Letter

Stochastic strong zero modes and their dynamical manifestations

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Strong zero modes (SZMs) are conserved operators localized at the edges of certain quantum spin chains, which give rise to long coherence times of edge spins. Here we define and analyze analogous operators in one-dimensional *classical stochastic* systems. For concreteness, we focus on chains with single occupancy and nearest-neighbor transitions, in particular particle hopping and pair creation and annihilation. For integrable choices of parameters we find the exact form of the SZM operators. Being in general nondiagonal in the classical basis, the dynamical consequences of stochastic SZMs are very different from those of their quantum counterparts. We show that the presence of a stochastic SZM is manifested through a class of exact relations between time-correlation functions, absent in the same system with periodic boundaries.

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Recent successes have transformed our understanding of how long relaxation times—and potential nonergodicity—emerge in quantum many-body systems (for reviews see, e.g., [1–6]). One simple mechanism in some systems with open boundaries is that of a *strong zero mode* (SZM) [7–18]. An SZM is an operator localized at the boundary that commutes with the Hamiltonian, up to exponentially small corrections. Its presence affects the structure of the whole spectrum of the Hamiltonian, resulting for example in boundary degrees of freedom having very long coherence times [11–15]. For certain integrable spin chains, SZMs can be constructed exactly and explicitly [9].

In a *classical stochastic* system, continuous-time Markov dynamics are defined by a stochastic generator, just like a Hamiltonian generates unitary dynamics in a quantum system. While being in general non-Hermitian, stochastic generators often share many properties with Hamiltonians, thus connecting classical stochastic and quantum problems at the technical level. An example of such a connection is between the simple exclusion process and the XXZ quantum chain, see, e.g., [19–22]. Therefore, a natural question to ask is whether SZMs exist in classical stochastic systems, and if they do, what consequences they have for the dynamics.

Here we answer this question. For simplicity we focus on systems of particles on a one-dimensional chain with at most single occupancy per site. We consider transitions between neighboring sites, including hopping and pair creation or annihilation. Detailed balance need not be obeyed. For certain choices of the transition rates the generators are integrable, and for these we find the explicit form of boundary localized operators that commute with the generator (either exactly or up to corrections that are exponentially small in the system size). These *stochastic SZMs* are nondiagonal in the classical basis, and as such do not correspond to classical observables. They represent "hidden" conservation laws which, as we show below, manifest themselves in the dynamics through a class of exact relations among time-correlation functions observable at finite times.

We study a system of particles stochastically hopping on a one-dimensional chain of length L, while obeying an exclusion constraint so that each site can be occupied by at most one particle. A particle can hop to a neighboring site (either left or right) if it is empty, two particles positioned on consecutive sites can evaporate from the lattice, and two particles can condense on a pair of empty sites. As illustrated in Fig. 1, the left- and right-hopping transitions have rates $D(1+\delta)$ and $D(1-\delta)$, respectively, while evaporation and condensation occur with rates $\gamma(1+\kappa)$, and $\gamma(1-\kappa)$. At the edges we typically assume open boundary conditions, where the first and last site each have only one nearest neighbor [23].

At each time the configuration of the system can be expressed in terms of an L-tuple $\underline{n}=(n_1,n_2,\ldots,n_L)\in\mathbb{Z}_2^L$, where $n_j=1$ if there is a particle on site j and $n_j=0$ when empty. To describe dynamics of *macroscopic* states (i.e., probability distributions) we use bra-ket notation, $|p\rangle=[p_0,p_1,\ldots,p_{2^L-1}]\in\mathbb{R}^{2^L}$, where each component $p_n\geqslant 0$ represents a probability of the configuration given by the binary representation of the subscript n, and the sum of all components is one, $\sum_n p_n=1$. Diagonal operators represent *observables*, i.e., quantities that can be measured.

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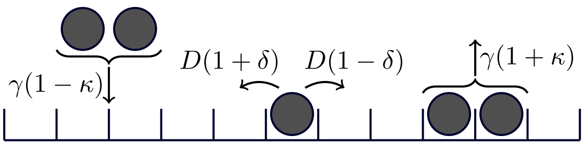

FIG. 1. Schematic representation of allowed transitions. A particle can only hop to a neighboring site if that site is empty. The rate to hop to the left is $D(1+\delta)$ and that to the right is $D(1-\delta)$. A pair of neighboring particles can evaporate with rate $\gamma(1+\kappa)$, while a pair of empty sites can condense a pair of particles with rate $\gamma(1-\kappa)$.

Their expectation values are by definition given by the sum $\langle a\rangle_p = \sum_n a_{n,n} p_n = \langle -|a|p\rangle$, where we introduced the *flat state* $\langle -| = [1\ 1]^{\otimes L}$. The normalization condition for $|p\rangle$ can then be equivalently expressed as $\langle -|p\rangle = 1$. Encoding the stochastic transitions with a generator $\mathbb{W}$ means that an initial state $|p\rangle$ evolves in time as $|p(t)\rangle = e^{t\mathbb{W}}|p\rangle$. Conservation of probability under time evolution requires $\langle -|e^{t\mathbb{W}} = \langle -|$. This notation gives convenient expressions for more complicated objects, such as the expectation value at time t after starting from some nonstationary initial state $\langle -|a e^{t\mathbb{W}}|p\rangle$, or correlation functions between multiple observables at different times, $\langle -|b\, e^{(t_2-t_1)\mathbb{W}}a\, e^{t_1\mathbb{W}}|p\rangle$.

We restrict the discussion to two different *integrable* limits of the generator $\mathbb{W}$ (see, e.g., [24]), for which the Hamiltonian counterparts are known to exhibit conserved edge modes [8,9,25]:

(i) In the regime $\gamma = D$, the generator is quadratic in fermionic operators (see Sec. A in [26]), so we refer to it as the *free-fermionic model*. The stochastic generator with open boundaries has the form

$$\mathbb{W}^{(\mathrm{FF})} = \sum_{j=1}^{L-1}\left[X_jX_{j+1} + \kappa Z_j + i\frac{\kappa+\delta}{2}X_jY_{j+1} + i\frac{\kappa-\delta}{2}Y_jX_{j+1} - 1\right] + \frac{\kappa+\delta}{2}(Z_L - Z_1), \quad (1)$$

where X_j, Y_j, and Z_j denote Pauli matrices acting on the site j. Without loss of generality, we rescaled the unit of time so that $D = 1$. Note that when $\delta \neq 0$, hopping is asymmetric and $\mathbb{W}$ does not obey detailed balance.

(ii) The second integrable regime arises when $\kappa = \delta = 0$, i.e., there is no asymmetry between the left and right hopping, and the rates for condensation and evaporation are the same. This model was studied with periodic boundaries in Ref. [27], and more recently, solutions to the boundary-driven setup have been found [28]. In this case the generator takes a form of a rotated anisotropic Heisenberg XXZ Hamiltonian,

$$\mathbb{W}^{(\mathrm{XZZ})} = \sum_{j=1}^{L-1}\left[\frac{1-\gamma}{2}(Y_jY_{j+1} + Z_jZ_{j+1}) + \frac{1+\gamma}{2}(X_jX_{j+1} - 1)\right], \quad (2)$$

so we refer to it as XZZ model. We again chose $D = 1$.

In analogy to the quantum setting, a *conserved edge mode* Ψ is an operator that commutes with the stochastic generator, $[\Psi, \mathbb{W}] = 0$; squares into identity, $\Psi^2 = 1$; and is localized at an edge—its local densities that involve sites far from the edge are exponentially suppressed.

In the case of $\mathbb{W}^{(\mathrm{FF})}$ we take advantage of the free-fermionic form to straightforwardly find the expression for $\Psi^{(\mathrm{FF})}$ (see Sec. A in [26] for the derivation),

$$\Psi^{(\mathrm{FF})} = \sum_{j=1}^{L}\lambda^{j-1}\mu_{j-1}(X_j + i\lambda Y_j), \quad (3)$$

where the disorder operator $\mu_j = \prod_{k=1}^{j} Z_k$ is a string of Z_k originating at the left edge, and the parameter λ is expressed in terms of κ and δ as

$$\lambda = \frac{1-\sqrt{1+\delta^2-\kappa^2}}{\delta+\kappa} \quad (4)$$

with $|\lambda| \leqslant 1$. This edge mode is *exactly* conserved, i.e., $[\mathbb{W}^{(\mathrm{FF})}, \Psi^{(\mathrm{FF})}] = 0$ with no corrections. For simplicity we neglect exponentially small corrections to the normalization: $\Psi^{(\mathrm{FF})2} = 1 + \mathcal{O}(\lambda^L)$.

The XZZ generator (2) is Hermitian and has exactly the same form as the XYZ Hamiltonian with appropriately chosen couplings, therefore we can directly adapt the exact form of Ref. [9] to obtain

$$\Psi^{(\mathrm{XZZ})} = \sum_{S=0}^{\infty}\sum_{1\leqslant a_1<\ldots<a_{2S}<b\leqslant L}\lambda^{2(b-1)}(1-\lambda^2)\left(1-\frac{1}{\lambda^2}\right)^S \times \lambda^{-\sum_{j=1}^{2S}(-1)^j a_j} X_b \prod_{j=1}^{S}\left(Y_{a_{2j-1}}Y_{a_{2j}} + Z_{a_{2j-1}}Z_{a_{2j}}\right), \quad (5)$$

where the value of λ, $|\lambda| \leqslant 1$, is now given by

$$\lambda = \frac{1-\gamma}{1+\gamma}. \quad (6)$$

Unlike the free-fermion case, the edge mode now no longer exactly commutes with the stochastic generator, but rather does so up to corrections of the order $\mathcal{O}(\lambda^L)$. In both cases (3) and (5), $|\lambda| \leqslant 1$ implies the exponential suppression of local densities on sites far from the edge of the lattice, making the SZM localized at the boundary.

Since neither SZM is diagonal, they cannot be directly interpreted as classical observables. Their effect on the dynamics therefore is not immediately obvious. A key observation is that the expectation value of an off-diagonal operator A can always be interpreted as an expectation value of a corresponding *diagonal* operator $\hat{A}$ defined by

$$\langle -|A = \langle -|\hat{A} \quad \Rightarrow \quad \langle -|A|p(t)\rangle = \langle -|\hat{A}|p(t)\rangle\,. \quad (7)$$

Pauli operators obey the two simple identities

$$[1\quad 1]X_j = [1\quad 1] \quad \text{and} \quad [1\quad 1]Y_j = \mathrm{i}[1\quad 1]Z_j, \quad (8)$$

which can be linearly extended to provide the diagonal operator $\hat{A}$ for an arbitrary A. Therefore, the existence of a nondiagonal operator Ψ commuting with $\mathbb{W}$ implies the existence of a *classical observable* $\hat{\Psi}$ whose expectation value does not change with time,

$$\langle -|\hat{\Psi}e^{t\mathbb{W}}|p\rangle = \langle -|\Psi e^{t\mathbb{W}}|p\rangle = \langle -|e^{t\mathbb{W}}\Psi|p\rangle = \langle -|\Psi|p\rangle = \langle -|\hat{\Psi}|p\rangle, \quad (9)$$

where we utilized the defining property (7) and the conservation of probability.

In our cases, a little more work is needed. Indeed, the identities (8) imply that $\langle -|$ is (up to terms exponentially small in L) a left eigenvector of both $\Psi^{(\mathrm{FF})}$ and $\Psi^{(\mathrm{XZZ})}$,

$$\langle -|\Psi = \langle -|, \tag{10}$$

and therefore the conservation of $\langle -|\hat{\Psi}|p(t)\rangle$ gives us no meaningful restriction on the dynamics.

Nonetheless, it is possible to define a *dynamical protocol*, under which the existence of the boundary mode gives nontrivial effects. We require the initial state $|\alpha\rangle$ to be an eigenvector of Ψ with eigenvalue 1: $\Psi|\alpha\rangle = |\alpha\rangle$ [29]. The conservation of Ψ implies the existence of observables whose expectation value remains constant after starting from these states. A general expectation value of an observable a at time t obeys

$$\begin{aligned}\langle -|a\,\mathrm{e}^{t\,\mathbb{W}}|\alpha\rangle &= \langle -|a\,\mathrm{e}^{t\,\mathbb{W}}\Psi^2|\alpha\rangle = \langle -|a\,\Psi\mathrm{e}^{t\,\mathbb{W}}|\alpha\rangle \\ &= -\langle -|\Psi a\,\mathrm{e}^{t\,\mathbb{W}}|\alpha\rangle + \langle -|\{a,\Psi\}\mathrm{e}^{t\,\mathbb{W}}|\alpha\rangle,\end{aligned} \tag{11}$$

which follows from the normalization $\Psi^2 = 1$, the definition of $|\alpha\rangle$, and the conservation of the edge mode. Because $\langle -|$ is the left eigenvector of Ψ [cf. Eq. (10)], we obtain a connection between the expectation value of a at any time t and that of its anticommutator $\{a, \Psi\}$:

$$\langle -|a\,\mathrm{e}^{t\,\mathbb{W}}|\alpha\rangle = \tfrac{1}{2}\langle -|\{a,\Psi\}\mathrm{e}^{t\,\mathbb{W}}|\alpha\rangle. \tag{12}$$

This general identity can now be used to obtain some nontrivial constraints on dynamics.

Let us start with the free-fermionic model, and consider $a = Z_j$. After a series of elementary manipulations similar to the ones of Eq. (10), one obtains

$$\tfrac{1}{2}\langle -|\{Z_j, \Psi^{(\mathrm{FF})}\} = \langle -|Z_j - \lambda^{j-1}(\langle -|\mu_j - \lambda\langle -|\mu_{j-1}),$$

which, together with (12) implies

$$\begin{aligned}\langle -|Z_1\mathrm{e}^{t\,\mathbb{W}^{(\mathrm{FF})}}|\alpha\rangle &= \lambda\langle -|\mathrm{e}^{t\,\mathbb{W}^{(\mathrm{FF})}}|\alpha\rangle = \lambda,\\ \langle -|\mu_k\mathrm{e}^{t\,\mathbb{W}^{(\mathrm{FF})}}|\alpha\rangle &= \lambda\langle -|\mu_{k-1}\mathrm{e}^{t\,\mathbb{W}^{(\mathrm{FF})}}|\alpha\rangle = \lambda^k.\end{aligned} \tag{13}$$

The second equality in both rows follows from the conservation of probabilities, $\langle -|\mathrm{e}^{t\,\mathbb{W}} = \langle -|$ and the normalization of the initial state, $\langle -|\alpha\rangle = 1$. The expectation values of $\mu_k = \prod_{j=1}^{k} Z_j$ are therefore constant in time, even though the initial state is *not* stationary and the system must undergo nontrivial dynamics before relaxing. For $t = 0$ the relation (13) is the property of the initial state and does not depend on whether or not $\Psi^{(\mathrm{FF})}$ is conserved: the surprising consequence of the existence of the edge mode is that it holds also when $t > 0$.

The XZZ regime can be treated analogously, with only the precise relations changing due to the different form of the edge mode. The left-action of the anticommutator on the flat state obeys

$$\tfrac{1}{2}\langle -|\{Z_j, \Psi^{(\mathrm{XZZ})}\} = \langle -|Z_j - (\lambda^{j-2} - \lambda^{j})\langle -|\chi, \tag{14}$$

where χ is a sum over the Z_j with coefficients decaying exponentially away from the edge:

$$\chi = \sum_{j=1}^{L}\lambda^j Z_j. \tag{15}$$

Inserting (14) into (12) immediately gives us the dynamical restriction for the XZZ case: the expectation value of χ is forced to be zero at all times, i.e.,

$$\langle -|\chi\,\mathrm{e}^{t\,\mathbb{W}}|\alpha\rangle = 0. \tag{16}$$

Equations (13) and (16) provide nontrivial dynamical constraints holding in the presence of the edge mode. A few remarks are in order. First, corrections exponentially small in the system size have been ignored. Therefore one might expect that these constraints only hold up to times of the order of magnitude $1/\lambda^L$. However, one can show (see Sec. C in [26]) that these expectation values coincide with the values in the stationary state, which implies the broader applicability of the constraints. Second, this dynamical protocol only makes sense if we can find appropriate eigenvectors $|\alpha\rangle$ that can be interpreted as valid probability distributions. Since they need to satisfy the nonnegativity condition, their existence is not *a priori* obvious. While we have not been able to characterize the full set of valid initial states, we have found several representative examples (see the discussion in Sec. B in [26]) that we use in the numerical demonstrations below.

To demonstrate explicitly that relations (13) and (16) represent nontrivial constraints on the time evolution, we simulate this dynamical protocol using Monte Carlo sampling of trajectories. For clarity, we restrict the discussion to the case of *symmetric hopping*—i.e., we assume $\delta = 0$ in *both* regimes. The stationary state is then the same for both periodic and open boundary conditions, while the edge mode is only conserved in the latter case. Changing boundary conditions therefore gives a direct probe of the validity of the dynamical constraints arising from the edge mode. The initial state in the free fermionic case is

$$|\alpha^{(\mathrm{FF})}\rangle = \frac{1+\Psi^{(\mathrm{FF})}}{2}\begin{bmatrix}1\\0\end{bmatrix}^{\otimes L}, \tag{17}$$

while the *interacting* initial state is chosen as

$$|\alpha^{(\mathrm{XZZ})}\rangle = \frac{1+\Psi^{(\mathrm{XZZ})}}{2}\begin{bmatrix}\frac{1}{4}\\ \frac{3}{4}\end{bmatrix}\otimes\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}^{\otimes L-1}. \tag{18}$$

We note that these are just two concrete choices, and the full family of possible initial states is very large due to the high degeneracy of the spectra of $\Psi^{(\mathrm{FF})}$ and $\Psi^{(\mathrm{XZZ})}$. See Sec. B in [26] for more details.

The behavior in the free-fermionic regime is shown in Fig. 2, where we compare the dynamics of the expectation value (13) between open and periodic boundaries. In both cases the initial value is equal to the stationary value, but the state itself is *not* stationary. Therefore for periodic boundaries the expectation value shows nontrivial dynamics, while in the open case the edge mode prevents it from changing. The dynamics of quantities not restricted by Eq. (13) does not strongly depend on the boundary conditions, as demonstrated in the inset, where we compare the expectation value of Z_j at two sites: one close to the edge and one in the bulk.

Analogous behavior can be observed in the interacting XZZ regime in Fig. 3. The existence of conserved $\Psi^{(\mathrm{XZZ})}$ forces the expectation value of χ to stay at zero, see Eq. (16), while in the case of periodic boundaries there is no such restriction and χ exhibits nontrivial dynamics. However, as shown in the inset, the dynamics of generic observables shows no qualitative difference between the different boundary conditions.

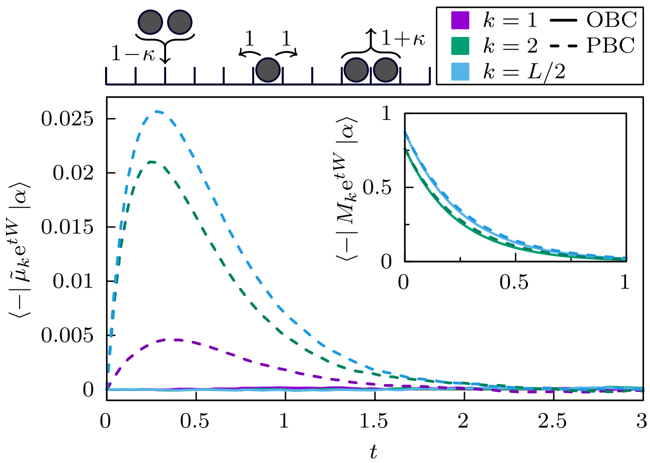

FIG. 2. Dynamics of disorder operators $\tilde{\mu}_k = \mu_k - \lambda^k = Z_1 Z_2 \cdots Z_k - \lambda^k$ in the *free-fermionic* model. The initial state $|\alpha\rangle$ is given in Eq. (17), while λ is the expectation value of Z_k in the stationary state. For open boundary conditions, the expectation values are restricted as in Eq. (13) due to the existence of the boundary mode, so that there is no evolution in such quantities. In contrast, the expectation values are unconstrained in the periodic case, and they undergo nontrivial time evolution. The time dependence of generic observables is not constrained, and they show qualitatively similar behavior in both cases, as is shown in the inset for the rescaled magnetization $M_k = Z_k - \lambda$. In this example we consider symmetric hopping ($\delta = 0$), the asymmetry between pair-annihilation and creation rates is $\kappa = 0.25$, the system size is $L = 20$, and the number of Monte Carlo trajectories is 10^9.

In this paper we have generalized the concept of strong zero modes from quantum spin chains to one-dimensional classical stochastic systems. For choices of parameters that make the stochastic generators integrable we were able to

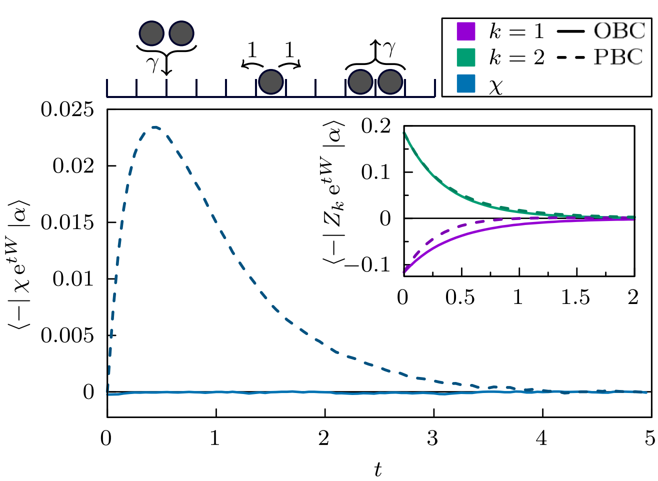

FIG. 3. Expectation values of χ [defined in Eq. (15)] in the *interacting* regime of the model. When the boundary conditions are open, the expectation value is constrained by the existence of the edge SZM [Eq. (16)], while the system with periodic boundaries exhibits nontrivial dynamics. For comparison, the dynamics of local magnetization Z_k in the inset show no qualitative difference between the two boundary conditions. The initial state $|\alpha\rangle$ is given in Eq. (18), the annihilation/creation rate is $\gamma = 0.35$, the system size is $L = 20$, and the number of Monte Carlo trajectories is 10^8.

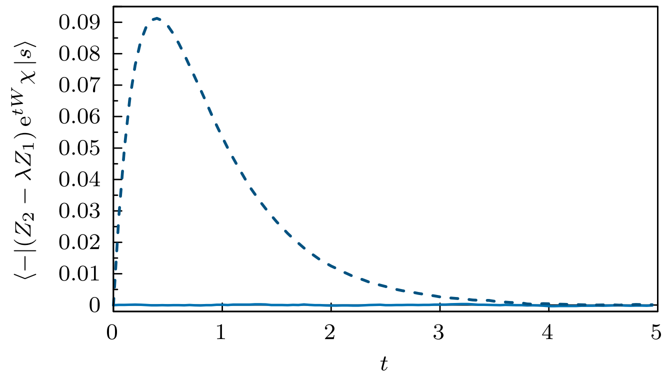

FIG. 4. Dynamical correlation function between χ and $\lambda^k Z_j - \lambda^j Z_j$ in the stationary state $|s\rangle = 2^{-L}|-\rangle$ for the interacting regime of the model (with the same numerical parameters as in Fig. 3). The vanishing at all times for the open boundaries case illustrates the identity (19) that follows from the stochastic zero mode.

obtain the SZMs exactly. In contrast to the quantum case, the conservation of a stochastic SZM cannot be observed directly in the dynamics, manifesting instead as specific constraints in time correlation functions. As far as we are aware these *hidden* conservation laws in systems with open boundaries were not identified before.

Relations (13) and (16) are just two examples of a large number of dynamical relation following from the existence of edge modes. For example, for the case of $W^{(\mathrm{XZZ})}$, a similar mechanism restricts the dynamics of a wide class of dynamical correlation functions in the stationary state. In particular, as we show in Sec. D in [26], the equilibrium time-correlation functions

$$\langle -|\{A, \Psi\} e^{tW} B|-\rangle = \langle -|A e^{tW} \{B, \Psi\}|-\rangle \tag{19}$$

are identical—up to times of the order of magnitude $1/\lambda^L$—for any two observables A and B. In Fig. 4 we plot the specific case of $A = Z_1$ and $B = Z_2$, where (19) reduces to $\langle -|(Z_2 - \lambda Z_1) e^{tW^{(\mathrm{XZZ})}} \chi|-\rangle = 0$.

Many questions remain. One is on the fate of SZMs away from integrability. Our results explicitly depend on the precise form of the SZMs, but typically the physics of these models shows no qualitative change when the stochastic rates are tuned to the integrable point. A related question is whether for nonintegrable stochastic spin chains, e.g., those in Ref. [30], SZMs are only conserved parametrically, as occurs in nonintegrable quantum systems [11,12], and if so, how these "almost" SZMs manifest themselves in the dynamics. A more general issue is to describe the dynamical consequences of other conserved nondiagonal operators in classical stochastic models. For instance, setting the condensation and evaporation rates to zero, our model reduces to the asymmetric simple exclusion process [31–33], which can be mapped to the XXZ Heisenberg Hamiltonian by a similarity transformation. This mapping implies the existence of an infinite number of nondiagonal local conserved operators that are obtained from the corresponding transfer matrix [34–36]. It would be very interesting to understand how they constrain the stochastic classical dynamics.

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[23] All transitions in the models we consider involve two neighboring sites. This means that in the case of open boudaries, the leftmost set of transitions (hopping/paircondensation/pairannihilation) are between sites 1 and 2, while the rightmost between sites $L-1$ and L. The local generator (that is, the one for transitions between sites i and $i+1$) is stochastic in itself. This means that we could even have chosen site dependent rates (see Fig. 1) while keeping the overall generator stochastic. See Eqs. (1) and (2).

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[25] Note that these two are not necessarily the only stochastic models that exhibit an almost conserved edge mode [11,12]. However, these are the only two regimes for which the explicit closed-form expression of the edge modes is known, and we rely on that.

[26] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.107.L042104 for the additional details on (i) the diagonal form of the free-fermionic generator, (ii) eigenvectors of edge modes, (iii) stationary states, and (iv) dynamical correlations in the stationary state of the interacting generator.

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