

# GORENSTEIN SPHERICAL FANO VARIETIES

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ABSTRACT. We obtain a combinatorial description of Gorenstein spherical Fano varieties in terms of certain polytopes, generalizing the combinatorial description of Gorenstein toric Fano varieties by reflexive polytopes and its extension to Gorenstein horospherical Fano varieties due to Pasquier. Using this description, we show that the rank of the Picard group of an arbitrary  $d$ -dimensional  $\mathbb{Q}$ -factorial Gorenstein spherical Fano variety is bounded by  $2d$ . This paper also contains an overview of the description of the natural representative of the anticanonical divisor class of a spherical variety due to Brion.

## 1. INTRODUCTION

A complete complex algebraic variety is called Gorenstein Fano if it is normal and its anticanonical divisor is Cartier and ample. For toric varieties one has a nice characterization of Gorenstein Fano varieties by convex geometry, namely one has a bijective correspondence between Gorenstein toric Fano varieties and reflexive polytopes, i. e. lattice polytopes whose dual is a lattice polytope as well (see [Bat94, Theorem 4.1.9]). Generalizing the notion of a reflexive polytope, Pasquier established a similar correspondence for horospherical varieties in [Pas08]. In this paper, we extend this correspondence to (arbitrary) spherical varieties.

In [Bri97], Brion determined a natural representative of the anticanonical divisor class for any spherical variety  $X$ , on which our combinatorial description of Gorenstein spherical Fano varieties essentially depends. For this reason, we have found it useful to give a self-contained overview of Brion's as well as some related results of Luna, together with some motivation. The new results of this paper, i. e. Theorem 1.9 and Theorem 1.10, are stated afterwards.

Let  $G$  be a connected reductive complex algebraic group and  $B \subseteq G$  a Borel subgroup. A closed subgroup  $H \subseteq G$  is called *spherical* provided that the homogeneous space  $G/H$  contains an open  $B$ -orbit. Let  $H \subseteq G$  be a spherical subgroup. A  $G$ -equivariant open embedding  $G/H \hookrightarrow X$  into a normal irreducible  $G$ -variety  $X$  is called a *spherical embedding*, and  $X$  is called a *spherical variety*.

An anticanonical divisor is a Weil divisor  $-K_X$  of  $X$  such that  $\mathcal{O}_X(-K_X) = \check{\omega}_X$  where  $\check{\omega}_X$  is the anticanonical sheaf of  $X$ , i. e. the reflexive sheaf coinciding with the top exterior power of the tangent sheaf on the smooth locus of  $X$ . It is equipped with a natural  $G$ -linearization (see Section 3).

It is known that the open  $B$ -orbit  $U$  is isomorphic to  $(\mathbb{C}^*)^r \times \mathbb{C}^s$  (see [Ros63, Theorem 5]) and hence has trivial divisor class group. In particular, the invertible sheaf  $\check{\omega}_{G/H}$  is trivial on  $U$ , and we might ask whether there is a natural choice of a generator  $s \in \Gamma(U, \check{\omega}_{G/H})$ .

We first recall the well-known case where  $G = B = T$  is a torus and  $H$  is trivial (see, for instance, [Ful93, Section 4.3]). A spherical embedding  $G/H \hookrightarrow X$  is then simply a toric variety  $X$  with embedded torus  $U = G/H \cong T$ , and it is possible to show that there is a unique generator  $s \in \Gamma(U, \check{\omega}_{G/H})$  which is  $T$ -invariant. This

generator can be explicitly written as

$$s = x_1 \frac{\partial}{\partial x_1} \wedge \dots \wedge x_n \frac{\partial}{\partial x_n}$$

where  $x_1, \dots, x_n$  is a choice of coordinates for the torus  $T$ . In addition to being  $T$ -invariant, this section has the important property that it has a zero of order 1 along every  $T$ -invariant divisor in  $X$ .

In general, however, there need not be any  $B$ -invariant section  $s \in \Gamma(U, \check{\omega}_{G/H})$  as the following example shows.

**Example 1.1.** Let  $G = \mathrm{SL}_2$  and  $H \subseteq G$  a Borel subgroup. Then the open  $B$ -orbit  $U$  in  $G/H$  is isomorphic to  $\mathbb{C}$ . It is not difficult to see that there is only one  $B$ -semi-invariant section in  $\Gamma(U, \check{\omega}_{G/H})$  and that it is not  $B$ -invariant.

As we cannot expect a  $B$ -invariant section, we have to look for something else. An equivalent characterization for a homogeneous space  $G/H$  to be spherical is that the  $G$ -module  $\Gamma(G/H, \mathcal{L})$  is multiplicity-free for every  $G$ -linearized invertible sheaf  $\mathcal{L}$ , i. e. the multiplicity of any simple  $G$ -module in the decomposition of the module of global sections is at most 1. We obtain

$$\Gamma(G/H, \check{\omega}_{G/H}) \cong \bigoplus V_\chi,$$

where  $\chi$  runs over pairwise different dominant weights of  $B$  and  $V_\chi$  is the simple  $G$ -module of highest weight  $\chi$ .

This decomposition is very simple when  $H \subseteq G$  is a parabolic subgroup since then  $\Gamma(G/H, \check{\omega}_{G/H})$  is a simple  $G$ -module by the Borel-Weil-Bott theorem (see [Dem68, Dem76]). In particular, exactly one dominant weight occurs (compare this with Example 1.1), and there is a unique choice (up to a constant factor) of a  $B$ -semi-invariant section  $s \in \Gamma(G/H, \check{\omega}_{G/H})$  which restricts to a generator  $s \in \Gamma(U, \check{\omega}_{G/H})$ .

Let  $G/H$  again be an arbitrary spherical homogeneous space. We denote by  $P \subseteq G$  the stabilizer of the open  $B$ -orbit  $U$ . If  $H$  contains a maximal unipotent subgroup of  $G$ , the homogeneous space  $G/H$  is called *horospherical*, and the normalizer of  $H$  in  $G$  is a parabolic subgroup conjugated to the opposite parabolic of  $P$ , which we denote by  $P^-$ . Hence there is a natural morphism  $\pi: G/H \rightarrow G/P^-$ , which is known to be a torus fibration. Therefore  $\pi^*(\check{\omega}_{G/P^-}) = \check{\omega}_{G/H}$ , the simple  $G$ -module  $\Gamma(G/P^-, \check{\omega}_{G/P^-})$  is a direct summand of  $\Gamma(G/H, \check{\omega}_{G/H})$ , and a unique  $B$ -semi-invariant section  $s \in \Gamma(G/P^-, \check{\omega}_{G/P^-}) \subseteq \Gamma(G/H, \check{\omega}_{G/H})$  exists, whose weight we denote by  $\kappa_P \in \mathfrak{X}(B)$ .

For arbitrary spherical homogeneous spaces  $G/H$ , there is no natural morphism  $\pi: G/H \rightarrow G/P^-$ , but the following statement is nevertheless valid.

**Theorem 1.2** ([Bri97, 4.1 and 4.2]). *The simple  $G$ -module  $\Gamma(G/P^-, \check{\omega}_{G/P^-})$  is a direct summand of  $\Gamma(G/H, \check{\omega}_{G/H})$ . Equivalently, there exists a  $B$ -semi-invariant section*

$$s \in \Gamma(G/H, \check{\omega}_{G/H})$$

*of weight  $\kappa_P$ , which restricts to a generator  $s \in \Gamma(U, \check{\omega}_{G/H})$ . For any spherical embedding  $G/H \hookrightarrow X$  the section  $s$  extends to a global section on  $X$ , and we have*

$$\mathrm{div} s = \sum_{i=1}^k m_i D_i + \sum_{j=1}^n X_j$$

*where  $D_1, \dots, D_k$  are the  $B$ -invariant prime divisors in  $G/H$  (identified with their closures in  $X$ ), the  $m_i$  are positive integers depending only on the homogeneous space  $G/H$ , and  $X_1, \dots, X_n$  are the  $G$ -invariant prime divisors in  $X$ .*

**Remark 1.3.** The weight  $\kappa_P$  is the weight  $\pi(\operatorname{div} s)$  in the sense of [Bri89, 3.3].

The fact from Theorem 1.2 that the section  $s$  has a zero of order 1 along any  $G$ -invariant prime divisor actually characterizes this section. This is a straightforward application of basic facts from the embedding theory of spherical homogeneous spaces.

**Theorem 1.4.** *Let  $s \in \Gamma(U, \check{\omega}_{G/H})$  be a generator (which is automatically  $B$ -semi-invariant). Then the following conditions are equivalent:*

- (1) *The section  $s$  is of  $B$ -weight  $\kappa_P$ .*
- (2) *For any spherical embedding  $G/H \hookrightarrow X$  the section  $s$  has a zero of order 1 along any  $G$ -invariant prime divisor.*

We denote by  $\mathcal{D} = \{D_1, \dots, D_k\}$  the set of  $B$ -invariant prime divisors in  $G/H$ , whose elements are called the *colors* of  $G/H$ . In order to obtain an explicit formula for the coefficients  $m_i$  in Theorem 1.2, Brion divided the colors into several types (see [Bri97, 4.2]). These types are in agreement with the definition of the types of colors due to Luna (see [Lun97, 2.7], [Lun01, 2.3], see also [Tim11, Section 30.10]), which we now explain. For additional information on the types of colors, we refer the reader to Section 5.

We choose a maximal torus  $T \subseteq B$ , denote by  $R \subseteq \mathfrak{X}(T) = \mathfrak{X}(B)$  the associated root system, and write  $S \subseteq R$  for the set of simple roots corresponding to  $B$ . For  $\alpha \in S$  we denote by  $P_\alpha \subseteq G$  the corresponding minimal parabolic subgroup containing  $B$ , and define

$$\mathcal{D}(\alpha) := \{D_i \in \mathcal{D} : P_\alpha \cdot D_i \neq D_i\}.$$

As the colors are not  $G$ -stable, every color is moved by at least one minimal parabolic subgroup, so that we have  $\mathcal{D} = \bigcup_{\alpha \in S} \mathcal{D}(\alpha)$ . Moreover, the stabilizer  $P \subseteq G$  of the open  $B$ -orbit is the parabolic subgroup containing  $B$  corresponding to the set  $S^p := \{\alpha \in S : \mathcal{D}(\alpha) = \emptyset\}$ .

We denote by  $\mathcal{M} \subseteq \mathfrak{X}(B)$  the weight lattice of  $B$ -semi-invariants in the function field  $\mathbb{C}(G/H)$  and by  $\mathcal{N} := \operatorname{Hom}(\mathcal{M}, \mathbb{Z})$  the dual lattice together with the natural pairing  $\langle \cdot, \cdot \rangle : \mathcal{N} \times \mathcal{M} \rightarrow \mathbb{Z}$ . We denote by  $\mathcal{V}$  the set of  $G$ -invariant discrete valuations on  $\mathbb{C}(G/H)$ , and define the map  $\iota : \mathcal{V} \rightarrow \mathcal{N}$  by  $\langle \iota(\nu), \chi \rangle := \nu(f_\chi)$  where  $f_\chi \in \mathbb{C}(G/H)$  is  $B$ -semi-invariant of weight  $\chi \in \mathcal{M}$  and unique up to a constant factor. As the map  $\iota$  is injective, we may consider  $\mathcal{V}$  as a subset of the vector space  $\mathcal{N}_{\mathbb{Q}} := \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ . It is known that  $\mathcal{V}$  is a cosimplicial cone (see [Bri90]), called the *valuation cone* of  $G/H$ . In particular, the valuation cone is full-dimensional. By  $\Sigma$  we denote the set of primitive generators in  $\mathcal{M}$  of the extremal rays of the negative of the dual of the valuation cone  $\mathcal{V}$ . The elements in  $\Sigma$  are called the *spherical roots* of  $G/H$ .

The type of a color  $D_i \in \mathcal{D}(\alpha)$  is defined as follows: If  $\alpha \in \Sigma$ , we say that  $D_i$  is of type  $a$ . If  $2\alpha \in \Sigma$ , we say that  $D_i$  is of type  $2a$ . Otherwise, we say that  $D_i$  is of type  $b$ . The type does not depend on the choice  $\alpha \in S$  such that  $D_i \in \mathcal{D}(\alpha)$ . Moreover, we have  $|\mathcal{D}(\alpha)| \leq 2$  with  $|\mathcal{D}(\alpha)| = 2$  if and only if  $\alpha \in \Sigma$  (i. e. the colors in  $\mathcal{D}(\alpha)$  are of type  $a$ ).

We can now state the explicit formula for the coefficients  $m_i$  in the expression for  $\operatorname{div} s$  due to Brion and Luna.

**Theorem 1.5** ([Bri97, Theorem 4.2], [Lun97, 3.6]). *We have*

$$\begin{aligned} m_i &= \frac{1}{2} \langle \alpha^\vee, \kappa_P \rangle = 1 && \text{for } D_i \text{ of type } a \text{ or } 2a, \\ m_i &= \langle \alpha^\vee, \kappa_P \rangle \geq 2 && \text{for } D_i \text{ of type } b. \end{aligned}$$

This concludes the overview, so that we are now able to explain the combinatorial description of Gorenstein spherical Fano varieties. Let  $m_1, \dots, m_k \in \mathbb{Z}_{>0}$  be the coefficients of the colors in the expression for the anticanonical divisor from

Theorem 1.2. We define the map  $\rho: \mathcal{D} \rightarrow \mathcal{N}$  by  $\langle \rho(D_i), \chi \rangle := \nu_{D_i}(f_\chi)$  where  $\nu_{D_i}$  is the discrete valuation associated to  $D_i \in \mathcal{D}$ .

**Definition 1.6.** Let  $G/H \hookrightarrow X$  be a complete spherical embedding, and let  $X_1, \dots, X_n$  be the  $G$ -invariant prime divisors in  $X$ . We define the polytope

$$Q_X := \text{conv} \left( \frac{\rho(D_1)}{m_1}, \dots, \frac{\rho(D_k)}{m_k}, \nu_{X_1}, \dots, \nu_{X_n} \right) \subseteq \mathcal{N}_{\mathbb{Q}}.$$

**Remark 1.7.** The polytope introduced in Definition 1.6 has been used by Alexeev and Brion to prove boundedness of spherical Fano varieties in [AB04].

The generalization of the notion of a reflexive polytope to the theory of spherical varieties is the following (it is a generalization of [Pas08, Définition 3.3]). For a polytope  $Q$  we denote by  $Q^*$  its dual polytope, and for a face  $F \preceq Q$  we denote by  $\widehat{F} \preceq Q^*$  its dual face.

**Definition 1.8.** A polytope  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  is called  $G/H$ -*reflexive* if the following conditions are satisfied:

- (1)  $\rho(D_i)/m_i \in Q$  for every  $i = 1, \dots, k$ .
- (2)  $0 \in \text{int}(Q)$ .
- (3) Every vertex of  $Q$  is contained in  $\{\rho(D_i)/m_i : i = 1, \dots, k\}$  or  $\mathcal{N} \cap \mathcal{V}$ .
- (4) Every vertex  $v \in Q^*$  satisfying  $\text{relint}(\text{cone}(\widehat{v})) \cap \mathcal{V} \neq \emptyset$  lies in the lattice  $\mathcal{M}$ .

Note that  $\text{cone}(\widehat{v})$  is a full-dimensional cone in  $\mathcal{N}_{\mathbb{Q}}$  for every vertex  $v \in Q^*$ .

**Theorem 1.9.** *The assignment  $X \mapsto Q_X$  induces a bijection between isomorphism classes of Gorenstein spherical Fano embeddings  $G/H \hookrightarrow X$  and  $G/H$ -reflexive polytopes.*

Using Theorem 1.9, one may translate questions about Gorenstein spherical Fano varieties into the realm of convex combinatorics. Applying this approach, we are going to prove the following bound on the Picard number.

**Theorem 1.10.** *Let  $X$  be a  $\mathbb{Q}$ -factorial Gorenstein spherical Fano variety of dimension  $d$  and Picard number  $\rho_X$ . Then we have*

$$\rho_X \leq 2d,$$

*with  $\rho_X = 2d$  if and only if  $d$  is even and  $X \cong (S_3)^{d/2}$  where  $S_3$  is the blowup of  $\mathbb{P}^2$  at three non-collinear points.*

Theorem 1.10 has been proven by Casagrande in the case of a toric variety  $X$  (see [Cas06]) and by Pasquier in the case of a horospherical variety  $X$  (see [Pas08]). Our proof is inspired by the two previous works. Observe that Theorem 1.10 does not hold for an arbitrary variety  $X$ , e. g. if  $S$  is the surface given by blowing-up  $\mathbb{P}^2$  in eight general points, the variety  $S^m$  has Picard number  $9m$  (see [Deb03]).

#### List of general notation.

$\mathfrak{X}(G)$	character lattice of a connected algebraic group $G$ ,
$Q^*$	dual polytope to a polytope $Q$ in a vector space $V$ , i. e. $Q^* = \{v \in V^* : \langle u, v \rangle \geq -1 \text{ for every } u \in Q\}$ ,
$\widehat{F}$	dual face to a face $F$ of a polytope $Q$ , i. e. $\widehat{F} := \{v \in Q^* : \langle u, v \rangle = -1 \text{ for every } u \in F\}$ ,
$\text{int}(A)$	topological interior of a subset $A$ in some finite-dimensional vector space,
$\text{relint}(A)$	relative interior of a subset $A$ in some finite-dimensional vector space, i. e. topological interior of $A$ in the affine span of $A$ .

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## 2. NOTATION AND GENERALITIES

Spherical embeddings admit a combinatorial description due to the Luna-Vust theory (see [LV83, Kno91]). Similarly to the theory of toric varieties, one obtains a description of spherical embeddings of  $G/H$  by colored fans, which are combinatorial objects living in the vector space  $\mathcal{N}_{\mathbb{Q}}$ .

**Definition 2.1.** A *colored cone* is a pair  $(\mathcal{C}, \mathcal{F})$  where  $\mathcal{F} \subseteq \mathcal{D}$  and  $\mathcal{C} \subseteq \mathcal{N}_{\mathbb{Q}}$  is a cone generated by  $\rho(\mathcal{F})$  and finitely many elements of  $\mathcal{V}$ . A colored cone is called *supported* if  $\text{relint}(\mathcal{C}) \cap \mathcal{V} \neq \emptyset$ . A colored cone is called *strictly convex* if  $\mathcal{C}$  is strictly convex and  $0 \notin \rho(\mathcal{F})$ .

**Definition 2.2.** A *face* of a colored cone  $(\mathcal{C}, \mathcal{F})$  is a colored cone  $(\mathcal{C}', \mathcal{F}')$  such that  $\mathcal{C}'$  is a face of  $\mathcal{C}$  and  $\mathcal{F}' = \mathcal{F} \cap \rho^{-1}(\mathcal{C}')$ . It is called a *supported face* if it is supported as a colored cone.

**Definition 2.3.** A *colored fan* is a nonempty finite collection  $\mathfrak{F}$  of strictly convex colored cones such that for every  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$  every face of  $(\mathcal{C}, \mathcal{F})$  is also in  $\mathfrak{F}$  and for every  $v \in \mathcal{N}_{\mathbb{Q}}$  there is at most one  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$  with  $v \in \text{relint}(\mathcal{C})$ . A colored fan  $\mathfrak{F}$  is called *complete* if  $\text{supp } \mathfrak{F} := \bigcup_{(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}} \mathcal{C} = \mathcal{N}_{\mathbb{Q}}$ .

**Definition 2.4.** A *supported colored fan* is a nonempty finite collection  $\mathfrak{F}$  of strictly convex supported colored cones such that for every  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$  every supported face of  $(\mathcal{C}, \mathcal{F})$  is also in  $\mathfrak{F}$  and for every  $v \in \mathcal{V}$  there is at most one  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$  with  $v \in \text{relint}(\mathcal{C})$ . A supported colored fan  $\mathfrak{F}$  is called *complete* if  $\text{supp } \mathfrak{F} \supseteq \mathcal{V}$ .

**Remark 2.5.** We have defined the terms “colored cone” and “colored fan” with and without the adjective “supported”. In the literature, only the supported versions are usually defined (and without the adjective “supported”).

**Remark 2.6.** There is a natural map

$$\begin{aligned} \{\text{colored fans}\} &\rightarrow \{\text{supported colored fans}\} \\ \mathfrak{F} &\mapsto \mathfrak{F}_{\text{supp}} := \{\sigma \in \mathfrak{F} : \sigma \text{ is supported}\}. \end{aligned}$$

If  $\mathfrak{F}$  is complete, every point in  $\mathcal{V}$  is contained in  $\text{relint}(\mathcal{C})$  for exactly one  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$ , which is supported, so  $\mathfrak{F}_{\text{supp}}$  is complete as well (as a supported colored fan).

**Theorem 2.7** ([Kno91, Theorem 3.3]). *Supported colored fans are in bijective correspondence with isomorphism classes of spherical embeddings  $G/H \hookrightarrow X$ . Moreover,  $X$  is complete if and only if the corresponding supported colored fan is complete.*

We now recall some results on divisors in spherical varieties due to Brion (see [Bri89]). We use [Tim11, Section 17] as general reference. Let  $G/H \hookrightarrow X$  be a complete spherical embedding and  $\mathfrak{F}$  the corresponding supported colored fan. Let  $E$  be a  $B$ -stable Weil divisor. The Weil divisor  $E$  is  $\mathbb{Q}$ -Cartier (resp. Cartier) if and only if for every maximal supported colored cone  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$  there exists  $v_{\mathcal{C}} \in \mathcal{M}_{\mathbb{Q}}$  (resp.  $v_{\mathcal{C}} \in \mathcal{M}$ ) such that for every  $B$ -stable prime divisor  $D$  which contains the  $G$ -orbit in  $X$  corresponding to  $(\mathcal{C}, \mathcal{F})$  the multiplicity of  $D$  in  $E$  is  $\langle \rho(D), v_{\mathcal{C}} \rangle$  if  $D$  is a color and  $\langle \nu_D, v_{\mathcal{C}} \rangle$  if  $D$  is  $G$ -stable. The linear functions  $\langle \cdot, v_{\mathcal{C}} \rangle : \mathcal{C} \cap \mathcal{V} \subseteq \mathcal{N}_{\mathbb{Q}} \rightarrow \mathbb{Q}$  may be pasted together to a piecewise linear function  $\psi_E : \mathcal{V} \rightarrow \mathbb{Q}$ .

**Proposition 2.8** ([Bri89, Proposition 3.1]).  *$X$  is  $\mathbb{Q}$ -factorial (resp. locally factorial) if and only if for every maximal  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$  the cone  $\mathcal{C}$  is spanned by a part of a  $\mathbb{Q}$ -basis of  $\mathcal{N}_{\mathbb{Q}}$  (resp. by a part of a  $\mathbb{Z}$ -basis of  $\mathcal{N}$ ) containing  $\rho(\mathcal{F})$  and  $\rho|_{\mathcal{F}}$  is injective.*

**Proposition 2.9** ([Bri89, Théorème 3.3], see also [Tim11, Corollary 17.24]). *Let  $E$  be a  $\mathbb{Q}$ -Cartier divisor on the complete spherical variety  $X$  and  $\psi_E : \mathcal{V} \rightarrow \mathbb{Q}$  the associated piecewise linear function. Then  $E$  is ample if and only if*

- (1) *the piecewise linear function  $\psi_E$  is strictly convex, i. e. for every  $u \in \mathcal{C} \setminus \mathcal{C}'$  we have  $\langle u, v_{\mathcal{C}} \rangle > \langle u, v_{\mathcal{C}'} \rangle$  for any two maximal  $(\mathcal{C}, \mathcal{F}), (\mathcal{C}', \mathcal{F}') \in \mathfrak{F}$ , and*
- (2) *for every maximal  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$  and every  $D \in \mathcal{D} \setminus \mathcal{F}$  the multiplicity of  $D$  in  $E$  is strictly greater than  $\langle \rho(D), v_{\mathcal{C}} \rangle$ .*

Finally, we will require the following definition.

**Definition 2.10.** Let  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  be a polytope with  $0 \in \text{int}(Q)$ . We associate to it the colored fan  $\mathfrak{F}(Q)$  consisting of the colored cones  $(\text{cone}(F), \rho^{-1}(F))$  for all proper faces  $F$  of  $Q$ . As  $Q$  is full-dimensional, the colored fan  $\mathfrak{F}(Q)$  is complete. The colored fan  $\mathfrak{F}(Q)$  is called the *colored face fan* of  $Q$ . We write  $\mathfrak{F}_{\text{supp}}(Q)$  for  $(\mathfrak{F}(Q))_{\text{supp}}$ .

### 3. THE CO- AND TANGENT SHEAVES OF A SMOOTH $G$ -VARIETY

In this section, let  $X$  be an arbitrary smooth  $G$ -variety for an arbitrary algebraic group  $G$ . Then  $G$  acts on  $X$  by an action morphism  $\alpha : G \times X \rightarrow X$ . We denote by  $\mu : G \times G \rightarrow G$  the multiplication morphism of the algebraic group  $G$  and by  $\pi_X : G \times X \rightarrow X$  (resp.  $\pi_{G \times X} : G \times G \times X \rightarrow G \times X$ ) the natural projection on the second (resp. on the second and third) factor. Let us repeat the definition of a  $G$ -linearization of a quasicoherent sheaf (see [Tim11, Definition C.2] or [MFK94, Definition 1.6]).

**Definition 3.1.** A  $G$ -linearization of a quasicoherent sheaf  $\mathfrak{F}$  on  $X$  is an isomorphism of quasicoherent sheaves  $\hat{\alpha} : \pi_X^* \mathfrak{F} \xrightarrow{\sim} \alpha^* \mathfrak{F}$  satisfying the cocycle condition, i. e. the diagram in Figure 1 commutes.

Recall that the *cotangent sheaf*  $\Omega_X$  of  $X$  is, locally on affine open neighbourhoods  $U$ , given as the sheaf associated to the module of Kähler differentials  $\Omega_{\mathcal{O}_X(U)/\mathbb{C}}$ . The pullback of differential forms with respect to the action morphism  $\alpha$  yields the inverse of a  $G$ -linearization of the cotangent sheaf, namely  $\hat{\alpha}^{-1} : \alpha^* \Omega_X \xrightarrow{\sim} \pi_X^* \Omega_X$ . As  $X$  is smooth, we may dualize  $\hat{\alpha}^{-1}$  and obtain a  $G$ -linearization of the tangent sheaf  $\mathcal{T}_X := \text{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ , namely  $\hat{\beta} := (\hat{\alpha}^{-1})^\vee : \pi_X^* \mathcal{T}_X \xrightarrow{\sim} \alpha^* \mathcal{T}_X$ .

For an affine open subset  $U \subseteq X$  and  $g \in G$ , let us denote the coordinate rings of  $U$  and  $g \cdot U$  by  $A$  and  $B$  respectively. The element  $g \in G$  acts on a local

$$\begin{array}{ccc}
(\pi_X \circ \pi_{G \times X})^* \mathfrak{F} & \xrightarrow{\pi_{G \times X}^* \widehat{\alpha}} & (\alpha \circ \pi_{G \times X})^* \mathfrak{F} \\
\parallel & & \parallel \\
& & (\pi_X \circ (\text{id}_G \times \alpha))^* \mathfrak{F} \xrightarrow{(\text{id}_G \times \alpha)^* \widehat{\alpha}} (\alpha \circ (\text{id}_G \times \alpha))^* \mathfrak{F} \\
\parallel & & \parallel \\
(\pi_X \circ (\mu \times \text{id}_X))^* \mathfrak{F} & \xrightarrow{(\mu \times \text{id}_X)^* \widehat{\alpha}} & (\alpha \circ (\mu \times \text{id}_X))^* \mathfrak{F}
\end{array}$$

FIGURE 1. The cocycle condition.

section  $\delta \in \text{Der}_{\mathbb{C}}(A, A) = \Gamma(U, \mathcal{T}_X)$  by restricting the  $G$ -linearization to  $\{g\} \times X$ , i. e.  $g \cdot \delta := \widehat{\beta}|_{\{g\} \times X}(\delta)$ . It is straightforward to check that

$$g \cdot \delta = \lambda_g^\# \circ \delta \circ \lambda_{g^{-1}}^\# \in \text{Der}_{\mathbb{C}}(B, B) = \Gamma(g \cdot U, \mathcal{T}_X),$$

where  $\lambda_g: X \rightarrow X$  is given by  $x \mapsto g^{-1} \cdot x$ .

Let  $\mathbb{G} = (\mathbb{C}, +)$  or  $\mathbb{G} = (\mathbb{C}^*, \cdot)$  be a one-dimensional connected algebraic group with neutral element  $e \in \mathbb{G}$ . We will recall how one can associate a global vector field  $\mathbf{u} \in \Gamma(X, \mathcal{T}_X)$  to a one-parameter subgroup  $u: \mathbb{G} \rightarrow G$ . The coordinate ring of  $\mathbb{G}$  is either the polynomial ring  $\mathbb{C}[t]$  or the Laurent polynomial ring  $\mathbb{C}[t^{\pm 1}]$ . In both cases, we have a natural choice of a basis of the tangent space of  $\mathbb{G}$  over the point  $e$ , namely  $\frac{\partial}{\partial t}|_e \in \mathcal{T}_{\mathbb{G}}|_e$ . Let  $U$  be an affine open subset of  $X$  with coordinate ring  $A$ . The restriction of  $\mathbf{u}$  to  $U$  lies in  $\text{Der}_{\mathbb{C}}(A, A)$  and is given by

$$(\mathbf{u}|_U f)(x) := \frac{\partial}{\partial t} \Big|_e f(u(t) \cdot x)$$

for  $f \in A$ ,  $x \in U$ . It is straightforward to check that these local sections are well-defined and glue to a global section  $\mathbf{u} \in \Gamma(X, \mathcal{T}_X)$ .

**Remark 3.2.** Assume that  $X = G$  and  $G$  acts on itself by left translation. It is then straightforward to check that  $\mathbf{u}$  is an *invariant* vector field, i. e.  $\rho_g^\# \circ \mathbf{u} \circ \rho_{g^{-1}}^\# = \mathbf{u}$  for all  $g \in G$ , where  $\rho_g: G \rightarrow G$  is given by  $h \mapsto hg$ . Moreover,  $\mathbf{u}|_e \in \text{Lie } \mathfrak{u}(\mathbb{G})$ .

#### 4. THE ANTICANONICAL SHEAF OF A SPHERICAL VARIETY

From now on, we continue to use the notation and the assumptions from the introduction. In this section, we reproduce (with more detail) the proof of [Bri97, 4.1].

Let  $G/H \hookrightarrow X$  be a spherical embedding. We denote the  $G$ -invariant prime divisors in  $X$  by  $X_1, \dots, X_n$ . As we are interested in the anticanonical sheaf of  $X$ , we may assume that  $X$  does not contain  $G$ -orbits of codimension two or greater. In particular,  $X$  is smooth toroidal and  $\text{Pic}(X) = \text{Cl}(X)$ .

Since  $X$  is a smooth toroidal variety, it has the following local structure (see [Tim11, Theorem 29.1]): The set  $X^0 := X \setminus \bigcup_{i=1}^k \overline{D_i}$  is stable by  $P$ . Let  $L$  be the Levi subgroup of  $P$  containing  $T$ . There exists a closed  $L$ -stable subvariety  $Z$  of  $X^0$  such that

$$\begin{aligned}
R_u(P) \times Z &\rightarrow X^0 \\
(u, z) &\mapsto u \cdot z
\end{aligned}$$

is a  $P$ -equivariant isomorphism. The kernel of the  $L$ -action on  $Z$ , which we denote by  $L_0$ , contains  $(L, L)$  and  $Z$  is a toric embedding of  $L/L_0$ . Every  $G$ -orbit intersects  $Z$  in a unique  $L/L_0$ -orbit.

Any  $x_0 \in U$  corresponds to some  $(u_0, z_0) \in R_u(P) \times Z$  under the isomorphism above. We fix  $x_0$  such that  $u_0$  is the neutral element of  $R_u(P)$ . By replacing  $H$  with a conjugate, we may assume that  $x_0$  has stabilizer  $H \subseteq G$ .

We denote by  $\check{\omega}_X := \bigwedge^{\dim X} \mathcal{T}_X$ , i.e. the top exterior power of the tangent sheaf, the anticanonical sheaf of  $X$ . It is an invertible sheaf and carries a natural  $G$ -linearization induced from the  $G$ -linearization of  $\mathcal{T}_X$ .

The variety  $Z$  is a toric variety for a quotient torus of  $T$ . Let  $T_0$  be the kernel of the  $T$ -action on  $Z$  and let  $T_1$  be a subtorus of  $T$  with  $T = T_0 T_1$  such that  $T_0 \cap T_1$  is finite. We have a commutative diagram of equivariant morphisms with respect to the action of  $B_0 := R_u(P)T_1$ :

$$\begin{array}{ccc} B_0 & \xrightarrow{\quad\quad\quad} & B \cdot x_0 \\ & \searrow & \nearrow \\ & & B_0/(T_0 \cap T_1) \end{array}$$

The arrows are finite coverings. In particular, the tangent space of  $X$  at  $(u_0, z_0)$  is isomorphic to the direct sum of the tangent spaces of  $R_u(P)$  and  $T_1$  at the corresponding neutral elements. This is the lie algebra of  $B_0$  which decomposes as

$$\mathrm{Lie} B_0 = \mathrm{Lie} T_1 \oplus \bigoplus_{\alpha \in R^+ \setminus \langle S^p \rangle} \mathfrak{g}_\alpha$$

where  $\langle S^p \rangle$  denotes the root system generated by  $S^p$  and  $\mathfrak{g}_\alpha$  denotes the subspace of  $T$ -semi-invariant vectors of weight  $\alpha$  in  $\mathfrak{g} := \mathrm{Lie} G$ . Choose a realization  $(u_\alpha)_{\alpha \in R}$  of the root system  $R$  (see [Spr09, §8.1]) and choose a basis  $\lambda_1, \dots, \lambda_r$  of the lattice of one-parameter multiplicative subgroups of  $T_1$ . In Section 3 we have seen how to associate a global vector field  $u_\alpha$  (resp.  $\iota_1, \dots, \iota_r$ ) to the one parameter subgroup  $u_\alpha$  (resp.  $\lambda_1, \dots, \lambda_r$ ). We obtain a global section

$$s := \left( \bigwedge_{\alpha \in R^+ \setminus \langle S^p \rangle} u_\alpha \right) \wedge \iota_1 \wedge \dots \wedge \iota_r \in \Gamma(X, \check{\omega}_X).$$

**Proposition 4.1** ([Bri97, Proposition 4.1]). *The zero set of  $s$  is exactly the union of the closures of the colors  $D_i$  and the boundary divisors  $X_j$ .*

*Proof.* Let  $u$  be a one-parameter subgroup used in the definition of  $s$ , i.e.  $u = u_\alpha$  or  $u \in \{\lambda_1, \dots, \lambda_r\}$ . Let  $Y$  be a  $B$ -stable divisor, i.e.  $Y = \overline{D_i}$  or  $Y = X_j$ . Since  $u$  maps into  $B$ , by construction,  $u|_y \in \mathcal{T}_Y|_y$  for all  $y \in Y$ . Since  $Y$  has codimension 1 in  $X$  and the number of vector fields which have been wedged to obtain  $s$  is  $\dim X$ , the global section  $s$  vanishes on  $Y$ .

We now show that  $s$  vanishes nowhere on the open  $B$ -orbit. Since  $u$  maps into  $B_0$ , we may define a global vector field  $u'$  on  $B_0$ . By the local structure theorem,  $B_0$  is a finite covering of  $B \cdot x_0$ . In particular, we have a well-defined pushforward of vector fields, and  $u|_{B \cdot x_0}$  is the pushforward of  $u'$ . Since  $u'$  is  $T_0 \cap T_1$ -invariant (see Remark 3.2), it suffices to show that  $s' \in \Gamma(B_0, \check{\omega}_{B_0})$ , i.e. the global section of the anticanonical sheaf of  $B_0$  which arises by wedging all the global vector fields  $u'$ , vanishes nowhere.

Since  $u'|_e \in \mathrm{Lie} u(\mathbb{G})$ , it follows that  $s'|_e$  arises by wedging a basis of  $\mathrm{Lie} B_0$ . In particular,  $s'$  does not vanish at  $e$ . Since  $s'$  is invariant, it follows that it vanishes nowhere.  $\square$

Recall from the introduction that, by the Borel-Weil-Bott theorem, the space  $\Gamma(G/P^-, \check{\omega}_{G/P^-})$  is a simple  $G$ -module, whose highest weight we denote by  $\kappa_P$ . Then the following result implies Theorem 1.2.



**Proposition 4.2** ([Bri97, Proof of Theorem 4.2]). *The global section  $s \in \Gamma(X, \check{\omega}_X)$  is  $B$ -semi-invariant of weight  $\kappa_P$ .*

*Proof.* Since  $U$  has trivial Picard group, every  $B$ -linearized invertible sheaf on  $U$  is  $B$ -equivariantly isomorphic to  $\mathcal{O}_U(\chi)$  for some  $\chi \in \mathfrak{X}(B)$ . We have maps  $\check{\omega}_U \rightarrow \mathcal{O}_U(\chi) \rightarrow \mathcal{O}_U$ , where the first map is chosen to be a  $B$ -equivariant isomorphism, and the second map is canonical, but not necessarily  $B$ -equivariant. Let  $f$  be the image of  $s$  under the composed map. By [KKV89, Proposition 1.3, (ii)], the regular function  $f$  is  $B$ -semi-invariant. It follows that  $s$  is  $B$ -semi-invariant as well, where the weight has to be corrected by the twist  $\chi$ .

Next, we determine the weight of  $s$ . Let  $w \in T$ ,  $f \in \Gamma(U, \mathcal{O}_U)$ , and  $x \in U$ . Then we have

$$\begin{aligned} ((w \cdot \mathbf{u}_\alpha)(f))(x) &= \left( \left( \lambda_w^\# \circ \mathbf{u}_\alpha \circ \lambda_{w^{-1}}^\# \right) (f) \right) (x) \\ &= \frac{\partial}{\partial t} \Big|_e f(wu(t)w^{-1} \cdot x) \\ &= \frac{\partial}{\partial t} \Big|_e f(u(\alpha(w)t) \cdot x) \\ &= \left( (\alpha(w)\mathbf{u}_\alpha)(f) \right) (x). \end{aligned}$$

Hence  $\mathbf{u}_\alpha$  is  $T$ -semi-invariant of weight  $\alpha$ . Analogously, one can show that  $\mathfrak{l}_i$  is  $T$ -invariant. In particular, the weight of  $s$  depends only on the set  $S^p$ . As  $G/P^-$  and  $X$  have the same stabilizer of the open  $B$ -orbit, it follows that  $s$  is  $B$ -semi-invariant of weight  $\kappa_P$ .  $\square$

**Remark 4.3.** For  $I \subseteq S$  we denote by  $\rho_I$  the half-sum of the positive roots in the root system generated by  $I$ . It is also known that  $\rho_I$  is the sum of the fundamental dominant weights of the root system generated by  $I$  (see [Hum78, Lemma 13.3A]). By Proposition 4.2, we have  $\kappa_P = 2\rho_S - 2\rho_{S^p}$ .

**Remark 4.4.** A nonzero  $B$ -semi-invariant rational section of  $\check{\omega}_X$  is uniquely determined by its weight  $\chi \in \mathfrak{X}(B)$  up to a constant factor, and such a rational section exists if and only if  $\chi \in \kappa_P + \mathcal{M}$  (see also Section 9).

**Corollary 4.5** ([Bri97, Proposition 4.1]). *We have*

$$\operatorname{div} s = \sum_{i=1}^k m_i D_i + \sum_{j=1}^n X_j$$

with  $m_i \in \mathbb{Z}_{>0}$ .

*Proof.* Since  $s$  is a  $B$ -semi-invariant section, it follows that its divisor is a linear combination of the  $B$ -invariant divisors of  $X$ , i. e.

$$\operatorname{div} s = \sum_{i=1}^k m_i D_i + \sum_{j=1}^n r_j X_j$$

for integers  $m_i$  and  $r_j$ . Since  $s$  vanishes on every  $B$ -invariant divisor, it follows that the  $m_i$  and  $r_j$  are positive. To show that  $r_j = 1$ , we consider the restriction  $s' := s|_{X^0}$ . Above, we have seen that the open subset  $X^0$  is isomorphic to the product variety  $R_u(P) \times Z$ . In particular,  $\check{\omega}_{X^0} \cong \pi_1^* \check{\omega}_{R_u(P)} \otimes \pi_2^* \check{\omega}_Z$  where  $\pi_i$  denotes the projection onto the  $i$ -th factor. The section  $s'$  behaves well under this product decomposition. Indeed, set  $s_1 := \bigwedge_{\alpha \in R^+ \setminus \langle S^p \rangle} \mathbf{u}_\alpha|_{R_u(P)}$  and  $s_2 := \mathfrak{l}_1|_Z \wedge \dots \wedge \mathfrak{l}_r|_Z$ .

Then under the isomorphism above the section  $s'$  corresponds to the section  $s_1 \otimes s_2$ . Since  $s_1$  does not vanish on  $R_u(P)$ , we obtain

$$\sum_{j=1}^n r_j X_j = (\operatorname{div} s)|_{X^0} = \operatorname{div} s' = \operatorname{div}(s_1 \otimes s_2) = \pi_2^* \operatorname{div} s_2.$$

Now,  $Z$  is a toric variety with respect to the quotient torus  $T_1/(T_0 \cap T_1)$ , and the pullback under  $\pi_2$  of its torus-invariant divisors are exactly the  $G$ -invariant divisors  $X_j$  of  $X$ . Since  $\sigma_2$  is invariant under the action of the torus  $T_1/(T_0 \cap T_1)$ , it follows, by toric geometry, that  $r_j = 1$ .  $\square$

*Proof of Theorem 1.4.* By Corollary 4.5, it follows that the  $B$ -semi-invariant section  $s$  of Theorem 1.2 has a zero of order 1 along any  $G$ -invariant prime divisor in any spherical embedding  $G/H \hookrightarrow X$ .

Now assume that  $s'$  is a generator of  $\Gamma(U, \check{\omega}_{G/H})$  which has a zero of order 1 along any  $G$ -invariant prime divisor in any spherical embedding  $G/H \hookrightarrow X$ . Since the open  $B$ -orbit  $U$  of  $G/H$  has trivial divisor class group, we have maps  $\check{\omega}_U \rightarrow \mathcal{O}_U(\chi) \rightarrow \mathcal{O}_U$ , where the first map is chosen to be a  $B$ -equivariant isomorphism, and the second map is canonical, but not necessarily  $B$ -equivariant. Under this isomorphism  $s$  and  $s'$  correspond to functions  $f, f' \in \Gamma(U, \mathcal{O}_{G/H})$ . Since  $s$  and  $s'$  are generators, the functions  $f$  and  $f'$  are invertible. In particular,  $f/f' \in \Gamma(U, \mathcal{O}_{G/H}^*)$ . By [KKV89, Proposition 1.3, (ii)],  $f/f'$  is  $B$ -semi-invariant and hence its  $B$ -weight is contained in  $\mathcal{M}$ .

Take any spherical embedding  $G/H \hookrightarrow X$  which contains exactly two orbits, namely  $G/H$  and a  $G$ -invariant prime divisor  $X_1$ . The colored fan corresponding to  $X$  is given by the ray  $\mathbb{Q}_{\geq 0}\nu_1$  where  $\nu_1 \in \mathcal{N}$  denotes the  $G$ -invariant valuation induced by  $X_1$ . Since  $s$  and  $s'$  have a zero of order 1 along  $X_1$ , we obtain  $\langle f/f', \nu_1 \rangle = 0$ .

It follows that the valuation cone  $\mathcal{V}$  of  $G/H$  is contained in the subspace  $\{w \in \mathcal{N}_{\mathbb{Q}} : \langle f/f', w \rangle = 0\}$ . Since  $\mathcal{V}$  is a full-dimensional cone, this is only possible if the  $B$ -weight of  $f/f'$  is 0, i. e.  $f$  and  $f'$  coincide up to a scalar multiple.  $\square$

## 5. TYPES OF COLORS

In [Bri97, 4.2], Brion has defined colors of types II, III, and IV. These types are equivalent to the types  $b$ ,  $a$ , and  $2a$  due to Luna respectively, which means that Theorem 1.5 is equivalent to [Bri97, Theorem 4.2]. We will, however, prove Theorem 1.5 by reducing it to the situation of [Lun97, 3.6], i. e. to the case of a wonderful variety (and afterwards it will be easy to see that the types of colors due to Brion and Luna coincide).

A *wonderful variety* is a spherical variety which is complete, smooth, simple, and toroidal. We explain (from [Lun01, 6.1]) how to associate to the spherical variety  $X$  a wonderful variety  $Y$ . We identify the  $G$ -equivariant automorphism group of  $G/H$  with  $N_G(H)/H$ . Then  $N_G(H)$  acts on  $\mathcal{D}$ , and we define  $\overline{H} \subseteq N_G(H)$  to be the kernel of this action. It contains  $H$  and is called the *spherical closure* of  $H$ . There exists a (unique) spherical embedding  $G/\overline{H} \hookrightarrow Y$  such that  $Y$  is a wonderful variety. We denote its set of colors by  $\overline{\mathcal{D}} = \{\overline{D}_1, \dots, \overline{D}_k\}$ , which is in bijection with the set of colors  $\mathcal{D} = \{D_1, \dots, D_k\}$  of  $G/H$  via  $\pi: G/H \rightarrow G/\overline{H}$ . We denote the stabilizer of the open  $B$ -orbit in  $Y$  by  $\overline{P}$  and the coefficients of the colors in the expression for the anticanonical divisor from Theorem 1.2 by  $\overline{m}_1, \dots, \overline{m}_k$ .

**Theorem 5.1** ([Lun97, 3.6]). *We have*

$$\begin{aligned} \overline{m}_i = r(\overline{D}_i) &:= \frac{1}{2} \langle \alpha^\vee, \kappa_{\overline{P}} \rangle = 1 && \text{for } \overline{D}_i \text{ of type } a \text{ or } 2a, \\ \overline{m}_i = r(\overline{D}_i) &:= \langle \alpha^\vee, \kappa_{\overline{P}} \rangle \geq 2 && \text{for } \overline{D}_i \text{ of type } b. \end{aligned}$$

*Proof.* By [Lun97, Proposition 3.6(2)], an anticanonical divisor of  $Y$  is given by  $\sum_{i=1}^k r(\overline{D}_i)\overline{D}_i + \sum_{j=1}^l Y_j$  where  $Y_1, \dots, Y_l$  are the  $G$ -invariant prime divisors in  $Y$ . The result now follows from the fact that  $\overline{D}$  is a basis of  $\text{Pic}(Y) = \text{Cl}(Y)$ .  $\square$

*Proof of Theorem 1.5.* Let  $s \in \Gamma(G/H, \check{\omega}_{G/H})$  and  $\overline{s} \in \Gamma(G/\overline{H}, \check{\omega}_{G/\overline{H}})$  be  $B$ -semi-invariant sections of weights  $\kappa_P$  and  $\kappa_{\overline{P}}$  respectively. As  $\overline{P} = P$ , we have  $\pi^*\overline{s} = s$  (up to a constant), and hence, for the pullback of Cartier divisors,  $\pi^* \text{div } \overline{s} = \text{div } s$ . On the other hand, we have the pullback of Cartier divisors  $\pi^*\overline{D}_i = D_i$  (apply [Fos98, Section 2.2, Theorem 2.2], see also [Tim11, Lemma 30.24]), hence  $m_i = \overline{m}_i$ .  $\square$

**Remark 5.2.** Replacing  $G$  with a finite cover, we may assume  $G = G^{ss} \times C$  where  $G^{ss}$  is semisimple simply-connected and  $C$  is a torus. Then, by [KKLV89, Proposition 2.4 and Remark after it], every invertible sheaf on the normal variety  $X$  can be  $G$ -linearized. There exist (unique)  $G$ -linearizations of the invertible sheaves  $\mathcal{O}_X(D_i)$  and  $\mathcal{O}_X(X_j)$  such that their canonical sections are  $C$ -invariant. With these linearizations, Brion's description of the anticanonical sheaf

$$\check{\omega}_X = \mathcal{O}_X(D_1)^{\otimes m_1} \otimes \dots \otimes \mathcal{O}_X(D_k)^{\otimes m_k} \otimes \mathcal{O}_X(X_1) \otimes \dots \otimes \mathcal{O}_X(X_n)$$

of an arbitrary spherical variety  $X$  is not only valid inside  $\text{Pic}(X_{\text{reg}})$ , but even inside the group of isomorphism classes of  $G$ -linearized invertible sheaves  $\text{Pic}^G(X_{\text{reg}})$ .

**Proposition 5.3.** *The types of colors II, III, and IV due to Brion coincide with the types of colors  $b$ ,  $a$ , and  $2a$  due to Luna respectively.*

*Proof.* A color  $D_i \in \mathcal{D}$  is of type II if and only if it is of type  $b$  as both situations are characterized by  $m_i \geq 2$ . Now let  $D \in \mathcal{D}$  be of type III (resp. of type IV), and let  $C$  be a basic curve representing  $D$  in the sense of [Bri97, 4.2]. The kernel of the  $B$ -action on  $C$  is the radical of a (unique) minimal parabolic subgroup  $P_\alpha$  (see [Bri97, 1.1]), and we have  $D \in \mathcal{D}(\alpha)$  (see [Bri93, Proposition 3.6]). According to [Bri97, Proposition 1.2], the spherical variety  $G \cdot C$  contains a color of type  $a$  (resp. of type  $2a$ ) which is moved by  $P_\alpha$ . By [GH15, Theorem 1.1], this is only possible if  $\alpha \in \Sigma$  (resp. if  $2\alpha \in \Sigma$ ), i. e.  $D$  is of type  $a$  (resp. of type  $2a$ ).  $\square$

Finally, it will be helpful to explain one further approach due to Knop to characterize the types of colors, which is easier to apply in certain situations (see the examples in Section 6). We fix a point  $x_0 \in U$  in the open  $B$ -orbit. Let  $\alpha \in S$  such that  $\mathcal{D}(\alpha) \neq \emptyset$ . We have  $P_\alpha/B \cong \mathbb{P}^1$  and the natural action of  $P_\alpha$  on  $\mathbb{P}^1$  yields a morphism  $\phi_\alpha: P_\alpha \rightarrow \text{PGL}_2$ . Let  $H_\alpha$  be the stabilizer of  $x_0$  inside  $P_\alpha$ . Then  $\phi_\alpha(H_\alpha)$  is a proper spherical subgroup of  $\text{PGL}_2$  (see [Kno95, Lemma 3.1]), i. e.  $\phi_\alpha(H_\alpha)$  is either a maximal torus, the normalizer of a maximal torus, or contains a maximal unipotent subgroup (see [Kno95, Lemma 3.2]).

**Theorem 5.4** ([Kno14, Section 2]). *A color  $D \in \mathcal{D}(\alpha)$  is*

- of type  $a$  if  $\Phi_\alpha(H_\alpha)$  is a maximal torus,*
- of type  $2a$  if  $\Phi_\alpha(H_\alpha)$  is the normalizer of a maximal torus,*
- of type  $b$  if  $\Phi_\alpha(H_\alpha)$  contains a maximal unipotent subgroup.*

## 6. EXAMPLES ILLUSTRATING THEOREM 1.5

In this section, we compute the coefficients  $m_i$  for several well-known spherical homogeneous spaces. In 6.1, 6.2, and 6.3 we give examples for every type of color. Example 6.4 illustrates  $S^p \neq \emptyset$  and  $m_i > 2$ . Observe that in 6.1, 6.3, and 6.4 the computation is simplified by applying Theorem 5.4.

**Example 6.1.** Consider  $G := \mathrm{SL}_n \times \mathrm{SL}_n$  and  $H := \mathrm{SL}_n \subseteq G$  diagonally embedded. Then  $G/H$  is isomorphic to  $\mathrm{SL}_n$  where  $G$  acts with the first factor from the left and with the second factor from the right after inverting. We denote by  $D_n \subseteq \mathrm{SL}_n$  the subgroup of diagonal matrices, by  $T_n \subseteq \mathrm{SL}_n$  the subgroup of upper triangular matrices, and by  $T_n^- \subseteq \mathrm{SL}_n$  the subgroup of lower triangular matrices. We define  $B := T_n^- \times T_n$  and  $T := D_n \times D_n$ , and obtain the set of simple roots  $S = \{\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}\}$ .

The homogeneous space  $G/H$  is spherical, and there are  $n - 1$  colors of  $G/H$ , i.e.  $\mathcal{D} = \{D_1, \dots, D_{n-1}\}$ , given by  $D_i = \mathbb{V}(f_i)$  where  $f_i \in \mathbb{C}[\mathrm{SL}_n]$  is the upper-left principal minor of size  $i \times i$ . It is not difficult to see that  $\mathcal{D}(\alpha_i) = \mathcal{D}(\beta_i) = \{D_i\}$ . Furthermore, all colors are of type  $b$  since for every  $\alpha_i$  the image of  $H \cap P_{\alpha_i}$  under  $\Phi_{\alpha_i} : P_{\alpha_i} \rightarrow \mathrm{PGL}_2$  is a Borel subgroup of  $\mathrm{PGL}_2$  and therefore contains a maximal unipotent subgroup.

The stabilizer of the open  $B$ -orbit in  $G/H$  is  $B$  itself. Therefore we have  $S^p = \emptyset$  and  $\kappa_P = 2\rho_S$ . Since  $\rho_S$  is equal to the sum of the fundamental dominant weights, we obtain  $m_i = \langle \alpha_i^\vee, 2\rho_S \rangle = 2$  for  $1 \leq i \leq n - 1$ .

**Example 6.2.** Consider  $G := \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$  and  $H := \mathrm{SL}_2 \subseteq G$  diagonally embedded. We denote by  $B \subseteq G$  the Borel subgroup of lower triangular matrices and by  $T \subseteq B$  the maximal torus of diagonal matrices. We obtain the set of simple roots  $\{\alpha, \beta, \gamma\}$  corresponding to the three factors in  $G$ .

The homogeneous space  $G/H$  is spherical, and there are 3 colors of  $G/H$ , i.e.  $\mathcal{D} = \{D_{12}, D_{13}, D_{23}\}$ . To be more precise, let  $A_{ij}$  for  $1 \leq i < j \leq 3$  be the  $2 \times 2$  matrix whose rows are given by the first rows of the  $i$ -th and  $j$ -th factor of  $G$ . Then  $\det A_{ij}$  is an equation for  $D_{ij}$ . It is not difficult to see that  $\mathcal{D}(\alpha) = \{D_{12}, D_{13}\}$ ,  $\mathcal{D}(\beta) = \{D_{12}, D_{23}\}$ , and  $\mathcal{D}(\gamma) = \{D_{13}, D_{23}\}$ . In particular, it follows that all colors are of type  $a$ , and therefore  $m_{ij} = 1$  for  $1 \leq i < j \leq 3$ .

By the definition of the types of colors due to Luna (see Section 1), it follows that  $\alpha, \beta, \gamma$  are contained in  $\mathcal{M}$ , and the valuation cone is given by

$$\mathcal{V} = \{v \in \mathcal{N}_{\mathbb{Q}} : \langle v, \alpha \rangle \leq 0, \langle v, \beta \rangle \leq 0, \langle v, \gamma \rangle \leq 0\}.$$

**Example 6.3.** Consider  $G := \mathrm{SL}_n$  for  $n \geq 3$  and  $H := \mathrm{SO}_n$ , the subgroup of orthogonal matrices. Let  $G$  act on the space  $\mathrm{Sym}(n)$  of symmetric  $n \times n$  matrices via  $A \cdot M = AMA^T$ . Then the stabilizer of the identity matrix is  $H$ . We denote by  $B \subseteq G$  the Borel subgroup of lower triangular matrices and by  $T \subseteq B$  the maximal torus of diagonal matrices. We obtain the set of simple roots  $\{\alpha_1, \dots, \alpha_{n-1}\}$ .

The homogeneous space  $G/H$  is spherical, and there are  $n - 1$  colors of  $G/H$ , i.e.  $\mathcal{D} = \{D_1, \dots, D_{n-1}\}$ , given by  $D_i = \mathbb{V}(f_i)$  where  $f_i \in \mathbb{C}[\mathrm{SL}_n]$  is again the upper-left principal minor of size  $i \times i$ . It is not difficult to see that  $\mathcal{D}(\alpha_i) = \{D_i\}$ . Furthermore, all colors are of type  $2a$  since for every  $\alpha_i$  the image of  $H \cap P_{\alpha_i}$  under  $\Phi_{\alpha_i} : P_{\alpha_i} \rightarrow \mathrm{PGL}_2$  is the normalizer of a maximal torus. It follows that  $m_i = 1$  for  $1 \leq i \leq n - 1$ .

As in the previous example, it follows that  $2\alpha_i$  is contained in  $\mathcal{M}$  for  $1 \leq i \leq n - 1$ , and the valuation cone is given by

$$\mathcal{V} = \{v \in \mathcal{N}_{\mathbb{Q}} : \langle v, 2\alpha_i \rangle \leq 0 \text{ for every } 1 \leq i \leq n - 1\}.$$

**Example 6.4.** Consider  $G := \mathrm{SL}_n$  for  $n \geq 3$  and  $H := \mathrm{SL}_{n-1}$  embedded as the block diagonal matrices with entries on the lower-right of  $\mathrm{SL}_n$ . Let  $B \subseteq G$  be the Borel subgroup of upper triangular matrices and  $T \subseteq B$  the subgroup of diagonal matrices. We obtain the set of simple roots  $S = \{\alpha_1, \dots, \alpha_{n-1}\}$ .

Let  $G$  act on  $\mathbb{C}^n \times \mathbb{C}^n$  by acting naturally on the first factor and with the contragredient action on the second factor. Denoting the coordinates of the first factor by  $X_1, \dots, X_n$  and the coordinates of the second factor by  $Y_1, \dots, Y_n$ , we

obtain

$$G/H \cong \mathbb{V}(X_1Y_1 + \dots + X_nY_n - 1) \subseteq \mathbb{C}^n \times \mathbb{C}^n.$$

There are two colors  $D_1 := \mathbb{V}(X_n)$  and  $D_2 := \mathbb{V}(Y_1)$ . It is not difficult to see that  $\mathcal{D}(\alpha_{n-1}) = \{D_1\}$ ,  $\mathcal{D}(\alpha_1) = \{D_2\}$ , and  $S^p = \{\alpha_2, \dots, \alpha_{n-2}\}$ . We see that both colors are of type  $b$  as in the previous example, but we have to suitably conjugate  $H$  first since the basepoint does not lie in the open  $B$ -orbit. We obtain

$$m_1 = \langle \alpha_{n-1}^\vee, 2\rho_S - 2\rho_{S^p} \rangle = \left\langle \alpha_{n-1}^\vee, \sum_{i=1}^{n-1} \sum_{j=1}^i \alpha_j + \sum_{i=1}^{n-2} \sum_{j=1}^i \alpha_{n-j} \right\rangle = n - 1,$$

and similarly  $m_2 = n - 1$ .

## 7. $\mathbb{Q}$ -GORENSTEIN SPHERICAL FANO VARIETIES

We continue to use the notation from the introduction. In particular,  $G/H$  is a spherical homogeneous space,  $\mathcal{D} = \{D_1, \dots, D_k\}$  is the set of colors, and  $m_i \in \mathbb{Z}_{>0}$  are the coefficients of the colors in the expression for the anticanonical divisor from Theorem 1.2.

**Definition 7.1.** A polytope  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  is called  $\mathbb{Q}$ - $G/H$ -*reflexive* if the following conditions are satisfied:

- (1)  $\rho(D_i)/m_i \in Q$  for every  $i = 1, \dots, k$ .
- (2)  $0 \in \text{int}(Q)$ .
- (3) Every vertex of  $Q$  is contained in  $\{\rho(D_i)/m_i : i = 1, \dots, k\}$  or a primitive element in  $\mathcal{N} \cap \mathcal{V}$ .

**Proposition 7.2.** Let  $G/H \hookrightarrow X$  be a  $\mathbb{Q}$ -Gorenstein spherical Fano embedding. Then the polytope  $Q_X \subseteq \mathcal{N}_{\mathbb{Q}}$  is  $\mathbb{Q}$ - $G/H$ -reflexive.

*Proof.* Let  $X_1, \dots, X_n$  be the  $G$ -invariant prime divisors in  $X$ . It follows from the completeness of  $X$  that

$$\text{cone}(\rho(D_1), \dots, \rho(D_k), \nu_{X_1}, \dots, \nu_{X_n}) = \mathcal{N}_{\mathbb{Q}},$$

and therefore  $0 \in \text{int}(Q_X)$ .  $\square$

**Proposition 7.3.** Let  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  be a  $\mathbb{Q}$ - $G/H$ -reflexive polytope. Then  $\mathfrak{F}_{\text{supp}}(Q)$  is a complete supported colored fan such that the associated spherical embedding  $G/H \hookrightarrow X_{\mathfrak{F}_{\text{supp}}(Q)}$  is  $\mathbb{Q}$ -Gorenstein Fano.

*Proof.* As  $Q$  is full-dimensional,  $\mathfrak{F}(Q)$  is complete (see Definition 2.10), and therefore, by Remark 2.6,  $\mathfrak{F}_{\text{supp}}(Q)$  is complete as well. Let  $X := X_{\mathfrak{F}_{\text{supp}}(Q)}$ . Every maximal cone  $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}_{\text{supp}}(Q)$  is given by  $\mathcal{C} = \text{cone}(\widehat{v_{\mathcal{C}}})$  and  $\mathcal{F} = \rho^{-1}(\widehat{v_{\mathcal{C}}})$  for a vertex  $v_{\mathcal{C}} \in \mathcal{Q}^*$ . We define a piecewise linear function  $\psi: \mathcal{V} \rightarrow \mathbb{Q}$  by  $\psi|_{\mathcal{C} \cap \mathcal{V}} := -\langle \cdot, v_{\mathcal{C}} \rangle$ . It is straightforward to check that  $\psi$  is the piecewise linear function corresponding to the  $\mathbb{Q}$ -Cartier divisor  $-K_X$ . As  $Q$  is a polytope, the piecewise linear function  $\psi$  is strictly convex. Together with property (1) of Definition 7.1, it follows from Proposition 2.9 that  $-K_X$  is ample, and hence  $X = X_{\mathfrak{F}_{\text{supp}}(Q)}$  is Fano.  $\square$

**Proposition 7.4.** The assignments  $X \mapsto Q_X$  and  $Q \mapsto X_{\mathfrak{F}_{\text{supp}}(Q)}$  define a bijection between isomorphism classes of  $\mathbb{Q}$ -Gorenstein spherical Fano embeddings of  $G/H$  and  $\mathbb{Q}$ - $G/H$ -reflexive polytopes.

*Proof.* The well-definedness of the two maps follows from Proposition 7.2 and Proposition 7.3. It remains to show that the maps are inverse to each other.

Let  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  be a  $\mathbb{Q}$ - $G/H$ -reflexive polytope and  $G/H \hookrightarrow X$  the spherical embedding corresponding to the colored fan  $\mathfrak{F}_{\text{supp}}(Q)$ . Let  $u \in Q$  be a vertex. By property (3) of Definition 7.1, we have  $u \in Q_X$ , hence  $Q \subseteq Q_X$ . Now, let  $u \in Q_X$

be a vertex. If  $u = \frac{\rho(D_i)}{m_i}$  for some  $i \in \{1, \dots, k\}$ , then  $u \in Q$  because of property (1) of Definition 7.1. Otherwise, the vertex  $u$  is the primitive generator of a ray in  $\mathfrak{F}_{\text{supp}}(Q)$ , in which case the definition of the face fan also implies  $u \in Q$ . Hence we have  $Q_X \subseteq Q$ .

Let  $G/H \hookrightarrow X$  be a  $\mathbb{Q}$ -Gorenstein spherical Fano embedding with associated supported colored fan  $\mathfrak{F}$  and  $\psi: \mathcal{V} \rightarrow \mathbb{Q}$  the strictly convex piecewise linear function associated to the anticanonical divisor  $-K_X$ . It suffices to show that the maximal cones of  $\mathfrak{F}_{\text{supp}}(Q_X)$  and  $\mathfrak{F}$  coincide. Let  $(\mathcal{C}, \mathcal{F})$  be a maximal cone in  $\mathfrak{F}$ . Then  $\psi|_{\mathcal{C} \cap \mathcal{V}} = -\langle \cdot, v_{\mathcal{C}} \rangle$  for some  $v_{\mathcal{C}} \in \mathcal{M}_{\mathbb{Q}}$ . We obtain that  $F := \{u \in Q_X : \langle u, v \rangle = -1\}$  is a facet of  $Q_X$  such that  $\text{cone}(F) = \mathcal{C}$  and  $\mathcal{F} = \rho^{-1}(F)$ . In particular, it follows that all maximal cones of  $\mathfrak{F}$  are in  $\mathfrak{F}_{\text{supp}}(Q_X)$ .  $\square$

## 8. GORENSTEIN SPHERICAL FANO VARIETIES

Let  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  be a  $\mathbb{Q}$ - $G/H$ -reflexive polytope. In this section, we investigate when the associated  $\mathbb{Q}$ -Gorenstein spherical Fano embedding  $G/H \hookrightarrow X$  is Gorenstein, i. e. the anticanonical divisor is Cartier. Recall that the vertices of  $Q^*$  correspond to the colored cones of maximal dimension in  $\mathfrak{F}(Q)$ .

**Definition 8.1.** A vertex  $v \in Q^*$  is called *supported* if the corresponding colored cone in  $\mathfrak{F}(Q)$  is supported. In this case, the corresponding facet  $\widehat{v} \preceq Q$  is called *supported* as well. The set of supported vertices of  $Q^*$  is denoted by  $V_{\text{supp}}(Q^*)$ .

**Lemma 8.2.** *Let  $\mathcal{C} \subseteq \mathcal{N}_{\mathbb{Q}}$  be a cone of maximal dimension. Then the following statements are equivalent:*

- (1)  $\text{relint}(\mathcal{C}) \cap \mathcal{V} = \emptyset$ .
- (2) *There exists  $v \in \mathcal{M}_{\mathbb{Q}}$  such that  $\langle \cdot, v \rangle|_{\mathcal{C}} \geq 0$  and  $\langle \cdot, v \rangle|_{\mathcal{V}} \leq 0$ .*
- (3)  $\text{cone}(\Sigma) \cap \mathcal{C}^{\vee} \neq \{0\}$ .

*Proof.* (1)  $\Rightarrow$  (2) follows from the Hahn-Banach separation theorem since  $\text{relint}(\mathcal{C})$  is open in  $\mathcal{N}_{\mathbb{Q}}$ , and (2)  $\Rightarrow$  (3) is obvious. In order to show (3)  $\Rightarrow$  (1), let  $0 \neq v \in \text{cone}(\Sigma) \cap \mathcal{C}^{\vee}$  and  $u \in \text{relint}(\mathcal{C})$ . As  $\mathcal{C}$  is full-dimensional, i. e.  $\mathcal{C}^{\perp} = \{0\}$ , we obtain  $\langle u, v \rangle > 0$ . It follows that  $u \notin \mathcal{V}$ .  $\square$

**Proposition 8.3.** *A vertex  $v \in Q^*$  is supported if and only if  $Q^* \cap (v + \text{cone}(\Sigma)) = \{v\}$ .*

*Proof.* Let  $v \in Q^*$  be a vertex and  $\mathcal{C} := \text{cone}(\widehat{v}) \subseteq \mathcal{N}_{\mathbb{Q}}$  the corresponding cone in  $\mathfrak{F}(Q)$ . We denote by

$$\mathcal{T}_v Q^* := \{v + \lambda \cdot (v' - v) : v' \in Q^*, \lambda \geq 0\}$$

the tangent cone of  $Q^*$  in  $v$ , which is an affine cone with apex  $v$ . By [HUL01, Corollary 5.2.5], we have  $\mathcal{T}_v Q^* - v = \mathcal{C}^{\vee}$  since  $\mathcal{C}$  is the normal cone of  $Q^*$  along the vertex  $v$ . The statement now follows from Lemma 8.2 since  $\mathcal{T}_v Q^* \cap (v + \text{cone}(\Sigma)) = \{v\}$  if and only if  $Q^* \cap (v + \text{cone}(\Sigma)) = \{v\}$ . The last statement follows since any  $v' \in \mathcal{T}_v Q^* \cap (v + \text{cone}(\Sigma))$  may be rescaled to be arbitrarily near to  $v$ .  $\square$

**Definition 8.4.** The  $\mathbb{Q}$ - $G/H$ -reflexive polytope  $Q$  is called  *$G/H$ -reflexive* if every supported vertex of  $Q^*$  lies in the lattice  $\mathcal{M}$ .

We will show in Proposition 8.6 that this definition is in agreement with Definition 1.8.

**Theorem 8.5.** *Recall that  $X$  is a  $\mathbb{Q}$ -Gorenstein spherical Fano variety by assumption. The variety  $X$  is Gorenstein if and only if the polytope  $Q$  is  $G/H$ -reflexive. In particular, there is a bijection between isomorphism classes of Gorenstein spherical Fano embeddings of  $G/H$  and  $G/H$ -reflexive polytopes.*

*Proof.* Let  $\psi: \mathcal{V} \rightarrow \mathbb{Q}$  be the piecewise linear function corresponding to the  $\mathbb{Q}$ -Cartier divisor  $-K_X$ . If  $\mathcal{C}_v \subseteq \mathcal{N}_{\mathbb{Q}}$  is the cone corresponding to the vertex  $v \in Q^*$ , we have  $\psi|_{\mathcal{C}_v \cap \mathcal{V}} = -\langle \cdot, v \rangle$ . Then  $-K_X$  is Cartier if and only if  $v \in \mathcal{M}$  for every  $v$  where  $\mathcal{C}_v$  is in  $\tilde{\mathfrak{F}}_{\text{supp}}(Q)$ , i. e. if and only if the supported vertices of  $Q^*$  lie in the lattice  $\mathcal{M}$ .  $\square$

**Proposition 8.6.** *A polytope  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  is  $G/H$ -reflexive if and only if the following conditions are satisfied:*

- (1)  $\rho(D_i)/m_i \in Q$  for every  $i = 1, \dots, k$ .
- (2)  $0 \in \text{int}(Q)$ .
- (3) Every vertex of  $Q$  is contained in  $\{\rho(D_i)/m_i : i = 1, \dots, k\}$  or  $\mathcal{N} \cap \mathcal{V}$ .
- (4) Every supported vertex of  $Q^*$  lies in the lattice  $\mathcal{M}$ .

*Proof.* Let  $Q \subseteq \mathcal{N}_{\mathbb{Q}}$  be a full-dimensional polytope satisfying the above conditions and  $u \in Q$  a vertex not contained in  $\{\rho(D_i)/m_i : i = 1, \dots, k\}$ . Then we have  $u \in \mathcal{V}$ , which implies that there exists a supported full-dimensional cone  $\mathcal{C}$  in  $\tilde{\mathfrak{F}}(Q)$  having  $\mathbb{Q}_{\geq 0}u$  as extremal ray. Let  $v \in Q^*$  be the supported vertex corresponding to  $\mathcal{C}$ . Then  $\langle u, v \rangle = -1$ , and, as  $u \in \mathcal{N}$  and  $v \in \mathcal{M}$ , we obtain that  $u$  is primitive.  $\square$

## 9. GLOBAL SECTIONS OF THE ANTICANONICAL SHEAF

In this section, we recall from [Bri89, 3.3] the  $G$ -module structure of the space of global sections of a  $B$ -invariant Cartier divisor on a spherical variety  $X$ . For simplicity, we assume that  $X$  is complete. We then investigate the special case of the anticanonical sheaf when  $X$  is Gorenstein Fano. We use [Tim11, Section 17.4] as a general reference.

Let  $G/H \hookrightarrow X$  be an arbitrary spherical embedding with associated supported colored fan  $\tilde{\mathfrak{F}}$ . We denote by  $D_1, \dots, D_k$  the colors and by  $X_1, \dots, X_n$  the  $G$ -invariant prime divisors in  $X$ . Consider a  $B$ -invariant Cartier divisor

$$\delta := \sum_{i=1}^k a_i D_i + \sum_{j=1}^n b_j X_j$$

on  $X$ . For every maximal supported colored cone  $(\mathcal{C}, \mathcal{F}) \in \tilde{\mathfrak{F}}$  we write  $v_{\mathcal{C}} \in \mathcal{M}$  as in the last part of Section 2. As in Remark 5.2, we may assume  $G = G^{ss} \times C$  where  $G^{ss}$  is semisimple simply-connected and  $C$  is a torus, and then the invertible sheaf  $\mathcal{O}_X(\delta)$  can be  $G$ -linearized. Let  $s_{\delta}$  be a rational section of  $\mathcal{O}_X(\delta)$  satisfying  $\text{div } s_{\delta} = \delta$ . As  $\delta$  is  $B$ -invariant, the section  $s_{\delta}$  is  $B$ -semi-invariant of some weight  $\kappa_{\delta} \in \mathfrak{X}(B)$  (not necessarily contained in  $\mathcal{M}$ ). We write  $\tilde{\mathfrak{F}}_{\text{max}}$  for the set of maximal cones of  $\tilde{\mathfrak{F}}$ , and we set  $\mathcal{D}_X := \bigcup_{(\mathcal{C}, \mathcal{F}) \in \tilde{\mathfrak{F}}_{\text{max}}} \mathcal{F}$ . If we define

$$P_{\delta} := \left\{ u \in \bigcap_{(\mathcal{C}, \mathcal{F}) \in \tilde{\mathfrak{F}}_{\text{max}}} (-v_{\mathcal{C}} + \mathcal{C}^{\vee}) : \langle \rho(D), u \rangle \geq -m_D \text{ for every } D \in \mathcal{D} \setminus \mathcal{D}_X \right\},$$

then  $(\kappa_{\delta} + P_{\delta}) \cap \mathfrak{X}(B)$  is contained in the set of dominant weights, and we have

$$\Gamma(X, \mathcal{O}_X(\delta)) \cong \bigoplus_{\chi \in (\kappa_{\delta} + P_{\delta}) \cap \mathfrak{X}(B)} V_{\chi}$$

where  $V_{\chi}$  denotes the irreducible  $G$ -module of highest weight  $\chi$ .

**Remark 9.1.** Let  $G/H \hookrightarrow X$  be a Gorenstein spherical Fano embedding with associated  $G/H$ -reflexive polytope  $Q$ , and let  $\delta$  be the anticanonical divisor of Theorem 1.2 equipped with the canonical  $G$ -linearization. It is straightforward to check that we have  $\kappa_{\delta} = \kappa_P$  and  $P_{\delta} = Q^*$ .

## 10. EXAMPLES ILLUSTRATING THEOREM 1.9

**Example 10.1.** Let  $G := \mathrm{SL}_2 \times \mathbb{C}^*$  and consider  $H := N \times \{1\}$  where  $N \subseteq \mathrm{SL}_2$  is the normalizer of a maximal torus. Fix some maximal torus contained in some Borel subgroup, and denote by  $\alpha$  the unique simple root of  $\mathrm{SL}_2$ . Denote by  $\varepsilon$  a primitive character of  $\mathbb{C}^*$ . Then there is exactly one spherical root  $\gamma := 2\alpha$  and  $(\gamma, \varepsilon)$  is a basis of the lattice  $\mathcal{M}$ . We denote by  $(\gamma^*, \varepsilon^*)$  the corresponding dual basis of the lattice  $\mathcal{N}$ . There is exactly one color  $D_1$  (of type 2a) with  $\rho(D_1) = 2\gamma^*$ . Then  $Q := \mathrm{conv}(2\gamma^*, \varepsilon^*, -\gamma^*, -\varepsilon^*) \subseteq \mathcal{N}_Q$  is a  $G/H$ -reflexive polytope, and its dual polytope is  $Q^* = \mathrm{conv}(\gamma - \varepsilon, \gamma + \varepsilon, -\frac{1}{2}\gamma + \varepsilon, -\frac{1}{2}\gamma - \varepsilon)$ . The polytopes  $Q$  and  $Q^*$  are illustrated in Figure 2. The valuation cone is shown in grey, and the dashed arrow is the image of the color under  $\rho$  in  $\mathcal{N}$ . The dotted arrows are translates of the spherical root  $\gamma \in \mathcal{M}$  showing that exactly the circled vertices of  $Q^*$  are supported (see Proposition 8.3).

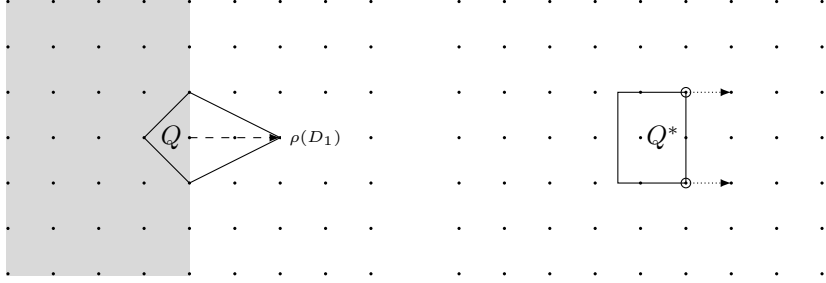


FIGURE 2. Illustration to Example 10.1.

**Example 10.2.** Let  $G := \mathrm{Spin}_5 \times \mathrm{Spin}_5$ . Fix some maximal torus contained in some Borel subgroup, and denote by  $\alpha_1, \alpha_2$  (resp.  $\alpha'_1, \alpha'_2$ ) the simple roots of the first (resp. the second) simple factor  $\mathrm{Spin}_5$  where  $\alpha_2$  (resp.  $\alpha'_2$ ) is the shorter root. According to the third entry in [Was96, Table B], there exists a spherical homogeneous space  $G/H$  with spherical roots  $\gamma_1 := \alpha_2 + \alpha'_2$  and  $\gamma_2 := \alpha_1 + \alpha'_1$ , such that  $(\gamma_1, \gamma_2)$  is a basis of the lattice  $\mathcal{M}$ . We denote by  $(\gamma_1^*, \gamma_2^*)$  the corresponding dual basis of the lattice  $\mathcal{N}$ . We have  $S^p = \emptyset$ , and there are exactly two colors  $D_1, D_2$  (both of type b) with  $\rho(D_1) = -\gamma_1^* + 2\gamma_2^*$  and  $\rho(D_2) = 2\gamma_1^* - 2\gamma_2^*$ . As  $S^p = \emptyset$ , we have  $\kappa_P = 2\rho_S$  (see Remark 4.3), so that the coefficients in the expression for the canonical divisor are  $m_1 = m_2 = 2$ . Then  $Q := \mathrm{conv}(\gamma_1^* - \gamma_2^*, -\frac{1}{2}\gamma_1^* + \gamma_2^*, -\gamma_1^*, -\gamma_2^*)$  is a  $G/H$ -reflexive polytope, and its dual polytope is  $Q^* = \mathrm{conv}(\gamma_1 - \frac{1}{2}\gamma_2, \gamma_1 + \gamma_2, \gamma_2, -4\gamma_1 - 3\gamma_2)$ . The polytopes  $Q$  and  $Q^*$  are illustrated in Figure 3. The valuation cone is shown in grey, and the dashed arrows are the images of the colors under  $\rho$  in  $\mathcal{N}$ . The dotted arrows are translates of the spherical roots  $\gamma_1, \gamma_2 \in \mathcal{M}$  showing that exactly the circled vertex of  $Q^*$  is supported (see Proposition 8.3).

## 11. POLYTOPES WITH SIMPLICIAL FACETS

The purpose of this section is to prove an auxiliary result on polytopes (Proposition 11.1), which will be used in the proof of Theorem 1.10.

Let  $V \cong \mathbb{Q}^n$  be a vector space of dimension  $n$  and  $Q \subseteq V$  a full-dimensional polytope with  $0 \in \mathrm{int}(Q)$ .

**Proposition 11.1.** *There exists a simplicial polytope  $Q_s \subseteq V$  containing  $Q$  such that the simplicial facets of  $Q$  are facets of  $Q_s$ .*



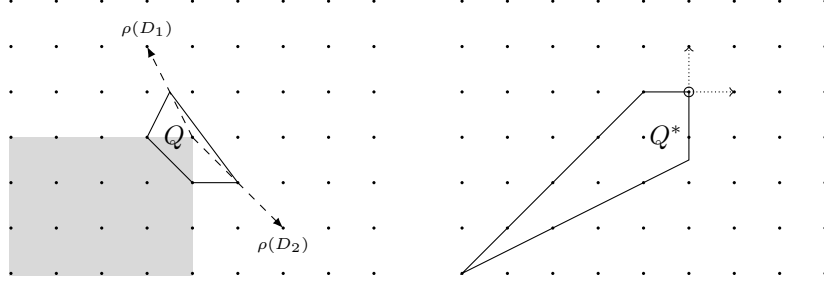


FIGURE 3. Illustration to Example 10.2.

We denote the facets of  $Q$  by  $F_1, \dots, F_r$ . For every facet  $F_i$  we choose a hyperplane  $H_i$  with normal vector  $n_i \in V^*$ , i. e.  $H_i = \{v \in V : \langle n_i, v \rangle = 1\}$  such that  $F_i = Q \cap H_i$ . Each hyperplane determines two open half-spaces

$$H_i^- := \{v \in V : \langle n_i, v \rangle < 1\} \text{ and}$$

$$H_i^+ := \{v \in V : \langle n_i, v \rangle > 1\},$$

such that  $H_1^- \cap \dots \cap H_r^- = \text{int}(Q)$ .

**Definition 11.2.** We say that  $v \in V$  is *beneath* (resp. is *beyond*)  $F_i$  if  $v$  belongs to  $H_i^-$  (resp. to  $H_i^+$ ).

We will use the following result.

**Theorem 11.3** ([Grü03, Theorem 5.2.1]). *Let  $v \in V$  such that  $v$  is a vertex of  $Q' := \text{conv}(\{v\} \cup Q)$ . Then*

- (1) *a face  $F$  of  $Q$  is a face of  $Q'$  if and only if there exists a facet  $E \preceq Q$  such that  $F \subseteq E$  and  $v$  is beneath  $E$ ,*
- (2) *if  $F$  is a face of  $Q$ , then  $F' := \text{conv}(\{v\} \cup F)$  is a face of  $Q'$  if and only if*
  - (a) *either  $v$  is contained in the affine span of  $F$ ,*
  - (b) *or among the facets of  $Q$  containing  $F$  there is at least one such that  $v$  is beneath it and at least one such that  $v$  is beyond it.*

Moreover, each face of  $Q'$  is of one and only one of those types.

The following definitions are taken from [Ewa96, Chapter III, Sections 1 and 2].

**Definition 11.4.** Let  $\mathfrak{F}$  be a fan in  $V$  and  $\sigma \in \mathfrak{F}$ . Then we set

$$\text{st}(\sigma, \mathfrak{F}) := \{\sigma' \in \mathfrak{F} : \sigma \subseteq \sigma'\} \quad (\text{the star of } \sigma \text{ in } \mathfrak{F}),$$

$$\overline{\text{st}}(\sigma, \mathfrak{F}) := \{\sigma'' \in \mathfrak{F} : \sigma'' \subseteq \sigma' \in \text{st}(\sigma, \mathfrak{F})\} \quad (\text{the closed star of } \sigma \text{ in } \mathfrak{F}).$$

**Definition 11.5.** Let  $\sigma \subseteq V$  be a cone and  $v \in V$  not contained in  $\sigma$ . Then we call  $v \cdot \sigma := \text{cone}(\{v\} \cup \sigma)$  the *join* of  $v$  and  $\sigma$ .

**Definition 11.6.** Let  $\mathfrak{F}$  be a fan in  $V$  and  $v \in V$ . Assume that  $v \cdot \sigma$  is defined for every  $\sigma \in \mathfrak{F}$  and that  $\text{relint}(v \cdot \sigma) \cap \text{relint}(v \cdot \sigma') = \emptyset$  whenever  $\sigma, \sigma' \in \mathfrak{F}$  are distinct. Then the fan  $v \cdot \mathfrak{F} := \{v \cdot \sigma : \sigma \in \mathfrak{F}\}$  is called the *join* of  $v$  and  $\mathfrak{F}$ .

**Definition 11.7.** Let  $\mathfrak{F}$  be a fan in  $V$  and  $v \in V$  a point such that there exists an (automatically uniquely determined) cone  $\sigma \in \mathfrak{F}$  with  $v \in \text{relint}(\sigma)$ . Then we call the transition

$$\mathfrak{F} \mapsto v \star \mathfrak{F} := (\mathfrak{F} \setminus \text{st}(\sigma, \mathfrak{F})) \cup v \cdot (\overline{\text{st}}(\sigma, \mathfrak{F}) \setminus \text{st}(\sigma, \mathfrak{F}))$$

the *stellar subdivision* of  $\mathfrak{F}$  in direction of  $v$ .

For a full-dimensional polytope  $Q' \subseteq V$  with  $0 \in \text{int}(Q')$  we denote by  $\mathfrak{F}(Q')$  its face fan in  $V$ .

**Lemma 11.8.** *Let  $F \preceq Q$  be a non-simplicial face and  $v \in \text{relint}(F)$ . Then there exists a polytope  $Q' \subseteq V$  containing  $Q$  such that*

$$\mathfrak{F}(Q') = v \star \mathfrak{F}(Q)$$

and the simplicial facets of  $Q$  are facets of  $Q'$ .

*Proof.* Let  $F_{s_1}, \dots, F_{s_k}$  be the facets not containing  $F$ . Choose  $t > 1$  such that  $\langle tv, n_{s_j} \rangle < 1$  for  $j = 1, \dots, k$  and set  $v' := tv$ . Then  $v'$  is beneath the facets not containing  $F$  and beyond the facets containing  $F$ . We set  $Q' := \text{conv}(\{v'\} \cup Q)$ . As simplicial facets of  $Q$  do not contain  $F$ , it follows from Theorem 11.3(1) that the simplicial facets of  $Q$  are facets of  $Q'$ .

We now verify that  $\mathfrak{F}(Q') = v \star \mathfrak{F}(Q)$ . It suffices to check that the sets of maximal cones coincide.

By Theorem 11.3(1), a facet  $F_i \preceq Q$  does not contain  $F$  if and only if  $F_i$  is a facet of  $Q'$ . Furthermore, the facets of  $Q$  not containing  $F$  are in correspondence with the maximal cones in  $\mathfrak{F}(Q) \setminus \text{st}(\mathbb{Q}_{\geq 0}F, \mathfrak{F}(Q))$ .

Let  $F'$  be a facet of  $Q'$  which is not a facet of  $Q$ . By Theorem 11.3(2), we have  $F' = \text{conv}(\{v'\} \cup F'')$  where  $F'' \preceq Q$  is a face of codimension 2 such that among the facets of  $Q$  containing  $F''$  there is at least one beneath and at least one beyond  $v'$ . Such faces  $F''$  are in correspondence with the maximal cones in  $\overline{\text{st}}(\mathbb{Q}_{\geq 0}F, \mathfrak{F}(Q)) \setminus \text{st}(\mathbb{Q}_{\geq 0}F, \mathfrak{F}(Q))$ . The result follows from the equality  $\mathbb{Q}_{\geq 0}F' = v \cdot \mathbb{Q}_{\geq 0}F''$ .  $\square$

*Proof of Proposition 11.1.* We can transform the fan  $\mathfrak{F}(Q)$  into a simplicial one by successively applying stellar subdivision to non-simplicial cones. By Lemma 11.8, we also obtain a corresponding polytope.  $\square$

## 12. PROOF OF THEOREM 1.10: THE INEQUALITY $\rho_X \leq 2d$

Let  $G/H \hookrightarrow X$  be a Gorenstein spherical Fano embedding with associated  $G/H$ -reflexive polytope  $Q$ . The condition for  $\mathbb{Q}$ -factoriality from Proposition 2.8 can be straightforwardly translated into the setting of  $G/H$ -reflexive polytopes as follows:

**Proposition 12.1.**  *$X$  is  $\mathbb{Q}$ -factorial if and only if every facet  $\widehat{v}$  of  $Q$  for  $v \in V_{\text{supp}}(Q^*)$  has exactly  $\text{rank } X$  vertices in  $V(Q)$ , where such a vertex can not be equal to  $\rho(D)$  for more than one  $D \in \mathcal{D}$ .*

Now assume that  $X$  is  $\mathbb{Q}$ -factorial, of rank  $r$ , and of dimension  $d$ . The proof of the inequality  $\rho_X \leq 2d$  appearing here is an extended version of the proof of the horospherical case in [Pas08]. Note that, in contrast to the horospherical case, not all facets of the polytope  $Q$  are necessarily simplicial (only the facets dual to the supported vertices of  $Q^*$  are).

**Lemma 12.2.** *Let  $v \in V_{\text{supp}}(Q^*)$  and  $u \in V(Q) \cap \mathcal{N}$ . If  $\langle u, v \rangle = 0$ , then there is a facet  $F \preceq Q$  containing  $u$  and intersecting  $\widehat{v}$  in a face of codimension 2 of  $Q$ , i. e.  $u$  is adjacent to  $\widehat{v}$ .*

*Proof.* Let  $e_1, \dots, e_r$  be the vertices of the facet  $\widehat{v} \preceq Q$ . By [Grü03, Theorem 3.1.6], for all  $j = 1, \dots, r$  there is exactly one facet  $F_j \preceq Q$  containing  $e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_r$  being distinct from  $\widehat{v}$ . Let  $u_j$  be a vertex of  $F_j$  not contained in  $\widehat{v} \cap F_j$ . Then  $(e_1, \dots, e_{j-1}, u_j, e_{j+1}, \dots, e_r)$  is a basis of  $\mathcal{N}_{\mathbb{Q}}$ .

Let  $(e_1^*, \dots, e_r^*)$  be the basis of  $\mathcal{M}_{\mathbb{Q}}$  dual to  $(e_1, \dots, e_r)$ . Let  $j = 1, \dots, r$ . Then  $\langle e_j^*, u_j \rangle \neq 0$  (otherwise  $u_j$  would be contained in the hyperplane spanned by  $\{e_i\}_{i \neq j}$ ). We define

$$\lambda_j := \frac{-1 - \langle u_j, v \rangle}{\langle u_j, e_j^* \rangle} \text{ and } v_j := v + \lambda_j e_j^*.$$

Then we have  $\langle e_i, v_j \rangle = -1$  for  $i \neq j$  and  $\langle u_j, v_j \rangle = -1$ . It follows that  $v_j$  is a (not necessarily supported) vertex of  $Q^*$  and  $F_j = \widehat{v}_j$ . We have  $\lambda_j > 0$  as  $-1 < \langle e_j, v_j \rangle$ . Then  $\langle u, v_j \rangle = \lambda_j \langle u, e_j^* \rangle$  and hence

$$u \notin F_j \text{ if and only if } \langle u, e_j^* \rangle \geq 0.$$

If  $u \notin F_j$  for all  $j = 1, \dots, r$ , then  $\langle u, e_j^* \rangle \geq 0$  for all  $j = 1, \dots, r$  and therefore  $u \in \text{cone}(e_1, \dots, e_r)$ , which implies  $u = e_j \in F_j$  for some  $j = 1, \dots, r$ , a contradiction. We obtain  $u \in F_j$  for some  $j = 1, \dots, r$ .  $\square$

**Corollary 12.3.** *Let  $v \in V_{\text{supp}}(Q^*)$ . Then we have  $|V(\widehat{v})| = r$  and*

$$|\{u \in V(Q) \cap \mathcal{V} \cap \mathcal{N} : \langle u, v \rangle = 0\}| \leq r.$$

*Proof.* As  $\widehat{v}$  is a simplex, it has  $r$  vertices, which we denote by  $u_1, \dots, u_n$ , and the first assertion follows. For the second assertion, by Proposition 11.1, we may replace the polytope  $Q$  by a simplicial polytope  $Q'$  such that

$$V(Q) \cap \mathcal{V} \cap \mathcal{N} = V(Q') \cap \mathcal{V} \cap \mathcal{N} \text{ and } V_{\text{supp}}(Q^*) \subseteq V((Q')^*).$$

Let  $u \in V(Q) \cap \mathcal{V} \cap \mathcal{N}$  with  $\langle u, v \rangle = 0$ . By Lemma 12.2, there exists a facet  $F_u \preceq Q$  adjacent to  $\widehat{v}$  and containing  $u$ . As  $Q$  is assumed simplicial,  $F_u$  has vertices  $u_1, \dots, u_{i-1}, u, u_{i+1}, \dots, u_r$  for some  $i = 1, \dots, r$ . Since there are  $r$  facets adjacent to  $\widehat{v}$ , the result follows.  $\square$

**Lemma 12.4.** *Let  $Q \subseteq \mathcal{N}_Q$  be a  $G/H$ -reflexive polytope. Then  $0$  is contained in the interior of the convex hull of the supported vertices of  $Q^*$  and  $-\Sigma$ .*

*Proof.* We set  $\sigma := \text{cone}(V_{\text{supp}}(Q^*) \cup (-\Sigma))$  and show  $\sigma^\vee = 0$ . Then  $\sigma = (\sigma^\vee)^\vee = 0^\vee = \mathcal{M}_Q$ , and hence  $0 \in \text{int}(\text{conv}(V_{\text{supp}}(Q^*) \cup (-\Sigma)))$ .

Assume  $0 \neq u \in \sigma^\vee$ . Since  $\langle u, -\Sigma \rangle \geq 0$ , we have  $u \in \mathcal{V}$ . Then there is a supported facet  $F \preceq Q$  and a rational number  $t > 0$  such that  $tu \in F$ . Let  $v$  be the supported vertex of  $Q^*$  such that  $\widehat{v} = F$ . Then we have  $0 \leq \langle tu, v \rangle = -1$ , a contradiction.  $\square$

It follows from Lemma 12.4 that there exist positive natural numbers  $m_v$  and  $l_\gamma$  such that

$$0 = \sum_{v \in V_{\text{supp}}(Q^*)} m_v v - \sum_{\gamma \in \Sigma} l_\gamma \gamma.$$

We define  $M := \sum_{v \in V_{\text{supp}}(Q^*)} m_v$ .

**Proposition 12.5.** *We have*

$$|V(Q) \cap \mathcal{V} \cap \mathcal{N}| \leq 3r + \sum_{\gamma \in \Sigma} \sum_{u \in V(Q) \cap \mathcal{V} \cap \mathcal{N}} \frac{l_\gamma \langle u, \gamma \rangle}{M}.$$

*Proof.* For  $u \in V(Q) \cap \mathcal{V} \cap \mathcal{N}$  we define

$$A(u) := \{v \in V_{\text{supp}}(Q^*) : \langle u, v \rangle = -1\} \text{ and } B(u) := \{v \in V_{\text{supp}}(Q^*) : \langle u, v \rangle = 0\}.$$

Then we have

$$\begin{aligned} 0 &= \sum_{v \in V_{\text{supp}}(Q^*)} m_v \langle u, v \rangle - \sum_{\gamma \in \Sigma} l_\gamma \langle u, \gamma \rangle \\ &\geq - \sum_{v \in A(u)} m_v + \sum_{v \notin A(u) \cup B(u)} m_v - \sum_{\gamma \in \Sigma} l_\gamma \langle \gamma, u \rangle \\ &= M - 2 \sum_{v \in A(u)} m_v - \sum_{v \in B(u)} m_v - \sum_{\gamma \in \Sigma} l_\gamma \langle u, \gamma \rangle \end{aligned}$$

and hence  $M \leq 2 \sum_{v \in A(u)} m_v + \sum_{v \in B(u)} m_v + \sum_{\gamma \in \Sigma} l_\gamma \langle u, \gamma \rangle$ . Summing up this inequality over all  $u \in V(Q) \cap \mathcal{V} \cap \mathcal{N}$ , we obtain

$$\begin{aligned} |V(Q) \cap \mathcal{V} \cap \mathcal{N}| M &\leq \sum_u \sum_{v \in A(u)} 2m_v + \sum_u \sum_{v \in B(u)} m_v + \sum_u \sum_{\gamma \in \Sigma} l_\gamma \langle u, \gamma \rangle \\ &= \sum_{v \in V_{\text{supp}}(Q^*)} \sum_{\langle u, v \rangle = -1} 2m_v + \sum_{v \in V_{\text{supp}}(Q^*)} \sum_{\langle u, v \rangle = 0} m_v + \sum_{\gamma \in \Sigma} \sum_{u \in V(Q) \cap \mathcal{V} \cap \mathcal{N}} l_\gamma \langle u, \gamma \rangle. \end{aligned}$$

For a fixed  $v$  the number of vertices  $u$  of  $V$  with  $\langle u, v \rangle = -1$  is equal to the number of vertices of the facet  $\widehat{v}$ , which is exactly  $r$ . By Corollary 12.3, the number of  $u \in V(Q) \cap \mathcal{V} \cap \mathcal{N}$  with  $\langle u, v \rangle = 0$  is less than or equal to  $r$ . Hence the result follows.  $\square$

**Proposition 12.6.** *We have  $\rho_X \leq 2r + |\mathcal{D}| \leq 2r + 2|S \setminus S^p| \leq 2d$ .*

*Proof.* As  $\langle u, \gamma \rangle \leq 0$  for every  $\gamma \in \Sigma$ ,  $u \in \mathcal{V}$ , it follows from Proposition 12.5 that  $|V(Q) \cap \mathcal{V} \cap \mathcal{N}| \leq 3r$ . Let  $r'$  be the number of  $G$ -invariant prime divisors in  $X$ . It follows from [Bri07, Proposition 4.1.1] that

$$\rho_X = r' + |\mathcal{D}| - r \leq |V(Q) \cap \mathcal{V} \cap \mathcal{N}| + |\mathcal{D}| - r \leq 2r + |\mathcal{D}|.$$

By [Tim11, Theorem 30.22], we obtain  $|\mathcal{D}| \leq 2|S \setminus S^p|$ , and by [Tim11, Corollary 15.18], we obtain  $r + |S \setminus S^p| \leq d$ .  $\square$

### 13. PROOF OF THEOREM 1.10: THE EXTREME CASE $\rho_X = 2d$

Let  $G/H \hookrightarrow X$  be a  $\mathbb{Q}$ -factorial Gorenstein spherical Fano embedding of rank  $r$  and of dimension  $d$  with  $\rho_X = 2d$ .

**Proposition 13.1.** *We have  $|S \setminus S^p| = |R^+ \setminus R_{S^p}^+|$  where  $R_{S^p}$  denotes the root system generated by  $S^p$ .*

*Proof.* According to Proposition 12.6, we have  $d = r + |S \setminus S^p|$ . On the other hand, we have  $d = r + \dim G/P = r + |R^+ \setminus R_{S^p}^+|$ .  $\square$

**Proposition 13.2.** *The root system  $R$  is of type  $A_1^k \times R_{S^p}$  for some  $k \in \mathbb{Z}_{\geq 0}$ .*

*Proof.* As  $S \subseteq R^+$  and  $S^p \subseteq R_{S^p}^+$ , it follows from Proposition 13.1 that  $R^+ \setminus R_{S^p}^+ = S \setminus S^p$ , which means that every positive root not contained in  $R_{S^p}^+$  is simple. Let  $\alpha \in S \setminus S^p$ . Then the irreducible factor of  $R$  which contains  $\alpha$  must be of type  $A_1$  as it can not contain any non-simple positive root.  $\square$

As in Remark 5.2, we may assume  $G = G^{ss} \times C$  where  $G^{ss}$  is semisimple simply-connected and  $C$  is a torus. According to Proposition 13.2, we have  $G^{ss} \cong \text{SL}_2^k \times G'$  where  $G'$  is the factor corresponding to  $R_{S^p}$ .

**Corollary 13.3.** *We may assume  $G = \text{SL}_2^k \times C$ .*

*Proof.* Let  $P = L \ltimes P_u$  be the Levi decomposition with  $T \subseteq L$ . According to [Tim11, Theorem 4.7],  $[L, L]$  acts trivially on the open  $B$ -orbit in  $X$ . The result follows from the observation  $G' = [L, L]$ .  $\square$

**Corollary 13.4.** *We have  $V(Q) \cap \mathcal{V} \subseteq \mathcal{N}$  consists of all primitive ray generators where the rays correspond to  $G$ -invariant prime divisors.*

**Proposition 13.5.** *We have  $\Sigma = \emptyset$ .*

*Proof.* According to the proof of Proposition 12.6, we must have  $|V(Q) \cap \mathcal{V} \cap \mathcal{N}| = 3r$ . Taking into account Proposition 12.5, this implies

$$\sum_{\gamma \in \Sigma} \sum_{u \in V(Q) \cap \mathcal{V} \cap \mathcal{N}} \frac{l_{\gamma} \langle u, \gamma \rangle}{M} = 0.$$

By Proposition 12.6 and 13.2, we obtain  $|\mathcal{D}| = 2|S \setminus S^p| = 2k$  and thus all colors are of type  $a$ ,  $\Sigma \subset S$  and  $\mathcal{D}(\alpha) \cap \mathcal{D}(\beta) = \emptyset$  for two distinct  $\alpha, \beta \in S$ . In particular,  $\langle \rho(D), \alpha \rangle \leq 1$  for all  $D \in \mathcal{D}$  and all  $\alpha \in \Sigma$  with equality if and only if  $D \in \mathcal{D}(\alpha)$ . As  $\rho(D') + \rho(D'') = \alpha^{\vee}$  for  $\mathcal{D}(\alpha) = \{D', D''\}$ , we obtain  $\langle \rho(D), \alpha \rangle = 0$  for all  $D \notin \mathcal{D}(\alpha)$ . Since  $X$  is complete, for every  $\alpha \in \Sigma$  there exists a primitive ray generator  $v \in \mathcal{V}$  such that  $\langle v, \alpha \rangle < 0$ . As  $l_{\alpha} > 0$ , we obtain  $\Sigma = \emptyset$ .  $\square$

By Proposition 13.5,  $X$  is horospherical, and therefore the last part of Theorem 1.10 follows from [Pas08, Théorème 1.2].

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