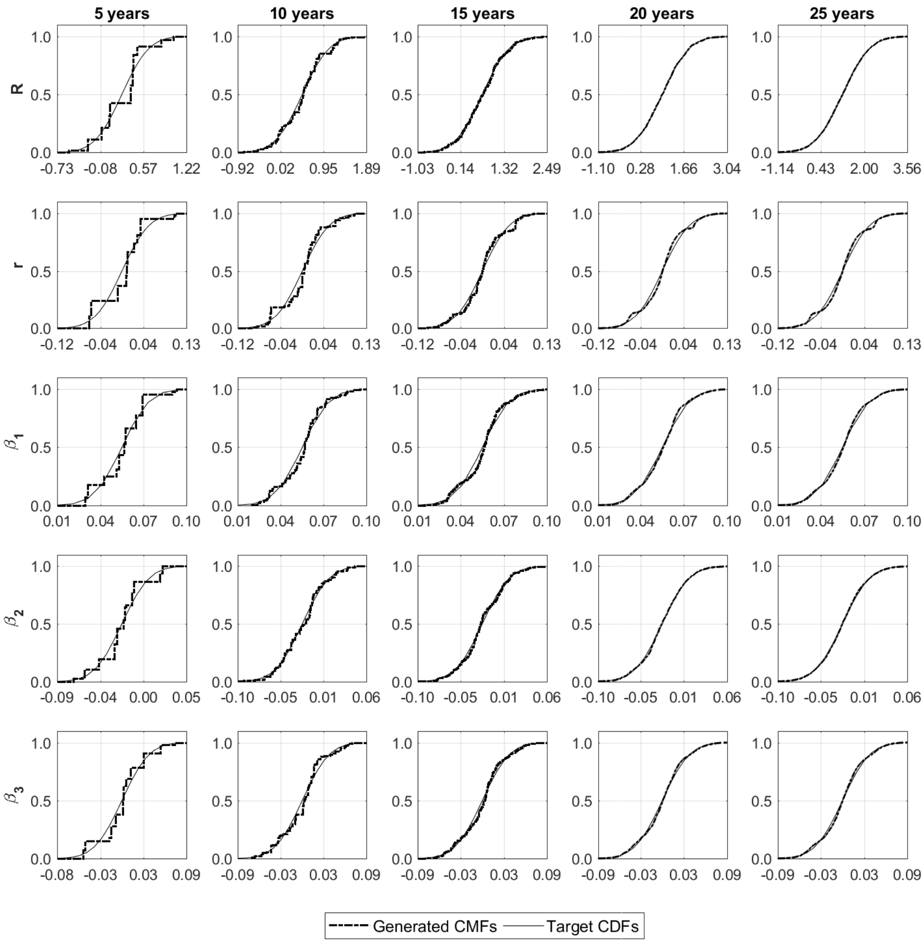

Fig. S-1 – Cumulative Density Functions (solid) from the financial market model and Cumulative Mass Functions (dot-dash) generated from the scenario tree for five variables at different times during the planning period.

where $o(x, e_{n^+}, pr_{n^+}) = \|e'_{n^+} diag(pr_{n^+})x\| := x \perp e_{n^+}$, and the function g is a deviation of the scenarios $\{\upsilon_{n^+}, \varepsilon_{n^+}\}$ from their target m-th distributional moment given conditional probabilities pr_{n^+}. This ensures that the real labour income shocks satisfy the desired distributional properties and are also as close to independent as possible from the financial asset returns.

Table S-2 shows the distributional properties of the generated scenarios of real labour income starting with a salary of £40,000 at time 0. Notice that the standard deviation (StDev) values of the real labour income growth rate, in the presence of the temporary real labour shock only, does not change with age (bottom half of Table S-2), whereas it increases with age when the permanent shock only is present (top half of Table S-2).

Finally, we note that generating the overall scenario structure takes about two hours with Matlab by using a parallel loop *parfor* on a HP desktop computer with Intel CPU i7-7700 3.60 Ghz and 32 GByte memory. We then use an efficient non-linear solver,

Table S-2 – Mean and standard deviation (StDev) of annual real labour income growth rate, and percentiles of annual real labour income (×£1,000), in the generated labour income scenarios. In the top half, only permanent real labour income shocks are present, and in the bottom half, only temporary real labour income shocks are present.

	permanent shock only $(\sigma_v^2\Delta t, \sigma_\varepsilon^2) = (0.10, 0.00)$				
age	40	45	50	55	60
log growth Mean	0.00000	0.01518	-0.01722	-0.10213	-0.25458
log growth StDev	0.00000	0.31623	0.44721	0.54772	0.63246
5th pctl	40.00000	29.72186	17.35576	15.24078	10.89122
	(1.00000)	(0.04483)	(0.03881)	(0.04565)	(0.04817)
median	40.00000	37.69431	38.51346	35.22053	30.43249
	(1.00000)	(0.48191)	(0.46767)	(0.49692)	(0.49982)
95th pctl	40.00000	57.27396	86.87215	93.52994	92.30769
	(1.00000)	(0.81311)	(0.91285)	(0.94930)	(0.94995)
	temporary shock only $(\sigma_v^2\Delta t, \sigma_\varepsilon^2) = (0.00, 0.15)$				
age	40	45	50	55	60
log growth Mean	0.00000	0.01518	-0.01722	-0.10213	-0.25458
log growth StDev	0.00000	0.38730	0.38730	0.38730	0.38730
5th pctl	40.00000	20.66305	23.92481	19.86972	16.47442
	(1.00000)	(0.18689)	(0.04556)	(0.04822)	(0.04944)
median	40.00000	34.88625	34.96046	35.62132	31.36666
	(1.00000)	(0.41040)	(0.46675)	(0.49904)	(0.49993)
95th pctl	40.00000	55.25718	80.69439	66.08680	56.81743
	(1.00000)	(0.78665)	(0.93111)	(0.94879)	(0.94999)

The scenarios are chosen as close as possible to a given percentile (5th percentile, median and 95th percentile). Actual cumulative probabilities are given in parentheses.

MOSEK, to find optimal investment and deferred annuity choices by maximizing the objective function in Eq. (1a) subject to the constraints in Eq. (1b) to Eq. (1j).

S-4. Calibrating on Different Historical Periods

The optimal investment and deferred annuity decisions depend on the estimated parameters of our financial market model. Nevertheless, our objective function drives the decisions in a consistent way, which is to maximize the expected utility of real retirement income from annuity. In order to confirm the consistency of our model, we investigate results with other distinctive datasets. Our original dataset, called DataSet1, is divided into three datasets: DataSet2 for Jan. 1985–Dec. 1997 (before the Asian financial crisis), DataSet3 for Jan. 1998–Dec. 2007 (between the Asian and global financial crisis), and DataSet4 for Jan. 2008–Jun. 2017 (after the global financial crisis). Distributional properties of the state variables for the original dataset are given in Table S-1 in S-2 and, for the new datasets, they are given in Table S-3.

Fig. 7 shows average asset allocations of the four datasets. The optimal asset allocations with DataSet2 (before the Asian financial crisis) in the first quadrant shows

Table S-3 – Mean, standard deviations and cross correlations of financial variables for 3 different periods.

DataSet2: January.1985 - December.1997 ($\lambda = 0.1332$ and $\tilde{\lambda} = 0.2345$)										
	R_t	Π_t	r_t	π_t	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$	$\widetilde{\beta}_{1,t}$	$\widetilde{\beta}_{2,t}$	$\widetilde{\beta}_{3,t}$
$\mathbb{E}$	0.5413	0.1866	0.0090	0.0031	0.0669	0.0130	0.0467	0.0399	-0.0121	0.0060
$\sqrt{\mathbb{V}}$	0.4099	0.0681	0.0496	0.0048	0.0121	0.0254	0.0366	0.0051	0.0118	0.0192
R_t	1.0000	-0.1262	0.1326	-0.0140	0.1018	-0.1030	-0.0566	-0.1045	0.0556	-0.0293
Π_t		1.0000	-0.0232	0.2266	0.0958	0.3547	-0.2653	0.2631	0.1022	-0.2051
r_t			1.0000	-0.0999	0.1865	-0.1449	-0.1628	-0.1238	0.0587	-0.0337
π_t				1.0000	-0.0953	0.3412	-0.0173	0.2418	-0.0510	-0.1188
$\beta_{1,t}$					1.0000	-0.4008	-0.7540	-0.1216	0.2165	-0.1158
$\beta_{2,t}$						1.0000	0.0309	0.5698	-0.1105	-0.0679
$\beta_{3,t}$							1.0000	0.0423	-0.2912	0.2279
$\widetilde{\beta}_{1,t}$								1.0000	-0.2425	-0.4648
$\widetilde{\beta}_{2,t}$									1.0000	-0.5353
$\widetilde{\beta}_{3,t}$										1.0000
DataSet3: January.1998 - December.2007 ($\lambda = 0.2989$ and $\tilde{\lambda} = 0.3667$)										
$\mathbb{E}$	0.2044	0.1902	0.0034	0.0032	0.0432	-0.0074	0.0056	0.0048	-0.0135	0.0554
$\sqrt{\mathbb{V}}$	0.3359	0.0312	0.0382	0.0034	0.0057	0.0122	0.0173	0.0077	0.0167	0.0264
R_t	1.0000	0.2533	0.1280	0.0082	-0.2501	0.2584	-0.0327	-0.4097	0.2351	0.1139
Π_t		1.0000	0.0603	0.1652	0.0177	-0.1621	-0.1556	-0.4029	-0.1911	0.4097
r_t			1.0000	0.0202	0.0119	0.0291	-0.0796	-0.0338	-0.0101	-0.0392
π_t				1.0000	0.0537	-0.0025	-0.0752	-0.1092	-0.0636	0.0923
$\beta_{1,t}$					1.0000	-0.4621	-0.6696	0.4343	-0.5138	0.1395
$\beta_{2,t}$						1.0000	0.2276	-0.0225	0.9047	-0.4680
$\beta_{3,t}$							1.0000	0.0785	0.2688	-0.1989
$\widetilde{\beta}_{1,t}$								1.0000	0.0362	-0.6722
$\widetilde{\beta}_{2,t}$									1.0000	-0.6286
$\widetilde{\beta}_{3,t}$										1.0000
DataSet4: January.2008 - June.2017 ($\lambda = 0.4297$ and $\tilde{\lambda} = 0.1876$)										
$\mathbb{E}$	0.2557	0.0521	0.0043	0.0009	0.0100	-0.0084	-0.0463	-0.0243	-0.0178	0.0010
$\sqrt{\mathbb{V}}$	0.2739	0.0430	0.0409	0.0042	0.0120	0.0148	0.0196	0.0092	0.0158	0.0205
R_t	1.0000	-0.0151	0.1331	-0.0875	-0.3064	0.2097	-0.0571	-0.2929	-0.0948	-0.0667
Π_t		1.0000	0.0444	0.1677	0.4036	-0.1209	0.2118	0.3188	0.1060	0.2346
r_t			1.0000	0.0670	-0.0445	-0.0808	-0.0498	-0.0072	-0.1413	0.0266
π_t				1.0000	0.1574	-0.0107	0.0286	0.1554	-0.3150	0.2367
$\beta_{1,t}$					1.0000	-0.7769	0.1592	0.8672	-0.2325	0.6199
$\beta_{2,t}$						1.0000	-0.1748	-0.6379	0.3254	-0.6287
$\beta_{3,t}$							1.0000	-0.2149	0.5491	0.4454
$\widetilde{\beta}_{1,t}$								1.0000	-0.4029	0.2512
$\widetilde{\beta}_{2,t}$									1.0000	-0.3001
$\widetilde{\beta}_{3,t}$										1.0000

Monthly data of FTSE 100 from Bloomberg, retail price index from the Office for National Statistics, fitted nominal and real yield curves from the Bank of England are used.

that inflation-protected deferred annuity takes priority over nominal deferred annuity. This is diametrically opposed to the optimal strategy with DataSet1. In addition, the average optimal investment strategy makes the cash fund dominant in the early stages and the equity fund in the later stages, among the financial assets. The equity fund is much more generously used than under DataSet1.

First of all, in DataSet2, the long-term real interest rate $\widetilde{\beta}_1$ starts at 0.0399 (-0.0012 in DataSet1) in which case the investor is more willing to purchase inflation-protected deferred annuity over nominal deferred annuity. The expected annual inflation rate ($0.0373 = 0.1866/5$) is greater than the implied inflation rate ($0.0270 = 0.0669 - 0.0399$) from the yield curves. Volatility of the expected inflation rate is the highest among the four datasets (0.0681). Thus, the investor needs a way to avoid volatile inflation in the distant future. For these reasons, the investor does not have to wait for the price of inflation-protected deferred annuity to drop: he buys inflation-protected deferred annuity in the early stages, and keep watch to time his purchase of nominal deferred annuity, in line with this time preference.

The nominal yield curve at the steady state of the VAR model with DataSet2 is downward slopping: $\beta_1 + \beta_2 > \beta_1$. The high short-term interest rate compensates its high volatility. This justifies why investing in cash occurs mainly in the early stages. Accumulated equity returns are negatively correlated with the long-term real interest rate $\widetilde{\beta}_1$. In other words, positively correlated with price changes in inflation-protected deferred annuity. On the other hand, the inflation rate Π_t is positively correlated with the long-term real interest rate. The equity fund here is a more suitable hedging tool for inflation-protected deferred annuity. The long-term nominal interest rate β_1 is also negatively correlated with the long-term real interest rate $\widetilde{\beta}_1$. This is one reason why nominal bonds suddenly disappear in the average optimal investment portfolio.

A similar interpretation can be applied to results from other datasets. For the period between the Asian financial crisis and global financial crisis, DataSet3 presents a barely positive long-term real interest rate (0.0048), so that the investor faces the risk of a price surge in inflation-protected deferred annuity, especially when the rate is negative. The accumulated inflation shows the highest mean and the lowest standard deviation levels among the four datasets. These factors push the investor to buy more inflation-protected deferred annuity much earlier. Inflation is also moderately negatively correlated with the long-term real interest rate. Because the inflation-linked bond prices depend on changes in the long-term real interest rate and inflation, under these financial market conditions, investing in the inflation-linked bond is risky. On the other hand, the long-term real interest rate, which determines inflation-protected annuity prices, is positively correlated with the long-term nominal interest rate and negatively correlated with equity returns. Furthermore, inflation is better hedged by equity than nominal bonds here. For this reason, investing in equity at later stages is not risky, but is a relatively safe option for purchasing inflation-protected annuity.

After the global financial crisis in DataSet4, the most notable features are interest rate levels. The long-term nominal interest rates are close to zero and the long-term real interest rates are strongly negative. The probability of positive long-term real interest rates is very low in our scenarios. Thus, purchasing inflation-protected annuity is not cost-effective, even though the investor desires real retirement income in the context of positive inflation (0.05325, mean of Π_t). As a result, the investor decides to spend a small portion of his wealth on the inflation-protected annuity at retirement. His portfolio is managed for buying nominal deferred annuity. Equity returns are moderately negatively correlated with long-term nominal yield β_1 (-0.3064), which means that the returns are somewhat positively correlated with price changes in nominal deferred annuity. The proportion of long-term nominal bonds increases over the planning horizon, and the optimal investment and annuitization strategy shifts from a risky to a less risky profile, in terms of securing retirement income.

The model therefore produces consistent decisions under different financial market conditions. This also reminds us that the parameterization of the model strongly influences the optimal decisions that are made by the investor.

References

Barberis, N. (2000). Investing for the Long Run when Returns Are Predictable. *The Journal of Finance*, *55*, 225–264.

Høyland, K., & Wallace, S. W. (2001). Generating scenario trees for multistage decision problems. *Management Science*, *47*, 295–307.

Klaassen, P. (2002). Comment on "Generating scenario trees for multistage decision problems". *Management Science*, *48*, 1512–1516.

Pedersen, A. M., B, Weissensteiner, A., & Poulsen, R. (2013). Financial planning for young households. *Annals of Operations Research*, *205*, 55–76.