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New trends in couple-stress hyperelasticity

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Abstract
This communication reconsiders and reorganises parts of existing knowledge that refers to Cosserat-type hyperelasticity formulations in a manner that respects and pays tribute to the early development pattern of the couple-stress theory and, also, associates with it more recent findings and relevant discoveries related to a certain type of polar hyperelastic behaviour of fibrous composites. It further shows that these different branches of couple-stress theory can both emanate, as special cases, from a common, more general, advanced theoretical hyperelasticity framework. It thus reveals that proper completion of any relevant theoretical formulation requires in advance specification of a pair of kinematic parameters, each of which (1) is related, in a virtual or actual manner, to the observed global, macroscopic elastic deformation of the material, but (2) is principally relevant to the polar part of the observed material response. The first of these parameters represents a pseudovector field whose gradient is energetically reciprocal to the emerging couple-stress field, while the second represents either a vector or a pseudovector field that serves specific constitutional needs characterising the source of the anticipated polar material response. Different versions of couple-stress theory formulations, thus, are obtained by appropriately choosing or suitably tuning that pair of kinematic parameters. It is also seen that, regardless of the employed couple-stress theoretical model, full solution of a relevant well-posed boundary value problem is generally achievable with use of a two-step solution process. The first step includes determination of the deviatoric couple-stress and the actual spin vector of the global material deformation. In the second step, these initial findings enable formation of an additional differential equation whose solution leads to determination of the spherical part of the couple-stress.

Keywords
Couple-stress theory, elasticity, generalised Cosserat theoretical framework, hyperelasticity, polar material elasticity

1. Introduction

The principal difference between the classical Cosserat-type couple-stress theory [1–4] and the relevant hyperelasticity formalism of fibre-reinforced materials with fibres resistant in bending [5] lies in the part of the constitutive considerations referring to couple-stress emergence and action. The former, basic theory [1–4] considers that couple-stresses are energetically and, therefore, constitutionally reciprocal to the gradient of a specific spin vector of the macroscopically observed deformation, namely the axial vector (pseudovector) of the antisymmetric rotation tensor. The latter, more recent theoretical development [5] refers to fibre-reinforced materials in which couple-stress develops due to deformation resistance of

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individual fibres, and is, therefore, reciprocal to the gradient of a fibre direction vector. Despite their noted fundamental difference, these essentially different theoretical formulations also exhibit substantial similarities. The most striking such similarity lies in the fact that both formulations leave the spherical part of the couple-stress tensor (for simplicity, spherical couple-stress in what follows) indeterminate.

Following the appearance of Spencer and Soldatos [5], further relevant investigation and study (e.g. works by Soldatos [6–12] and references therein) gradually led to better understanding of the origin and the nature of the observed similarities of and differences between those two theoretical formulations. Every step followed on that curiosity-driven research route was naturally underpinned by the knowledge available and the relevant understanding that was achieved at the time. The gradually accumulated new information suggests now that (1) there exists a more general couple-stress theoretical framework, which both models can individually emanate from as special cases, and (2) the logical order of development or presentation of that framework does not necessarily coincide with the chronological order that Spencer and Soldatos [5–12] developed and presented their findings.

The present communication, thus, aims to reconsider and reorganise the existing theoretical knowledge in a manner that (1) respects and pays tribute to the early development and pattern of the couple-stress theory [1–4], and (2) suitably connects and associates with it the more recent findings emerging from the works by Spencer and Soldatos [5] and Soldatos [6–12]. In this context, it (3) describes the aforementioned, more general and still-developing couple-stress hyperelasticity framework, (4) promotes its relevance and connection with both the conventional Cosserat-type theory [1–4] and its fibre-reinforced material counterpart [5,11], and, thus, (5) indicates or makes clearer several topics and research routes that remain open for further exploration.

Towards these purposes, section 2 outlines the principal equations and concepts of the conventional couple-stress theory [1–4] and thus highlights the bounds or limits of interest of this communication. Historically, parts of section 2 may therefore be associated with or even compressed within the present introductory section. However, by forming an independent unit, section 2 also sets a structure plan that is largely followed throughout the remaining of this paper. In that context, section 2.3 pays separate attention to infinitesimally small deformations of polar elastic materials and, thus, (1) uncovers the controversial fact that the elastic energy stored internally in a polar elastic material differs from its displacement gradient counterpart, and (2) makes it easier to understand the reasons that lead to the well-known spherical couple-stress indeterminacy. Hence, section 2 (3) introduces and justifies reasons that necessitate the refinement of the conventional couple-stress theory, detailed afterwards in section 3 (see also the work by Soldatos [12]).

Section 4 sets up the foundations of the aforementioned, general, couple-stress hyperlasticity framework, which holds regardless of whether the couple-stress field is energetically reciprocal to the gradient of a vector [5,11] or a pseudovector field [1–4,12]. The latter case, where the couple-stress field is reciprocal to the gradient of some appropriate axial (spin-type) vector field, is next considered in sections 5 and 6. These sections demonstrate the manner that the proposed general framework produces, as special cases, relevant isotropic and anisotropic versions, respectively, of polar material hyperelasticity. A direct connection thus is established between the newly established general framework detailed in section 4 and the refined [12] and conventional [1–4] couple-stress formalisms considered earlier in section 3 and 2, respectively. Section 7 then establishes a corresponding kind of connection between the same general formalism (section 4) and the theory of fibre-reinforced materials with fibres resistant in bending [5,10], by considering reciprocity of the couple-stress field with the gradient of a fibre direction vector. Section 8 summarises the progress made with this communication, highlights its principal findings, and underlines the most important conclusions drawn.

2. Basic equations and concepts of the conventional couple-stress theory

The 1960s marked a substantial interest in the revival and further development of the Cosserat's couple-stress theory [1]. At the beginning of that decade, Mindlin and Tiersten published their influential relevant contribution [2], which noted that, at the time, the Cosserat brothers had left the theory in the form of four equations. Namely, the equations labelled (205.2), (205.10), (205.17), and (241.4) in the now classic work of Truesdell and Toupin [3]. The first three of these equations refer to the state of equilibrium of the elastic polar material of interest, while the fourth is a power balance equation that makes use of the material velocity (and relevant vorticity/spin vector) and relates to energy conservation.

2.1. Equilibrium

By here neglecting, for simplicity, the non-mechanical terms included in the work by Truesdell and Toupin [3], as well as the influence of inertia and body forces, the first two of those four equations meet the familiar form of the equations of static equilibrium

$$\sigma_{ij,\,i} = 0, m_{\ell k,\,\ell} + \varepsilon_{kji}\sigma_{ji} = 0, \tag{1a, b}$$

where $\boldsymbol{\sigma}$ and $\mathbf{m}$ denote the stress and couple-stress tensor, respectively, $\boldsymbol{\varepsilon}$ represents the alternating tensor, indices take the values 1, 2, and 3 within a suitable three-dimensional Cartesian co-ordinate framework Ox_i, and, in the usual manner, a comma between successive indices signifies partial differentiation with respect to one or more spatial co-ordinates.

Equation (1b) makes it evident that, due to its interaction with the gradient of the couple-stress tensor, the stress tensor is generally non-symmetric. With use of the standard decomposition of the stress tensor,

$$\sigma_{ij} = \sigma_{(ij)} + \sigma_{[ij]}, \tag{2}$$

into symmetric and antisymmetric parts, the second equilibrium equation converts into

$$\sigma_{[ij]} = \frac{1}{2}\varepsilon_{kji}m_{\ell k,\,\ell}, \tag{3}$$

which, from now on, is regarded as a constitutive equation for the antisymmetric part of the stress tensor. In this context, constitutive equations are required to either be provided or be sought and found only for the couple-stress tensor and the symmetric part of the stress tensor.

Upon inserting equations (2) and (3) into equation (1a), and assuming that the couple-stress components are at least twice differentiable functions, the pair of equations (1) reduces into a single equilibrium equation,

$$\sigma_{(ij),\,i} + \frac{1}{2}\varepsilon_{kji}m_{\ell k,\,\ell i} = 0, \tag{4}$$

which is the third of the four Cosserat's equations noted by Truesdel and Toupin [3] and, subsequently, Mindlin and Tiersten [2].

It is fitting at this point to note that, in the usual manner, the components of the traction and couple-traction vectors acting on any internal or bounding surface of the material are, respectively, given as follows:

$$T_i^{(n)} = \sigma_{ji}n_j, L_i^{(n)} = m_{ji}n_j, \tag{5a, b}$$

where $\boldsymbol{n}$ denotes the outward unit normal of that surface. It is then further fitting to recall Koiter's finding [4], according to which only two of the three boundary conditions implied through equation (5b) can be set independently of the deformation.

2.2. Energy balance—indeterminacy of the spherical part of the couple-stress

The fourth of the aforementioned Cosserat's equations [2,3] is the power balance equation

$$\frac{\rho}{\rho_0}\dot{W} = \sigma_{(ji)}v_{i,j} + \frac{1}{2}m_{ji}\varepsilon_{i\ell k}v_{k,\,\ell j} = \sigma_{(ji)}d_{ij} + m_{ji}\omega_{i,j} \tag{6}$$

where ρ and ρ_0 represent material density in the deformed and the undeformed configuration, respectively; W stands for the internal energy per unit mass; a superposed dot signifies total derivative with respect to time t; v_i are the components of the material velocity vector v; and the standard relations [13]

$$d_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right), \omega_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) = \varepsilon_{jik}\omega_k, \omega_i = \frac{1}{2}\varepsilon_{ijk}\omega_{kj} = \frac{1}{2}\varepsilon_{ijk}v_{k,j}, \quad \text{(7a, b, c)}$$

provide the components of the appearing rate of deformation tensor, vorticity tensor, and its relevant axial/spin vector, respectively, in terms of the velocity gradients.

Equation (6) will be rederived in section 3.1, as a special case of a more general power balance equation which, by making use of virtual velocity and spin vectors, underpins a refined version of the couple-stress theory. It is recalled in this context that any axial or spin vector, like the one defined in equation (7c), is in fact a pseudovector because it does not change its sign, as vectors do, under reflection-type orthogonal transformations of the co-ordinate system [14].

It is also noted at this point that Mindlin and Tiersten [2] were apparently first to mention that the spherical part of the couple-stress tensor, m_{rr}, (1) fails to be accounted for in the equilibrium equation (4), (2) does not mark its contribution into the energy balance equation (6), and, hence, (3) remains indeterminate in the conventional couple-stress theory.

Indeed, by denoting with $\bar{m}_{\ell k}$ the components of the deviatoric part of the couple-stress tensor, and noting that

$$m_{\ell k} = \bar{m}_{\ell k} + \frac{1}{3}m_{rr}\delta_{\ell k}, \tag{8}$$

the assumption of twice differentiable couple-stresses enables reduction of the equilibrium equation (4) into the following:

$$\sigma_{(ij),i} + \frac{1}{2}\varepsilon_{kji}\bar{m}_{\ell k,\ell i} = 0. \tag{9}$$

It, thus, is seen that the spherical couple-stress, m_{rr}, does not influence the equilibrium.

Moreover, upon inserting equations (8) and (7c) into equation (6), and making also use of the identity

$$\omega_{i,i} = 0, \tag{10}$$

one obtains

$$\frac{\rho}{\rho_0}\dot{W} = \sigma_{(ji)}d_{ij} + \bar{m}_{ji}\omega_{i,j}, \tag{11}$$

which reveals that m_{rr} does not influence the energy balance equation either. The spherical couple-stress, thus, is, indeed, indeterminate in the conventional couple-stress theory, and this fact will re-emerge in a special case of the analysis presented in section 3.

The equilibrium and the power balance equations (9) and (11), respectively, hold regardless of whether the polar elastic material of interest undergoes finite or infinitesimally small elastic deformations. However, more specific attention on the magnitude of the deformation will be paid later. Namely, after the symmetry group of the material and its influence in the formation of the internal energy density, W, will also be considered and lead to formulation of appropriate sets of constitutive equations.

In dealing, for instance, with finite elastic deformations of isotropic polar materials, such a set of constitutive equations is obtained in section 5. Nevertheless, it is fitting at this point to refer separately to the special case of infinitesimally small elastic deformations, which received substantial attention in the works by Mindlin and Tiersten [2] and Koiter [4] and, traditionally, makes use of linear constitutive equations.

2.3. Infinitesimally small elastic deformations—displacement gradient energy

The special case of linear polar elasticity is based on the concept of infinitesimally small deformations and, thus, anticipates that W is accurately approximated by a quadratic form or the involved deformation gradients. Since Mindlin and Tiersten [2] paid considerable, and Koiter [4] exclusive, attention to the special case of linear polar isotropy, that case has been regarded as a natural point of reference in many subsequent publications and will also attract specific attention later.

It is appropriate in this context to recall that, when dealing with linear elasticity, equation (11) is approximated as follows:

$$\dot{W} = \sigma_{(ji)}\dot{e}_{ij} + \bar{m}_{ji}\dot{\Omega}_{i,j}. \tag{12}$$

where the components of the appearing small strain tensor, displacement-generated spin vector, and small rotation tensor, respectively, are defined, in terms of the gradients of the displacement vector $\mathbf{u}$, in the usual manner,

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \Omega_i = \frac{1}{2}\varepsilon_{ijk}\Omega_{kj} = \frac{1}{2}\varepsilon_{ijk}u_{k,j} \quad \Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = \varepsilon_{jik}\Omega_k. \tag{13a, b, c}$$

It is further recalled that the postulation of practically identical deformed and reference configurations justifies here the approximation $\mathbf{v} = \dot{\mathbf{u}}$ (see also section 5.2) and underpins the concept of practically unchanged material density ($\rho = \rho_0$).

Since W is customarily considered quadratic in the deformation gradients, the polar elasticity extension of Clapeyron's theorem [8,9] enables equation (12) to attain the following alternative form:

$$W = W^e + W^\Omega, W^e = \frac{1}{2}\sigma_{(ji)}e_{ij} \geqslant 0, W^\Omega = \frac{1}{2}m_{ji}\Omega_{i,j} = \frac{1}{2}\bar{m}_{ji}\Omega_{i,j} \geqslant 0, \tag{14a, b, c}$$

where W^e is the standard strain energy function met in non-polar linear elasticity, and W^Ω is a spin-energy contribution stemming from the observed polar material response. The latter depends quadratically solely on the appearing spin-vector gradients, which are assumed of the same order of magnitude with the displacement gradients and, therefore, with the strains.

It is pointed out that, in deriving equation (14c), use is also made of the identity

$$\Omega_{i,i} = 0, \tag{15}$$

which stems from the spin-vector definition (equation (13b)) and is naturally analogous to equation (10). It is, in fact, through action of this pair of identities that conventional couple-stress theory is deprived ability to record any kind of influence that the spherical couple-stress may exert on the energy balance of the polar material of interest.

It is recalled in passing that this well-known indeterminacy drawback of the conventional couple-stress theory has always been an issue of considerable controversy and debate. That debate may be traced through relevant publications cited elsewhere (e.g. works by Soldatos [11,12] and references therein) and needs not be referred here to any further. However, along with the indeterminacy problem itself, that debate provides another reason supporting the search for refinement of the conventional theory.

It may now be argued that a third energy term may appear in the right-hand side of equation (14a). Such an energy term, which may potentially involve mixed products of displacement and spin gradients, had indeed initially been included in, but subsequently eliminated from either version of the conventional polar linear elasticity presented by Mindlin and Tiersten [2] and Koiter [4]. Reasons that underpin elimination of that energy term are provided in both publications [2,4], and will also be shown valid in what follows, wherever this is appropriate and necessary.

For reasons that will be clarified in subsequent sections, a reminder becomes now necessary of the concept of the displacement gradient energy function [9,11],

$$U\left(e_{ij}, \Omega_{ij}\right) = \frac{1}{2}\sigma_{ji}u_{i,j} = \frac{1}{2}\left(\sigma_{(ji)} + \sigma_{[ji]}\right)\left(e_{ij} + \Omega_{ij}\right) = \frac{1}{2}\left(\sigma_{(ij)}e_{ij} + \sigma_{[ji]}\Omega_{ij}\right) = W^e\left(e_{ij}\right) + W^\omega\left(\Omega_{ij}\right), \tag{16}$$

where

$$W^{\omega}(\Omega_{ij}) = \frac{1}{2}\sigma_{[ji]}\Omega_{ij} = \frac{1}{4}\varepsilon_{kij}m_{\ell k,\ell}\Omega_{ij} = -\frac{1}{2}\Omega_k m_{\ell k,\ell} = \frac{1}{2}\Omega_{k,\ell}m_{\ell k} - \frac{1}{2}(\Omega_k m_{\ell k})_{,\ell} = W^{\Omega} - \frac{1}{2}(\Omega_k m_{\ell k})_{,\ell}, \quad (17)$$

represents an amount of energy that is due to the interaction of the antisymmetric part of the stress with the small rotation field (13c).

As is well known, the displacement gradient energy (16) and the internal energy (14a) are identical in the special case of non-polar linear elasticity, where $W^{\Omega} = W^{\omega} = 0$. Since the second equilibrium equation (1b) is now found equivalent to the constitutive equation (3), which provides the antisymmetric part of stress in terms of the couple-stresses, the same must also be true in the present, more general polar elasticity case. However, these amounts of internal energy are apparently different within the conventional couple-stress elasticity framework, where the positive values of the appearing spin- and rotation-generated energies give rise to the energy difference:

$$W^m = W^{\Omega}(\Omega_{i,j}) - W^{\omega}(\Omega_{ij}) = \frac{1}{2}(m_{\ell k}\Omega_k)_{,\ell}. \quad (18)$$

It is though further observed that W^{ω} is an amount of stored energy that takes into consideration the action of the couple-stress tensor, $\mathbf{m}$, while W^{Ω} accounts only for relevant action of the deviatoric part, $\bar{\mathbf{m}}$, of that tensor. This observation then suggests that the apparent energy difference (18) relates, if is not fully due to action of the spherical couple-stress. Hence, this observation is regarded as additional evidence supporting the need for refinement of the conventional couple-stress theory.

It is accordingly also observed that integration of equation (18) over an arbitrary volume, V, of the material, followed by application of the divergence theorem and a subsequent use of equation (5b), yields

$$\int_V W^m dV = \frac{1}{2}\int_V (m_{\ell k}\Omega_k)_{,\ell} dV = \frac{1}{2}\int_S m_{\ell k}\Omega_k n_\ell dS = \frac{1}{2}\int_S L_k^{(n)}\Omega_k dS. \quad (19)$$

This result reveals that the contribution of W^m into the total elastic energy stored in an arbitrary volume V of the material equals one half of the total work done through the interaction of the actual spin vector, Ω_i, with the set of couple-tractions, $L_i^{(n)}$, acting on the bounding surface of V.

Hence, by considering a special case in which S represents the part of the outer boundary that is subjected to couple-traction boundary conditions, one concludes that equation (19) implies that there exists some connection between the total energy difference, which W^m stores elastically within the polar material of interest, and the action of the couple-tractions applied externally on its outer boundary, including their spherical part.

3. Refined couple-stress theory

While the identities (10) and (15) result as consequences of purely algebraic definitions and handling, their involvement affects the physics efficiency of the outlined conventional polar material model. This fact naturally raises a question regarding the validity of the postulation that declares the spherical couple-stress energetically reciprocal to the divergence of the displacement-generated spin vector. Along with the relevant observations noted in the preceding section, that question prompts and motivates a search for potential improvement, or refinement of the conventional couple-stress theory.

The fact that the spherical couple-stress does not influence the state of equilibrium seems unsusceptible to such improvement or refinement, because neither equation (10) nor (15) affects the main part of the analysis detailed in section 2.1. However, this pair of identities does affect the energy balance of the system, which thus invites attention for potential reconsideration.

3.1. Reconsideration of the energy balance equation for elastic deformations of arbitrary magnitude

The outlined observations underpin introduction of an unspecified, auxiliary, virtual spin-type vector field, ψ_i. This is generally considered of the same order of magnitude with, but not independent of the rate of the actual deformation gradient, and is generally considered such that

$$\psi_i \neq \alpha Q_{ij}\omega_j, \tag{20}$$

where α is a non-zero real constant, and $\mathbf{Q}$ is a constant orthogonal matrix that represents an arbitrary orthogonal transformation of the co-ordinate system. Condition (20), thus, implies that ψ_i differs not only from ω_i, but also from any vector field that may result through a uniform extension/contraction of ω_i, combined with an arbitrary rotation/reflection of the same.

Since

$$\psi_{i,i} \neq \alpha Q_{ij}\omega_{j,i} = \alpha\omega_{j,j} = 0, \tag{21}$$

the divergence of such a virtual vector is generally non-zero. Hence, ψ_i is qualified to replace the pseudovector ω_i in the role of a vector field whose gradient is energetically reciprocal to the couple-stress. In line with the relevant definitions quoted in equations (7b, c) and (13b, c), ψ_i is still considered as pseudovector, in the sense that it is regarded as the axial vector of a corresponding, antisymmetric, virtual rotation tensor, ψ_{ij}, to which it relates through the standard relationships

$$\psi_i = \frac{1}{2}\varepsilon_{ijk}\psi_{kj},\ \psi_{kj} = \varepsilon_{ijk}\psi_i. \tag{22a, b}$$

Within the standard framework that underpins the principle of virtual work [13,15], it is further recalled that, wherever appropriate, virtual velocities and, hence, virtual rotation and spin fields are replaceable by their virtual displacement counterparts, regardless of the magnitude of the deformation. In the light of the relevant relationship established between equations (7b, c) and (13b, c), the virtual fields (22) are accordingly anticipated stemming from a pair of dual virtual fields

$$\Phi_i = \frac{1}{2}\varepsilon_{ijk}\Phi_{kj},\ \Phi_{kj} = \varepsilon_{ijk}\Phi_i, \tag{23a, b}$$

to which they relate as follows:

$$\psi_i = \dot{\Phi}_i,\ \psi_{ij} = \dot{\Phi}_{ij}. \tag{24a, b}$$

Φ_i, thus, is considered as an auxiliary, virtual pseudovector field that is not independent from and, therefore, of the same order of magnitude with the displacement gradients. It is instructive at this point to recall that Φ_i had been initially proposed to enter the theory as an actual vector field, such as a fibre-rotation field [10], which, if present, is different to its displacement counterpart, Ω_i. However, that considerations were soon afterwards found unnecessary [11,12] because, as will also be verified in what follows, Φ_i does not need to finally be determined. Instead, it remains unspecified and therefore auxiliary, even after the ultimate solution of any well-posed polar elasticity boundary value problem is completely determined.

Under these considerations, the standard process of energy balance that led to equation (6) is refined as follows:

$$\frac{D}{Dt}\int_V W dV = \int_S \left(T_i^{(n)}v_i + L_i^{(n)}\psi_i\right)dS, \tag{25}$$

where the action of the velocity-spin vector ω_i is here replaced by that of its virtual counterpart, ψ_i. In the usual manner, V represents an arbitrary volume of the polar elastic solid of interest, S is the bounding surface of V, and equations (5) still hold.

Application of Reynolds transport theorem, followed by consideration of the equilibrium equation (1a) and application of the divergence theorem, leads to

$$\frac{\rho}{\rho_0}\dot{W} = \sigma_{ji}v_{i,j} + \left(m_{ji}\psi_i\right)_{,j}. \tag{26}$$

With use of equations (2), (7), and (1b), this equation reduces to

$$\frac{\rho}{\rho_0}\dot{W} = \sigma_{(ji)}d_{ij} + m_{ji,j}(\psi_i - \omega_i) + m_{ji}\psi_{i,j} = \sigma_{(ji)}d_{ij} + \left[m_{ji}(\psi_i - \omega_i)\right]_{,j} + m_{ji}\omega_{i,j}, \tag{27}$$

which generalises and, thus, replaces the power balance equation (6).

Hence, with further use of equations (8) and (10), equation (27) is seen equivalent to

$$\frac{\rho}{\rho_0}\dot{W} = \sigma_{(ji)}d_{ij} + \left[m_{ji}(\psi_i - \omega_i)\right]_{,j} + \bar{m}_{ji}\omega_{i,j}, \tag{28}$$

which generalises its conventional counterpart (equation (11)). Evidently, equation (28) reduces to equation (11) in the special case that dismissal of equation (20) allows replacement of the virtual spin field ψ_i with its actual counterpart, ω_i.

Due to the requirement (20) and, therefore, to the subsequent condition (21), the customary decomposition (8) of the couple-stress, into deviatoric and spherical parts, does not prevent anymore the spherical couple-stress, m_{rr}, to mark its contribution into the refined power balance equation (28). It is also observed that, since ψ_i is auxiliary and represents a general class of virtual spin vectors, a combination of equation (20) with the additional condition

$$\left(m_{ji}\psi_i\right)_{,j} = \left(m_{ji}\omega_i\right)_{,j} \tag{29}$$

identifies a subclass of virtual spin vectors that makes equation (28) identical with its conventional counterpart (11). Validity of equation (29) thus enables equation (11) to remain also valid.

It is emphasised that, on its own, the single equation (29) is insufficient to uniquely determine all three components of the auxiliary spin vector ψ_i. Since equation (29), thus, is satisfied by a doubly infinite number of such vectors, both ψ_i and Φ_i remain unspecified virtual spin vectors. As will be seen in section 5, even the coupling of equation (29) with a second relevant condition (see equation (56) is still insufficient to determine uniquely either of these vectors.

3.2. Infinitesimally small elastic deformations

In the special case of small elastic deformations, W is again represented by a quadratic form of the displacement and the spin-vector gradients. In the light of equations (12), (13), and (24), the balance equation (27), thus, is approximated as follows:

$$\dot{W} = \sigma_{(ji)}\dot{e}_{ij} + m_{ji,j}(\dot{\Phi}_i - \dot{\Omega}_i) + m_{ji}\dot{\Phi}_{i,j}, \tag{30}$$

where the gradients of the appearing virtual spin vector are generally assumed of the same order of magnitude with the displacement gradients.

With use of the polar elasticity extension of Clapeyron's theorem [8,9], the implied quadratic form of W then enables equation (30) to acquire the equivalent energy balance form

$$W = W^e + W^{\Phi},\ W^e = \frac{1}{2}\sigma_{(ji)}e_{ij} \geqslant 0,\ W^{\Phi} = \frac{1}{2}\left[m_{ji,j}(\Phi_i - \Omega_i) + m_{ji}\Phi_{i,j}\right] \geqslant 0, \tag{31a, b, c}$$

where W^e is still the standard, quadratic strain energy function met in non-polar linear elasticity. Section 5.2 makes later clear that (1) the internal energy contribution that is due to the polar material response, W^{Φ}, depends quadratically on the appearing spin vectors and their gradients only, and (2) no energy terms that involve mixed products of displacement and spin gradients are present in the right-hand side of equation (31a)

With use of equation (8), equation (31c) can conveniently be expressed in the following alternative form:

$$W^{\Phi} = \frac{1}{2}\{[m_{\ell i}(\Phi_i - \Omega_i)]_{,\ell} + \bar{m}_{\ell i}\Omega_{i,\ell}\} \geqslant 0. \tag{32}$$

In agreement with the relevant comment that follows equation (28), the choice $\Phi_i = \Omega_i$ enables equations (31) and (32), to naturally reduce into the forms equations (14) and (14c), respectively, of their conventional theory counterparts.

More generally though, since equation (32) can obtain the form

$$W^{\Omega} - W^{\Phi} = \frac{1}{2}[m_{\ell i}(\Omega_i - \Phi_i)]_{,\ell}, \tag{33}$$

and, therefore,

$$W^{\Phi} = W^{\Omega} \Leftrightarrow \frac{1}{2}[m_{\ell i}(\Phi_i - \Omega_i)]_{,\ell} = 0, \tag{34a, b}$$

the value $\Phi_i = \Omega_i$ of the virtual spin vector may alternatively be regarded as a trivial solution of equation (34b). Solutions to this equation thus enable the otherwise unequal internal energies (31) and (14) to attain the same value. In terms of the analysis detailed in section 3.1, equation (34b) does accordingly represent the small deformation counterpart of equation (29).

In this regard, a comparison of equation (34) with equation (18) makes it understood that some other, non-trivial solution of equation (34b) may now be sought by requiring from Φ_i and Ω_i to satisfy the additional equation

$$(m_{\ell i}\Phi_i)_{,\ell} = (m_{\ell i}\Omega_i)_{,\ell} = 2W^m. \tag{35}$$

Validity of this equation implies that, by virtue of equation (34a), both the conventional (section 2.3) and the present refined versions of linear couple-stress theory account for the same amount of internal energy. It is emphasised though that, in the light of equations (31) and (14), W^{Φ} does account for energy contributions emerging through action of the spherical couple-stress, while W^{Ω} does not. Hence, replacement of W^{Ω} in equation (17) with the equal amount of internal energy W^{Φ} reinforces the earlier observation that the energy difference W^m is due to action of the spherical couple-stress. While involvement of equation (15) deprives the expression of W^{Ω} from noticing explicitly the W^m contribution, that extra energy contribution is now expected to appear explicitly in the corresponding expression of W^{Φ}, through its association with the divergence of the virtual spin vector, Φ_i.

Under these considerations, use of equation (8) enables the second part of equation (35) to acquire the following form:

$$\Omega_{\ell} m_{rr,\ell} = 6W^m - 3(\bar{m}_{\ell i}\Omega_i)_{,\ell}. \tag{36}$$

This is essentially a partial differential equation (PDE) for the otherwise indeterminate spherical couple-stress, m_{rr}, because the appearing components of the actual spin vector, Ω_i, and the deviatoric part of couple-stress, $\bar{m}_{ij}$, can be made available in any well-posed boundary value problem by solving the governing equations of the conventional theory.

Nevertheless, potential solution of this PDE also requires previous identification or determination of the energy term W^m, whose earlier noted unaccountability justifies the difference between the values of W^{Ω} and W^{ω}, noted in equation (18). The role and value of that term depend on the constitutional properties of the polar material of interest and, thus, become an issue of substantial further interest in subsequent sections. After W^m is identified, or becomes known, successful solution of the PDE Equation (36) will thus enable determination of the spherical couple-stress.

It is re-emphasised that the remaining, unused part of equation (35) is insufficient for unique determination of all three components of the auxiliary, spin vector Φ_i. Since there exists a doubly infinite number of such vectors that satisfy equation (35), Φ_i, thus, remains unspecified and, thus, retains its virtual spin-vector character.

4. Founding principles of couple-stress hyperelasticity

Formulation of couple-stress hyperelasticity may begin with the standard consideration that a generic material particle is initially at a position **X**, with co-ordinates X_R in the reference configuration, and moves to the position **x**, with co-ordinates x_i in the deformed configuration. The deformation is considered describable by an equation of the form

$$x_i = x_i(X_R), \tag{37}$$

which gives rise to the deformation gradient tensor, **F**, and the Cauchy–Green deformation tensors, $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, with components [13,16]

$$F_{iR} = \frac{\partial x_i}{\partial X_R}, C_{RS} = \frac{\partial x_i}{\partial X_R}\frac{\partial x_i}{\partial X_S} = F_{iR}F_{iS}, B_{ij} = \frac{\partial x_i}{\partial X_R}\frac{\partial x_j}{\partial X_R} = F_{iR}F_{jR}, \tag{38a, b, c}$$

respectively.

It is here postulated that, in couple-stress hyperelasticity, the tensor **F** is accompanied by some additional deformation gradient tensor, say **G**, with components

$$G_{iR} = \frac{\partial g_i}{\partial X_R} = g_{i,R}, \tag{39}$$

which is energetically reciprocal to the couple-stress tensor **m**. The appearing vector **g** or, in components, $g_i(X_R)$ represents a corresponding vector or pseudovector field whose specific definition is a subject of the constitutional features of the specific polar material of interest. Hence, dependent on the precise nature of the vector field **g**, the nature of the gradient field **G** may be that of either a tensor or a pseudotensor.

Precise determination of g_i is not required at present. However, it must be helpful for a reader to consider that, in sections 5 and 6, g_i is chosen to represent the pseudovector Φ_i introduced earlier in equation (23), while in section 7, which will be dealing with polar behaviour of a specific class of fibre-reinforced materials, it is represented by a fibre direction vector.

It will be seen in this context that the present, generalised couple-stress theoretical framework embraces the analysis detailed in the works by Spencer and Soldatos [5,11] as well. It is thus appropriate to parenthetically note that if g_i is chosen to represent some specific vector (rather than a pseudovector) field, that vector must be such that

$$g_i \neq \hat{Q}_{ij}x_j(X_R), \tag{40}$$

where $\hat{\mathbf{Q}}$ is a constant orthogonal matrix that represents an arbitrary rotation/reflection of the co-ordinate system.

Under these considerations, the well-known introductory postulates and definitions met in non-polar material hyperelasticity are now complemented by further introducing the constitutive assumption,

$$W = W(\mathbf{F}, \mathbf{G}), \tag{41}$$

which is considered adequate for the present purposes, at least within the bounds of polar material isotropy. Nevertheless, as also happens in non-polar hyperelasticity, additional agencies that identify possible preference material directions may still be included into the general form of W in cases of polar material anisotropy. It, thus, is anticipated that most of the analysis presented in this section is still applicable in such cases.

Within the framework of the refined couple-stress formulation developed in section 3.1, the constitutive assumption (41) leads to

$$\dot{W} = \frac{\partial W}{\partial F_{iR}}\dot{F}_{iR} + \frac{\partial W}{\partial G_{iR}}\dot{G}_{iR} = \frac{\partial W}{\partial F_{iR}}\frac{\partial v_i}{\partial X_R} + \frac{\partial W}{\partial G_{iR}}\frac{\partial \dot{g}_i}{\partial X_R} = \frac{\partial W}{\partial F_{iR}}\frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial X_R} + \frac{\partial W}{\partial G_{iR}}\frac{\partial \dot{g}_i}{\partial x_j}\frac{\partial x_j}{\partial X_R},$$

and, henceforth, to

$$\dot{W} = F_{jR}\left(\frac{\partial W}{\partial F_{iR}}\frac{\partial v_i}{\partial x_j} + \frac{\partial W}{\partial G_{iR}}\frac{\partial \dot{g}_i}{\partial x_j}\right) = F_{jR}\left[\frac{\partial W}{\partial F_{iR}}\left(d_{ij}+\omega_{ij}\right) + \frac{\partial W}{\partial G_{iR}}\dot{g}_{i,j}\right], \tag{42}$$

which is to be compared with its counterpart stemming from equation (26).

It is next recalled that invariance of equation (41) under rigid body rotation requires from W to depend on the scalar products of the vectors F_{iR} and G_{iR}, for each fixed value of R. The relevant process that enables satisfaction of this requirement is essentially detailed in section 5 of Spencer and Soldatos [5] and need not be repeated here.

By thus ignoring the fibre direction vector involved in the work by Spencer and Soldatos [5], that process requires reduction of equation (41) into the form

$$W = W(\mathbf{C}, \mathbf{\Lambda}), \tag{43}$$

where the components of the appearing symmetric right Cauchy–Green deformation tensor are given in equation (38b), and $\mathbf{\Lambda}$ is a non-symmetric tensor with components

$$\Lambda_{RS} = F_{jR}G_{jS}. \tag{44}$$

It is emphasised though that unlike the work by Spencer and Soldatos [5], where $\mathbf{G}$ and, therefore, $\mathbf{\Lambda}$ are specified as a second-order proper tensors, the fact that $\mathbf{G}$ is here regarded as either a proper tensor or a pseudo-tensor implies that $\mathbf{\Lambda}$ must also be regarded as a proper tensor or a pseudo-tensor, respectively.

The necessary equivalence of the energy representations (41) and (43) thus requires from W to depend on $\mathbf{F}$ and $\mathbf{G}$ only through the components of $\mathbf{C}$ and $\mathbf{\Lambda}$. In this context, Appendix 1 quotes a set of auxiliary formulas that assist reduction of equation (41) into equation (43), as well as the relevant transformation of subsequent constitutive equations, such as those obtained in the next section. Nevertheless, further progress towards derivation of such constitutive equations becomes possible only after the general vector field $\mathbf{g}$ is defined more specifically.

It is noted in passing that, dependent on the constitutive features of a polar elastic material, no reason prevents existence, and, hence, incorporation into this general theoretical framework, of two or more $\mathbf{g}$-type vector agencies, say $g_i^{(1)}, g_i^{(2)}, \ldots$, and so on. For instance, no reason could oppose potential existence of a class of fibre-reinforced elastic materials whose polar material behaviour is influenced by the spin vector Φ_i introduced in equation (23), as well as by one or more fibre direction vectors (see sections 6 and 7). However, such an additional complication is felt unnecessary for the principal purposes of this communication.

5. Couple-stress hyperelasticity of isotropic materials: connection with the virtual vector $\mathbf{\Phi}$

5.1. Finite deformations

It is now postulated that the general vector field g_i, introduced in equation (39), is represented by the virtual pseudovector Φ_i that entered the refined couple-stress theory with use of the relationships (24). In this case, it is

$$g_i \equiv \Phi_i, G_{iR} = \frac{\partial \Phi_i}{\partial X_R} = \Phi_{i,R}, \tag{45a, b}$$

and $\mathbf{G}$ thus is a pseudotensor. By virtue of equations (23a) and (24a), equation (42) then reduces to

$$\dot{W} = F_{jR}\left[\frac{\partial W}{\partial F_{iR}}\left(d_{ij}+\omega_{ij}\right) + \frac{\partial W}{\partial G_{iR}}\psi_{i,j}\right]. \tag{46}$$

This equation must coincide with the power balance equation (28) which, by virtue of equation (29), is anticipated equivalent with its conventional counterpart (11). Hence, a comparison of equation (46) with equation (11) yields

$$\left(\sigma_{(ji)} - \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial F_{iR}}\right) d_{ij} + \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial F_{iR}} \omega_{ij} + \left(\bar{m}_{ji}\omega_{i,j} - \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial G_{iR}} \psi_{i,j}\right) = 0. \tag{47}$$

Since d_{ij} and ω_{ij} are arbitrary, equation (47) requires

$$\begin{aligned} \sigma_{(ji)} &= \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial F_{iR}}, \\ \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial F_{iR}} \omega_{ij} &= 0, \\ \bar{m}_{ji}\omega_{i,j} - \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial G_{iR}} \psi_{i,j} &= 0. \end{aligned} \tag{48a, b, c}$$

The first of these equations is a constitutive equation for the symmetric part of the stress and its form looks identical to that of its counterpart met in non-polar hyperelasticity. However, its apparent simplicity is deceptive because, as is implied by equation (43), it must be reduced into an equivalent form that depends on **C** and **Λ**.

The second of equation (48) requires from the coefficient of ω_{ij} to be symmetric with respect to interchanges of the indices i and j. This requirement leads to

$$F_{jR} \frac{\partial W}{\partial F_{iR}} = F_{iR} \frac{\partial W}{\partial F_{jR}}, \tag{49}$$

which is essentially a restriction on the admissible forms of W that guarantees the symmetry of $\sigma_{(ji)}$.

The last of equations (48) is an augmented form of a constitutive equation for the couple-stress. This fact becomes more evident and informative by considering the couple of special cases that the present analysis is fundamentally interested on.

Accordingly, by initially ignoring equation (20) and, thus, selecting

$$\psi_i \equiv \omega_i, \Phi_i \equiv \Omega_i, G_{iR} = \Omega_{i,R}, \tag{50a, b, c}$$

the outlined analysis is engaged with the conventional couple-stress theory briefed earlier in section 2. In this special case, equation (48c) reduces to

$$\left(\bar{m}_{ji} - \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial G_{iR}}\right)\omega_{i,j} = 0, (G_{iR} = \Omega_{i,R}), \tag{51}$$

and, due to the arbitrariness of $\omega_{i,j}$, leads to the following constitutive equation for the deviatoric couple-stress:

$$\bar{m}_{ji} = \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial \Omega_{i,R}}. \tag{52}$$

As is detailed in section 2, a combination of this constitutive equation with its symmetric stress counterpart (48a) and the equilibrium equation (9) suffices to determine the deviatoric couple-stress components in any well-posed boundary value problem. However, that combination still leaves the spherical couple-stress, m_{rr}, indeterminate.

On the other hand, though, upon restoring validity of equation (29), one distinguishes a second special case in which the virtual vector ψ_i is selected to satisfy

$$\bar{m}_{ji}\omega_{i,j} = m_{ji}\psi_{i,j} \Leftrightarrow m_{rr}\psi_{k,k} = 3\bar{m}_{ji}\left(\omega_{i,j} - \psi_{i,j}\right), \tag{53a, b}$$

where use is also made of equation (8).

In that case, equation (48c) reduces to

$$\left(m_{ji} - \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial G_{iR}} \right) \psi_{i,j} = 0, \tag{54}$$

which, due to the arbitrariness of $\psi_{i,j}$, associates with the refined couple-stress formalism detailed in section 3 the following couple-stress constitutive equation:

$$m_{ji} = \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial G_{iR}} = \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial \Phi_{i,R}}, (G_{iR} = \Phi_{i,R} \neq \Omega_{i,R}). \tag{55}$$

It is recalled (see end of section 3.1) that the implied combination of conditions (53) and (29) is already anticipated insufficient for unique specification of Φ_i (or, equivalently, ψ_i), which, thus, remains a virtual spin vector. A practical verification of this observation is demonstrated in Appendix 2, which employs a further simplified situation that requires from equation (53) to hold along with the restriction

$$\psi_{i,j} = \omega_{i,j}, (i \neq j). \tag{56}$$

Interest on the latter simplification stems from the fact that, since $\bar{\mathbf{m}}$ is generally determinable through use of the conventional constitutive equation (52), a combination of equation (53a) with equation (56) implies that the constitutive equation (55) may be employed and used only for determination of the diagonal components of **m**, and, henceforth, of the remaining, unknown spherical couple-stress. By considering separately the simplified case (56), Appendix 2 succeeds to identify the corresponding form of ψ_i, which, thus, is indeed shown to be an arbitrary and, therefore, virtual pseudovector.

As is implied in equation (43), W must depend on **F** and **G** through the components of **C** and Λ, and, in this regard, the obtained constitutive equations (48a), (52), and (55) are next expressed in terms of the latter pair of tensors. With use of the auxiliary calculations detailed in Appendix 1, the constitutive equations (48a) and (55) of the refined theory thus reduce to

$$\begin{aligned} \sigma_{(ji)} &= \frac{\rho}{\rho_0} F_{jR} F_{iM} \left(\frac{\partial W}{\partial C_{RM}} + \frac{\partial W}{\partial C_{MR}} \right) + F_{jR} G_{iN} \frac{\partial W}{\partial \Lambda_{RN}}, \\ m_{ji} &= \frac{\rho}{\rho_0} F_{jR} F_{iM} \frac{\partial W}{\partial \Lambda_{MR}}, (G_{iR} = \Phi_{i,R} \neq \Omega_{i,R}), \end{aligned} \tag{57a, b}$$

while, by virtue of (57a), the symmetry restriction (49) reduces to

$$\left(F_{jR} G_{iN} - F_{iR} G_{jN} \right) \frac{\partial W}{\partial \Lambda_{RN}} = 0. \tag{58}$$

It is noted that equation (57a) still holds in the case of the conventional version of the theory, where dismissal of equation (20) enables the outlined calculations to take place after equations (45b) and (48c) are replaced by equations (50c) and (51), respectively. However, the reduced form of the couple-stress constitutive equation (52) of conventional couple-stress hyperelasticity is found to be

$$\bar{m}_{ji} = \frac{\rho}{\rho_0} F_{jR} F_{iM} \frac{\partial W}{\partial \Lambda_{MR}}, (G_{iR} = \Omega_{i,R}). \tag{59}$$

Regardless of whether conventional or refined couple-stress hyperelasticity is attended to, W thus is required to be an isotropic invariant of the tensors **C** and $\mathbf{\Lambda}$. The most general such form of W is accordingly required to be a function of a complete set of isotropic invariants of these tensors, such as [17]:

$$I_1 = \text{tr}\,\mathbf{C}, I_2 = 1/2\{(\text{tr}\,\mathbf{C})^2 - \text{tr}\,\mathbf{C}^2\}, I_3 = \det\mathbf{C}, I_4 = \text{tr}\boldsymbol{\Lambda}^s = \text{tr}\boldsymbol{\Lambda}, I_5 = \text{tr}\boldsymbol{\Lambda}_s^2,$$
$$I_6 = \text{tr}\boldsymbol{\Lambda}_a^2, I_7 = \text{tr}\boldsymbol{\Lambda}_s^3, I_8 = \text{tr}\,\mathbf{C}\boldsymbol{\Lambda}_s = \text{tr}\,\mathbf{C}\boldsymbol{\Lambda}, I_9 = \text{tr}\,\mathbf{C}^2\boldsymbol{\Lambda}_s = \text{tr}\,\mathbf{C}^2\boldsymbol{\Lambda}, I_{10} = \text{tr}\,\mathbf{C}\boldsymbol{\Lambda}_s^2,$$
$$I_{11} = \text{tr}\,\mathbf{C}^2\boldsymbol{\Lambda}_s^2, I_{12} = \text{tr}\,\mathbf{C}\boldsymbol{\Lambda}_a^2, I_{13} = \text{tr}\,\mathbf{C}^2\boldsymbol{\Lambda}_a^2, I_{14} = \text{tr}\,\mathbf{C}^2\boldsymbol{\Lambda}_a^2\mathbf{C}\boldsymbol{\Lambda}_a, I_{15} = \text{tr}\boldsymbol{\Lambda}_s\boldsymbol{\Lambda}_a^2, \tag{60}$$
$$I_{16} = \text{tr}\,\boldsymbol{\Lambda}_s^2\boldsymbol{\Lambda}_a^2, I_{17} = \text{tr}\boldsymbol{\Lambda}_s^2\boldsymbol{\Lambda}_a^2\boldsymbol{\Lambda}_s\boldsymbol{\Lambda}_a, I_{18} = \text{tr}\,\mathbf{C}\boldsymbol{\Lambda}_s\boldsymbol{\Lambda}_a, I_{19} = \text{tr}\,\mathbf{C}^2\boldsymbol{\Lambda}_s\boldsymbol{\Lambda}_a,$$
$$I_{20} = \text{tr}\,\mathbf{C}\boldsymbol{\Lambda}_s^2\boldsymbol{\Lambda}_a, I_{21} = \text{tr}\,\mathbf{C}\boldsymbol{\Lambda}_a^2\boldsymbol{\Lambda}_s\boldsymbol{\Lambda}_a,$$

where, the appearing symmetric the antisymmetric parts of the pseudotensor Λ, namely

$$\boldsymbol{\Lambda}_s = \frac{1}{2}\left(\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T\right),\ \boldsymbol{\Lambda}_a = \frac{1}{2}\left(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^T\right), \tag{61a, b}$$

respectively, are evidently also pseudotensors.

The constitutive equations (57) of the refined theory can then be expressed as follows:

$$\sigma_{(ji)} = \frac{\rho}{\rho_0}\sum_{\alpha=1}^{21}\frac{\partial W}{\partial I_\alpha}\left\{F_{jR}F_{iM}\left(\frac{\partial I_\alpha}{\partial C_{RM}} + \frac{\partial I_\alpha}{\partial C_{MR}}\right) + F_{jR}G_{iN}\frac{\partial I_\alpha}{\partial \Lambda_{RN}}\right\},$$
$$m_{ji} = \frac{\rho}{\rho_0}\sum_{\alpha=1}^{21}F_{jR}F_{iM}\frac{\partial W}{\partial I_\alpha}\frac{\partial I_\alpha}{\partial \Lambda_{MR}}, (G_{iR} = \Phi_{i,R} \neq \Omega_{i,R}). \tag{62a, b}$$

It is noted, though, that the invariants I_4, I_7, I_8, I_9, I_{14}, I_{15}, and I_{20} are odd functions of the pseudotensors (61) and, therefore, can enter W only in terms of their squares and products.

It is further observed that in the case of the conventional theory, where (50) holds, it is

$$I_4 = tr\boldsymbol{\Lambda} = tr\left(F_{jR}G_{jS}\right) = F_{jR}\Omega_{j,R} = \Omega_{j,j} = 0. \tag{63}$$

Hence, the set (57a) and (59) of the constitutive equations of the conventional theory reduces to the following:

$$\sigma_{(ji)} = \frac{\rho}{\rho_0}\sum_{\alpha=1,\neq 4}^{21}\frac{\partial W}{\partial I_\alpha}\left\{F_{jR}F_{iM}\left(\frac{\partial I_\alpha}{\partial C_{RM}} + \frac{\partial I_\alpha}{\partial C_{MR}}\right) + F_{jR}G_{iN}\frac{\partial I_\alpha}{\partial \Lambda_{RN}}\right\},$$
$$\bar{m}_{ji} = \frac{\rho}{\rho_0}\sum_{\alpha=1,\neq 4}^{21}F_{jR}F_{iM}\frac{\partial W}{\partial I_\alpha}\frac{\partial I_\alpha}{\partial \Lambda_{MR}}, (G_{iR} = \Omega_{i,R}). \tag{64a, b}$$

It is now recalled that validity of equation (29) implies that the power balance equations of the conventional and the refined theory, namely, equations (11) and (28), coincide. Hence, the difference noted between the couple-stress constitutive equations (62b) and (64b) supports the feeling that the spherical couple-stress contribution, which is missing in the left hand (64b), is related with the generally non-zero value that the invariant $I_4(=\Phi_{j,j} \neq \Omega_{j,j} = 0)$ attains in the case of the refined theory.

Further clarification regarding the origin and the nature of the anticipated connection between I_4 and the spherical couple-stress may become easily understood by distinguishing and studying separately simple special forms of the internal energy function. The special case of polar isotropic linear elasticity is underpinned by one of the simplest such forms of W and, thus, enters naturally into the next part of the present analysis.

5.2. Infinitesimally small deformations

The special case of polar, isotropic linear elasticity has been considered and studied independently in the work by Soldatos [12], where comparisons are also made with early fundamental publications referring to the conventional couple-stress theory [2,4]. It is therefore adequate in this section to (1) describe the way that appropriate linearisation of the outlined finite deformation equations leads to relevant equations detailed in the work by Soldatos [12], and (2) briefly recall and quote some relevant concepts and results serving the purpose of the present communication.

The starting point of the implied linearisation process is the standard approximation of W with a quadratic function of the deformation gradients, which, in turn, anticipates ultimate derivation of linear constitutive equations. It is recalled in this context that, in the case of polar material behaviour, the quadratic form sought for W is a reasonable approximation of the internal energy function because the gradients of both the actual and the virtual spin vectors are of the same order of magnitude with the displacement gradients.

It is accordingly observed that only 6 of the 21 invariants listed in equation (60), namely, I_1, I_2, I_4, I_5, I_6, and I_8, are of the first or second order in the implied deformation gradients. These are, therefore, the only invariants that may contribute into a quadratic form of W. However, the second-order invariant I_8 is an odd function of the pseudo-tensor (61a). Hence, since it can contribute into the internal energy only through its square, I_8 is dropped from a quadratic representation of W. For the same reason, the product of the first-order invariants I_1 and I_4 is also dropped.

It further happens that I_8 and I_1I_4 were the only potential second-order invariant contributors that involve mixed products of displacement and spin gradients. It follows that, as is anticipated in equation (31a), the quadratic form sought for W will necessarily split, into a term W^e that depends solely on the linear strains, and a term W^{Φ} that depends solely on the relevant polar response features of the material. Hence, the long-known linearised version of I_1 and I_2, namely [13,18]

$$\hat{I}_1 = tr\boldsymbol{e} = e_{ii}, \hat{I}_2 = tr\boldsymbol{e}^2 = e_{ij}e_{ji}, \tag{65}$$

reveals that

$$W^e = \frac{1}{2}\lambda \hat{I}_1^2 + \mu \hat{I}_2 = \frac{1}{2}\lambda (e_{ii})^2 + \mu e_{ij}e_{ji} \geqslant 0, \tag{66}$$

where λ and μ are the standard Lamé moduli.

On the other hand, infinitesimally small strains and spin gradients justify the approximations

$$\begin{aligned} F_{jR} &= \delta_{jR} + u_{j,R} \simeq \delta_{jR} \\ \Lambda_{RS} &= F_{jR}G_{jS} = \left(\delta_{jR} + u_{j,R}\right)\Phi_{j,S} \simeq \Phi_{R,S}, \end{aligned} \tag{67}$$

where the distinction made earlier between deformed and undeformed material configuration and, therefore, between the capital and the low-case indices can now be dropped. It follows that the linearised form of the remaining three invariants is as follows:

$$I_4 \simeq J_1 = \Phi_{(i,i)} \equiv \Phi_{i,i}, I_5 \simeq J_2 = \Phi_{(i,j)}\Phi_{(j,i)}, I_6 \simeq J_3 = \Phi_{[i,j]}\Phi_{[j,i]}. \tag{68}$$

The most general, quadratic form of the polar part of the internal energy function (31a) thus is

$$W^{\Phi}\left(\Phi_{m,n}\right) = \frac{1}{2}\left(\eta_0 J_1^2 + \eta_1 J_2 + \eta_2 J_3\right) = \frac{1}{2}\left[\eta_0(\Phi_{m,m})^2 + \eta_1 \Phi_{(m,n)}\Phi_{(n,m)} + \eta_2 \Phi_{[m,n]}\Phi_{[n,m]}\right], \tag{69}$$

where η_0, η_1, and η_2 are appropriate material moduli having dimensions of force.

It is next observed that, due to the involvement of the tensor $\mathbf{G}$, the second term within the curled bracket appearing in the constitutive equation (62a) is of higher order in the deformation gradients when compared with the first relevant term. Hence, that higher-order term is dropped during the outlined linearisation process. By thus also considering that infinitesimal elastic deformations leave practically unchanged the material density ($\rho = \rho_0$), equation (62a) reduces to

$$\sigma_{(ij)} = \frac{\partial W^e}{\partial e_{ij}} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, \tag{70}$$

which naturally coincides with the standard constitutive equation met in non-polar isotropic linear elasticity.

The corresponding linear couple-stress constitutive equation is similarly obtained by inserting the approximations (67), as well as $\rho = \rho_0$, into equation (62b), thus leading to

$$m_{ji} = \frac{\partial W^{\Phi}}{\partial \Phi_{i,j}} = \eta_0 \Phi_{m,m} \delta_{ij} + 2\eta_1 \Phi_{(i,j)} + 2\eta_2 \Phi_{[j,i]} = \eta_0 \Phi_{m,m} \delta_{ij} + (\eta_1 + \eta_2)\Phi_{j,i} + (\eta_1 - \eta_2)\Phi_{i,j}. \tag{71}$$

This constitutive equation is evidently identical with its counterpart obtained in the work by Soldatos [12] solely on purely linear elasticity considerations. Hence, by contracting the appearing free indices, and taking equation (20) into consideration, one further obtains the following value of the spherical couple-stress:

$$m_{rr} = \frac{\partial W^{\Phi}}{\partial \Phi_{r,r}} = (3\eta_0 + 2\eta_1)\Phi_{r,r} \neq 0. \tag{72}$$

It is observed that, in the case of the conventional theory, where equation (15) holds, equation (69) reduces to

$$W^{\Omega}\left(\Omega_{i,j}\right) = \frac{1}{2}\left(\eta_1 \Omega_{(i,j)} \Omega_{(j,i)} + \eta_2 \Omega_{[i,j]} \Omega_{[j,i]}\right). \tag{73}$$

This observation leads to the conclusion that the term $\eta_0 (\Phi_{m,m})^2/2$ emerges in equation (69) as an extra energy term that escapes the attention of and, therefore, is unaccountable by the conventional theory (see also section 2.3). The additional modulus, η_0, introduced in equation (69), thus, clearly represents energy contribution that is due to interaction of the virtual spin divergence with the spherical couple-stress. More information and relevant discussions are provided in the work by Soldatos [12] where it is also concluded that the conventional couple-stress theory emerges as an exceptional, singular version of its refined counterpart.

Since all information that the conventional theory makes available through the solution of a relevant, well-posed boundary value problem is still valid (see also section 3.2), the spherical couple-stress remains the only physical quantity that still needs to be determined. By accordingly requiring from the unaccounted energy part, $\eta_0 (\Phi_{m,m})^2/2$, to balance the unrecorded energy contribution W^m, noted in equations (35) and (36), the latter equation obtains the more specific form:

$$\Omega_{\ell} m_{rr,\ell} = 3\eta_0 (\Phi_{m,m})^2 - 3\left(\bar{m}_{\ell i} \Omega_i\right)_{,\ell}. \tag{74}$$

Nevertheless, inversion of equation (72) and subsequent elimination of the appearing unknown virtual spin divergence enables conversion of equation (74) into the non-linear first-order PDE

$$\Omega_{\ell} m_{rr,\ell} = \eta m_{rr}^2 - 3\left(\bar{m}_{\ell i} \Omega_i\right)_{,\ell}, \; \eta = \frac{3\eta_0}{(3\eta_0 + 2\eta_1)^2} > 0, \tag{75}$$

for the unknown spherical couple-stress.

Hence, for any well-posed boundary value problem, solution of the PDE (75) can be pursued immediately after the solution is completed of the conventional couple-stress theory equations. Unique specification of the, thus, obtained spherical couple-stress will also require association of equation (75) with some appropriate boundary condition. The latter will emerge either from the unused part of the traction boundary conditions (5b) or from the corresponding geometrical counterpart of the same. A pair of relatively simple relevant examples and their solutions is available in the work by Soldatos [12].

5.3. The role of the invariant $I_4 = tr\Lambda$ and its internal energy contribution

The invariant $I_4 = \Lambda_{RR}$ is associated with deformation features stemming from polar material behaviour. However, in the special case of the conventional couple-stress theory, that invariant fails to mark its contribution into the internal energy and, hence, into the relevant constitutive equations (64). While, on the other hand, I_4 does contribute into the internal energy of the refined couple-stress theory and, hence, into the constitutive equations (62), the virtual nature of the spin vector Φ_i prevents in that case its direct evaluation.

Nevertheless, in the case of infinitesimally small deformations, the relatively simple material properties of isotropy enable direct connection of $I_4 = \Lambda_{RR}(\simeq \Phi_{r,r} = J_1)$ with the spherical couple-stress, m_{rr}, and, henceforth, formation of the PDE (75). While solution of that PDE is anticipated sufficient for determination of m_{rr}, a question may arise on whether this process is still possible in cases that the isotropic polar material of interest is subjected to finite deformations.

It is recalled in this context that the form of any strain energy density employed in finite elasticity applications is required to be consistent with its own approximate form that serves the purposes of the linear, small deformation version of the theory. The noted connection of I_4 with the energy contribution of the spherical couple-stress is therefore anticipated generally present not only within the infinitesimal, but also within the finite deformation regime.

In other words, the quadratic form

$$W^m(I_4) \simeq \frac{\eta_0 (\Phi_{m,m})^2}{2}, \tag{76}$$

which equation (35) associates with the extra energy contribution (18) in the small deformation regime, is regarded as a leading-order approximation of the influence that the spherical couple-stress exerts on any admissible form of a corresponding internal energy density employed in relevant isotropic finite hyperelasticity applications. It follows that a term of the form (76) is anticipated present in any admissible internal energy density or in its equivalent polynomial expansion in terms of the invariants (60).

Alternatively, a W^m-term must be present in any admissible, relevant internal energy density, and its leading-order (infinitesimal deformation) approximation must necessarily be quadratic in I_4. The presence of such a term makes thus certain that formation of a PDE of the form (36) and, hence, determination of the spherical couple-stress is always possible, regardless of the magnitude of the deformation. In cases of finite deformation though, the implied non-linearity involved in W^m may extend beyond the quadratic term appearing in equation (76) and, thus, increase difficulty in searching for solutions of equation (36).

6. Polar material anisotropy

Basic principles and equations of couple-stress hyperelasticity are already set and discussed in the last couple of sections, though section 5 focused on the material symmetries of isotropy only. Relevant extensions that incorporate into the theory effects of material anisotropy can now be pursued in the manner that non-polar anisotropic hyperelasticity extends its isotropic material counterpart. The present section aims to initiate that theoretical extension process by demonstrating it only in the special case that material symmetries are those of transverse isotropy. The symmetries of more advanced anisotropic material configurations can be handled in a similar manner, though the analysis will be getting increasingly cumbersome with increasing the number of the involved deformation invariants. Detailed consideration of more advanced material anisotropy, thus, is expected to become the subject of future communications and studies.

6.1. Finite elastic deformations of polar transverse isotropic materials

While equations and definitions (37)–(40) still hold, it is now postulated that the specific internal energy density of the transverse isotropic polar material of interest is of the form

$$W = W(\mathbf{F}, \mathbf{G}, \mathbf{A}), \tag{77}$$

where the deformation gradient tensors $\mathbf{F}$ and $\mathbf{G}$ are still defined by equations (38a) and (45b), respectively, and $\mathbf{A}(\mathbf{X})$ represents the unit vector field that defines the direction of transverse isotropy in the reference configuration. The corresponding unit vector field is denoted with $\mathbf{a}(x)$ in the deformed configuration, and the following standard formulas thus hold [5,11,13]:

$$\mathbf{b} = \hat{\lambda}\mathbf{a} = \mathbf{FA},\ b_i = \hat{\lambda}a_i = F_{iR}A_R,\ \hat{\lambda}^2 = A_R C_{RS} A_S, \tag{78}$$

where $\hat{\lambda}$ denotes the stretch in the direction of transverse isotropy.

As happens in non-polar hyperelasticity, the augmented form equation (77) of the internal energy density leaves unaffected the principal analysis detailed in section 4. Hence, while equation (42) still holds, invariance under rigid body rotation requires reduction of equation (77) into the form

$$W = W(\mathbf{C}, \mathbf{\Lambda}, \mathbf{A}), \tag{79}$$

where the components of the tensors $\mathbf{C}$ and $\mathbf{\Lambda}$ are still defined by equations (38b) and (44), respectively. The analysis that follows equation (44), thus, is still valid and leads again to the constitutive equations (57) for the refined couple-stress theory, and the pair (57a) and (59) for its conventional counterpart.

Nevertheless, the influence that $\mathbf{A}$ now exerts on W requires from the set of invariants listed in (60) is essentially a long list of equations to be augmented, and complemented by the following additional members [17]:

$$\begin{aligned} &I_{22} = \mathbf{ACA}, I_{23} = \mathbf{AC}^2\mathbf{A}, I_{24} = \mathbf{A\Lambda}_s\mathbf{A} = \mathbf{A\Lambda A}, I_{25} = \mathbf{A\Lambda}_s^2\mathbf{A},\\ &I_{26} = \mathbf{A\Lambda}_a^2\mathbf{A}, I_{27} = \mathbf{AC\Lambda}_s\mathbf{A}, I_{28} = \mathbf{AC\Lambda}_a\mathbf{A}, I_{29} = \mathbf{AC}^2\mathbf{\Lambda}_a\mathbf{A},\\ &I_{30} = \mathbf{A\Lambda}_a\mathbf{C\Lambda}_a^2\mathbf{A}, I_{31} = \mathbf{A\Lambda}_s\mathbf{\Lambda}_a\mathbf{A}, I_{32} = \mathbf{A\Lambda}_s^2\mathbf{\Lambda}_a\mathbf{A}, I_{33} = \mathbf{A\Lambda}_a\mathbf{\Lambda}_s\mathbf{\Lambda}_a^2\mathbf{A}, \end{aligned} \tag{80}$$

where equations (61) still hold.

It is noted that the invariants I_{24}, I_{27}, I_{28}, I_{29}, I_{30}, and I_{32} are odd functions of the pseudotensors (61) and, along with I_4, I_7, I_8, I_9, I_{14}, I_{15}, and I_{20} defined in (60), can enter the form of W only in terms of their squares and products. Under these considerations, the final form (62) of the constitutive equations of the refined theory, as well as the form (64) of their conventional theory counterparts, is seen also valid, provided that the upper limit of the appearing summation index, α, is modified and becomes 33.

6.2. Infinitesimal deformations

The process leading to the corresponding version of transversely isotropic linear elasticity resembles its counterpart employed earlier in section 5.2. That process begins with the search for a quadratic approximation of the form (79) of W which, as is detailed section 5.2, must depend on (1) the invariants I_1, I_2, I_4, I_5, and I_6 appearing in (60), and (2) no more than the eight additional invariants I_{22}, I_{23}, I_{24}, I_{25}, I_{26}, I_{27}, I_{28}, and I_{31}, which are listed in (80) and are of the first or the second order in the deformation gradients.

Among those eight invariants, I_{27} and I_{28} are the only second-order invariants that involve mixed products of displacement and spin gradients. However, these are both odd functions of the pseudotensors (61) and, since they can thus enter W only in their squares and product, are dropped from the quadratic form sought for W. It follows that the ultimate form of W will again split in the manner implied in (31a), namely into a term W^e that depends solely on the linear strains, and a term W^Φ that is solely due to the observed features of polar material response.

Hence, along with equation (65), the well-known [18] linearised versions of I_{22} and I_{23}, namely

$$\hat{I}_3 = \mathrm{tr}\mathbf{AeA} = A_i e_{ij} A_j,\ \hat{I}_4 = \mathrm{tr}\mathbf{Ae}^2\mathbf{A} = A_i e_{ij} e_{jk} A_k, \tag{81}$$

respectively, will now form the group of four invariants contributing into W^e. The most general form of the latter thus is

$$W^e = \frac{1}{2}\lambda\hat{I}_1^2 + \mu\hat{I}_2 + \frac{1}{2}\kappa_1\hat{I}_3^2 + \kappa_2\hat{I}_1\hat{I}_3 + \kappa_3\hat{I}_4, \tag{82}$$

where κ_1, κ_2, and κ_3 correspond to the additional, standard independent material moduli that accompany the Lamé moduli in non-polar transverse isotropic linear elasticity [18–20].

It is now emphasised that the arguments employed in section 5.2 for the derivation of the linearised constitutive equation,

$$\sigma_{(ij)} = \frac{\partial W^e}{\partial e_{ij}}, \tag{83}$$

hold regardless of the material symmetries of the present polar elastic solid. It follows that not only equation (82) but also the corresponding linear constitutive equation of the symmetric part of the stress is identical with its counterpart met in non-polar transverse isotropic linear elasticity [18–20].

The last four remaining invariants are linearised with use of the small strain approximations (67), which leads to

$$\begin{aligned} I_{24} &\simeq J_4 = A_i\Phi_{(i,j)}A_j = A_i\Phi_{i,j}A_j, \; I_{25} \simeq J_5 = A_i\Phi_{(i,j)}\Phi_{(j,k)}A_k, \\ I_{26} &\simeq J_6 = A_i\Phi_{[i,j]}\Phi_{[j,k]}A_k, \; I_{31} \simeq J_7 = A_i\Phi_{(i,j)}\Phi_{[j,k]}A_k. \end{aligned} \tag{84a, b}$$

Along with equation (68), these form the group of seven invariants that contribute to W^{Φ}.

The most general such form of W^{Φ} thus is

$$W^{\Phi}\left(\Phi_{i,j}\right) = \frac{1}{2}\left[\eta_0 J_1\left(J_1 + 2\hat{\eta}_0 J_4\right) + \eta_1 J_2 + \eta_2 J_3 + \eta_3 J_4^2 + \eta_4 J_5 + \eta_5 J_6 + 2\eta_6 J_7\right], \tag{85}$$

where η_3 to η_6 and the product $\eta_0\hat{\eta}_0$ represent five additional material moduli that have dimensions of force and complement their isotropic material counterparts appearing in equation (68). It is further noted that, in the light of the relevant discussions detailed in section 5, $\hat{\eta}_0$ is a nondimensional parameter that is associated with energy contribution due to interaction of the spherical couple-stress with J_4. From its definition (84a), the latter is seemingly an invariant that represents a stretching type of deformation along the direction of transverse isotropy.

While J_4 has a double term entry into equation (85), its entry that involves $\hat{\eta}_0$ does not contribute into the internal energy recorded by the conventional couple-stress theory, where validity of equation (15) nullifies the term $\eta_0 J_1(J_1 + 2\hat{\eta}_0 J_4)$. In that case, equation (85) yields

$$\begin{aligned} W^{\Omega}\left(\Omega_{i,j}\right) = \frac{1}{2}&\left[\eta_1\Omega_{(i,j)}\Omega_{(j,i)} + \eta_2\Omega_{[i,j]}\Omega_{[j,i]} + \eta_3\left(a_i\Omega_{(i,j)}a_j\right)^2.\right. \\ &\left. + \eta_4 a_i\Omega_{(i,j)}\Omega_{(j,k)}a_k + \eta_5 a_i\Omega_{[i,j]}\Omega_{[j,k]}a_k + 2\eta_6 a_i\Omega_{(i,j)}\Omega_{[j,k]}a_k\right], \end{aligned} \tag{86}$$

which, in analogy with its isotropic counterpart (73), fails to record energy contributions that may be due to spherical couple-stress action.

In contrast, equation (85) does account for such energy contributions and by thus requiring from $\eta_0 J_1(J_1 + 2\hat{\eta}_0 J_4)$ to balance the extra energy contribution noted in equation (35), enables replacement of equation (74) with the following:

$$\Omega_\ell m_{rr,\ell} = 3\eta_0\Phi_{r,r}\left(\delta_{mn} + 2\hat{\eta}_0 a_m a_n\right)\Phi_{(m,n)} - 3\left(\bar{m}_{\ell i}\Omega_i\right)_{,\ell}. \tag{87}$$

The transverse isotropic counterpart of the constitutive equation (71) is then found to be

$$\begin{aligned} m_{ji} = \frac{\partial W^{\Phi}}{\partial \Phi_{i,j}} &= \eta_0(\delta_{ij} + \hat{\eta}_0 a_i a_j)\Phi_{m,m} + 2\eta_1\Phi_{(i,j)} + 2\eta_2\Phi_{[j,i]} + 2\eta_3\Phi_{(m,n)}a_m a_n a_i a_j \\ &+ 2\eta_4\Phi_{(j,k)}a_i a_k + 2\eta_5\Phi_{[j,k]}a_i a_k + 2\eta_6(a_i\Phi_{[j,k]} + \Phi_{(k,i)}a_j)a_k, \end{aligned} \tag{88}$$

and a contraction of the appearing free indices produces the relationship

$$m_{rr} = \frac{\partial W^{\Phi}}{\partial \Phi_{r,r}} = [(3 + \hat{\eta}_0)\eta_0 + 2\eta_1]\Phi_{r,r} + 2(\eta_3 + \eta_4 + \eta_6)a_m a_n\Phi_{(m,n)}. \tag{89}$$

Equations (87) and (89) are algebraically considerably more complicated than their isotropic material counterparts (74) and (72), respectively. A straightforward conversion of equation (87) into a single PDE for the unknown spherical couple-stress does not seem feasible in this case, and m_{rr} may not be determined in the relatively simpler manner described in the case of polar material isotropy.

However, conversion of equation (87) into a relevant PDE may still become possible through a rather complicated and computationally cumbersome process. That process, which will not be pursued much further in this communication, is described by initially defining the nine-component vectors,

$$\hat{\mathbf{M}} = (m_{11}, m_{22}, m_{33}, m_{23}, m_{32}, m_{13}, m_{31}, m_{12}, m_{21})^T,$$
$$\hat{\boldsymbol{\Phi}} = \left(\Phi_{1,1}, \Phi_{2,2}, \Phi_{3,3}, \Phi_{(2,3)}, \Phi_{[2,3]}, \Phi_{(1,3)}, \Phi_{[1,3]}, \Phi_{(1,2)}, \Phi_{[1,2]}\right)^T, \quad (90a, b)$$

and, thus, rearranging the linear constitutive equation (88) into the matrix form,

$$\hat{\mathbf{M}} = \hat{\mathbf{H}}\hat{\boldsymbol{\Phi}}, \quad (91)$$

where the components of the appearing 9×9 stiffness matrix $\hat{\mathbf{H}}$ are formed through appropriate combination of the η-moduli appearing in equation (88).

Inversion of the matrix equation (91),

$$\hat{\boldsymbol{\Phi}} = \hat{\mathbf{H}}^{-1}\hat{\mathbf{M}}, \quad (92)$$

will thus enable replacement of the unknown gradients (90b) of Φ_i, appearing in equations (87) and (89), with the components (90a) of the non-symmetric couple-stress tensor, m_{ji}. In a final step, the described conversion process is required to make appropriate use of equation (8). In this manner, every term appearing in equation (87) will finally be expressed in terms of the unknown spherical couple-stress, m_{rr}, and/or components of the deviatoric couple-stress, $\bar{m}_{\ell k}$, which become known by solving the governing equations of the conventional couple-stress theory.

This conversion process of equation (87) and the final form of the thus obtained PDE are anticipated more cumbersome and complicated in cases of advanced material anisotropy. This is evidently due to the emergence of additional deformation invariants (see, for instance, Appendix of Soldatos [12]), which complicates further the form of the constitutive equation (88) and, hence, of the stiffness matrix $\hat{\mathbf{H}}$.

It is noted in that context that studies related to anisotropic forms of the matrix $\hat{\mathbf{H}}$ are already available in the literature [21,22]. However, Gourgiotis and Bigoni [21] directed its interest on the conventional couple-stress theory (section 2), where the spherical couple-stress remains indeterminate. Hence, the forms of the matrix $\hat{\mathbf{H}}$ that Gourgiotis and Bigoni [21] referred to need to be updated and include additional elastic moduli that reflect the contribution of the spherical couple-stress into the implied constitutive equations.

On the other hand, forms of the matrix $\hat{\mathbf{H}}$ that Ilkewicz et al. [22] is referring to are also unsuitable for the present formulation, as they contain several extra elastic moduli. This is because Ilkewicz et al. [22] is interested on the micro-polar elasticity formalism [23,24], where the components of the pseudo-vector field that correspond to Φ_i (or $\hat{\boldsymbol{\Phi}}$) represent a spin vector that may emerge at a micro-, probably molecular-scale independently of the macro-scale displacement components and gradients. Implication of some extra elastic moduli, thus, is required in that case, to match the resulting decoupling of the pair (1) of governing equilibrium equations or, more generally, of their dynamic equivalents (see also [2,3]).

It is recalled in this context that, unlike the present formulation, micro-polar elasticity [23,24] is deprived the ability to present the latter pair of balance of momentum and balance of couple-stress governing equations in some alternative, reduced form that is analogous to equation (9) or (1.12) in the work by Mindlin and Tiersten [2] and equation (205.17) in the work by Truesdell and Toupin [3]. Instead, the formalism presented in the work by Eringen [23] necessarily considers that all six individual equations that are stemming from equations (205.2) and (205.10) of Truesdell and Toupin [3] are mechanically independent, and thus enables matching of an augmented number of six independent degrees of freedom with the augmented number of six independent governing equations.

7. Polar material behaviour of fibre-reinforced materials with fibres resistant in bending

It will now be seen that the couple-stress theory of hyperelastic fibre-reinforced materials with fibres resistant in bending [5,11] can also emerge as a special case of the theoretical framework introduced in section 4. In continuum mechanics, the fibres are assumed distributed through the bulk of the material [25] and their direction then coincides with a direction of material preference. In the case that a fibrous composite has embedded a single family of unidirectional fibres [5,11], the fibre direction thus defines a direction of transverse isotropy, and the general background underpinning section 6 might appear adequate for the developments outlined in the present section.

However, section 6 considers that not only the energy balance is sufficiently described with use of the actual and the virtual spin vectors Ω_i and Φ_i (see section 3) but, as happens in the relevant case of polar material isotropy (see section 5), the couple-stress tensor is still energetically reciprocal to the gradient of the virtual pseudovector, Φ_i. In this context, section 6 considers transverse isotropy as a direction of material preference that does not necessarily resist, but essentially follows the bulk material deformation.

In other words, the polar material behaviour modelled in section 6 is regarded inherent in the anisotropic material of interest, as inherent also is regarded in the case of polar material isotropy (section 5) where its source remains macroscopically not observable or virtual and unknown. If the implied direction of material preference is felt representative of a single family of unidirectional fibres, the model developed in section 6 thus considers that family as an essentially perfectly flexible fibre phase of a polar elastic material. It is recalled in that context that fibres are naturally considered perfectly flexible in non-polar elasticity [5,26].

In the present case of interest, where the embedded unidirectional fibres resist the imposed deformation locally [5,11], the fibre direction and the direction of transverse isotropy are still considered identical and, hence, equation (77) still represents a suitable choice of the internal energy density. However, while the virtual pseudovector Φ_i remains engaged with and, hence, involved in the energy balance considerations (section 3), the couple-stress is now considered energetically reciprocal to the gradient of the deformed fibre direction vector, rather than to the gradient of Φ_i. The couple-stress tensor then becomes reciprocal to a relevant fibre-curvature tensor, and, in loose terms, the implied constitutive postulation honours the moment-curvature relationship that is long known in structural mechanics [20,27,28].

In terms of the present notation, this postulation requires from the vector field **g**, introduced in equation (39), to represent the deformed fibre direction. It follows that a principal mathematical difference between the present theoretical development and its counterparts detailed earlier, in sections 2, 3, 5 and 6, stems from the fact that (1) g_i now represents a proper vector, rather than a pseudovector field and, therefore, (2) the corresponding tensorial quantities **G** and **Λ** will both be proper second-order tensors, rather than pseudotensors.

As is detailed in the work by Spencer and Soldatos [5], the components of the material derivative of the unit fibre direction vector, **a**, appearing in equation (78) attain the rather complicated form (see also [29])

$$\dot{a}_i = \left(\delta_{ij} - a_i a_j\right) a_k v_{j,k}, \tag{93}$$

which makes the search for an analytically manageable set of constitutive equations particularly challenging, if possible. In contrast, the choice

$$g_i \equiv b_i \Rightarrow G_{iR} = b_{i,R}, \tag{94}$$

where **b** is defined in (78), leads to the following expressions [5]:

$$\begin{aligned} &\dot{g}_i = \dot{b}_i = b_j v_{i,j}, \\ &F_{jR}\dot{g}_{i,j} = x_{j,R} b_{k,j} v_{i,k} + F_{jR} b_k v_{i,kj} = G_{kR} v_{i,k} + F_{jR} b_k v_{i,kj}, \end{aligned} \tag{95a, b}$$

which enable easier handling of equation (42) and, hence, of parts of the subsequent analysis that resembles its counterpart detailed in section 6.

It is noted in passing that replacement of equation (93) with equation (95a) is justified by the fact that this makes no difference in two special cases of substantial practical importance. Namely, the case of practically inextensible fibres of any shape [26,29,6,30], and the case of straight extensible fibres. In the last remaining special case, where fibres may be extensible but curved, use of equations (95) leads to approximate constitutive equations, with the magnitude of the implied approximation being dependent on the magnitude of the involved fibre curvature. Alternatively, use of the cumbersome expression (93) may potentially lead to a set of constitutive equations which, if feasible, will be (1) generally different from those obtained in the work by Spencer and Soldatos [5] and Soldatos [11], but (2) reducible to the latter when fibres are either inextensible or extensible and straight.

Under these considerations, the use of equation (95b) enables equation (42) to obtain the form

$$\dot{W} = F_{jR}\frac{\partial W}{\partial F_{iR}}\left(d_{ij}+\omega_{ij}\right) + \frac{\partial W}{\partial G_{iR}}\left(G_{jR}v_{i,j}+F_{jR}b_k v_{i,kj}\right), \tag{96}$$

and, henceforth, the form

$$\dot{W} = \left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right)\left(d_{ij}+\omega_{ij}\right) + F_{jR}\frac{\partial W}{\partial G_{iR}}b_k v_{i,kj}, \tag{97}$$

which is identical with the equation labelled (5.15) in the work by Spencer and Soldatos [5] or (4.4) in the work by Soldatos [11]. It follows that all relevant results detailed in the works by Spencer and Soldatos [5] and Soldatos [11] can still be obtained here, though in a slightly different manner.

Accordingly, since equation (29) is pre-assumed valid throughout the present analysis, equation (28) remains equivalent with equation (11) and its comparison with equation (97) leads to

$$\left\{\sigma_{(ji)} - \frac{\rho}{\rho_0}\left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right)\right\}d_{ij} - \frac{\rho}{\rho_0}\left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right)\omega_{ij} + \bar{m}_{ji}\omega_{i,j} - \frac{\rho}{\rho_0}F_{jR}\frac{\partial W}{\partial G_{iR}}b_k v_{i,kj} = 0. \tag{98}$$

Due to the arbitrariness of d_{ij} and ω_{ij}, satisfaction of this equation requires

$$\begin{aligned} &\sigma_{(ji)} = \frac{\rho}{\rho_0}\left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right), \\ &\left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right)\omega_{ij} = 0, \\ &\bar{m}_{ji}\omega_{i,j} - \frac{\rho}{\rho_0}F_{jR}\frac{\partial W}{\partial G_{iR}}b_k v_{i,kj} = 0. \end{aligned} \tag{99a, b, c}$$

The first two of equation (99) are identical with their counterparts obtained in the work by Spencer and Soldatos [5]. Hence, equation (99a) is the relevant constitutive equation for the symmetric part of the stress, while by requiring from the coefficient of ω_{ij} to be symmetric with respect to the indices i and j, equation (99b) leads to the condition

$$F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}} = F_{iR}\frac{\partial W}{\partial F_{jR}} + G_{iR}\frac{\partial W}{\partial G_{jR}}, \tag{100}$$

which guarantees the symmetry of $\sigma_{(ji)}$ and has ultimately shown valid in the work by Spencer and Soldatos [5].

Equation (99c) is slightly different from its counterpart obtained in the work by Spencer and Soldatos [5], because it contains the deviatoric part of the couple-stress, rather than the full couple-stress tensor. Nevertheless, with use of equation (7c), that equation reduces to

$$\left(\varepsilon_{pki}\bar{m}_{jp} - 2\frac{\rho}{\rho_0}F_{jR}\frac{\partial W}{\partial G_{iR}}b_k\right)v_{i,kj} = 0, \tag{101}$$

which requires from the symmetric part, with respect to the indices k and j, of the coefficient of $v_{i,kj}$ to be zero, thus leading to

$$\varepsilon_{pki}\bar{m}_{jp} + \varepsilon_{pji}\bar{m}_{kp} = 2\frac{\rho}{\rho_0}\frac{\partial W}{\partial G_{iR}}\left(F_{jR}b_k + F_{kR}b_j\right). \tag{102}$$

Hence, by multiplying both sides of equation (102) with ε_{rki}, and using standard $\varepsilon - \delta$ identities, one obtains

$$\bar{m}_{jp} = \frac{2}{3}\varepsilon_{pik}\frac{\rho}{\rho_0}\frac{\partial W}{\partial G_{iR}}\left(F_{jR}b_k + F_{kR}b_j\right), \tag{103}$$

which, naturally, is identical with the ultimate form of the deviatoric couple-stress constitutive equation obtained in the work by Spencer and Soldatos [5].

The remaining of the analysis involves the transformation of the constitutive equations (99a) and (103) into their reduced form stemming from equation (79), namely from the reduced form of W that guarantees invariance of its value under rigid body rotation. That process is detailed in the work by Spencer and Soldatos [5] and, while need not be repeated here, leads to the following reduced form of the obtained constitutive equations:

$$\begin{aligned}\sigma_{(ji)} &= \frac{\rho}{\rho_0}\left\{F_{iR}F_{iS}\left(\frac{\partial W}{\partial C_{RS}} + \frac{\partial W}{\partial C_{SR}}\right) + \left(G_{iR}F_{jS} + G_{jR}F_{iS}\right)\frac{\partial W}{\partial \Lambda_{SR}}\right\},\\ \bar{m}_{ji} &= \frac{2}{3}\varepsilon_{ikm}\frac{\rho}{\rho_0}\frac{\partial W}{\partial \Lambda_{PR}}F_{mP}\left(F_{jR}b_k + F_{kR}b_j\right), \left(G_{iR} = b_{i,R}\right).\end{aligned} \tag{104}$$

It is fitting at this point to parenthetically commend on a suggestion made recently in the work by Shariff et al. [31], regarding the special case of curved extensible fibres and the approximation that equations (104) may involve in that case through the replacement of equations (93) with (95a). According to Shariff et al. [31], that approximation is claimed there removed through use of an artificial alteration of the definition (44) of the tensor $\mathbf{\Lambda}$. Namely, by just replacing the right-hand side of equation (44) with $F_{iR}G_{iS} - A_{R,S}$, and, thus, subtracting from the formal definition of the tensor Λ the curvature of non-straight extensible fibres that may be embedded in a fibrous composite. Inexplicably though, the remaining of the analysis presented in the work by Shariff et al. [31] continues by controversially employing and making use of the constitutive equations (104), first derived in the work by Spencer and Soldatos [5].

However, if it is assumed that the replacement of $\mathbf{\Lambda}$ proposed in the work by Shariff et al. [31] is correct, such an alteration of equation (44) should necessarily be perceived equivalent with employing equation (93), rather than equation (95a), in the analysis. As is explained after equation (95) though, some different set of constitutive equations are ought to be derived in that case, and thus also used in the work by Shariff et al. [31], instead of the corresponding set derived previously in the work by Spencer and Soldatos [5].

Alternatively, a proper explanation is missing in the work by Shariff et al. [31] of the reason that the employed artificial alteration of $\mathbf{\Lambda}$ did not pass first through some equivalent alteration of the tensor $\mathbf{G}$ appearing in the right-hand side of the definition (44). As it stands, such an a posteriori–imposed alteration of equation (44) invalidates the necessary equivalence of the internal energy expressions (77) and (79). Since it thus invalidates the relationships (108) quoted in Appendix 1, it further invalidates the equivalence of the set of constitutive equations (99a) and (103) with their (104) counterpart.

The lack of proper relevant explanations leads to the conclusion that, on its own, the artificial alteration of the $\mathbf{\Lambda}$-tensor definition, introduced in the work by Shariff et al. [31] and perhaps used in possible subsequent or future relevant publications, must be denounced unfortunate if not erroneous. As erroneous is also the fact that Shariff et al. [31] claims and treats (10) as a material constraint, rather than as a mathematical identity that holds true regardless of material constitution or deformation features. It must be noted, though, that the spectral analysis that Shariff et al.[31] applied afterwards on the constitutive equations (104) may be found useful in the future (see also [32]).

It is now recalled that the strain energy density (79) is an isotropic invariant of the tensors $\mathbf{C}$ and $\mathbf{\Lambda}$, and the vector $\mathbf{A}$, and thus is expressible in the form

$$W = W(I_1, I_2, \ldots, I_{33}), \tag{105}$$

as a function of the invariants listed in equations (60) and (80). It is though further recalled that in the present case of interest, Λ and, therefore, its symmetric and antisymmetric parts (61) are proper tensors, rather than pseudotensors. Hence, all invariants I_1,..., I_{33} may now enter equation (105) freely, regardless of whether they are even or odd combinations of the tensors (61). However, if, as is usually the case, the sense of the fibres is not significant, the subgroup of 13 invariants mentioned after equation (80), namely, I_4, I_7, I_8, I_9, I_{14}, I_{15}, I_{20}, I_{24}, I_{27}, I_{28}, I_{29}, I_{30}, and I_{32}, must still enter W in their squares or products only, because these are the only invariants which are odd function of the fibre direction.

In view of equation (105), and as is also detailed in the work by Spencer and Soldatos [5], the constitutive equations (104) can further be expressed in the following form:

$$\sigma_{(ji)} = \frac{\rho}{\rho_0}\sum_{\ell=1}^{33}\frac{\partial W}{\partial I_\ell}\left\{F_{iR}F_{iS}\left(\frac{\partial I_\ell}{\partial C_{RS}} + \frac{\partial I_\ell}{\partial C_{SR}}\right) + \left(G_{iR}F_{jS} + G_{jR}F_{iS}\right)\frac{\partial I_\ell}{\partial \Lambda_{SR}}\right\},$$
$$\bar{m}_{ji} = \frac{2}{3}\varepsilon_{ikm}\frac{\rho}{\rho_0}\sum_{\ell=1,\neq 20}^{33}\frac{\partial W}{\partial I_\ell}\frac{\partial I_\ell}{\partial \Lambda_{PR}}F_{mP}\left(F_{jR}b_k + F_{kR}b_j\right), \left(G_{iR} = b_{i,R}\right). \tag{106a, b}$$

It is noted that, as is made clear in the works by Spencer and Soldatos [5] and Soldatos [11], the couple-stress-generated invariant

$$I_{20} = \mathbf{A\Lambda A} \tag{107}$$

fails to relate its internal energy contribution with the couple-stress constitutive equation (106b).

Association of the constitutive equations (107) with the equations of equilibrium (9) thus provides all the principal information needed for determination of the deformation, as well as of the stress and the deviatoric part of the couple-stress. On the other hand, Soldatos [11] discusses in detail the role of I_{20}, whose failure to influence the couple-stress constitutive equation relates to the internal energy contribution due to the spherical couple-stress.

In this context, the purposes of the present communication are sufficiently served by just observing, or recalling [11], that the implied role of $I_{20} = \mathbf{A\Lambda A}$ is here completely analogous to the role that the invariant $I_4 = \mathrm{tr}\mathbf{\Lambda}$ played earlier in sections 5 and 6. It is accordingly recalled (see also section 5.3 above, as well as section 4.2 of Soldatos [11]) that the form of any relevant admissible internal energy density must necessarily contain a term $W^m(I_{20})$, whose appropriate connection with equation (29) (see also equation (35)) provides the additional PDE required for determination of the spherical couple-stress.

Substantial amount of further relevant information is provided in the work by Soldatos [11] and need not be repeated here. This is because the main distinction between the theoretical formulation presented in this section and its counterpart detailed in the work by Soldatos [11] lies only in the manner that couple-stress hyperelasticity of fibre-reinforced materials with fibres resistant in bending is formulated. While, like its first part [5], Soldatos [11] develops the theory from scratch as an independent entity, the present section has essentially presented the same theory as a special case of the more general couple-stress theoretical framework detailed in section 4.

8. Further remarks and conclusion

A connection route is established between the conventional couple-stress theory [1–4] and its counterpart referring to hyperelastic behaviour of fibrous composites with embedded fibres resistant in bending [5,11]. That route emanates from the constitutive assumption (41), which implies that the internal energy density of a polar elastic material depends not only on the two-point global deformation gradient tensor, **F**, met in non-polar hyperelasticity but also on some additional such tensor, **G**, which accounts for the gradients of local deformation features that give rise to polar material response and behaviour (section 4). Completion of any relevant theoretical formulation requires in advance specification of a pair of vector quantities, both of which (1) emerge in addition to the global deformation/displacement vector, (2) are still related with the implied global elastic deformation in either a determined or a virtual manner,

(3) are principally relevant to the polar part of the observed material response and, in general, (4) are both of the same order of magnitude with the displacement gradients.

The first of these quantities is represented by a pseudovector (spin-type) field, Φ_i, that pairs energetically with the couple-traction field encountered in any internal or external surface of the polar material of interest and, along with the corresponding pair of displacement and traction fields, applies direct control on the energy balance of the system. The second additional kinematic quantity is represented by a vector or pseudovector g_i, which (1) relates to specific constitutional features that give rise to polar material response and, through its gradient, (2) defines the local deformation gradient tensor, **G**.

In this context, the conventional couple-stress formulation [1–4] assumes that both Φ_i and g_i are adequately represented by the axial vector (spin-type pseudovector), Ω_i, of the antisymmetric rotation tensor of the global deformation. This assumption makes unavoidable use of the identities (10) and (15) which, in turn, deprive from the theory ability to account for the action and the energy contribution of the spherical couple-stress and, hence, to determine the same.

This well-known drawback of the spherical couple-stress indeterminacy remains intact in the initial version [5] of the model that refers to polar material behaviour of fibrous composites containing fibres with bending stiffness. This is because, while g_i is now represented by the deformed fibre direction vector b_i, defined in equation (78), the pseudovectors Φ_i and Ω_i are still considered identical. It, however, happens that, in this case, the adopted replacement of g_i with the fibre direction vector b_i becomes the source of some extra internal energy contribution which, although relates to the observed polar material response, does not influence the obtained deviatoric couple-stress constitutive equation.

The emergence of that extra energy contribution is necessarily related to action of the indeterminate spherical couple-stress and, thus, enables resolution of that well-known indeterminacy problem. In the case of the composites model [5,11] where fibres are resistant in bending, that indeterminacy is removed by keeping g_i identical to the fibre direction vector b_i, but considering Φ_i as a virtual macro-rotation pseudovector that differs from its actual macro-rotation counterpart Ω_i (see section 7). The, thus, resulting refined formalism enables conversion of additional information, emerging through appropriate use of the aforementioned extra energy term, into an extra PDE whose solution determines the spherical couple-stress.

In the case of the conventional couple-stress theory [1–4], the couple-stress indeterminacy is removed in an analogous, though slightly different manner. Namely, by refining the theory in a manner that specifies Φ_i and g_i as identical pseudovectors, but assigning to them the status of a virtual rather than actual macro-rotation vector (see sections 3 and 5). The, thus, resulting refined formulation furnishes the couple-stress theory with some extra internal energy contribution which, through a mechanism analogous to that implied in the previous paragraph, provides the extra PDE needed for determination of the spherical couple-stress.

It is re-emphasised in this regard that, in either case, full solution of a relevant well-posed boundary value problem is generally achievable by means of a two-step solution process. This is because the variable coefficients of the implied extra PDE are dependent on the components of the deviatoric couple-stress tensor, $\bar{m}_{ij}$, and the actual spin vector of the global deformation, Ω_i.

Regardless of the version of the employed couple-stress theory, the $\bar{m}_{ij}$- and Ω_i-components are always determinable in a first step devoted to the solution of the governing equations of the initial version of the theory, namely the version that leaves the spherical couple-stress indeterminate. The second solution step of the implied boundary value problem thus requires formulation, and subsequent solution of that extra PDE, thus leading to the determination of the spherical couple-stress.

The, thus, established route of theoretical connection reveals that both the classical couple-stress theory [1–4,12] and its fibrous composites counterpart [5,11] emanate from the same origin and have the same source. However, there also exist substantial differences between those two couple-stress theories, and these deserve further exploration and study.

The most striking such difference stems from the fact that the latter theory [5,11] is an anisotropic elasticity theory that models polar material behaviour emanating, in a specific type of composites, through resistance that strong and stiff embedded fibres may exhibit when subjected to bending, splay, and/or twist deformation modes. In contrast, by focussing to the Cosserat equations [3], the early relevant publications (Mindlin and Tiersten [2] and, especially, Koiter [4]) mainly studied polar material response that is inherent in isotropic elastic materials, where the source of such response is macroscopically unknown or unclear.

In that context, a transversely isotropic material extension of the implied isotropic couple-stress model [4] is presented in section 6 (see also [12]), which essentially initiates interest towards potential, further extension of that kind of inherent polar material behaviour within the region of more advanced anisotropic elasticity. It is worth noting in this regard that the constitutive equations of the latter, transversely isotropic version of conventional couple-stress theory, namely equation (64) with the upper limit of the appearing summation symbol raised to 33, differ substantially from their counterparts (106) obtained for the specific fibrous composite considered in section 7.

Moreover, the noted specific interest on composites containing very strong and stiff fibres enabled the theory developed in the work by Spencer and Soldatos [5] and Soldatos [11] to establish a couple of restricted, simplified theoretical models, which can handle more effectively relevant boundary value problems dominated by either their bending or their splay fibre deformation mode. In either of those special cases, the implied theory [5,11] achieves substantial reduction in the number of invariants employed in its constitutive equations (106) and, thus, attains substantial analytical simplification. It is, though, currently unclear whether a similar kind of theoretical simplification may become possible in cases of boundary value problems referring to inherent type of polar material behaviour.

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Appendix I

Useful transformation formulas

The energy representations (41) and (43) imply that W depends on $\mathbf{F}$ and $\mathbf{G}$ only through the components of the tensors $\mathbf{C}$ and $\mathbf{\Lambda}$, defined in equations (38b) and (44), respectively. Hence, it is initially noted that

$$\frac{\partial W}{\partial F_{iR}} = \frac{\partial W}{\partial C_{MN}}\frac{\partial C_{MN}}{\partial F_{iR}} + \frac{\partial W}{\partial \Lambda_{MN}}\frac{\partial \Lambda_{MN}}{\partial F_{iR}}, \quad \frac{\partial W}{\partial G_{iR}} = \frac{\partial W}{\partial C_{MN}}\frac{\partial C_{MN}}{\partial G_{iR}} + \frac{\partial W}{\partial \Lambda_{MN}}\frac{\partial \Lambda_{MN}}{\partial G_{iR}}, \tag{108}$$

where, by virtue of equations (38b) and (44),

$$\begin{aligned}
\frac{\partial C_{MN}}{\partial F_{iR}} &= \frac{\partial\left(F_{jM}F_{jN}\right)}{\partial F_{iR}} = \delta_{ij}\left(\delta_{RM}F_{jN} + \delta_{RN}F_{jM}\right) = \delta_{RM}F_{iN} + \delta_{RN}F_{iM}, \\
\frac{\partial \Lambda_{MN}}{\partial F_{iR}} &= \frac{\partial\left(F_{jM}G_{jN}\right)}{\partial F_{iR}} = \delta_{ij}\delta_{RM}G_{jN} = \delta_{RM}G_{iN}, \\
\frac{\partial C_{MN}}{\partial G_{iR}} &= \frac{\partial\left(F_{jM}F_{jN}\right)}{\partial G_{iR}} = 0, \quad \frac{\partial \Lambda_{MN}}{\partial G_{iR}} = \frac{\partial\left(F_{jM}G_{jN}\right)}{\partial G_{iR}} = F_{jM}\delta_{ij}\delta_{RN} = \delta_{RN}F_{iM}.
\end{aligned} \tag{109}$$

Hence, with use of equations (108) and (109), one obtains

$$\begin{aligned}
F_{jR}\frac{\partial W}{\partial F_{iR}} &= F_{jR}\left\{\frac{\partial W}{\partial C_{MN}}\frac{\partial C_{MN}}{\partial F_{iR}} + \frac{\partial W}{\partial \Lambda_{MN}}\frac{\partial \Lambda_{MN}}{\partial F_{iR}}\right\} = F_{jR}\left\{\frac{\partial W}{\partial C_{MN}}\left(\delta_{RM}F_{iN} + \delta_{RN}F_{iM}\right) + \frac{\partial W}{\partial \Lambda_{MN}}\delta_{RM}G_{iN}\right\} \\
&= F_{jR}F_{iM}\left(\frac{\partial W}{\partial C_{RM}} + \frac{\partial W}{\partial C_{MR}}\right) + F_{jR}G_{iN}\frac{\partial W}{\partial \Lambda_{RN}}, \\
F_{jR}\frac{\partial W}{\partial G_{iR}} &= F_{jR}\left\{\frac{\partial W}{\partial C_{MN}}\frac{\partial C_{MN}}{\partial G_{iR}} + \frac{\partial W}{\partial \Lambda_{MN}}\frac{\partial \Lambda_{MN}}{\partial G_{iR}}\right\} = F_{jR}F_{iM}\frac{\partial W}{\partial \Lambda_{MR}},
\end{aligned} \tag{110}$$

which are the final formulas that enable the constitutive equations (48a), (52), and (55) to be expressed in terms of $\mathbf{C}$ and $\mathbf{\Lambda}$.

Appendix 2

Arbitrary form of the vector ψ_i stemming from a combination of equations (55) and (56)

In the special case that the couple-stress constitutive equation (55) holds in association with the additional requirement (56), the explicit form of the predominant condition (53b) acquires the relatively simpler form

$$m_{rr}\psi_{k,k} = 3\left[\bar{m}_{11}\left(\omega_{1,1} - \psi_{1,1}\right) + \bar{m}_{22}\left(\omega_{2,2} - \psi_{1,1}\right) + \bar{m}_{33}\left(\omega_{3,3} - \psi_{3,3}\right)\right]. \tag{111}$$

Condition (56) then implies that

$$\begin{aligned} \psi_{1,2} &= \omega_{1,2} \Rightarrow \psi_1 = \omega_1 + \bar{f}_1(x_1, x_3), \\ \psi_{1,3} &= \omega_{1,3} \Rightarrow \psi_1 = \omega_1 + \tilde{f}_1(x_1, x_2), \end{aligned} \tag{112}$$

where $\bar{f}_1$ and $\tilde{f}_1$ are arbitrary functions of their arguments. It follows that

$$\psi_{1,1} = \omega_{1,1} + \bar{f}_{1,1}(x_1, x_3) = \omega_{1,1} + \tilde{f}_{1,1}(x_1, x_2), \tag{113}$$

and, henceforth,

$$\bar{f}_{1,1}(x_1, x_3) = \tilde{f}_{1,1}(x_1, x_2) = f_{1,1}(x_1). \tag{114}$$

It then further follows, from equations (112), that

$$\psi_1 = \omega_1 + f_1(x_1) + \alpha_2 x_2 + \alpha_3 x_3, \tag{115}$$

where $f_1(x_1)$ is an arbitrary function of its single argument, while α_2 and α_3 are arbitrary constants of integration. Evidently, similar expressions hold for ψ_2 and ψ_3. It is, thus, shown that the most general form of the vector ψ_i stemming from the condition (53) is as follows:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \omega_1 + f_1(x_1) \\ \omega_2 + f_2(x_2) \\ \omega_3 + f_3(x_3) \end{pmatrix} + \begin{bmatrix} 0 & \alpha_2 & \alpha_3 \\ \beta_1 & 0 & \beta_3 \\ \gamma_1 & \gamma_2 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{116}$$

where each of $f_2(x_2)$ and $f_3(x_3)$ is also an arbitrary function of its single argument, and β_1, β_3, γ_1 and γ_2 are further arbitrary constants.

The appearing arbitrary functions and constants reveal that (1) ψ_i and ω_i are of the same order of magnitude but, in general, different vectors, and (2) unless some additional condition(s) somehow emerge, ψ_i remains an unidentified, virtual spin vector. Moreover,

$$\psi_{i,i} = \omega_{i,i} + f_1'(x_1) + f_2'(x_2) + f_3'(x_3) = f_1'(x_1) + f_2'(x_2) + f_3'(x_3), \tag{117}$$

where a prime denotes ordinary differentiation, with respect to the indicated co-ordinate variable, and equation (10) is also accounted for.

Hence, non-constant but otherwise arbitrary forms of one or more of the functions $f_1(x_1)$,$f_2(x_2)$ and $f_3(x_3)$ suffice to satisfy the necessary conditions (20) and (21). Moreover, in this special case, the dominant condition (53) or (111) acquires the form

$$m_{rr}\left[f_1'(x_1) + f_2'(x_2) + f_3'(x_3)\right] = 3\left[\bar{m}_{11}\, f_1'(x_1) + \bar{m}_{22}\, f_2'(x_2) + \bar{m}_{33}\, f_3'(x_3)\right]. \tag{118}$$