”I would (not) teach proof, because it is (not) relevant to exams”: changing beliefs about teaching proof

Gabriel Chun-Yeung Lee

“I would (not) teach proof, because it is (not) relevant to exams”: changing beliefs about teaching proof

Gabriel Chun-Yeung Lee

University of Oxford, United Kingdom; chun.lee@education.ox.ac.uk

Experts in mathematics education agree that reasoning and proof are essential and should be made central to learning mathematics. However, some school teachers tend to focus on procedural skills because of different beliefs unfavourable for the teaching of proof. To encourage teachers to teach proof, I developed and studied an intervention for preservice teachers in Hong Kong. In this paper, I report the findings of this study regarding changing Hong Kong preservice teachers' beliefs about teaching proof, particularly their beliefs about the relevance of proof to examinations.

Keywords: Beliefs, intervention, preservice teachers, proof.

Introduction

Proof can play different important roles in learning mathematics, including, but not limited to, justifying, communicating, and explaining mathematical knowledge (Knuth, 2002). Consequently, many experts in mathematics education suggest that learning activities involving proof and proof-related reasoning should spread through school mathematics at different levels and in different content areas. Noting that teachers are the key decision-makers for what students will experience in classrooms, researchers have carried out studies on teachers' proof-related knowledge and beliefs (e.g., Knuth, 2002) and have identified different difficulties that the teachers often have with (teaching) proof (e.g., Stylianides et al., 2013). Few intervention studies have been conducted to explore a resolution to these difficulties (e.g., Buchbinder & McCrone, 2020). To address the need to encourage teachers to teach proof, I conducted a 4-cycle design-based research study (The Design-Based Research Collective, 2003) that aimed to develop an intervention to change Hong Kong preservice teachers' beliefs about teaching proof, through iterations of implementation, evaluation, and revision (McKenney & Reeves, 2014). In this paper, I report the findings of the fourth research cycle, focusing on the change in the participants' beliefs about teaching proof, particularly their beliefs about relevance of proof to examinations.

The context: Hong Kong

Depending on different factors (e.g., classroom culture, curriculum & policy), students' opportunities to learn proof can vary in different contexts. In Hong Kong, the place of proof in school mathematics is ambiguous. On the one hand, Hong Kong students seem to have more opportunities to be exposed to proof than their counterparts in other countries (Leung, 2005). For example, proof is introduced early in junior secondary mathematics, under the topics of “Pythagorean theorem” and “deductive geometry” in Year 8 (CDC, 2017). On the other hand, there are indications that proof has a marginal place in Hong Kong classrooms. The majority of the tasks that appear in textbooks or exam papers are non-proof-related and focus on students' routine procedures, for example, applying formulae and solving equations. Few tasks focus on developing and/or assessing students' proof and argumentation skills (e.g., Lee, 2021; Wong & Sutherland, 2018). Moreover, some Hong Kong teachers prioritise the development of students' skills in routine procedures over proof (e.g., Lee, 2019). The observed hesitation of Hong Kong teachers in teaching proof indicates that there is a need for interventions with (preservice) teachers. As the findings regarding the place of proof in the Hong Kong mathematics curriculum are also consistent with the results of studies conducted in other countries, it is believed that a study in developing an intervention that changes preservice teachers' beliefs about teaching proof is in the international community's interest.

Theoretical background

Beliefs refer to a statement of relation among ideas or objects that an individual holds to be true (Philipp, 2007), which might have influences on the way a teacher teaches mathematics (Furinghetti & Morselli, 2011). Beliefs about teaching proof are multidimensional. They include, but are not limited to, beliefs about the nature and roles of proof in mathematics and in education, beliefs about teaching, and beliefs about students (e.g., ability, motivation). Different dimensions of beliefs are interrelated and related to emotions, such as anxiety about producing proof and interest in proof; the more positively a teacher views proof, the more likely s/he values proof and provides students with related activities (e.g., Fraiser, 2010; Kotelawala, 2016). Research studies reported that many teachers agree on the importance of proof in mathematics but they often have difficulty in teaching proof because: (a) they, themselves, have cognitive and/or attitudinal difficulties with proof (e.g., Stylianides et al., 2017), (b) they believe that proof is not accessible to *all* students (e.g., Knuth, 2002), and/or (c) they believe that proof is not indispensable in school mathematics (e.g., Lee, 2019).

Preservice teachers enter teacher education programmes with pre-existing beliefs about mathematics and its teaching, which are often based on their previous learning experiences. Although beliefs are often considered to be resistant or slow to change, there is evidence (e.g., Yoo, 2008) that preservice teachers' beliefs about teaching proof can be changed after a programme, a course or even an intervention of relatively short duration, if they are provided with alternative learning experiences (Liljedahl et al., 2021). In this study, I developed an intervention that provided preservice teachers with alternative experiences of learning and teaching proof, which were different from the teacher-talk approach they had mainly experienced in the past.

Intervention design

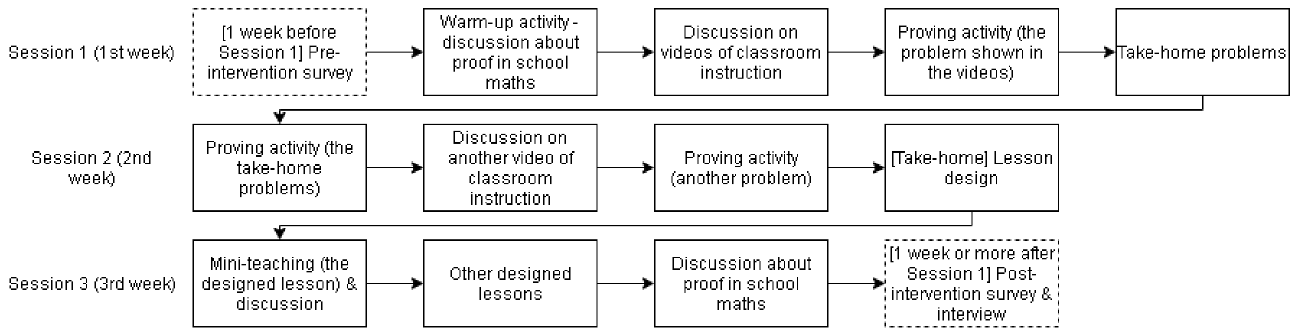

Figure 1: Design of the intervention

In this study, an intervention was designed for Hong Kong preservice teachers. The intervention consisted of three weekly extracurricular workshops. Each of the workshops lasted two hours. In the first three research cycles (July 2018–August 2019), the intervention design was trialled with different groups of preservice teachers. The trials enabled evaluations of the intervention design, informing revisions of the design. The intervention design in the fourth research cycle (Figure 1) differed from that in the first research cycle; in general, the participants (a) engaged in proving activities that emphasised learning mathematics through proof and argumentation, (b) discussed the role of proof in school mathematics, (c) watched and discussed videos of classroom instruction involving proof and argumentation, and (d) planned a lesson and trialled it with peers (Table 1).

Table 1: Examples of activities of the intervention

Type of activities	Example
Proving activities	What is the divisor of the sum of any *three* consecutive integers? What about *four*, *five*, and so on?
Discussions on the role of proof	When and how does proof appear in the previous task? What is the role of proof in this task?
Videos of classroom instruction	[*TIMSS 1999 Video Study – HK3 Polygons*] What did the teacher do in the video? Why? When did proof appear?
Lesson design and mini-teaching	Use what you have learnt in the workshops so far to develop a less on "Area of a circle"

Methods

The fourth research cycle took place between September and November 2019. The intervention was not conducted with one single group, but multiple small groups. Twenty-seven Hong Kong preservice teachers attended the intervention. Among them, 18 participants attended all sessions of the intervention whereas 6 attended only one session.

Table 2: Examples of questionnaire items

Type of items	Example
Likert items: Beliefs about the importance of proof	Making proofs improves mathematical thinking.
Likert items: Beliefs about the relevance of proof to examinations	Knowing how to make proofs is very important in excelling an examination/a test in school mathematics.
Likert items: Enjoyment and interest in proof	Making proofs arouses my curiosity.
Likert items: Anxiety about proof	I feel myself under pressure when I make proofs in mathematics lessons.
Likert items: Negative reactions when failing to produce proofs	Not being able to prove upsets me.
Open-ended questions	Should we implement tasks of proof and proving in mathematics class? Please briefly explain.

A questionnaire, consisting of Likert items and open-ended questions (Table 2), was designed, and used to gather information about preservice teachers' beliefs about teaching proof before and after the intervention. A subset of the teachers was invited to attend individual interviews after the intervention so that they could explain their beliefs about teaching proof, elaborate on their responses to the questionnaire and give their feedback about the intervention. The intervention was also audio-recorded and fieldnotes were made as supplementary data. Among all participants, 23 completed the pre-intervention questionnaire, 7 completed the post-intervention questionnaire, and 12 attended the post-intervention interviews. Pre- and post-intervention data were coded separately according to different dimensions of teachers' beliefs about teaching proof. The coded data were then compared to identify differences and, in turn, changes in their beliefs about teaching proof after the intervention.

More data were expected, but due to pro-democracy protests since June 2019 and the coronavirus pandemic since December 2019, several planned interventions were deferred and eventually cancelled. However, despite a reduced amount of data, analysis of the available data was sufficient to show the effects of the intervention on the participants' beliefs about teaching proof.

Results

I analysed 12 sets of pre-post data, indicating the corresponding 12 participants' beliefs about teaching proof and their changes after the intervention. In this paper, I first discuss the general findings in relation to the changes in the participants' beliefs about teaching proof, particularly their beliefs about the relevance of proof to examinations, then use selected data excerpts to illustrate the changes in the participants' beliefs about the relevance of proof to examinations and discuss how the intervention facilitated such belief changes.

In general, the participants developed more positive beliefs about teaching proof after the intervention. The analysis of the Likert items showed the participants maintained agreement that proof is essential to mathematics and developed more positive and less negative emotional dispositions towards proof, and more participants reported agreeing that proof is related to examinations. The analysis of the open-ended questions also showed that more participants reported valuing the explanatory power of proof and being willing to teach proof regularly. Having said that, a few participants reported and maintained some of their worries about teaching proof (e.g., students lacked ability and interest, class time was not enough). Worries about students' ability to learn proof remained influential in the participants' beliefs about teaching proof. After the intervention, however, these worries no longer discouraged these participants from teaching proof, but rather prompted them to consider different approaches for teaching proof.

In the following, I use Participant 4's data as an example of my analysis. Before the intervention, although Participant 4 was able to relate proof to the development of mathematical understanding, he believed that teaching proof was not related to examinations and was therefore not essential:

> Participant 4: […] I think, for secondary school students, proof is not so important to them. In relation to exams, [proof] is not so important, but proof can allow them to learn why this [mathematical statement/ concept/ idea] is true, making them not to learn by rote, but to understand a theorem and its principle, for applying the theorem more easily or knowing when to apply it.

At the beginning of the intervention, Participant 4 also revealed his belief about the relationships between school mathematics, proof, students, and examinations. He believed that a number of students learn mathematics merely because it is a compulsory subject for all students in Hong Kong and prefer learning by rote over learning proof, so he believed that these students are relatively weak with proof:

> Participant 4: I think there must be a portion of students who are not interested in mathematics. To them, learning mathematics is for taking [the Hong Kong public exam]. So, there must actually be a portion of students who rely on [learning] formulae by rote [and learning] examples by rote, for taking exams. Therefore, when seeing non-proof tasks, and when [seeing] tasks that can be solved by simply applying methods [that

they] have learnt by rote, s/he [the student] can easily solve [the tasks], answer correctly. However, when [seeing] proof tasks, [particularly] those that are not standard, they [the students] do not know how to apply [the methods] to solve [the tasks], so […] these students might be weaker with proof tasks.

Yet, after the intervention, Participant 4 developed a deeper understanding of the relationship between proof and the development of mathematical understanding and became aware that proof can play important roles in preparing students for examinations. Particularly, he developed a belief that learning proof is helpful for students to develop knowledge and skills in solving proving and non-proving tasks:

Participant 4: [Before,] I thought there were not many proving tasks in [exams], so [students] could still achieve good results without proof. However, [now] I think [proof] can not only be [helpful in] proving tasks, but also helpful in other tasks. I think proof can be related to other domains.

Participant 4 attributed his learning in and changes after the intervention to the discussion about proof and the videos of classroom instruction, which allowed him to explore and reflect on different views about proof and teaching. Particularly, Participant 4's experience of the intervention provided alternatives to his pre-existing beliefs about what proof is and about approaches for teaching proof. Moreover, after seeing how primary students could learn mathematics through argumentation and proof in a video, Participant 4 became aware that students have the ability to learn mathematics through proof and argumentation. To him, the ideas of teaching proof conveyed in the intervention were positive and he wanted to apply them into his future teaching:

Interviewer: What did you experience and learn in [the intervention]?

Participant 4: In fact… at the beginning [of the intervention], [the instructor] asked everyone's views about proof. Using [the discussion about which tasks in exams are proof-related] as an example, it could be seen that there are some people [who hold views] that are different from my view. This might, in turn, reflect [an idea that] students [have different views about what proof is]. Second, video… [I] watched two videos; [I] saw more, different approaches for teaching, [for example,] that [video] of teaching children even and odd [numbers]. That teaching approach [was something that] I have never experienced. I [started to] thinking about my teaching approach, and whether some of [my] lessons can involve that approach. [I also] saw that they [the primary students] were able to think [argue the meanings of even and odd numbers], [indicating that] probably older, secondary students should have this ability as well. [After the intervention, I started to] consider more about this aspect.

Participant 4's change exemplifies one possible way how the intervention facilitated changes in the participants' beliefs about teaching proof, particularly their beliefs about the relevance of proof to examinations. The intervention, through discussions about roles of proof in learning mathematics and videos of classroom instruction, provided the participants with alternative experiences of learning and teaching proof. These experiences challenged the participants' pre-existing beliefs about teaching proof (e.g., "proof is always difficult, algebraic and formally presented", "proof is separate from other domains of school mathematics") and broadened their horizons by conveying that proof can be accessible to most (if not all) students and can be communicated via different representations as long as the representations are accessible to and accepted by the students. Consequently, the revised ideas of proof and teaching mathematics allowed the participants to discover more connections between proof, learning mathematics and examinations, and in turn to develop more positive beliefs about the relevance of proof to examinations.

I do not claim that all participants became convinced of the relevance of proof to examinations after the intervention. Rather, I assert that the participants gained *more positive* beliefs about the relevance of proof to examinations. In other words, after the intervention, whilst there were participants becoming convinced that proof is relevant to examinations (as exemplified by the case of Participant 4), others continued to believe that proof is not indispensable but helpful to students when preparing for and taking an exam. For the latter case, since the majority of tasks appearing in exam papers require students to apply formulae and solve equations and only a few require students to produce a proof or an argument (Lee, 2021), the participants continued to believe that proof is not necessary for students to pass an exam. Yet, after the intervention, they became aware that proof is somewhat related to examinations: proof can promote students' mathematical understanding and reasoning, implying that teaching proof can help students do better in examinations.

In summary, during the intervention, the participants were provided with alternative experiences of proof (different from their past experiences), which challenged their pre-existing beliefs that discourage teaching proof (e.g., proof is not related to examinations, proof confuses students). The participants had positive experiences with proof (e.g., they deepened mathematical understanding via proof) and developed positive emotions about proof (e.g., excitement, interest), helping them replace their pre-existing, discouraging beliefs by beliefs that encourage teaching proof (e.g., proof can promote students' mathematical understanding and reasoning, proof can help students prepare for examinations). Having said that, beliefs that have basis in external factors, particularly in quantitative information (e.g., the number of proving tasks in exam papers and textbooks), seemed to be difficult to change (e.g., it is possible to pass an exam without proof).

Summary and discussion

There is some evidence that teachers' beliefs about and practices of teaching proof are affected by examinations (e.g., Frasier & Panasuk, 2013; Lee, 2019; Nyaumwe & Buzuzi, 2007), in Hong Kong and in other countries. This paper demonstrates that whilst (a) some counterproductive beliefs of preservice teachers about the relevance of proof to examinations (e.g., "Students can pass exams without proof") are difficult to change if the composition of tasks appearing in exam papers is not changed (curricular aspect), (b) more positive beliefs (e.g., "Proof can promote students' mathematical understanding, which in turn helps preparing for and doing better in exams") can be developed after an intervention that is carefully designed to provide preservice teachers with positive experiences of proof. In other words, for an effective intervention that aims to change preservice teachers' beliefs about teaching proof, it is important to create alternative experiences of learning and teaching proof in which preservice teachers can deepen their mathematical understanding through proof and can translate such experiences into their future teaching.

It is believed that this study is in the international community's interest for three reasons. First, the findings of this study are consistent with other studies that involved interventions with preservice teachers in which the participants developed more positive beliefs and emotional dispositions towards (teaching) proof (e.g., Buchbinder & McCrone, 2020; Yoo, 2008). These consistent findings in different settings not only provide evidence that it is possible to change preservice teachers' beliefs about teaching proof (which are often considered difficult to change) during teacher training, but also show that it is possible to generalise each finding to other settings.

Second, in contrast with other interventions that were implemented in regular courses and lasted about a semester, the intervention of this study was designed as extracurricular workshops and took place within a month. The short-duration format allowed the intervention design to be tested and revised multiple times within a relatively short period of time, thereby enhancing the validity of the study's findings. This study also explored and showed the possibility of changing preservice teachers' beliefs about teaching proof after an intervention of a short duration, responding to a "challenging but important question for mathematics education researchers: *Would it be possible to design classroom-based interventions of short duration in mathematics classrooms that could help alleviate significant problems of students' learning in mathematics?*" (Stylianides & Stylianides, 2013, p. 339).

Third, I reported different changes in the participants' beliefs about teaching proof after the intervention in this study. The findings also demonstrated why some beliefs (e.g., belief about roles of proof in learning mathematics) changed and some (e.g., belief that students can pass exams without proof) remained unchanged. In other words, some beliefs about teaching proof are dependent on external factors (e.g., the proportion of proof to tasks that require only procedural skills in exam papers) and often remain unchanged if the corresponding external factors remain unchanged, and some are dependent on one's past experiences and can be revised if positive and inspiring experiences of proof are provided (e.g., during courses of university mathematics and mathematics education).

Future work should continue to design (and improve) positive and inspiring experiences of proof for (preservice) teachers and explore and validate the effects of different designs on beliefs about teaching proof (ideally, both immediate and delayed effects) with larger samples (ideally, together with comparison or control groups). In order that dimensions of beliefs about teaching proof can be quantitatively measured for pre-post analysis and statistical modelling, it is also important to conduct studies to develop practical instruments.

References

Buchbinder, O., & McCrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. *The Journal of Mathematical Behavior*, *58*, 1–19. https://doi.org/10.1016/j.jmathb.2020.100779

Curriculum Development Council (CDC). (2017). *Mathematics education: Key learning area curriculum guide (Primary 1 - Secondary 6)*. The Curriculum Development Council.

Frasier, B. J., & Panasuk, R. (2013). Proofs in Mathematics Education: Expectations and Reality. *Journal of Curriculum and Teaching*, *2*(2), 76–85. https://doi.org/10.5430/jct.v2n2p76

Furinghetti, F., & Morselli, F. (2011). Beliefs and beyond: hows and whys in the teaching of proof. *ZDM*, *43*(4), 587–599. https://doi.org/10.1007/s11858-011-0316-7

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, *33*(5), 379–405. https://doi.org/10.2307/4149959

Kotelawala, U. (2016). The Status of Proving Among US Secondary Mathematics Teachers. *International Journal of Science and Mathematics Education*, *14*(6), 1113–1131. https://doi.org/10.1007/s10763-015-9638-1

Lee, C.-Y. (2019). Factors influencing teachers' decision making on reasoning-and-proving in Hong Kong. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *CERME11* (pp. 257–264). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Lee, C.-Y. (2021, July 12–18). *Proof and reasoning in high-stakes testing systems: The senior secondary mathematics curricula in Hong Kong and International Baccalaureate Diploma Programme* [Paper presentation]. *ICME14*, Shanghai, PRC.

Leung, F. K.-S. (2005). Some Characteristics of East Asian Mathematics Classrooms Based on Data from the Timss 1999 Video Study. *Educational Studies in Mathematics*, *60*(2), 199–215. https://doi.org/10.1007/s10649-005-3835-8

Liljedahl, P. G., Rösken, B., & Rolka, K. (2021). Changes to preservice elementary teachers' beliefs about mathematics and the teaching and learning of mathematics: How and why? *Journal of Adult Learning, Knowledge and Innovation*, *4*(1), 20–30. https://doi.org/10.1556/2059.03.2019.09

McKenney, S. E., & Reeves, T. C. (2014). Methods of evaluation and reflection in design research. *Zeitschrift Für Berufs- Und Wirtschaftspädagogik*, *27*, 141–153.

Nyaumwe, L., & Buzuzi, G. (2007). Teachers' attitudes towards proof of mathematical results in the secondary school curriculum: The case of Zimbabwe. *Mathematics Education Research Journal*, *19*(3), 21–32. https://doi.org/10.1007/BF03217460

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (2nd ed., pp. 257–315). Information Age.

Stylianides, A. J., & Stylianides, G. J. (2013). Seeking research-grounded solutions to problems of practice: classroom-based interventions in mathematics education. *ZDM*, *45*(3), 333–341. https://doi.org/10.1007/s11858-013-0501-y

Stylianides, G. J., Stylianides, A. J., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education*, *11*(6), 1463–1490. https://doi.org/10.1007/s10763-013-9409-9

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237–266). National Council of Teachers of Mathematics.

The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, *32*(1), 5–8. https://doi.org/10.3102/0013189X032001005

Yoo, S. (2008). *Effects of traditional and problem-based instruction on conceptions of proof and pedagogy in undergraduates and prospective mathematics teachers* [Unpublished doctoral dissertation]. The University of Texas at Austin.

Wong, K.-C., & Sutherland, R. (2018). Reasoning-and-proving in algebra in school mathematics textbooks in Hong Kong. In A. J. Stylianides & G. Harel (Eds.), *Advances in mathematics education research on proof and proving: An international perspective* (pp. 185–198). Springer.