Paul Houston, Ignacio Muga, Sarah Roggendorf*, and Kristoffer G. van der Zee*

The convection-diffusion-reaction equation in non-Hilbert Sobolev spaces: A direct proof of the inf-sup condition and stability of Galerkin's method

May 19, 2019

https://doi.org/..., Received ...; accepted ...

Abstract: While it is classical to consider the solution of the convection-diffusion-reaction equation in the Hilbert space $H_0^1(\Omega)$, the Banach Sobolev space $W_0^{1,q}(\Omega)$, $1 < q < \infty$, is more general allowing more irregular solutions. In this paper we present a well-posedness theory for the convection-diffusion-reaction equation in the $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$ functional setting, $1/q + 1/q' = 1$. The theory is based on directly establishing the inf-sup conditions. Apart from a standard assumption on the advection and reaction coefficients, the other key assumption pertains to a subtle regularity requirement for the standard Laplacian. An elementary consequence of the well-posedness theory is the stability and convergence of Galerkin's method in this setting, for a diffusion-dominated case and under the assumption of $W^{1,q'}$-stability of the H_0^1-projector.

Keywords: convection-diffusion equation, inf-sup condition, Galerkin methods, FEM, well-posedness, Banach spaces, elliptic regularity, stability of H_0^1-projector

PACS: ...

Communicated by: ...
Dedicated to ...

***Corresponding author: Sarah Roggendorf, Kristoffer G. van der Zee,** School of Mathematical Sciences, The University of Nottingham, University Park, NG7 2RD, UK, email: Sarah.Roggendorf@nottingham.ac.uk, KG.vanderZee@nottingham.ac.uk

Paul Houston, School of Mathematical Sciences, The University of Nottingham, University Park, NG7 2RD, UK, email: Paul.Houston@nottingham.ac.uk

Ignacio Muga, Instituto de Mathemáticas, Pontificia Universidad Católica de Valparaíso, email: Ignacio.Muga@pucv.cl

1 Introduction

In this paper we prove the inf-sup conditions for the convection-diffusion-reaction problem in a nonsymmetric Sobolev-space setting; in particular, we consider the $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$-setting, $1 < q < \infty$, $1/q + 1/q' = 1$. The main motivation for considering this non-Hilbert setting is to allow more irregular solutions, if, e.g., the right-hand side is not in $H^{-1}(\Omega)$ but in $W^{-1,q'}(\Omega)$ for some $1 < q < 2$. Apart from a standard assumption on the advection and reaction coefficients, the other key assumption pertains to a subtle regularity requirement for the standard Laplacian. We furthermore prove an elementary stability result of Galerkin's method in that setting.

In the context of finite element methods (FEMs), well-posedness of the convection-diffusion-reaction variational problem is traditionally established by proving coercivity and continuity of the underlying bilinear form in $H_0^1(\Omega)$. Due to the Lax-Milgram theorem this then implies well-posedness of the continuous problem and its discrete counterpart. The concept of coercivity, however, requires that the trial and test spaces should be the same.[1] This is not always possible or desirable, e.g., in the context of Petrov-Galerkin methods or mixed methods, or if the continuous problem itself is stated in a non-symmetric manner such as in the $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$-setting.

A generalization of the Lax-Milgram theorem originally due to Nečas [27] replaces the coercivity requirement by two inf-sup conditions and can be formulated for Banach spaces with distinct test and trial spaces. A version of this theorem for approximations in finite dimensional subspaces is due to Babuška [3]. It is important to note that the inf-sup conditions in the continuous formulation does not imply an inf-sup condition in the discrete setting. In fact, for the convection-diffusion-reaction equation in $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$, we will establish discrete well-posedness under much stronger assumptions than in the continuous setting.

In the remainder of the introduction we specify the weak formulation of the underlying problem, stipulate the assumptions and announce our main results.

1.1 Notation

Throughout this paper, we denote by $L^q(\Omega)$, $1 \leq q < \infty$, the Lebesgue space of q-integrable functions on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3, \ldots\}$;

1 Moreover, the concept of coercivity is a notion relevant only in Hilbert spaces [12, Section A.2.4].

$L^\infty(\Omega)$ is the Lesbeque space of functions on Ω with finite essential supremum; and $W^{1,q}(\Omega)$, $1 \le q \le \infty$, is the Sobolev space of functions that are in $L^q(\Omega)$ such that their gradient is in $L^q(\Omega)^d$. Furthermore, $W_0^{1,q}(\Omega) \subset W^{1,q}(\Omega)$ is the subspace of all functions with zero trace on the boundary $\partial\Omega$. The corresponding norms are denoted by $\|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{W^{1,q}(\Omega)}$, respectively, and the Sobolev-seminorm on $W^{1,q}(\Omega)$ by $|\cdot|_{W^{1,q}(\Omega)}$. For $q = 2$, we furthermore use the usual notation $H^1(\Omega) := W^{1,2}(\Omega)$ and $H_0^1(\Omega) := W_0^{1,2}(\Omega)$. For $1 \le q \le \infty$ we write q' to denote the dual exponent such that $1/q + 1/q' = 1$. The space of smooth functions with compact support in Ω is denoted by $C_c^\infty(\Omega)$. For any Banach space V, its dual space is denoted by V'; the dual space of $W_0^{1,q}(\Omega)$ is given by $W^{-1,q'}(\Omega)$ and $H^{-1}(\Omega) := W^{-1,2}(\Omega)$. For $v \in V$ and $\varphi \in V'$, we have the duality pairing

$$\langle \varphi, v \rangle_{V',V} := \varphi(v). \tag{1.1}$$

1.2 Problem statement

We consider the following model problem: find u such that

$$-\varepsilon\Delta u + \boldsymbol{b} \cdot \nabla u + cu = f \qquad \text{in } \Omega, \tag{1.2a}$$

$$u = 0 \qquad \text{on } \Gamma = \partial\Omega, \tag{1.2b}$$

where $\varepsilon > 0$, $\boldsymbol{b} : \Omega \to \mathbb{R}^d$, and $c : \Omega \to \mathbb{R}$ are the (positive) diffusion parameter, convection field and reaction coefficient, respectively, and $f : \Omega \to \mathbb{R}$ is a given source.

Multiplying (1.2a) by a test function $v \in C_c^\infty(\Omega)$ and integrating by parts yields the bilinear form

$$\mathcal{B}_\varepsilon(u, v) = \varepsilon \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega u \nabla \cdot (\boldsymbol{b} v) \,\mathrm{d}\boldsymbol{x} + \int_\Omega cuv \,\mathrm{d}\boldsymbol{x}. \tag{1.3}$$

This allows us to state the following variational problem for some $q \in (1, \infty)$: find $u \in U := W_0^{1,q}(\Omega)$ such that

$$\mathcal{B}_\varepsilon(u, v) = \langle f, v \rangle_{V',V} \qquad \forall v \in V := W_0^{1,q'}(\Omega), \tag{1.4}$$

where f is allowed to be any element in $V' = W^{-1,q}(\Omega)$. Furthermore, we endow U and V with the following norms

$$\|u\|_U^q := c_0 \|u\|_{L^q(\Omega)}^q + \varepsilon \|\nabla u\|_{L^q(\Omega)}^q, \tag{1.5a}$$

$$\|v\|_V^{q'} := c_0 \|v\|_{L^{q'}(\Omega)}^{q'} + \varepsilon \|\nabla v\|_{L^{q'}(\Omega)}^{q'}, \tag{1.5b}$$

with c_0 given by Assumption 1.3 below.

The proof of well-posedness of this problem with $q = 2$ is well known, see, e.g., [12, Section 3.1]. Furthermore, well-posedness of the problem for $q \neq 2$ in smooth domains (e.g., domains with C^1- or $C^{1,1}$-boundary) is also well-established [16]. For $q \neq 2$ and general Lipschitz domains on the other hand, even simply proving well-posedness of the Poisson problem is far more challenging. It is no longer sufficient to require that Ω is a bounded Lipschitz domain. In fact, if $d \geq 3$, then for any $q > 3$ there exists a Lipschitz domain Ω and a right-hand side $f \in C^\infty(\bar{\Omega})$ such that the solution to the Poisson problem is not in $W^{1,q}(\Omega)$,cf., [20]; in two dimensions the same result holds for $q > 4$.

The novelty of our approach is that we provide a direct proof of the inf-sup conditions for the convection-diffusion-reaction equation illustrating that the so-called duality map is invaluable as a replacement for the Riesz isometry in the context of Banach spaces. The duality map is a fundamental operator in Banach spaces and will be recalled in Section 2.2. Furthermore, our proof only relies on standard assumptions on $\boldsymbol{b}$ and c, and $W^{1,r}$-regularity of the standard H_0^1-Poisson problem for all r up to q'. The latter assumption can be interpreted as an assumption on the domain Ω for a given q, or, for a given bounded Lipschitz domain Ω it restricts the values that can be chosen for q. Another important technique in our proof (for the second inf-sup condition) is a bootstrapping argument involving repeated use of elliptic regularity and Sobolev embeddings to gain sufficient regularity. To prove the *discrete* inf-sup condition, as needed for the stability of Galerkin's method, we will further assume a diffusion-dominated case and the $W^{1,q'}$-stability of the H_0^1-projector.

1.3 Assumptions

We now present the assumptions in detail. One of the main assumptions we rely on is (adjoint-) regularity of the standard $H_0^1(\Omega)$-Poisson problem.

Assumption 1.1 (Regularity for Laplacian). *Let $1 < q < 2$. For each $r \in (2, q']$, there is a stability constant $C_{r,\Omega}$, such that for all $g \in W^{-1,r}(\Omega) \subset H^{-1}(\Omega) := [H_0^1(\Omega)]'$, the unique $z_g \in H_0^1(\Omega)$ that solves the problem*

$$\int_\Omega \nabla w \cdot \nabla z_g \, \mathrm{d}\boldsymbol{x} = \langle g, w \rangle_{W^{-1,r}(\Omega), W_0^{1,r'}(\Omega)} \qquad \forall w \in H_0^1(\Omega), \tag{1.6}$$

satisfies the a priori estimate:

$$\|\nabla z_g\|_{L^r(\Omega)} \leq C_{r,\Omega} \|g\|_{W^{-1,r}(\Omega)}; \tag{1.7}$$

hence $z_g \in W_0^{1,r}(\Omega)$.

Remark 1.2. Assumption 1.1 holds for any $1 < q < 2$ if Ω has a C^1 boundary, cf., e.g., [16, 20, 17]. It is also true for any q if Ω is a convex Lipschitz domain, cf., [1, 15, 24]. The latter result can be extended to Lipschitz domains that satisfy an exterior ball condition [19]. For bounded Lipschitz domains, there exists $1 < q_0 < 2$ such that the assumption holds for all $q \geq q_0$, where $q_0 \leq 3/2$ if $d \geq 3$ and $q_0 \leq 4/3$ if $d = 2$. The precise value for q_0 depends on the domain Ω. It is possible to construct counterexamples for any $q < 3/2$ if $d \geq 3$ and for any $q < 4/3$ if $d = 2$. For more details we refer the reader to [20]. For $1 < q \leq 2$ it is possible to obtain higher regularity for the solution, i.e., $z_g \in W^{2,q}(\Omega)$, provided the right-hand side g is in $L^q(\Omega)$. □

Similar to the analysis of the advection-reaction equation [6, 25], we also need some requirements on the advective field and the reaction coefficient.

Assumption 1.3 (Friedrich's positivity assumption). *Let $1 < q < \infty$. The advective field $\boldsymbol{b} \in L^\infty(\Omega)^d$ satisfies $\nabla \cdot \boldsymbol{b} \in L^\infty(\Omega)$, the reaction coefficient $c \in L^\infty(\Omega)$, and there exists a constant $c_0 > 0$ such that*

$$c(\boldsymbol{x}) - \frac{1}{q}\nabla \cdot \boldsymbol{b}(\boldsymbol{x}) \geq c_0, \qquad \text{a.e. in } \Omega. \tag{1.8}$$

In order to prove well-posedness in the *discrete* setting, we request that the $H_0^1(\Omega)$-projector onto the chosen h-parametrized family of finite-dimensional subspaces is stable in the following sense.

Assumption 1.4 (Stability of $H_0^1(\Omega)$-projector). *Let $q' > 2$ and $U_h \subset W_0^{1,\infty}(\Omega)$ be a finite-dimensional subspace. The $H_0^1(\Omega)$-projector $P_h : H_0^1(\Omega) \to U_h$ is uniformly stable in $W^{1,q'}(\Omega)$, i.e., there is a stability constant $C_P \geq 1$, independent of h, such that for all $z \in W_0^{1,q'}(\Omega) \subset H_0^1(\Omega)$, the unique $H_0^1(\Omega)$-projection $P_h z \in U_h$ that solves the discrete problem*

$$\int_\Omega \nabla v_h \cdot \nabla(P_h z)\, \mathrm{d}\boldsymbol{x} = \int_\Omega \nabla v_h \cdot \nabla z\, \mathrm{d}\boldsymbol{x} \qquad \forall v_h \in U_h, \tag{1.9}$$

satisfies the a priori bound:

$$|P_h z|_{W^{1,q'}(\Omega)} \leq C_P |z|_{W^{1,q'}(\Omega)}. \tag{1.10}$$

We have not assumed any specific properties of the (finite element) space U_h or the underlying mesh in order to keep the above assumption as general as possible. Note, however, that the validity of Assumption 1.4 has only been proven in very specific settings and thus it can be expected that this assumption implies certain restrictions. If Ω is a bounded interval in $\mathbb{R}$, a convex polygonal domain in $\mathbb{R}^2$, or a convex polyhedral domain in $\mathbb{R}^3$, Assumption 1.4 is known to hold for all $q' > 2$ for standard finite element spaces on quasi-uniform meshes, see [28, 18] and [4, Chapter 8], and certain graded meshes, see [9, 22]. Furthermore, the assumption is valid for all $q' > 2$ if U_h is a spectral space of d-variate polynomials of fixed degree on a (finite union of) star-shaped domain(s), see [10].

1.4 Main results

We can now state the main results of this paper, the proofs of which are given in Sections 3 and 4. We prove the inf-sup conditions, both in the continuous and the discrete settings, under the assumptions stated in the previous section. To this end, firstly, we have the well-posedness in the continuous case.

Theorem 1.5. *Let $1 < q < \infty$. If Assumption 1.1 holds and additionally Assumption 1.3 is satisfied, then the variational problem* (1.4) *is well-posed. In particular, the bilinear form $\mathcal{B}_\varepsilon$ defined in* (1.3) *is bounded, i.e.,*

$$\mathcal{B}_\varepsilon(w, v) \leq M \|v\|_V \|w\|_U \tag{1.11}$$

and satisfies the following inf-sup *conditions: there exists $\gamma > 0$ such that*

$$\inf_{w \in U} \sup_{v \in V} \frac{\mathcal{B}_\varepsilon(w, v)}{\|w\|_U \|v\|_V} \geq \gamma, \tag{1.12a}$$

and

$$\forall v \in V, \quad \big(\forall w \in W,\ \mathcal{B}_\varepsilon(w, v) = 0\big) \Rightarrow (v = 0). \tag{1.12b}$$

Both M and γ are positive constants that may depend on ε, c_0, $\|c\|_{L^\infty(\Omega)}$ and $\|\boldsymbol{b}\|_{L^\infty(\Omega)}$.

Remark 1.6. The proof of (1.12a) given in Section 3 shows the following estimate for γ:

$$\gamma \geq \frac{\min\left(\frac{C_1^2}{2\varepsilon c_0^{2/q'}}, \frac{\varepsilon^{1-2/q'}}{2} \right)}{2C_{q',\Omega}(c_0 C_{F,q}^q + \varepsilon)^{1/q} \varepsilon^{-1/q} + C_1^2 c_0^{-1} \varepsilon^{-1} 2^{\frac{1}{q} - \frac{1}{q'}}}. \tag{1.13}$$

Here, $C_{F,q}$ is the constant in the Poincaré-Friedrichs inequality in $W^{1,q}(\Omega)$, $C_{q',\Omega}$ is the constant given in Assumption 1.1 for $r = q'$ and

$$C_1 = C_{q',\Omega}\Big(\|b\|_{L^\infty(\Omega)} + C_{F,q}\|c\|_{L^\infty(\Omega)}\Big).$$

For very small ε the dominating term in the numerator scales like $\varepsilon^{1-\frac{2}{q'}}$. Note that this is equal to 1 for $q = q' = 2$. The denominator scales like ε^{-1}; the overall scaling for very small ε is therefore $\varepsilon^{2-\frac{2}{q'}}$. For very small c_0 the lower bound for the inf-sup constant γ scales like c_0. It can be assumed that this estimate is not optimal, since the scaling in ε and c_0 for $q = q' = 2$ is less favourable than in the standard proof when $q = q' = 2$. □

Secondly, consider the (Bubnov-) Galerkin approximation for a (finite-element) space $U_h \subset W_0^{1,\infty}(\Omega)$: find $u_h \in U_h$ such that

$$\mathcal{B}_\varepsilon(u_h, v_h) = \langle f, v_h \rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} \qquad \forall v_h \in U_h. \tag{1.14}$$

In this discrete setting, we have the following result.

Theorem 1.7. *Let* $1 < q < 2$, $\boldsymbol{b} \in L^\infty(\Omega)^d$, $\nabla \cdot \boldsymbol{b} \in L^\infty(\Omega)$ *and* $c \in L^\infty(\Omega)$. *If Assumption 1.1 and Assumption 1.4 hold, then the following discrete inf-sup condition holds true:*

$$\sup_{v_h \in U_h} \frac{\mathcal{B}_\varepsilon(w_h, v_h)}{|v_h|_{W^{1,q'}(\Omega)}} \geq \widehat{\gamma} |w_h|_{W^{1,q}(\Omega)} \qquad \textit{for all } w_h \in U_h, \tag{1.15}$$

where

$$\widehat{\gamma} \geq \left(\varepsilon C_P^{-1} C_{q',\Omega}^{-1} - C_{F,q'} (\|\boldsymbol{b}\|_\infty + C_{F,q} \|c\|_\infty) \right), \tag{1.16}$$

with $C_{F,q}$ *and* $C_{F,q'}$ *the Poincaré-Friedrichs constant for* $W_0^{1,q}(\Omega)$ *and* $W_0^{1,q'}(\Omega)$, *respectively (see* (3.6)*).*

Note that it can be guaranteed that the constant $\widehat{\gamma}$ in (1.16) is strictly positive. Indeed, this is true if the convection-diffusion-reaction problem is sufficiently *diffusion-dominated*, i.e., the advection and the reaction coefficients are sufficiently small compared to the diffusion parameter ε. From this perspective, Theorem 1.7 is an elementary result: We strongly believe that (1.16) is a suboptimal result, and we conjecture that the discrete inf-sup condition holds true for any choice of the parameters (although not robustly).

In summary, by standard arguments (see, e.g., [12, Chapter 2]), Theorem 1.5 implies the existence of a unique solution $u \in W_0^{1,q}(\Omega)$ to the convection-diffusion-reaction problem, which satisfies the a priori bound:

$$\|u\|_U \leq \frac{1}{\gamma} \|f\|_{W^{-1,q}(\Omega)}.$$

Recall from (1.5a) that $\|\cdot\|_U$ is a norm on $W_0^{1,q}(\Omega)$ equivalent to, e.g., $|\cdot|_{W^{1,q}(\Omega)}$. Theorem 1.7 further implies that in the diffusion-dominated case, Galerkin's method is a stable and convergent method in the $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$-setting. In particular, the following a priori bound holds

$$\|u_h\|_U \leq \frac{1}{\widehat{\gamma}}\|f\|_{W^{-1,q}(\Omega)}, \tag{1.17}$$

as well as the following a priori error estimate

$$\|u-u_h\|_U \leq \big(1+\frac{M}{\widehat{\gamma}}\big) \inf_{w_h\in U_h} \|u-w_h\|_U .$$

This last result can be sharpened, by invoking the error estimate due to Stern [29]:

$$\|u-u_h\|_U \leq \min\left\{2^{|\frac{2}{q}-1|}\frac{M}{\widehat{\gamma}}\, ,\, 1+\frac{M}{\widehat{\gamma}}\right\} \inf_{w_h\in U_h} \|u-w_h\|_U .$$

To illustrate the results for Galerkin's method, we verify the bound in (1.17) by performing several numerical experiments in two- and three-dimensional domains on convection-diffusion problems, for which the solution is not in $H_0^1(\Omega)$ but in $W_0^{1,r}(\Omega)$ for suitable $r < 2$. These numerical experiments show that Galerkin's method is indeed stable for the anticipated values $q \leq r$.

1.5 Outline of the paper

The rest of the paper is organized as follows. In Section 2, we present several necessary preliminaries, in particular, we present well-posedness results for the Poisson problem in the $W_0^{1,q}$-$W_0^{1,q'}(\Omega)$ setting, fundamental properties of duality mappings, and a brief discussion on difficulties in proving the continuous inf-sup conditions. We then present in Section 3 the proof of Theorem 1.5, relying heavily on the preliminaries in the preceding section. Section 3 discusses in the following order: the proof of the continuity of $\mathcal{B}_\varepsilon$, the first inf-sup condition, and finally, the second inf-sup condition. In Section 4, we give the proof of Theorem 1.7 (discrete inf-sup condition). Section 5 contains the numerical experiments for Galerkin's method considering the approximation of irregular solutions $u \notin H_0^1(\Omega)$. Finally, the Appendix A contains the proof of Proposition 2.1 (well-posedness of Poisson problem in the $W_0^{1,q}$-$W_0^{1,q'}(\Omega)$ setting).

2 From Well-Posedness in $H_0^1(\Omega)$ to Well-Posedness in the $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$ setting

In the context of finite element methods the analysis of variational formulations of PDEs is traditionally undertaken in subspaces of $L^2(\Omega)$ such as $H_0^1(\Omega)$. Even though we are considering a formulation in more general Sobolev spaces that are no longer Hilbert spaces, we are still using many standard techniques that have been developed for the numerical analysis of finite element methods. In this section we focus on illustrating how concepts and techniques that rely on a Hilbert space setting with identical test and trial spaces can be extended to our more general setting. To this end, we first consider the Poisson problem in the $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$-setting as a simplified model problem, which will be crucial later when dealing with the full convection-diffusion-reaction problem.

As part of the proof of the inf-sup-conditions, we will construct a very specific functional in $W^{-1,q'}(\Omega)$ that is an example of a duality mapping. We thus continue in Section 2.2 with introducing duality mappings as a general concept. Duality mappings have proven to be a very useful concept for mimicking certain techniques that rely on properties of Hilbert spaces in more general Banach spaces. In some sense duality mappings are a suitable nonlinear replacement for the Riesz map. This becomes particularly evident in the context of residual minimization problems; see, for example, [26].

Finally, we conclude this section by explaining how we intend to include the lower order terms in the convection-diffusion-advection equation. The next sections will then focus on rigorous proofs of Theorem 1.5 and Theorem 1.7.

2.1 The Poisson Problem in the $W_0^{1,q}(\Omega)$-$W_0^{1,q'}(\Omega)$ setting

The goal is to extend the well-posedness proof of the Poisson problem to the convection-diffusion-reaction equation provided that Assumption 1.1 holds. We will now see that Assumption 1.1 indeed implies the following well-posedness result for the Poisson problem.

Proposition 2.1 (Well-posedness of the Poisson problem in Banach Sobolev spaces). *Let $1 < q < 2$. If Assumption 1.1 holds, then for every $f \in W^{-1,q}(\Omega)$ there exists* *a unique* $u_f \in W_0^{1,q}(\Omega)$ *that satisfies the Poisson problem:*

$$\int_\Omega \nabla u_f \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \langle f, v \rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} \qquad \forall v \in W_0^{1,q'}(\Omega). \tag{2.1}$$

Furthermore, u_f *satisfies the a priori estimate*

$$\|\nabla u_f\|_{L^q(\Omega)} \le C_{q',\Omega} \|f\|_{W^{-1,q}(\Omega)}. \tag{2.2}$$

Proof. Although the above well-posedness result is known (cf. [20]), we provide in Appendix A an elementary self-contained proof of the inf-sup conditions (2.6a)–(2.6b) using the duality map. We also demonstrate that when $d = 1$, Assumption 1.1 is not needed. □

Combining Proposition 2.1 again with Assumption 1.1 immediately implies the following general regularity result; this will be required in the proof of Theorem 1.5 instead of Assumption 1.1 directly.

Corollary 2.2 (Elliptic regularity for Laplacian in Banach Sobolev spaces). *Let* $1 < q < 2$ *and* $q \le r \le q'$. *If Assumption 1.1 holds, then for all* $g \in W^{-1,r}(\Omega) \subset W^{-1,q}(\Omega)$, *the unique solution* $u_g \in W_0^{1,q}(\Omega)$ *that solves the problem*

$$\int_\Omega \nabla u_g \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \langle g, v \rangle_{W^{-1,r}(\Omega), W_0^{1,r'}(\Omega)} \qquad \forall v \in W_0^{1,q'}(\Omega) \tag{2.3}$$

satisfies the a priori estimate:

$$\|\nabla u_g\|_{L^r(\Omega)} \le C_{r,\Omega} \|g\|_{W^{-1,r}(\Omega)}; \tag{2.4}$$

hence $u_g \in W_0^{1,r}(\Omega)$. *The constant* $C_{r,\Omega} = C_{r',\Omega}$ $(r \neq 2)$ *is the same as in Assumption 1.1, and* $C_{2,\Omega} = 1$.

Proof. This result is a direct consequence of Assumption 1.1 and Proposition 2.1. Suppose $r \in (q, 2)$, then apply Proposition 2.1 (replacing q by r and f by g) to obtain that $u_g \in W_0^{1,r}(\Omega) \subset W_0^{1,q}(\Omega)$; by duality $C_{r',\Omega} = C_{r,\Omega}$. Suppose that $r = 2$, then use the standard H_0^1-Poisson problem with right-hand side $g \in H^{-1}(\Omega)$ to obtain that $u_g \in H_0^1(\Omega) \subset W_0^{1,q}(\Omega)$ and $\|\nabla u_g\|_{L^2(\Omega)} \le \|g\|_{H^{-1}(\Omega)}$. Suppose $r \in (2, q')$, then apply Assumption 1.1 (replacing z_g by u_g) to obtain that $u_g \in W_0^{1,r}(\Omega) \subset H_0^1(\Omega) \subset W_0^{1,q}(\Omega)$. □

Let us recall that for $q = 2$ well-posedness follows immediately from the Lax-Milgram Theorem since both continuity and coercivity are obvious in this case. The same approach is obviously no longer applicable if $q \neq 2$ since it requires the trial and test spaces to be identical. We thus apply a generalized Lax-Milgram Theorem (or BNB Theorem [12, Chapter 2]) that is originally due to Nečas [27]. A version of this theorem for approximations in finite dimensional subspaces was derived by Babuška [3].

Theorem 2.3 (Banach-Nečas-Babuška). *Let U and V be Banach spaces, and assume additionally that V is reflexive. Let $b : U \times V \to \mathbb{R}$ be a continuous bilinear form and $\ell \in V'$. Furthermore, consider the problem: find $u \in U$ such that*

$$b(u, v) = \ell(v) \qquad \forall v \in V. \tag{2.5}$$

Then, (2.5) is well-posed for all $\ell \in V'$ if and only if

$$\exists \gamma > 0, \qquad \inf_{w \in U} \sup_{v \in V} \frac{b(w, v)}{\|w\|_U \|v\|_V} \geq \gamma, \tag{2.6a}$$

$$\forall v \in V, \quad (\forall w \in U,\, b(w, v) = 0) \Rightarrow (v = 0). \tag{2.6b}$$

Moreover, we have the a priori estimate:

$$\forall \ell \in V', \qquad \|u\|_U \leq \frac{1}{\gamma} \|\ell\|_{V'}. \tag{2.7}$$

For the Poisson problem with $q = 2$, the inf-sup condition (2.6a) is again immediate since in this case

$$\sup_{v \in V = U} \frac{b(u, v)}{\|v\|_V} \geq \frac{b(u, u)}{\|u\|_U} = \frac{\|u\|_U^2}{\|u\|_U} = \|u\|_U. \tag{2.8}$$

The key observation here is that $b(u, u)$ is equal to the square of the norm on U.

For $q < 2$, we can no longer use u itself as a test function. But if we suppose that there exists a v_u such that

$$b(u, v_u) = |u|^q_{W^{1,q}(\Omega)}, \tag{2.9a}$$

$$|v_u|_{W^{1,q'}(\Omega)} \leq C |u|^{q-1}_{W^{1,q}(\Omega)}, \tag{2.9b}$$

we would again immediately obtain an inf-sup condition.

Remark 2.4. Later we will instead use $\tilde{v}_u$ such that

$$b(u, \tilde{v}_u) = |u|^2_{W^{1,q}(\Omega)},$$

$$|\tilde{v}_u|_{W^{1,q'}(\Omega)} \leq C |u|_{W^{1,q}(\Omega)}.$$

Note that if v_u satisfies (2.9a) and (2.9b), we can define $\tilde{v}_u = |u|^{2-q}_{W^{1,q}(\Omega)} v_u$. □

The question now is whether it is possible to construct such a test function $v_u \in W_0^{1,q'}(\Omega)$ for any $u \in W_0^{1,q}(\Omega)$. Let us start with the right-hand side of (2.9a). As a first step we determine a $\boldsymbol{\xi} : \Omega \to \mathbb{R}^d$ such that

$$|u|^q_{W^{1,q}(\Omega)} = \int_\Omega \sum_{i=1}^d |\partial_i u|^q \,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla u \cdot \boldsymbol{\xi} \,\mathrm{d}\boldsymbol{x}. \tag{2.10}$$

If we divide each summand by $\partial_i u$, we can see that $\boldsymbol{\xi}$ has to be of the form

$$\xi_i = |\partial_i u|^{q-1} \mathrm{sgn}(\partial_i u). \tag{2.11}$$

Ideally, we would like to construct $v_u \in W_0^{1,q'}(\Omega)$ such that $\nabla v_u = \boldsymbol{\xi}$, which, however, seems to be impossible in general. Instead we identify $\boldsymbol{\xi}$ with $\ell_{\boldsymbol{\xi}} \in W^{-1,q'}(\Omega)$ as follows

$$\langle \ell_{\boldsymbol{\xi}}, w \rangle_{W^{-1,q'}(\Omega), W_0^{1,q}(\Omega)} = \int_\Omega \nabla w \cdot \boldsymbol{\xi} \,\mathrm{d}\boldsymbol{x}. \tag{2.12}$$

It is easy to check that we have $\xi_i \in L^{q'}(\Omega)$ and as a consequence $\ell_{\boldsymbol{\xi}} \in W^{-1,q'}(\Omega)$. Furthermore, we can compute $\|\ell_{\boldsymbol{\xi}}\|_{W^{-1,q'}(\Omega)} = |u|^{q-1}_{W^{1,q}(\Omega)}$ using the duality of the spaces $L^q(\Omega)$ and $L^{q'}(\Omega)$. We can now define v_u as the solution to the following Poisson problem:

$$\int_\Omega \nabla w \cdot \nabla v_u \,\mathrm{d}\boldsymbol{x} = \langle \ell_{\boldsymbol{\xi}}, w \rangle_{W^{-1,q'}(\Omega), W_0^{1,q}(\Omega)} \qquad \forall w \in W_0^{1,q}(\Omega). \tag{2.13}$$

Note that the existence of $v_u \in H_0^1(\Omega)$ is due to the well-posedness of the Poisson problem in $H_0^1(\Omega)$. The higher regularity of v_u then follows from Assumption 1.1 and the a priori estimate in Assumption 1.1 implies (2.9b). Furthermore note that by definition $\ell_{\xi}(u) = |u|^q_{W^{1,q}(\Omega)}$, which implies (2.9a).

If we instead use

$$\tilde{\xi}_i = |u|^{2-q}_{W^{1,q}(\Omega)} |\partial_i u|^{q-1} \mathrm{sgn}(\partial_i u), \tag{2.14}$$

we can analogously define a linear functional $\ell_{\tilde{\boldsymbol{\xi}}}$ and a test function $\tilde{v}_u$ that satisfies the conditions in Remark 2.4.

2.2 Duality Mappings

In the previous section we have heuristically constructed the linear functional $\ell_{\tilde{\boldsymbol{\xi}}}$ that allowed us to define a suitable test function. In [26, Section 2] the same idea is undertaken in a rather more abstract setting. The functional $\ell_{\tilde{\boldsymbol{\xi}}}$ can also be defined as the functional $\ell_{\tilde{\boldsymbol{\xi}}} \in W^{-1,q'}(\Omega)$ such that

$$\begin{aligned} \langle \ell_{\tilde{\boldsymbol{\xi}}}, u \rangle_{W^{-1,q'}(\Omega), W_0^{1,q}(\Omega)} &= |u|_{W^{1,q}(\Omega)} \|\ell_{\tilde{\boldsymbol{\xi}}}\|_{W^{-1,q'}(\Omega)}, \\ \|\ell_{\tilde{\boldsymbol{\xi}}}\|_{W^{-1,q'}(\Omega)} &= |u|_{W^{1,q}(\Omega)}. \end{aligned} \tag{2.15}$$

The Hahn-Banach Theorem implies existence of a functional with these properties. In order to obtain uniqueness, we have to require strict convexity of the underlying space, which is true for $1 < q < \infty$. A linear functional with the above properties is called a *duality mapping* and we will now give a general definition of duality mappings in Banach spaces.

Definition 2.5. Let V be a Banach space and denote by $\mathcal{P}(V')$ the power set of V'. Then the multivalued map $\mathcal{J}_V : V \to \mathcal{P}(V')$, defined by

$$\mathcal{J}_V(v) := \left\{ v' \in V' \,:\, \langle v', v \rangle_{V',V} = \|v\|_V \|v'\|_{V'}, \|v'\|_{V'} = \|v\|_V \right\} \tag{2.16}$$

is called a *duality mapping*[2]. □

Due to a corollary of the Hahn-Banach Theorem (see, e.g., [5, Corollary 1.3]) the set $\mathcal{J}_V(v)$ is non-empty. Furthermore, note that we have $\mathcal{J}_V(v) = \{R_V(v)\}$, where $R_V : V \to V'$ is the Riesz map if V is a Hilbert space due to the Riesz Representation Theorem.
In general, duality mappings are multivalued and neither injective nor surjective. If the underlying spaces have certain properties, however, we can obtain an invertible single valued map. The following proposition summarizes some of the properties of duality mappings related to certain properties of the underlying Banach spaces.

Proposition 2.6. *Let V be a Banach space and denote by $\mathcal{J}_V : V \to \mathcal{P}(V')$ the duality map on V. Then the following statements are true:*

1. *V' is strictly convex if and only if $\mathcal{J}_V$ is single valued, cf., [8, Prop. 12.3]. In this case we define the duality map $J_V : V \to V'$ such that $\mathcal{J}_V(v) = \{J_V(v)\}$ for all $v \in V$.*
2. *If V is strictly convex, then $\mathcal{J}_V(v) \cap \mathcal{J}_V(w) = \emptyset$ for all $w \neq v$; cf. [26, Section 2]. In particular, $\mathcal{J}_V$ is injective.*

---

2 This is the defnition used in [5, 8, 31]. In [7], a more general notion of duality mappings is used and the duality mapping defined here is referred to as the *normalized* duality mapping. For the purposes of this paper, there is no benefit to using this more general notion of duality mappings. However, for some practical aspects of using the method introduced in [26], a duality mapping with a different so-called weight is useful. An example of such a duality mapping is the functional $\ell_{\boldsymbol{\xi}}$.

3. *V is reflexive if and only if $\mathcal{J}_V$ is surjective in the sense that for every $v' \in V'$ there is a $v \in V$ such that $v' \in \mathcal{J}_V(v)$, cf., [7, Theorem 3.4, Chapter II].*
4. *If V is a reflexive Banach space and $\mathcal{J}_V$ is a duality mapping of weight φ, then $(\mathcal{J}_V)^{-1}$ is a duality mapping on V^*, cf., [7, Cor. 3.5, Ch. II].*

The following theorem is a special case of Theorem 4.4 in [7, Chapter I] and states that the duality map on V can be characterized using the subdifferential of the norm on V. This is a key property of the duality map that will allow us to derive the duality map for some specific Banach spaces in the special case that the subdifferential is essentially the Gâteaux or Fréchet derivative of the norm.

Theorem 2.7 (Asplund, cf., [7, Ch. I, Theorem 4.4]). *Let V be a Banach space and define $F_V : V \to \mathbb{R}$ by $F_V(\cdot) := \frac{1}{2}\|\cdot\|_V^2$. Then for any $v \in V$, we have*

$$\mathcal{J}_V(v) = \partial F_V(v), \tag{2.17}$$

where $\partial F_V(v)$ denotes the subdifferential of F_V at v.

If V' is strictly convex and thus the duality mapping is single valued, the subdifferential exists as a Gâteaux derivative. This allows us to explicitly compute the duality mappings for Sobolev spaces with exponent $1 < q < \infty$.

Let us denote the duality mapping on $L^q(\Omega)$ by J_q. We compute

$$\begin{aligned} \langle J_q(v), w\rangle_{L^{q'}(\Omega),L^q(\Omega)} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|v+tw\|_{L^q(\Omega)}^2\right)\Bigg|_{t=0} \\ &= \|v\|_{L^q(\Omega)}^{2-q}\int_\Omega |v|^{q-1}\operatorname{sgn}(v)w\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{2.18}$$

As a second example consider the space $W_0^{1,q}(\Omega)$ with the (semi-)norm $|v|_{W^{1,q}(\Omega)}$. In the same way as before we obtain

$$\left\langle J_{W_0^{1,q}(\Omega)}(v), w\right\rangle_{W^{-1,q'}(\Omega),W_0^{1,q}(\Omega)} = |v|_{w^{1,q}(\Omega)}^{2-q}\int_\Omega \sum_{i=1}^d |\partial_i v|^{q-1}\operatorname{sgn}(\partial_i v)\partial_i w\,\mathrm{d}\boldsymbol{x}. \tag{2.19}$$

Note that $J_{W_0^{1,q}(\Omega)}$ is identical to the functional $\ell_{\tilde{\boldsymbol{\xi}}}$.

2.3 Lower Order Terms

We now aim to extend the result we have seen for the Poisson problem to the convection-diffusion-reaction equation. For $q = 2$, well-posedness again follows from the Lax-Milgram Theorem due to the continuity and coercivity of the underlying bilinear form.

Let us take a brief look at the well-known proof of coercivity for the bilinear form $\mathcal{B}_\varepsilon$ under Assumption 1.3 with $q = 2$:

$$\begin{aligned}\mathcal{B}_\varepsilon(u,u) &= \varepsilon \int_\Omega \nabla u \cdot \nabla u \,\mathrm{d}\boldsymbol{x} - \int_\Omega u \nabla \cdot (\boldsymbol{b} u) \,\mathrm{d}\boldsymbol{x} + \int_\Omega c u^2 \,\mathrm{d}\boldsymbol{x} \\ &= \varepsilon \int_\Omega \nabla u \cdot \nabla u \,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\nabla \cdot \boldsymbol{b}) u^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega c u^2 \,\mathrm{d}\boldsymbol{x} \\ &\geq \varepsilon \int_\Omega \nabla u \cdot \nabla u \,\mathrm{d}\boldsymbol{x} + c_0 \int_\Omega u^2 \,\mathrm{d}\boldsymbol{x} \\ &= \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + c_0 \|u\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.20}$$

As for the Poisson problem, coercivity also immediately implies the inf-sup condition (2.6a).

For $q \neq 2$, we have already seen how the Laplace operator can be treated; thereby, we will now focus on the lower order terms. In (2.20), there are two steps that are applied to the lower order terms: the first one is rearranging everything by integration by parts such that Assumption 1.3 can be applied and the second one is the rather trivial observation that the resulting term can be bounded by the square of the $L^2(\Omega)$-norm of u. If $q \neq 2$, we again have the problem that we cannot simply test with u itself. The idea is to start from the end and mimic the steps in the proof for the Poisson problem by constructing a test function v such that

$$\int_\Omega uv \,\mathrm{d}\boldsymbol{x} = \|u\|^2_{L^q(\Omega)}, \quad \|v\|_{L^{q'}(\Omega)} = \|u\|_{L^q(\Omega)}. \tag{2.21}$$

Due to the duality of the spaces $L^q(\Omega)$ and $L^{q'}(\Omega)$ this construction actually becomes easier than for the Poisson problem. Either by using the abstract concept of duality mappings or by explicitly constructing v, we can see immediately that v must be defined as

$$v := J_q(u) = \|u\|^{q-2}_{L^q(\Omega)} |u|^{q-1} \operatorname{sgn}(u). \tag{2.22}$$

It is easy to verify that $v \in L^{q'}(\Omega)$. However, to use v as a test function we require $v \in W^{1,q'}_0(\Omega)$. It turns out, that this is not true for $q < 2$, but it is the case for $q > 2$ (a proof of this will be given in Section 3.3). This suggests to prove the following inf-sup conditions on the adjoint instead:

$$\inf_{v \in V} \sup_{w \in U} \frac{\mathcal{B}_\varepsilon(w,v)}{\|w\|_U \|v\|_V} \geq \gamma, \tag{2.23a}$$

$$\forall w \in U, \quad (\forall v \in V, \mathcal{B}_\varepsilon(w,v) = 0) \Rightarrow (w = 0). \tag{2.23b}$$

This is equivalent to (1.12) (for an elementary proof, see, e.g., [23, Prop. A.2]). In particular, the inf-sup constant γ is the same.

Another observation is the following. We need to be able to mimic the steps in the proof of coercivity in the L^q-setting:

$$\boldsymbol{b} \cdot \nabla v |v|^{q'-1} \operatorname{sgn}(v) = \frac{1}{q'} \nabla \cdot (\boldsymbol{b}|v|^{q'}) - \frac{1}{q'} (\nabla \cdot \boldsymbol{b}) |v|^{q'}. \tag{2.24}$$

The last issue that has to be resolved is that we have two different test functions — one for the Laplace operator and one for the advection operator. In order to obtain an inf-sup condition for the full bilinear form we thus consider a linear combination of both functions and additionally estimate the Laplace term tested with the second test function and the advection-reaction term with the first. The rather technical details of this are presented in the next section.

3 Proof of Theorem 1.5: Well-posedness of the Convection-Diffusion-Reaction Equation in $W_0^{1,q}(\Omega)$

In this section we now present a rigorous proof of Theorem 1.5. We start by showing continuity of the bilinear form $\mathcal{B}_\varepsilon$ in Section 3.1. Sections 3.2 and 3.3 then contain all necessary estimates for the two test functions introduced in Sections 2.1 and 2.3, respectively. In Section 3.4 we then consider a linear combination of the two test functions and combine all of the estimates to conclude the proof of the inf-sup condition (2.23a). We then finish the proof of Theorem 1.5 in Section 3.5 by proving the second inf-sup condition (2.23b). For this last step we employ a so-called bootstrap argument that is more commonly used in the PDE literature, cf., Remark 3.2 below.

Remark 3.1. If $q = 2$ it is possible to obtain ε-independent estimates for the inf-sup constant and the continuity constant if we choose a different norm on V, namely

$$\|v\|_{\tilde{V}} := \|v\|_V + \|\nabla \cdot (\boldsymbol{b}v)\|_{U'}, \tag{3.1}$$

where we consider $\nabla \cdot (\boldsymbol{b}v) \in L^2(\Omega)$ as an element in $U' = H^{-1}(\Omega) \supset L^2(\Omega)$ with its norm on U' given by

$$\|\nabla \cdot (\boldsymbol{b}v)\|_{U'} = \sup_{u \in U} \frac{\langle \nabla \cdot (\boldsymbol{b}v), u \rangle_{U',U}}{\|u\|_U} \tag{3.2}$$

A proof of this result is presented in [30, Section 4.4.1]. There are essentially three steps to the proof: firstly, it is easy to prove that the continuity constant with respect to the norms $\|\cdot\|_U$ and $\|\cdot\|_{\tilde V}$ is independent of ε; secondly, an ε-independent estimate for the inf-sup constant can be proven using the norm (1.5a) on V; thirdly, it is then possible to bound the term (3.2) from above as well without losing robustness of the inf-sup constant. The first and the third step can easily be extended to $q \neq 2$. However, the generalization of the second step to Banach spaces remains an open problem. □

3.1 Continuity of the bilinear form

In order to prove continuity of the bilinear form (1.11), we apply the Hölder inequality and obtain

$$\begin{aligned}\mathcal{B}_\varepsilon(w,v) &= \varepsilon \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega w \nabla \cdot (\boldsymbol{b} v)\,\mathrm{d}\boldsymbol{x} + \int_\Omega cwv \,\mathrm{d}\boldsymbol{x} \\ &\le \varepsilon \|\nabla w\|_{L^q(\Omega)} \|\nabla v\|_{L^{q'}(\Omega)} + \frac{\|c\|_{L^\infty(\Omega)}}{c_0} c_0 \|w\|_{L^q(\Omega)} \|v\|_{L^{q'}(\Omega)} \\ &\quad + \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{c_0^{1/q'} \varepsilon^{1/q}} \varepsilon^{1/q} \|\nabla w\|_{L^q(\Omega)} c_0^{1/q'} \|v\|_{L^{q'}(\Omega)} \\ &\le \left(\max\left(1, \frac{\|c\|_{L^\infty(\Omega)}}{c_0}\right) + \frac{\|b\|_{L^\infty(\Omega)}}{c_0^{1/q'}\varepsilon^{1/q}} \right) \|w\|_U \|v\|_V .\end{aligned} \tag{3.3}$$

Here we chose to integrate the advection term by parts and then apply the Hölder inequality. This is a rather arbitrary choice, but note that some ε-dependence of the continuity constant cannot be avoided even if the advection term is estimated in its current form.

3.2 A test function for the Laplace operator

We now start with proving the inf-sup condition (2.23a) by establishing estimates for a test function that is tailored for the diffusion part of the bilinear form. Let $u_v \in W_0^{1,q}(\Omega)$ be the unique solution to the problem

$$\int_\Omega \nabla u_v \cdot \nabla z \,\mathrm{d}\boldsymbol{x} = \left\langle J_{W_0^{1,q'}(\Omega)}(v), z \right\rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} \qquad \forall z \in W_0^{1,q'}(\Omega). \tag{3.4}$$

Hence, by definition

$$\varepsilon \int_\Omega \nabla u_v \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \varepsilon \langle J_{W_0^{1,q'}(\Omega)}(v), v \rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} = \varepsilon \|\nabla v\|^2_{L^{q'}(\Omega)}. \quad (3.5)$$

Next we look at the advection-reaction term; to bound these terms we integrate by parts, and employ Hölder's inequality, the Poincaré-Friedrichs inequality:

$$\|w\|_{L^q(\Omega)} \le C_{F,q} \|\nabla w\|_{L^q(\Omega)} \qquad \forall w \in W_0^{1,q}(\Omega), \quad (3.6)$$

and the a priori estimate for ∇u_v (1.7). Thereby, we get

$$\begin{aligned} \left| -\int_\Omega u_v \nabla \cdot (\boldsymbol{b} v) \, \mathrm{d}\boldsymbol{x} + \int_\Omega c u_v v \, \mathrm{d}\boldsymbol{x} \right| &= \left| \int_\Omega (\boldsymbol{b} \cdot \nabla u_v) v \, \mathrm{d}\boldsymbol{x} + \int_\Omega c u_v v \, \mathrm{d}\boldsymbol{x} \right| \\ &\le \|\boldsymbol{b}\|_{L^\infty(\Omega)} \|\nabla u_v\|_{L^q(\Omega)} \|v\|_{L^{q'}(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u_v\|_{L^q(\Omega)} \|v\|_{L^{q'}(\Omega)} \\ &\le \Big(\|\boldsymbol{b}\|_{L^\infty(\Omega)} + C_{F,q} \|c\|_{L^\infty(\Omega)} \Big) \|\nabla u_v\|_{L^q(\Omega)} \|v\|_{L^{q'}(\Omega)} \\ &\le C_{q',\Omega} \Big(\|\boldsymbol{b}\|_{L^\infty(\Omega)} + C_{F,q} \|c\|_{L^\infty(\Omega)} \Big) \|\nabla v\|_{L^{q'}(\Omega)} \|v\|_{L^{q'}(\Omega)}. \end{aligned} \quad (3.7)$$

To simplify the notation we define

$$C_1 = C_{q',\Omega} \Big(\|\boldsymbol{b}\|_{L^\infty(\Omega)} + C_{F,q} \|c\|_{L^\infty(\Omega)} \Big). \quad (3.8)$$

Using Young's inequality gives

$$\left| -\int_\Omega u_v \nabla \cdot (\boldsymbol{b} v) \, \mathrm{d}\boldsymbol{x} + \int_\Omega c u_v v \, \mathrm{d}\boldsymbol{x} \right| \le \left(\frac{\varepsilon}{2} \|\nabla v\|^2_{L^{q'}(\Omega)} + \frac{C_1^2}{2\varepsilon} \|v\|^2_{L^{q'}(\Omega)} \right). \quad (3.9)$$

3.3 A test function for the advection-reaction operator

We now consider the second test function introduced in Section 2.3 to obtain a lower bound for the advection-reaction term. As we have already mentioned, we require $W_0^{1,q}(\Omega)$-regularity for the test function and hence we first have to prove $J_{q'}(v) \in W_0^{1,q}(\Omega)$. By definition $J_{q'}(v) \in L^q(\Omega)$; we will now show that $\nabla J_{q'}(v) = \|v\|^{2-q'}_{L^{q'}(\Omega)} |v|^{q'-2} \nabla v \in [L^q(\Omega)]^d$. Indeed, since $v \in W_0^{1,q'}(\Omega)$, we deduce that

$$\begin{aligned}
\|\nabla J_{q'}\|^q_{L^q(\Omega)} &= \|v\|^{(2-q')q}_{L^{q'}(\Omega)} \int_\Omega \left| |v|^{q'-2}\nabla v\right|^q \mathrm{d}\boldsymbol{x} \\
&= \|v\|^{(2-q')q}_{L^{q'}(\Omega)} \int_\Omega |v|^{(q'-2)q}|\nabla v|^q \,\mathrm{d}\boldsymbol{x} \\
&\le \|v\|^{(2-q')q}_{L^{q'}(\Omega)} \||v|^{(q'-2)q}\|_{L^{\frac{q'-1}{q'-2}}(\Omega)} \||\nabla v|^q\|_{L^{q'-1}(\Omega)} \qquad (3.10)\\
&= \|v\|^{(2-q')q}_{L^{q'}(\Omega)} \left(\int_\Omega |v|^{q'}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{q'-2}{(q'-1)}} \left(\int_\Omega |\nabla v|^{q'}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{(q'-1)}} \\
&= \|v\|^{(2-q')q}_{L^{q'}(\Omega)} \|v\|^{(q'-2)q}_{L^{q'}(\Omega)} \|\nabla v\|^q_{L^{q'}(\Omega)} = \|\nabla v\|^q_{L^{q'}(\Omega)} < \infty,
\end{aligned}$$

where it is essential that $q' > 2$. Now we can apply (2.24) to obtain

$$\begin{aligned}
-\int_\Omega J_{q'}(v)\nabla\cdot(\boldsymbol{b}v)\,\mathrm{d}\boldsymbol{x} + \int_\Omega cJ_{q'}(v)v\,\mathrm{d}\boldsymbol{x}& \\
&= \|v\|^{2-q'}_{L^{q'}(\Omega)} \int_\Omega |v|^{q'-1}\operatorname{sgn}(v)(cv - \nabla\cdot(\boldsymbol{b}v))\,\mathrm{d}\boldsymbol{x} \\
&= \|v\|^{2-q'}_{L^{q'}(\Omega)} \int_\Omega |v|^{q'-1}\operatorname{sgn}(v)\big(cv - (\nabla\cdot\boldsymbol{b})v - \boldsymbol{b}\cdot\nabla v\big)\,\mathrm{d}\boldsymbol{x} \\
&= \|v\|^{2-q'}_{L^{q'}(\Omega)} \int_\Omega |v|^{q'}\left(c - \frac{1}{q}\nabla\cdot\boldsymbol{b}\right) - \frac{1}{q'}\nabla\cdot(\boldsymbol{b}|v|^{q'})\,\mathrm{d}\boldsymbol{x} \\
&\ge c_0\|v\|^2_{L^{q'}(\Omega)} - \frac{\|v\|^{2-q'}_{L^{q'}(\Omega)}}{q'} \int_{\partial\Omega} (\boldsymbol{b}\cdot\boldsymbol{n})|v|^{q'}\,\mathrm{d}s = c_0\|v\|^2_{L^{q'}(\Omega)}.
\end{aligned} \tag{3.11}$$

Next, we have to bound the diffusion term when testing with $J_{q'}(v)$. Here, we observe that

$$\int_\Omega \nabla(J_{q'}(v))\cdot\nabla v\,\mathrm{d}\boldsymbol{x} = \|v\|^{2-q'}_{L^{q'}(\Omega)} \int_\Omega |v|^{q'-2}\nabla v\cdot\nabla v\,\mathrm{d}\boldsymbol{x} \ge 0; \tag{3.12}$$

thereby, we can disregard this term in the inf-sup analysis.

3.4 Combining the estimates

We now combine all the above estimates and prove a lower bound for the bilinear form when testing with a linear combination of the two test functions above, i.e., we test with

$$w_v := u_v + \beta J_{q'}(v), \tag{3.13}$$

where β is a constant to be chosen. Combining the estimates (3.11), (3.12), (3.5) and (3.9), we obtain

$$\begin{aligned} \mathcal{B}_\varepsilon(w_v, v) \geq\ & \varepsilon\|\nabla v\|^2_{L^{q'}(\Omega)} + \beta c_0\|v\|^2_{L^{q'}(\Omega)} \\ & - \left(\frac{\varepsilon}{2}\|\nabla v\|^2_{L^{q'}(\Omega)} + \frac{C_1^2}{2\varepsilon}\|v\|^2_{L^{q'}(\Omega)}\right). \end{aligned} \tag{3.14}$$

If we now choose $\beta = C_1^2 c_0^{-1}\varepsilon^{-1}$, we have

$$\begin{aligned} \mathcal{B}_\varepsilon(w_v, v) &\geq \frac{C_1^2}{2\varepsilon}\|v\|^2_{L^{q'}(\Omega)} + \frac{\varepsilon}{2}\|\nabla v\|^2_{L^{q'}(\Omega)} \\ &\geq \min\left(\frac{C_1^2}{2\varepsilon c_0^{2/q'}}, \frac{\varepsilon^{1-2/q'}}{2}\right)\left(c_0^{2/q'}\|v\|^2_{L^{q'}(\Omega)} + \varepsilon^{2/q'}\|\nabla v\|^2_{L^{q'}(\Omega)}\right) \\ &\geq \frac{1}{2}\min\left(\frac{C_1^2}{2\varepsilon c_0^{2/q'}}, \frac{\varepsilon^{1-2/q'}}{2}\right)\|v\|^2_V. \end{aligned} \tag{3.15}$$

To conclude, we now only need to estimate $\|w_v\|_U$. According to Proposition 2.1 we have

$$\|\nabla u_v\| \leq C_{q',\Omega}\|J_{W_0^{1,q'}(\Omega)}(v)\|_{W^{-1,q}(\Omega)} = C_{q',\Omega}\|\nabla v\|_{L^{q'}(\Omega)}. \tag{3.16}$$

Thus, recalling the Poincaré-Friedrichs inequality (3.6), we obtain

$$\begin{aligned} \|u_v\|_U &= \left(c_0\|u_v\|^q_{L^q(\Omega)} + \varepsilon\|\nabla u_v\|^q_{L^q(\Omega)}\right)^{1/q} \\ &\leq \left(c_0 C^q_{F,q} + \varepsilon\right)^{1/q}\|\nabla u_v\|_{L^q(\Omega)} \\ &\leq C_{q',\Omega}\left(c_0 C^q_{F,q} + \varepsilon\right)^{1/q}\|\nabla v\|^{q'-1}_{L^{q'}(\Omega)} \\ &\leq C_{q',\Omega}\left(c_0 C^q_{F,q} + \varepsilon\right)^{1/q}\varepsilon^{-1/q'}\|v\|_V. \end{aligned} \tag{3.17}$$

Similarly, using (3.10), gives

$$\begin{aligned} \|\beta J_{q'}(v)\|_U &= \beta\left(c_0\|J_{q'}(v)\|^q_{L^q(\Omega)} + \varepsilon\|\nabla J_{q'}(v)\|^q_{L^q(\Omega)}\right)^{1/q} \\ &\leq \beta\left(c_0\|v\|^q_{L^{q'}(\Omega)} + \varepsilon\|\nabla v\|^q_{L^q(\Omega)}\right)^{1/q} \\ &\leq C_1^2 c_0^{-1}\varepsilon^{-1}2^{1/q-1/q'}\|v\|_V. \end{aligned} \tag{3.18}$$

Dividing (3.15) by $\|w_v\|_U$ and using the above estimates finally yields the inf-sup condition (2.23a). We refer the reader to (1.13) for the final estimate of the inf-sup constant γ.

3.5 The second inf-sup condition

The final step is the proof of the second inf-sup condition (2.23b). To this end, assume there exists $0 \neq w \in W_0^{1,q}(\Omega)$ such that

$$\begin{aligned} 0 = \mathcal{B}_\varepsilon(w,v) &= \varepsilon \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega w \nabla \cdot (\boldsymbol{b} v)\,\mathrm{d}\boldsymbol{x} + \int_\Omega cwv \,\mathrm{d}\boldsymbol{x} \\ &= \varepsilon \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{b} \cdot \nabla w v \,\mathrm{d}\boldsymbol{x} + \int_\Omega cwv \,\mathrm{d}\boldsymbol{x} \qquad \forall v \in W_0^{1,q'}(\Omega). \end{aligned}$$

The idea is to use the above equation to gain sufficient regularity in order to test with w itself. A priori this is not possible since $W_0^{1,q}(\Omega) \not\subset W_0^{1,q'}(\Omega)$ for $q' > q$. We will consider w as a solution to a Poisson problem and then use the regularity of the right-hand side in order to employ Corollary 2.2 and the Sobolev embedding theorem to gain higher regularity for w. We then iterate this until we have gained sufficient regularity.

Given that $\boldsymbol{b} \cdot \nabla w + cw \in L^q(\Omega)$, we can consider w as the solution of a Poisson problem with right hand side $g = \boldsymbol{b} \cdot \nabla w + cw$. Denote by q^* the Sobolev conjugate of q given by

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}. \tag{3.19}$$

Furthermore, let s be the dual exponent to q^*, i.e., $1 = 1/q^* + 1/s$. Then q' is the Sobolev conjugate to s. Indeed,

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d} \Leftrightarrow 1 - \frac{1}{s} = 1 - \frac{1}{q'} - \frac{1}{d} \Leftrightarrow \frac{1}{q'} = \frac{1}{s} - \frac{1}{d}. \tag{3.20}$$

Thus we obtain from the Sobolev embedding theorem that $W_0^{1,s}(\Omega) \subset L^{q'}(\Omega)$ and therefore $L^q(\Omega) = [L^{q'}(\Omega)]' \subset [W_0^{1,s}(\Omega)]' = W^{-1,q^*}(\Omega)$.

Suppose that $q^* \geq q'$, then Corollary 2.2 with $r = q'$ can be used (indeed $L^q(\Omega) \subset W^{-1,q^*}(\Omega) \subset W^{-1,q'}(\Omega)$) to obtain $w \in W_0^{1,q'}(\Omega)$, and we can select $v = w$ as a test function.

Otherwise, for $q^* < q'$, by Corollary 2.2 with $r = q^*$, we obtain $w \in W_0^{1,q^*}(\Omega)$. We subsequently observe that w satisfies a Poisson problem with a right-hand side $\boldsymbol{b} \cdot \nabla w + cw \in L^{q^*}(\Omega)$ and we can define iteratively

$$\frac{1}{r_{i+1}} = \max\left(\frac{1}{q'}, \frac{1}{r_i} - \frac{1}{d}\right), \quad r_1 := q^*. \tag{3.21}$$

We then obtain iteratively $w \in W_0^{1,\min\{r_i,q'\}}(\Omega)$ and $\boldsymbol{b} \cdot \nabla w + cw \in L^{\min\{r_i,q'\}}(\Omega)$. Note that

$$\frac{1}{r_i} = \max\left(\frac{1}{q'}, \frac{1}{q} - \frac{i}{d}\right).$$

Thus if we choose an integer $i \geq d/q - d/q'$, we observe that $w \in W_0^{1,q'}(\Omega)$ and hence we can choose $v = w$ as a test function.

Therefore, $0 = \mathcal{B}_\varepsilon(w, w) = \varepsilon\|\nabla w\|^2_{L^2(\Omega)} + c_0\|w\|^2_{L^2(\Omega)}$ and thus $w = 0$, which is a contradiction.

Remark 3.2. The idea of exploiting the Sobolev embedding multiple times iteratively is often referred to as a bootstrapping argument. These types of arguments are commonly used to obtain improved integrability or regularity in the context of elliptic partial differential equations. One example is the proof of Lemma 9.16 in [16]. There the same argument is used locally to improve a previously obtained L^q-estimate (cf., [16, Theorem 9.13]) to an $L^{q'}$-estimate. The fact that we can apply this argument globally, i.e., on the whole domain Ω, in our case is due to the regularity assumption on the domain (Assumption 1.1). For sufficiently smooth boundaries (e.g., C^1 or $C^{1,1}$), local estimates can often be turned into global estimates by combining them with boundary estimates (see, e.g., [16, Chapter 6] in the context of Schauder estimates). □

Remark 3.3 (The case $q > 2$). Theorem 1.5 can be analogously proven for $q > 2$. Indeed, instead of proving the inf-sup condition on the adjoint, we can prove the inf-sup condition directly. The steps of the proof in Sections 3.1 and 3.2 are identical with q and q' swapped and the constant C_1 has to be adjusted slightly to account for the fact that the advection term is not symmetric. To imitate the argument in Section 3.3, note that this time $J_q(w) \in W^{1,q'}(\Omega)$ for any $w \in W^{1,q}(\Omega)$, since $q > 2$ by the same argument as before and we can thus again apply (2.24) to obtain

$$\begin{aligned} -\int_\Omega w\nabla\cdot(\boldsymbol{b}J_q(w))\,\mathrm{d}\boldsymbol{x} + \int_\Omega cJ_q(w)w\,\mathrm{d}\boldsymbol{x} &= \int_\Omega (\boldsymbol{b}\cdot\nabla w)J_q(w)\,\mathrm{d}\boldsymbol{x} + \int_\Omega cJ_{q'}(w)w\,\mathrm{d}\boldsymbol{x} \\ &= \|w\|^{2-q}_{L^q(\Omega)}\int_\Omega\left(c|w|^q - \frac{1}{q}(\nabla\cdot\boldsymbol{b})|w|^q + \frac{1}{q}\nabla\cdot(|w|^q)\right)\mathrm{d}\boldsymbol{x} \\ &\geq c_0\|w\|^2_{L^q(\Omega)} + \frac{\|w\|^{2-q}_{L^q(\Omega)}}{q}\int_{\partial\Omega}(\boldsymbol{b}\cdot\boldsymbol{n})|w|^q\,\mathrm{d}s = c_0\|w\|^2_{L^q(\Omega)}. \end{aligned} \tag{3.22}$$

To prove the second inf-sup condition, we observe that for $v \in W^{1,q'}(\Omega)$ we have $cv - \nabla\cdot(\boldsymbol{v}) \in L^{q'}(\Omega)$ and we can apply the same bootstrap argument as in Section 3.5 after swapping q and q'. □

4 Proof of Theorem 1.7: Discrete inf-sup condition

In this section we prove Theorem 1.7. Note that Proposition 2.1 implies an inf-sup condition

$$\sup_{v\in W_0^{1,q'}(\Omega)} \frac{\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x}}{|v|_{W^{1,q'}(\Omega)}} \geq \frac{1}{C_{q',\Omega}}|u|_{W^{1,q}(\Omega)} \qquad \forall u\in W_0^{1,q}(\Omega). \tag{4.1}$$

Using Assumption 1.4, we have for $u_h \in U_h$

$$\begin{aligned}\sup_{v_h\in U_h} \frac{\int_\Omega \nabla u_h\cdot\nabla v_h\,\mathrm{d}\boldsymbol{x}}{|v_h|_{W^{1,q'}(\Omega)}} &\geq \sup_{v\in V} \frac{\int_\Omega \nabla u_h\cdot\nabla P_h v\,\mathrm{d}\boldsymbol{x}}{|P_h v|_{W^{1,q'}(\Omega)}} = \sup_{v\in V} \frac{\int_\Omega \nabla u_h\cdot\nabla v\,\mathrm{d}\boldsymbol{x}}{|P_h v|_{W^{1,q'}(\Omega)}}\\ &\geq C_P^{-1} \sup_{v\in V} \frac{\int_\Omega \nabla u_h\cdot\nabla v\,\mathrm{d}\boldsymbol{x}}{|v|_{W^{1,q'}(\Omega)}} \geq C_P^{-1}C_{q',\Omega}^{-1}|u_h|_{W^{1,q}(\Omega)}.\end{aligned} \tag{4.2}$$

This yields a discrete inf-sup condition for the Laplacian. In the next step we rearrange terms in order to use this estimate to control the remaining terms in the convection-diffusion-reaction equation. To this end, we note that

$$\begin{aligned}C_P^{-1}C_{q',\Omega}^{-1}|u_h|_{W^{1,q}(\Omega)} &\leq \varepsilon^{-1} \sup_{v_h\in U_h} \frac{\mathcal{B}_\varepsilon(u_h,v_h) - \int_\Omega \boldsymbol{b}\cdot\nabla u_h v_h\,\mathrm{d}\boldsymbol{x} - \int_\Omega c u_h v_h\,\mathrm{d}\boldsymbol{x}}{|v_h|_{W^{1,q'}(\Omega)}}\\ &\leq \varepsilon^{-1} \sup_{v_h\in U_h} \frac{\mathcal{B}_\varepsilon(u_h,v_h)}{|v_h|_{W^{1,q'}(\Omega)}} + \varepsilon^{-1} \sup_{v_h\in U_h} \frac{\int_\Omega \boldsymbol{b}\cdot\nabla u_h v_h\,\mathrm{d}\boldsymbol{x} + \int_\Omega c u_h v_h\,\mathrm{d}\boldsymbol{x}}{|v_h|_{W^{1,q'}(\Omega)}}.\end{aligned} \tag{4.3}$$

Next, we estimate the advection-reaction term using Hölder's inequality and the Poincaré-Friedrichs inequality (3.6); thereby, we get

$$\begin{aligned}&\sup_{v_h\in U_h} \frac{\int_\Omega \boldsymbol{b}\cdot\nabla u_h v_h\,\mathrm{d}\boldsymbol{x} + \int_\Omega c u_h v_h\,\mathrm{d}\boldsymbol{x}}{|v_h|_{W^{1,q'}(\Omega)}}\\ &\quad\leq \sup_{v_h\in U_h} \frac{\|b\|_{L^\infty(\Omega)}|u_h|_{W^{1,q}(\Omega)}\|v_h\|_{L^{q'}(\Omega)} + \|c\|_{L^\infty(\Omega)}\|u_h\|_{L^q(\Omega)}\|v_h\|_{L^{q'}(\Omega)}}{|v_h|_{W^{1,q'}(\Omega)}}\\ &\quad\leq C_{F,q'}(\|b\|_{L^\infty(\Omega)} + C_{F,q}\|c\|_{L^\infty(\Omega)})|u_h|_{W^{1,q}(\Omega)}.\end{aligned}$$

Thus,

$$\sup_{v_h\in U_h} \frac{\mathcal{B}_\varepsilon(u_h,v_h)}{|v_h|_{W^{1,q'}(\Omega)}} \geq \left(\varepsilon C_P^{-1}C_{q',\Omega}^{-1} - C_{F,q'}(\|b\|_{L^\infty(\Omega)} + C_{F,q}\|c\|_{L^\infty(\Omega)})\right)|u_h|_{W^{1,q}(\Omega)},$$

as required.

5 Numerical Illustration

In this section, we consider illustrative numerical examples. To this end, we first consider an example with smooth right-hand side on a square domain with a smooth solution to show optimal convergence rates in the $W^{1,q}(\Omega)$ norm. Secondly, we will illustrate why it is useful to consider a non-Hilbert setting for the convection-diffusion equation by considering examples in two and three dimensions with right-hand sides with very low regularity. Note that Assumptions 1.1 and 1.4 are satsified since all domains are convex Lipschitz domains.

5.1 Convergence Rates for a Simple Example in Two Dimensions

In order to illustrate the quasi-optimality estimates given at the end of Section 1.4, we consider the following simple example (which is essentially the Eriksson–Johnson test case, but with a reaction term):

$$\frac{\partial u}{\partial x} - \varepsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + u = f(x,y) \quad \text{in } (0,1)^2,$$
$$u = 0 \text{ on } x = 1, y = 0, 1, \qquad u = \sin(\pi y) \text{ on } x = 0,$$

where

$$f(x,y) = \frac{\exp(r_1(x-1)) - \exp(r_2(x-1))}{\exp(-r_1) - \exp(-r_2))} \sin(\pi y),$$

with

$$r_1 = \frac{1 + \sqrt{1 + 4\pi^2\varepsilon^2}}{2\varepsilon}, \qquad r_2 = \frac{1 - \sqrt{1 + 4\pi^2\varepsilon^2}}{2\varepsilon}.$$

Note that the exact solution of this problem is also given by $f(x,y)$. Figure 1 shows the error in the $W^{1,q}(\Omega)$-norm for $q = 1.5, 2, 3, 5$, $\varepsilon = 1$, and with the polynomial degree chosen uniformly as $p = 1, 2, 3, 4$, on a uniform triangluation of the domain. Here, we observe the optimal rate of convergence $\mathcal{O}(h^p)$, as the mesh is uniformly refined for each fixed p. This illustrates that in the diffusion-dominated case and sufficiently smooth right-hand side the underlying finite element method performs similarly to the case when $q = 2$.

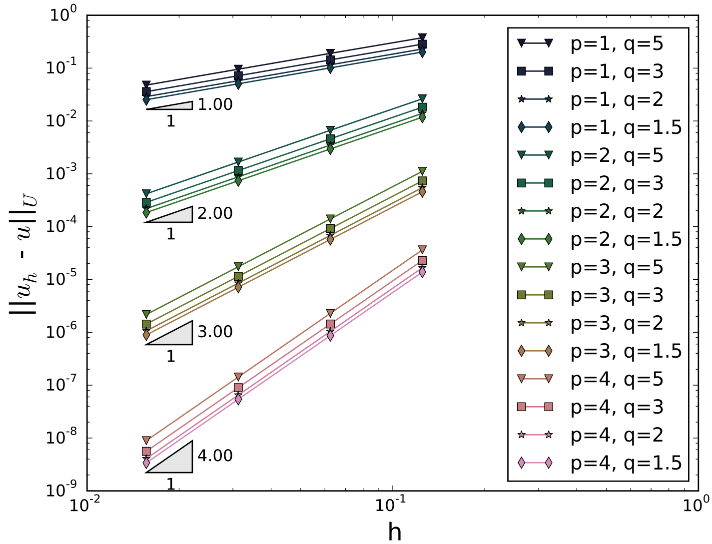

Fig. 1: Eriksson-Johnson model problem with $\varepsilon = 1$: Error in the $W^{1,q}(\Omega)$-norm for $q = 1.5, 2, 3, 5$, with the polynomial degree chosen uniformly as $p = 1, 2, 3, 4$, on a uniform triangulation of the domain.

5.2 Examples with Rough Right-hand Sides

To motivate looking at the $W^{1,q}(\Omega)$-$W^{1,q'}(\Omega)$-setting instead of the standard $H_0^1(\Omega)$-setting, we consider examples with rough right-hand sides, viz. Dirac delta distributions, both in two and three dimensions.

To this end, consider the problem

$$-\Delta u + \partial_{x_1} u + u = \delta_0 \quad \text{in } \Omega = [-0.5, 0.5]^d, \quad d = 2, 3, \tag{5.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{5.1b}$$

For the weak formulation to be well-defined, we require that $\delta_0 \in \left[W^{1,q'}(\Omega)\right]'$. In other words, we require $W^{1,q'}(\Omega) \subset C^0(\Omega)$. According to the Sobolev embedding result (see, e.g., [2]), we have for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ that $W^{1,q'}(\Omega) \subset C^0(\Omega)$ if $q' > d$. Thus we require $q' > 2$ for $d = 2$ and $q' > 3$ for $d = 3$ or, equivalently, $q < 2$ for $d = 2$ and $q < 1.5$ for $d = 3$. Furthermore, the solution u is not in $W^{1,q}(\Omega)$ for $q \geq 2$ in two dimensions and $q \geq 1.5$ in three dimensions.

Indeed, it is well-known, see, e.g., [13], that the fundamental solution of the Poisson problem contains a singularity of the form $1/|x|$ in three dimensions and

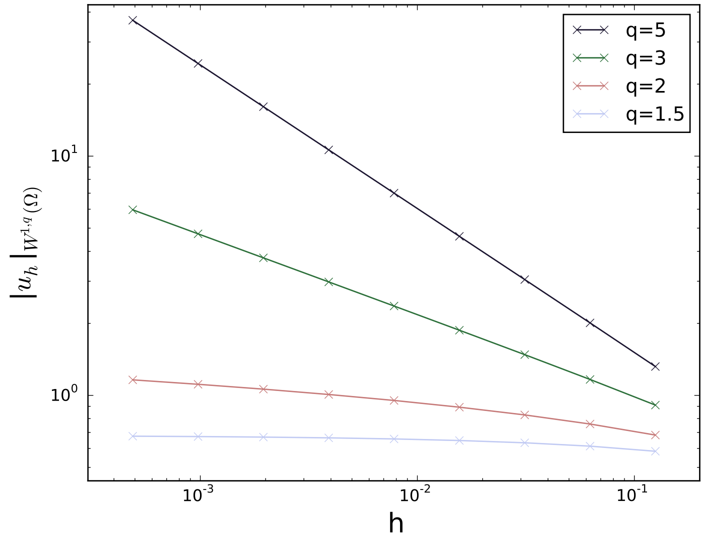

Fig. 2: 2D test case: Convergence ($q = 1.5$) and divergence ($q = 2, 3, 5$) of $|u_h|_{W^{1,q}(\Omega)}$ for linear finite-element approximations u_h. The irregular exact solution u is not in $W_0^{1,r}(\Omega)$ for any $r \geq 2$.

$\ln(|x|)$ in two dimensions. Thus the fundamental solution is not contained in $W^{1,q}(\Omega)$ for $q \geq 1.5$ in three dimensions and for $q \geq 2$ in two dimensions. The same applies to the convection-diffusion equation; near the origin the singularity of the fundamental solution behaves like $\mathcal{O}(1/|x|)$ for $d = 3$ and like $\mathcal{O}(\ln(|x|))$ for $d = 2$, cf. [14, 11, 21].

We can thus only expect convergence of $|u_h|_{W^{1,q}(\Omega)}$ to a finite value if $q < 2$ for $\Omega \subset \mathbb{R}^2$ and if $q < 1.5$ for $\Omega \subset \mathbb{R}^3$ and should observe divergence otherwise. This is illustrated for the 2D-case in Fig. 2 and for the 3D-case in Fig. 3. These figures plot $|u_h|_{W^{1,q}(\Omega)}$ for the finite element method using linear finite elements on a mesh (triangles in 2D, or tetrahedra in 3D) of mesh size h. In two dimensions, we can observe that $|u_h|_{W^{1,q}(\Omega)}$ diverges for $q = 2, 3, 5$. For $q = 2$, i.e., the borderline case, divergence is very slow; the values converge for $q = 1.5$. Similarly, in three dimensions, we clearly observe divergence for $q = 1.5, 1.7, 2$, while the values converge for $q = 1.3$.

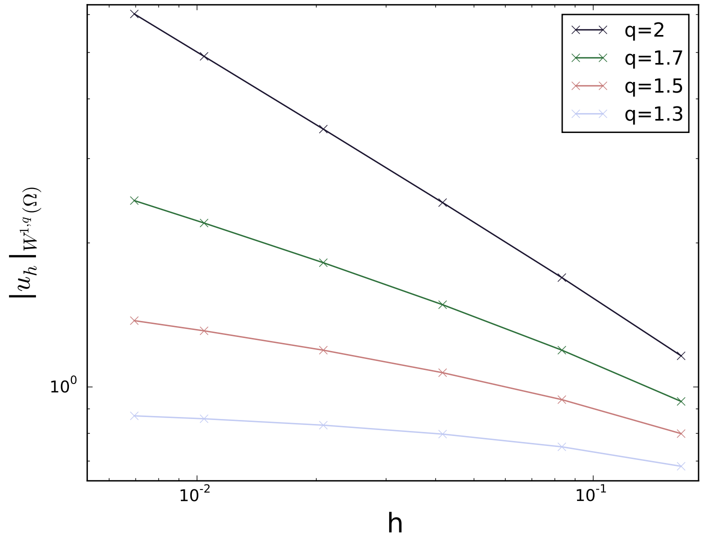

Fig. 3: 3D test case: Convergence ($q = 1.3$) and divergence ($q = 1.5, 1.7, 2$) of $|u_h|_{W^{1,q}(\Omega)}$ for linear finite-element approximations u_h. The irregular exact solution u is not in $W_0^{1,r}(\Omega)$ for any $r \geq 1.5$.

A Proof of Proposition 2.1

In this section, we give the proof of Proposition 2.1. We establish the inf-sup conditions (2.6a) and (2.6b), employing the standard duality technique that invokes the assumed regularity (i.e., Assumption 1.1). In the 1-D case, the latter assumption is not needed; see below. The proof is brief, since we employ straightforward properties of duality maps.

To proof (2.6a), let $J_{W_0^{1,q}}$ denote the duality map of $W_0^{1,q}(\Omega)$ endowed with the (semi-)norm $|\cdot|_{W^{1,q}(\Omega)}$ (see (2.19) for the expression of $J_{W_0^{1,q}}$). Take $g = J_{W_0^{1,q}}(u) \in W^{-1,q'}(\Omega)$ and let z_g solve (1.6) with $r = q' > 2$. By Assumption 1.1, we get that $z_g \in W_0^{1,q'}(\Omega)$ and

$$|z_g|_{W^{1,q'}(\Omega)} = \|\nabla z_g\|_{L^{q'}(\Omega)} \leq C_{q',\Omega} \big\| J_{W_0^{1,q}}(u) \big\|_{W^{-1,q'}} = C_{q',\Omega} |u|_{W^{1,q}(\Omega)} \,.$$

Furthermore, by density, z_g satisfies

$$\int_\Omega \nabla w \cdot \nabla z_g \,\mathrm{d}\boldsymbol{x} = \big\langle J_{W_0^{1,q}}(u), w \big\rangle_{W^{-1,q'},W_0^{1,q}} \,, \qquad \forall w \in W_0^{1,q}(\Omega) \supset H_0^1(\Omega).$$

Next,

$$\sup_{v\in W_0^{1,q'}(\Omega)} \frac{\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x}}{|v|_{W^{1,q'}(\Omega)}} \geq \frac{\int_\Omega \nabla u\cdot\nabla z_g\,\mathrm{d}\boldsymbol{x}}{|z_g|_{W^{1,q'}(\Omega)}} = \frac{\left\langle J_{W_0^{1,q}}(u),u\right\rangle_{W^{-1,q'},W_0^{1,q}}}{|z_g|_{W^{1,q'}(\Omega)}}$$
$$= \frac{|u|^2_{W^{1,q}(\Omega)}}{|z_g|_{W^{1,q'}(\Omega)}} \tag{A.1}$$
$$\geq C^{-1}_{q',\Omega}|u|_{W^{1,q}(\Omega)}\,,$$

which implies the condition (2.6a) with $\gamma = C^{-1}_{q',\Omega}$.

To prove condition (2.6b), let $v\in W_0^{1,q'}(\Omega)$ satisfy

$$\int_\Omega \nabla w\cdot\nabla v\,\mathrm{d}\boldsymbol{x}=0,\qquad \forall w\in W_0^{1,q}(\Omega)\,.$$

In particular, for $w=v\in W_0^{1,q'}(\Omega)\subset W_0^{1,q}(\Omega)$ we get $\|\nabla v\|^2_{L^2(\Omega)}=0$, which implies $v=0$. The a priori estimate (2.2) follows as usual from the inf-sup condition (A.1).

On the other hand, in the 1-D case ($d=1$), one can prove well-posedness *without* invoking Assumption 1.1. Indeed, let $\Omega\subset\mathbb{R}$ be any open bounded set. Since open sets are composed of a countable union of disjoint open intervals, we focus on the case of only one open interval, say for simplicity $I=(0,1)$.

Let $\rho>1$ and notice that the image of the derivative operator $(\cdot)'$ applied to $W_0^{1,\rho}(I)$ gives the space (see [12, Lemma B.69])

$$L^\rho_{\int=0}(I) := \left\{\phi\in L^\rho(I) : \int_I \phi\,\mathrm{d}x = 0\right\}.$$

Let $\sigma=\frac{\rho}{\rho-1}$, and for any $w\in W_0^{1,\rho}(I)$, define $\phi_w = J_\rho(w')-\int_\Omega J_\rho(w')\,\mathrm{d}x$ using the duality map $J_\rho : L^\rho(I)\to L^\sigma(I)$ (see (2.18)). Observe that $\phi_w\in L^\sigma_{\int=0}(I)$, $w'\in L^\rho_{\int=0}(I)$, and

$$\|\phi_w\|_{L^\sigma(\Omega)} \leq 2\|J_\rho(w')\|_{L^\sigma(\Omega)} = 2\|w'\|_{L^\rho(\Omega)}\,. \tag{A.2}$$

Thus,

$$
\begin{aligned}
\sup_{0\neq v\in W_0^{1,\sigma}(\Omega)} \frac{\int_\Omega w'v'\,\mathrm{d}x}{\|v'\|_{L^\sigma(\Omega)}} &= \sup_{0\neq \phi\in L^\sigma_{\int=0}(I)} \frac{\int_\Omega w'\phi\,\mathrm{d}x}{\|\phi\|_{L^\sigma(\Omega)}} && \text{(by surjectivity)}\\
&\geq \frac{\int_\Omega w'\phi_w\,\mathrm{d}x}{\|\phi_w\|_{L^\sigma(\Omega)}} && (\text{since } \phi_w \in L^\sigma_{\int=0}(I))\\
&\geq \frac{\int_\Omega w' J_\rho(w')\,\mathrm{d}x - \int_\Omega J_\rho(w')\,\mathrm{d}x \int_\Omega w'\,\mathrm{d}x}{2\|w'\|_{L^\rho(\Omega)}} && \text{(by (A.2))}\\
&= \frac{1}{2}\|w'\|_{L^\rho(\Omega)}\,. && (\text{since } w' \in L^\rho_{\int=0}(I))
\end{aligned}
$$

In conclusion,

$$
\sup_{0\neq v\in W_0^{1,\sigma}(\Omega)} \frac{\int_\Omega w'v'\,\mathrm{d}x}{\|v'\|_{L^\sigma(\Omega)}} \geq \frac{1}{2}\|w'\|_{L^\rho(\Omega)} \qquad \forall \rho > 1. \tag{A.3}
$$

In particular, taking $\rho = q$, (A.3) implies inf-sup condition (2.6a) for problem (2.1) with $\gamma = \frac{1}{2}$. On the other hand, taking $\rho = q'$, (A.3) implies that inf-sup condition (2.6b) is fulfilled. ■

Remark A.1 (Elliptic regularity in 1-D). The proof of Proposition 2.1 shows that when $d = 1$, problem (2.1) is well-posed in a $W_0^{1,\rho}$-$W_0^{1,\sigma}$ setting for *any* $\rho > 1$, without any elliptic-regularity assumption. Of course, in particular this implies that when $d = 1$, the elliptic-regularity Assumption 1.1 holds for any $r > 2$. □

Acknowledgment: The work by IM was done in the framework of Chilean FONDECYT research project #1160774. IM was also partially supported by the European Union's Horizon 2020, research and innovation program under the Marie Sklodowska-Curie grant agreement No 777778.

References

[1] V. Adolfsson. L^p-integrability of the second order derivatives of Green potentials in convex domains. *Pacific J. Math.*, 159(2):201–225, 1993.

[2] H. W. Alt. *Linear functional analysis.* Universitext. Springer-Verlag London, Ltd., London, 2016. An application-oriented introduction, Translated from the German edition by Robert Nürnberg.

[3] I. Babuška. Error-bounds for finite element method. *Numer. Math.*, 16:322–333, 1970/1971.

[4] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2002.

[5] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[6] P. Cantin. Well-posedness of the scalar and the vector advection-reaction problems in Banach graph spaces. *C. R. Math. Acad. Sci. Paris*, 355(8):892–902, 2017.

[7] I. Cioranescu. *Geometry of Banach spaces, duality mappings and nonlinear problems*, volume 62 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[8] K. Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[9] A. Demlow, D. Leykekhman, A. H. Schatz, and L. B. Wahlbin. Best approximation property in the W^1_∞ norm for finite element methods on graded meshes. *Math. Comp.*, 81(278):743–764, 2012.

[10] D. A. Di Pietro and J. Droniou. $W^{s,p}$-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a hybrid high-order discretisation of Leray-Lions problems. *Math. Models Methods Appl. Sci.*, 27(5):879–908, 2017.

[11] K. Eriksson. Improved accuracy by adapted mesh-refinements in the finite element method. *Math. Comp.*, 44(170):321–343, 1985.

[12] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Element Methods*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[13] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[14] S. Franz and N. Kopteva. On the sharpness of green's function estimates for a convection-diffusion problem. *arXiv preprint arXiv:1102.4520*, 2011.

[15] S. J. Fromm. Potential space estimates for Green potentials in convex domains. *Proc. Amer. Math. Soc.*, 119(1):225–233, 1993.

[16] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[17] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[18] J. Guzmán, D. Leykekhman, J. Rossmann, and A. H. Schatz. Hölder estimates

for Green's functions on convex polyhedral domains and their applications to finite element methods. *Numer. Math.*, 112(2):221–243, 2009.

[19] T. Jakab, I. Mitrea, and M. Mitrea. Sobolev estimates for the Green potential associated with the Robin-Laplacian in Lipschitz domains satisfying a uniform exterior ball condition. In *Sobolev spaces in mathematics. II*, volume 9 of *Int. Math. Ser. (N. Y.)*, pages 227–260. Springer, New York, 2009.

[20] D. Jerison and C. E. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.

[21] J. P. Krasovskiĭ. Isolation of singularities of the green's function. *Mathematics of the USSR-Izvestiya*, 1(5):935, 1967.

[22] H. Li. The W_p^1 stability of the Ritz projection on graded meshes. *Math. Comp.*, 86(303):49–74, 2017.

[23] J. M. Melenk and C. Schwab. An hp finite element method for convection-diffusion problems. Tech. Rep. 97-05, Eidgenössische Technische Hochschule Zürich (ETH). Seminar für Angewandte Mathematik, 1997.

[24] D. Mitrea, M. Mitrea, and L. Yan. Boundary value problems for the Laplacian in convex and semiconvex domains. *J. Funct. Anal.*, 258:2507–2585, 2010.

[25] I. Muga, M. J. W. Tyler, and K. G. van der Zee. The discrete-dual minimimal-residual method (DDMRes) for weak advection–reaction problems in Banach spaces. *submitted*, 2018.

[26] I. Muga and K. G. van der Zee. Discretization of Linear Problems in Banach Spaces: Residual Minimization, Nonlinear Petrov-Galerkin, and Monotone Mixed Methods. *arXiv preprint arXiv:1511.04400v3*, 2018.

[27] J. Nečas. Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 16(4):305–326, 1962.

[28] R. Rannacher and R. Scott. Some optimal error estimates for piecewise linear finite element approximations. *Math. Comp.*, 38(158):437–445, 1982.

[29] A. Stern. Banach space projections and Petrov–Galerkin estimates. *Numer. Math.*, 130:125–133, 2015.

[30] R. Verfürth. *A posteriori error estimation techniques for finite element methods*. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2013.

[31] E. Zeidler. *Nonlinear functional analysis and its applications. III.* Springer-Verlag, New York, 1985.