



Strictification theorems for the homotopy time-slice axiom

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Abstract

It is proven that the homotopy time-slice axiom for many types of algebraic quantum field theories (AQFTs) taking values in chain complexes can be strictified. This includes the cases of Haag–Kastler-type AQFTs on a fixed globally hyperbolic Lorentzian manifold (with or without time-like boundary), locally covariant conformal AQFTs in two spacetime dimensions, locally covariant AQFTs in one spacetime dimension, and the relative Cauchy evolution. The strictification theorems established in this paper prove that, under suitable hypotheses that hold true for the examples listed above, there exists a Quillen equivalence between the model category of AQFTs satisfying the homotopy time-slice axiom and the model category of AQFTs satisfying the usual strict time-slice axiom.

Keywords Algebraic quantum field theory · Gauge theory · Homotopical algebra · Chain complexes · Operads · Localizations

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1 Introduction and summary

The time-slice axiom is one of the central axioms of algebraic quantum field theory (AQFT). It introduces a notion of time evolution on globally hyperbolic Lorentzian manifolds, which is the key ingredient for analyzing the physical behavior of an AQFT, for instance through the relative Cauchy evolution (RCE) and its associated stress-energy tensor [12, 18], or through the local measurement schemes introduced in [19].

At a more technical level, an AQFT is described by an algebra $\mathfrak{A} : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathbf{T}$ over a suitable colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$, called the AQFT operad [9], that is associated to a category of spacetimes $\overline{\mathbf{C}}$. (See Sect. 2.1 for the relevant mathematical background.) The symmetric monoidal target category \mathbf{T} is arbitrary, but the traditional examples of AQFTs are based on the closed symmetric monoidal category $\mathbf{T} = \mathbf{Vec}_{\mathbb{K}}$ of vector spaces over a field \mathbb{K} of characteristic 0. In this context the time-slice axiom is implemented by demanding that \mathfrak{A} sends a certain subset W of the 1-ary operations in $\mathcal{O}_{\overline{\mathbf{C}}}$, the so-called Cauchy morphisms, to isomorphisms in \mathbf{T} .

A modern research stream in AQFT is the exploration of homotopical phenomena associated with quantum gauge theories, which has been initiated in [10] by the development of model categories that describe AQFTs taking values in the closed symmetric monoidal model category $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ of (possibly unbounded) chain complexes of \mathbb{K} -vector spaces. This provides a suitable axiomatic framework for quantum gauge theories on globally hyperbolic Lorentzian manifolds, which captures the explicit examples obtained from the BRST/BV formalism for AQFT [20, 21] or from employing the techniques of derived geometry [4, 7]. A new phenomenon of such homotopical AQFTs is that the time-slice axiom gets relaxed to what we call the *homotopy time-slice axiom*, which demands that a $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFT $\mathfrak{A} : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathbf{Ch}_{\mathbb{K}}$ sends every Cauchy morphism in W to a *quasi-isomorphism* of chain complexes. This homotopical relaxation from the strict time-slice axiom (involving isomorphisms) to the homotopy time-slice axiom (involving quasi-isomorphisms) is not only natural from a homotopical algebra perspective to obtain an axiom that is stable under weak equivalences of AQFTs, but it is also the variant of the time-slice axiom that is fulfilled by the typical examples of quantum gauge theories constructed, for instance, in [4, 7, 20, 21].

While from an abstract point of view the relaxation from the strict to the homotopy time-slice axiom does not cause any serious complications, it unfortunately does have a considerable impact on the concrete applicability of $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs to

physically motivated problems. For instance, formulating the relative Cauchy evolution [12, 18] or setting up local measurement schemes [19] in this richer homotopical context is considerably more involved than in the case of ordinary AQFTs, because the maps $\mathfrak{A}(f) : \mathfrak{A}(M) \rightarrow \mathfrak{A}(N)$ associated with Cauchy morphisms $f : M \rightarrow N$ do in general not admit strict inverses and working instead with quasi-inverses produces a tower of homotopy coherence data that is hard to control. A suitable strategy to circumvent these issues is to establish *strictification theorems* that allow one to replace the homotopy time-slice axiom by the strict one. In fact, if it would be possible to replace the $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFT \mathfrak{A} satisfying the homotopy time-slice axiom by a weakly equivalent AQFT \mathfrak{A}_{st} that satisfies this axiom strictly, one could avoid all the practical complications mentioned above by working simply with the equivalent model \mathfrak{A}_{st} instead of \mathfrak{A} . A first example of such strictification theorems, which is valid for the special case where $\overline{\mathbf{C}}$ is the RCE category and \mathfrak{A} is a linear homotopy AQFT, has been proven recently in [11].

The aim of the present paper is to prove a variety of strictification theorems for the homotopy time-slice axiom of AQFTs, including in particular the following relevant cases:

- (i) $\mathbf{Ch}_{\mathbb{K}}$ -valued Haag–Kastler-type AQFTs on a fixed globally hyperbolic Lorentzian manifold M (with or without time-like boundary);
- (ii) $\mathbf{Ch}_{\mathbb{K}}$ -valued locally covariant *conformal* AQFTs in two spacetime dimensions;
- (iii) $\mathbf{Ch}_{\mathbb{K}}$ -valued locally covariant AQFTs in one spacetime dimension;
- (iv) $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs on the RCE category, generalizing the result in [11].

Unfortunately, we currently do not know if there exists a strictification theorem for the homotopy time-slice axiom of $\mathbf{Ch}_{\mathbb{K}}$ -valued locally covariant AQFTs in spacetime dimension $m \geq 2$. Our strictification theorems are not only more general than the one in [11], but they are also stronger and more powerful in the following sense: Instead of asking the object-wise question whether the homotopy time-slice axiom of an individual $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFT \mathfrak{A} can be strictified, we address the global strictification problem that asks whether the model category $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ of AQFTs that satisfy the homotopy time-slice axiom (see [13] and Theorem 2.21) is Quillen equivalent to the model category $\mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}])$ of AQFTs that satisfy the strict time-slice axiom (see Corollary 2.15). Such global strictification theorems not only provide a solution of the object-wise strictification problem, which as mentioned above is very useful for physical applications such as RCE or measurement schemes, but they are also conceptually very interesting as they show that the homotopy time-slice axiom has no higher homotopical content in these cases. The availability of strictification theorems seems to be a phenomenon that is strongly tied to AQFTs on globally hyperbolic *Lorentzian* manifolds and, in particular, it is linked to the one-dimensional behavior of time evolution. Analogous results are not available for topological QFTs, formulated in the context of locally constant prefactorization algebras as in [15, 16], where the higher homotopical content of m -dimensional isotopy equivalences gives rise to the homotopically non-trivial E_m -operads, see [30, Theorem 5.4.5.9], [1] or [14].

The outline of the remainder of this paper is as follows: In Sect. 2, we collect the relevant preliminaries for this work. Section 2.1 recalls some key aspects of orthogonal categories and their associated AQFT operads from [9]. Section 2.2 develops

systematically a theory of localizations of orthogonal categories and explains how these can be used to implement the strict time-slice axiom, see in particular Corollary 2.15. Section 2.3 recalls the projective model structure for $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs from [10] and also the left Bousfield localized model structure from [13] that captures the homotopy time-slice axiom. The Quillen adjunction established in Proposition 2.24 is the key result that allows us to formulate and prove our strictification theorems for the homotopy time-slice axiom. Our first strictification theorem applies to reflective localizations of orthogonal categories (in the sense of Definition 3.1) and it is proven in Sect. 3, see in particular Theorem 3.6. This strictification theorem covers the examples (i), (ii) and (iii) from the itemization above. In Sect. 4, we focus on the case in which the orthogonal category $\overline{\mathbf{C}}$ carries an empty orthogonality relation $\perp_{\mathbf{C}} = \emptyset$. We then prove in Theorem 4.1 that in this case a sufficient condition for a strictification theorem for the homotopy time-slice axiom of AQFTs is that a simpler strictification problem at the level of $\mathbf{Ch}_{\mathbb{K}}$ -valued functors can be solved. Building on earlier results from [11], we then show that this is the case for the RCE category, which leads to example (iv) from the itemization above. Appendix A proves that the localization of the category \mathbf{Loc}_1 of connected globally hyperbolic Lorentzian 1-manifolds at all Cauchy morphisms is reflective, which we need to deduce example (iii) from Theorem 3.6.

2 Preliminaries

2.1 Orthogonal categories, colored operads and AQFTs

Orthogonal categories [9] are an abstraction of the concept of a category of spacetimes with a notion of causally independent pairs of subregions $f_1 : M_1 \rightarrow N \leftarrow M_2 : f_2$. The relevant definitions are as follows:

Definition 2.1 (a) An *orthogonal category* is a pair $\overline{\mathbf{C}} := (\mathbf{C}, \perp_{\mathbf{C}})$ consisting of a small category \mathbf{C} and a subset $\perp_{\mathbf{C}} \subseteq \text{Mor } \mathbf{C}_{\mathbf{t}} \times_{\mathbf{t}} \text{Mor } \mathbf{C}$ (called orthogonality relation) of the set of pairs of morphisms to a common target, such that the following conditions hold true:

- (i) Symmetry: $(f_1, f_2) \in \perp_{\mathbf{C}}$ implies $(f_2, f_1) \in \perp_{\mathbf{C}}$.
- (ii) Composition stability: $(f_1, f_2) \in \perp_{\mathbf{C}}$ implies $(g \circ f_1 \circ h_1, g \circ f_2 \circ h_2) \in \perp_{\mathbf{C}}$ for all composable morphisms g, h_1 and h_2 .

We often write $f_1 \perp_{\mathbf{C}} f_2$ instead of $(f_1, f_2) \in \perp_{\mathbf{C}}$ to denote orthogonal pairs of morphisms.

- (b) An *orthogonal functor* $F : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{D}}$ is a functor $F : \mathbf{C} \rightarrow \mathbf{D}$ between the underlying categories that preserves orthogonal pairs, i.e. $F(f_1) \perp_{\mathbf{D}} F(f_2)$ for all $f_1 \perp_{\mathbf{C}} f_2$.
- (c) We denote by \mathbf{Cat}^{\perp} the 2-category whose objects are orthogonal categories, 1-morphisms are orthogonal functors and 2-morphisms are natural transformations between orthogonal functors.

We would like to note in passing that the equivalences in the 2-category \mathbf{Cat}^{\perp} can be characterized explicitly.

Lemma 2.2 [6, Lemma 2.7] *An orthogonal functor $F : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{D}}$ is an equivalence in the 2-category \mathbf{Cat}^\perp if and only if the following two conditions hold true:*

- (i) *The underlying functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is fully faithful and essentially surjective.*
- (ii) *The orthogonality relation $\perp_{\mathbf{C}} = F^*(\perp_{\mathbf{D}})$ agrees with the pullback along F (see [9, Lemma 3.19]) of the orthogonality relation $\perp_{\mathbf{D}}$, i.e. $f_1 \perp_{\mathbf{C}} f_2$ if and only if $F(f_1) \perp_{\mathbf{D}} F(f_2)$.*

It was shown in [9] that, associated to each orthogonal category $\overline{\mathbf{C}}$, there is a colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$ that codifies the algebraic structure of AQFTs on $\overline{\mathbf{C}}$. See also [8] for a concise review. Recall that a colored operad (aka multicategory) is a generalization of the concept of a category in which morphisms may have multiple inputs. More precisely, a colored operad \mathcal{P} consists of the following data:

- (i) a class of objects, sometimes also called colors;
- (ii) for each tuple $(\underline{c}, t) = ((c_1, \dots, c_n), t)$ of objects, a set of operations $\mathcal{P}(\underline{c})$ from \underline{c} to t ;
- (iii) composition maps $\gamma : \mathcal{P}(\underline{c}) \times \prod_{i=1}^n \mathcal{P}(\underline{d}_i) \rightarrow \mathcal{P}(\underline{d}_1, \dots, \underline{d}_n)$;
- (iv) unit elements $\mathbb{1} \in \mathcal{P}(t)$;
- (v) permutation actions $\mathcal{P}(\sigma) : \mathcal{P}(\underline{c}) \rightarrow \mathcal{P}(\underline{c}\sigma)$, for each $\sigma \in \Sigma_n$, where $\underline{c}\sigma = (c_{\sigma(1)}, \dots, c_{\sigma(n)})$.

These data have to satisfy the usual associativity, unitality and equivariance axioms, see e.g. [33] for the details. Similarly to categories, we often denote an operation $\phi \in \mathcal{P}(\underline{c})$ by an arrow $\phi : \underline{c} \rightarrow t$. Given also a tuple of operations $\underline{\psi} = (\psi_1, \dots, \psi_n)$, with $\psi_i : \underline{d}_i \rightarrow c_i$, we denote the composition by juxtaposition $\underline{\phi} \underline{\psi} := \gamma(\phi, (\psi_1, \dots, \psi_n)) : \underline{d} \rightarrow t$, where $\underline{d} = (\underline{d}_1, \dots, \underline{d}_n)$. The permutation actions will be denoted by dots $\underline{\phi} \cdot \sigma := \mathcal{P}(\sigma)(\underline{\phi}) : \underline{c}\sigma \rightarrow t$. Colored operads assemble into a 2-category, see e.g. [17, 32] for the details.

Definition 2.3 We denote by \mathbf{Op} the 2-category whose objects are colored operads, 1-morphisms are multifunctors and 2-morphisms are multinatural transformations.

The AQFT operad $\mathcal{O}_{\overline{\mathbf{C}}}$ admits the following explicit description, see [8, 9].

Definition 2.4 Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp_{\mathbf{C}}) \in \mathbf{Cat}^\perp$ be an orthogonal category. The associated AQFT operad $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}$ is the colored operad that is defined by the following data:

- (i) the objects are the objects of \mathbf{C} ;
- (ii) the set of operations from $\underline{M} = (M_1, \dots, M_n)$ to N is the quotient set

$$\mathcal{O}_{\overline{\mathbf{C}}}(\underline{M}) := \left(\Sigma_n \times \prod_{i=1}^n \mathbf{C}(M_i, N) \right) / \sim_{\perp_{\mathbf{C}}}, \tag{2.1}$$

where $\mathbf{C}(M_i, N)$ denotes the set of \mathbf{C} -morphisms from M_i to N and the equivalence relation is defined as follows: $(\sigma, \underline{f}) \sim_{\perp_{\mathbf{C}}} (\sigma', \underline{f}')$ if and only if $\underline{f} = \underline{f}'$ and the right permutation $\sigma\sigma'^{-1} : \underline{f}\sigma^{-1} \rightarrow \underline{f}\sigma'^{-1}$ is generated by transpositions of adjacent orthogonal pairs;

- (iii) the composition of $[\sigma, \underline{f}] : \underline{M} \rightarrow N$ with $[\sigma_i, \underline{g}_i] : \underline{K}_i \rightarrow M_i$, for $i = 1, \dots, n$, is

$$[\sigma, \underline{f}] [\underline{\sigma}, \underline{g}] := [\sigma(\sigma_1, \dots, \sigma_n), \underline{f} \underline{g}] : \underline{K} \longrightarrow N, \tag{2.2a}$$

where $\sigma(\sigma_1, \dots, \sigma_n)$ denotes the composition in the unital associative operad and

$$\underline{f} \underline{g} := (f_1 g_{11}, \dots, f_1 g_{1k_1}, \dots, f_n g_{n1}, \dots, f_n g_{nk_n}) \tag{2.2b}$$

is given by compositions in the category \mathbf{C} ;

- (iv) the unit elements are $\mathbb{1} := [e, \text{id}_N] : N \rightarrow N$, where $e \in \Sigma_1$ is the identity permutation;
- (v) the permutation action of $\sigma' \in \Sigma_n$ on $[\sigma, \underline{f}] : \underline{M} \rightarrow N$ is

$$[\sigma, \underline{f}] \cdot \sigma' := [\sigma \sigma', \underline{f} \sigma'] : \underline{M} \sigma' \longrightarrow N, \tag{2.3}$$

where $\underline{f} \sigma' = (f_{\sigma'(1)}, \dots, f_{\sigma'(n)})$ and $\underline{M} \sigma' = (M_{\sigma'(1)}, \dots, M_{\sigma'(n)})$ denote the permuted tuples and $\sigma \sigma'$ is given by the group operation of the permutation group Σ_n .

Remark 2.5 There exists a useful presentation of the colored operad $\mathcal{O}_{\mathbf{C}}$ in terms of generators and relations, see [9, Section 3.3] and also [8, Section 2.2] for a concise review. Very briefly, there are three types of generators

$$\begin{array}{ccc}
 \begin{array}{c} N \\ | \\ f \\ | \\ M \end{array} &
 \begin{array}{c} N \\ | \\ 1_N \\ \circ \\ \emptyset \end{array} &
 \begin{array}{c} N \\ \swarrow \mu_N \searrow \\ N \quad N \end{array}
 \end{array} \tag{2.4}$$

that, in AQFT terminology, describe the pushforward of observables along morphisms $f : M \rightarrow N$ in \mathbf{C} , the unit observable on N , and the multiplication of observables on N . These generators have to satisfy various relations, which can be classified into functoriality relations, algebra relations, compatibility relations and $\perp_{\mathbf{C}}$ -commutativity relations. The latter may be visualized as

$$\begin{array}{c} N \\ \swarrow \mu_N \searrow \\ f_1 \quad f_2 \\ M_1 \quad M_2 \end{array} = \begin{array}{c} N \\ \swarrow \mu_N \searrow \\ f_2 \quad f_1 \\ M_1 \quad M_2 \end{array}, \tag{2.5}$$

for all $(f_1 : M_1 \rightarrow N) \perp_{\mathbf{C}} (f_2 : M_2 \rightarrow N)$, and their role is to implement the quotient in the set of operations from Definition 2.4. The physical interpretation of the $\perp_{\mathbf{C}}$ -commutativity relations is that they enforce ‘Einstein causality’ for all orthogonal pairs of morphisms.

The assignment $\overline{\mathbf{C}} \mapsto \mathcal{O}_{\overline{\mathbf{C}}}$ of AQFT operads can be upgraded to a 2-functor. Given an orthogonal functor $F : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{D}}$, we define the multifunctor

$$\begin{aligned} \mathcal{O}_F : \mathcal{O}_{\overline{\mathbf{C}}} &\longrightarrow \mathcal{O}_{\overline{\mathbf{D}}}, \\ M &\longmapsto F(M), \\ ([\sigma, \underline{f}] : \underline{M} \rightarrow N) &\longmapsto ([\sigma, F(\underline{f})] : F(\underline{M}) \rightarrow F(N)), \end{aligned} \tag{2.6}$$

where F is defined to act component-wise on tuples, i.e. $F(\underline{M}) = (F(M_1), \dots, F(M_n))$ and also $F(\underline{f}) = (F(f_1), \dots, F(f_n))$. Given further a natural transformation $\chi : F \rightarrow G$ between orthogonal functors $F, G : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{D}}$, we define the multi-natural transformation $\mathcal{O}_\chi : \mathcal{O}_F \rightarrow \mathcal{O}_G$ between the corresponding multifunctors $\mathcal{O}_F, \mathcal{O}_G : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathcal{O}_{\overline{\mathbf{D}}}$ by setting

$$(\mathcal{O}_\chi)_M := [e, \chi_M] : F(M) \longrightarrow G(M), \tag{2.7}$$

for all components $M \in \mathbf{C}$. Summing up, we obtain

Proposition 2.6 *The above defines a 2-functor*

$$\mathcal{O} : \mathbf{Cat}^\perp \longrightarrow \mathbf{Op} \tag{2.8}$$

from the 2-category of orthogonal categories to the 2-category of colored operads.

AQFTs on $\overline{\mathbf{C}}$ are by definition algebras over the AQFT operad $\mathcal{O}_{\overline{\mathbf{C}}}$ with values in a suitable closed symmetric monoidal category \mathbf{T} . For ordinary AQFTs, one usually takes $\mathbf{T} = \mathbf{Vec}_{\mathbb{K}}$ to be the category of vector spaces over a field \mathbb{K} of characteristic 0, while for homotopy AQFTs one takes $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ to be the category of chain complexes. (The relevant model categorical aspects in the latter case will be discussed later in Sect. 2.3.) From our 2-categorical perspective, there is the following slick definition of the category of AQFTs on $\overline{\mathbf{C}}$. Recall that to each symmetric monoidal category \mathbf{T} one can assign a colored operad (which we denote with abuse of notation by the same symbol) that has the same objects and whose sets of operations are given by

$$\mathbf{T}\left(\frac{Y}{\underline{X}}\right) := \mathbf{T}\left(\bigotimes_{i=1}^n X_i, Y\right), \tag{2.9}$$

for all tuples $(\underline{X}, Y) = ((X_1, \dots, X_n), Y)$ of objects in \mathbf{T} . Operadic composition is defined by compositions and tensor products of \mathbf{T} -morphisms, the unit elements correspond to the identity morphisms $\mathbb{1} = \text{id}_Y \in \mathbf{T}\left(\frac{Y}{Y}\right) = \mathbf{T}(Y, Y)$ and the permutation actions are defined via the symmetric braiding.

Definition 2.7 Let $\overline{\mathbf{C}} \in \mathbf{Cat}^\perp$ be an orthogonal category and \mathbf{T} a closed symmetric monoidal category. The *category of \mathbf{T} -valued AQFTs on $\overline{\mathbf{C}}$* is defined as the Hom-category

$$\mathbf{AQFT}(\overline{\mathbf{C}}) := \mathbf{Alg}_{\mathcal{O}_{\overline{\mathbf{C}}}}(\mathbf{T}) := \mathbf{Hom}_{\mathbf{Op}}(\mathcal{O}_{\overline{\mathbf{C}}}, \mathbf{T}) \tag{2.10}$$

in the 2-category **Op** from the AQFT operad $\mathcal{O}_{\bar{\mathbf{C}}}$ to the colored operad associated to **T**. More explicitly, an object in this category is a multifunctor $\mathfrak{A} : \mathcal{O}_{\bar{\mathbf{C}}} \rightarrow \mathbf{T}$ and a morphism between two objects $\mathfrak{A}, \mathfrak{B} : \mathcal{O}_{\bar{\mathbf{C}}} \rightarrow \mathbf{T}$ is a multinatural transformation $\zeta : \mathfrak{A} \rightarrow \mathfrak{B}$.

Observe that the assignment $\bar{\mathbf{C}} \mapsto \mathbf{AQFT}(\bar{\mathbf{C}})$ admits a canonical upgrade to a 2-functor

$$\mathbf{AQFT} : (\mathbf{Cat}^\perp)^{\text{op}} \longrightarrow \mathbf{Cat} \tag{2.11}$$

to the 2-category **Cat** of (not necessarily small) categories, functors and natural transformations. Indeed, for each orthogonal functor $F : \bar{\mathbf{C}} \rightarrow \bar{\mathbf{D}}$, we can define a pullback functor

$$F^* := (-)\mathcal{O}_F : \mathbf{AQFT}(\bar{\mathbf{D}}) \longrightarrow \mathbf{AQFT}(\bar{\mathbf{C}}) \tag{2.12}$$

that acts on objects $(\mathfrak{A} : \mathcal{O}_{\bar{\mathbf{D}}} \rightarrow \mathbf{T}) \mapsto (\mathfrak{A} \mathcal{O}_F : \mathcal{O}_{\bar{\mathbf{C}}} \rightarrow \mathbf{T})$ by pre-composition and on morphisms $\zeta \mapsto \zeta \mathcal{O}_F$ by whiskering in the 2-category **Op**. The action of the 2-functor (2.11) on 2-morphisms is given by whiskering too, i.e. for each natural transformation $\chi : F \rightarrow G$ between orthogonal functors we define $\chi^* := (-)\mathcal{O}_\chi : (-)\mathcal{O}_F \rightarrow (-)\mathcal{O}_G$.

We conclude this subsection by recording the following standard result about operadic left Kan extensions, see e.g. [9, Theorem 2.11] for a spelled out proof.

Proposition 2.8 *Suppose that the closed symmetric monoidal category **T** is cocomplete, i.e. it admits all small colimits. Then, for each orthogonal functor $F : \bar{\mathbf{C}} \rightarrow \bar{\mathbf{D}}$, the pullback functor in (2.12) admits a left adjoint, i.e. we obtain an adjunction*

$$F_! : \mathbf{AQFT}(\bar{\mathbf{C}}) \rightleftarrows \mathbf{AQFT}(\bar{\mathbf{D}}) : F^* . \tag{2.13}$$

2.2 Time-slice axiom and localizations

Note that our Definition 2.7 of AQFTs does *not* explicitly refer to one of the central physical axioms, namely the time-slice axiom. The goal of this subsection is to clarify, in the case where the target **T** is an ordinary category (in contrast to a model category such as $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$), the precise sense in which the time-slice axiom may be implemented by a localization of orthogonal categories (see Corollary 2.15).

Let us start by making precise the concept of localization of orthogonal categories, which has previously appeared in a more *ad hoc* fashion in [9, Section 4.2]. Similarly to ordinary category theory, localizations in \mathbf{Cat}^\perp are characterized by the following universal property.

Definition 2.9 Let $\bar{\mathbf{C}}$ be an orthogonal category and $W \subseteq \text{Mor } \bar{\mathbf{C}}$ a subset. A *localization* of $\bar{\mathbf{C}}$ at W is an orthogonal category $\bar{\mathbf{C}}[W^{-1}]$ together with an orthogonal functor $L : \bar{\mathbf{C}} \rightarrow \bar{\mathbf{C}}[W^{-1}]$ satisfying the following properties:

- (i) For all morphisms $(f : M \rightarrow N) \in W$, the morphism $L(f) : L(M) \rightarrow L(N)$ is an isomorphism in $\bar{\mathbf{C}}[W^{-1}]$.

- (ii) For any orthogonal category $\overline{\mathbf{D}}$ and any orthogonal functor $F : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{D}}$ that sends morphisms in W to isomorphisms in $\overline{\mathbf{D}}$, there exists an orthogonal functor $F_W : \overline{\mathbf{C}}[W^{-1}] \rightarrow \overline{\mathbf{D}}$ and a natural isomorphism $F \cong F_W L$.
- (iii) For all orthogonal functors $G, H : \overline{\mathbf{C}}[W^{-1}] \rightarrow \overline{\mathbf{D}}$, the whiskering map

$$(-)L : \text{Nat}(G, H) \longrightarrow \text{Nat}(GL, HL) \tag{2.14}$$

between the sets of 2-morphisms (i.e. natural transformations) is a bijection.

Remark 2.10 Denoting by $\text{Hom}_{\text{Cat}^\perp}(\overline{\mathbf{C}}, \overline{\mathbf{D}}) \in \text{Cat}$ the Hom-category between two objects $\overline{\mathbf{C}}$ and $\overline{\mathbf{D}}$ in the 2-category Cat^\perp of orthogonal categories, the properties from Definition 2.9 can be rephrased more concisely as follows: For every orthogonal category \mathbf{D} , the functor

$$(-)L : \text{Hom}_{\text{Cat}^\perp}(\overline{\mathbf{C}}[W^{-1}], \overline{\mathbf{D}}) \longrightarrow \text{Hom}_{\text{Cat}^\perp}(\overline{\mathbf{C}}, \overline{\mathbf{D}})^W \tag{2.15a}$$

is an equivalence of categories, where

$$\text{Hom}_{\text{Cat}^\perp}(\overline{\mathbf{C}}, \overline{\mathbf{D}})^W \subseteq \text{Hom}_{\text{Cat}^\perp}(\overline{\mathbf{C}}, \overline{\mathbf{D}}) \tag{2.15b}$$

denotes the full subcategory of orthogonal functors that send W to isomorphisms.

It is easy to prove that the *ad hoc* concept of localization of orthogonal categories from [9, Section 4.2] defines a localization in the sense of Definition 2.9.

Proposition 2.11 *Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp_{\mathbf{C}})$ be an orthogonal category and $W \subseteq \text{Mor } \mathbf{C}$ a subset. Consider the localization $L : \mathbf{C} \rightarrow \mathbf{C}[W^{-1}]$ of the underlying category \mathbf{C} at W and endow $\mathbf{C}[W^{-1}]$ with the pushforward orthogonality relation $\perp_{\mathbf{C}[W^{-1}]} := L_*(\perp_{\mathbf{C}})$ (see [9, Lemma 3.19]), i.e. the minimal orthogonality relation such that $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}[W^{-1}]}$ is an orthogonal functor. Then the resulting orthogonal functor $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}[W^{-1}]}$ is a localization in the sense of Definition 2.9.*

Proof Items (i) and (iii) hold true because the underlying functor $L : \mathbf{C} \rightarrow \mathbf{C}[W^{-1}]$ is a localization of categories. For item (ii), we obtain at the level of the underlying categories a functor $F_W : \mathbf{C}[W^{-1}] \rightarrow \mathbf{D}$ and a natural isomorphism $F \cong F_W L$. It remains to show that F_W is orthogonal with respect to $\perp_{\mathbf{C}[W^{-1}]} = L_*(\perp_{\mathbf{C}})$. By composition stability of $\perp_{\mathbf{D}}$ and the definition of the pushforward orthogonality relation (see [9, Lemma 3.19]), this is the case if and only if $F_W L(f_1) \perp_{\mathbf{D}} F_W L(f_2)$, for all $f_1 \perp_{\mathbf{C}} f_2$. The latter holds true because $F \cong F_W L$ are naturally isomorphic and $\perp_{\mathbf{D}}$ is composition stable. Let us spell out this last step in more detail: Writing $(f_1 : M_1 \rightarrow N) \perp_{\mathbf{C}} (f_2 : M_2 \rightarrow N)$ and $\chi : F \rightarrow F_W L$ for the given natural isomorphism, we obtain that

$$(F_W L(f_1), F_W L(f_2)) = (\chi_N F(f_1) \chi_{M_1}^{-1}, \chi_N F(f_2) \chi_{M_2}^{-1}) \in \perp_{\mathbf{D}} \tag{2.16}$$

because $\perp_{\mathbf{D}}$ is composition stable and $F(f_1) \perp_{\mathbf{D}} F(f_2)$ as a consequence of F being an orthogonal functor. □

We shall now prove that the 2-functor $\mathcal{O} : \mathbf{Cat}^\perp \rightarrow \mathbf{Op}$ from Proposition 2.6 that assigns to an orthogonal category $\overline{\mathbf{C}}$ its AQFT operad $\mathcal{O}_{\overline{\mathbf{C}}}$ maps localizations of orthogonal categories to localizations of operads. The latter are characterized by the following universal property.

Definition 2.12 Let \mathcal{P} be a colored operad and $W \subseteq \text{Mor}_1 \mathcal{P}$ a subset of 1-ary operations in \mathcal{P} . A *localization* of \mathcal{P} at W is a colored operad $\mathcal{P}[W^{-1}]$ together with a multifunctor $L : \mathcal{P} \rightarrow \mathcal{P}[W^{-1}]$ satisfying the following properties:

- (i) For all operations $(\phi : c \rightarrow t) \in W$, the 1-ary operation $L(\phi) : L(c) \rightarrow L(t)$ is an isomorphism in $\mathcal{P}[W^{-1}]$.
- (ii) For any colored operad \mathcal{Q} and any multifunctor $F : \mathcal{P} \rightarrow \mathcal{Q}$ that sends operations in W to isomorphisms in \mathcal{Q} , there exists a multifunctor $F_W : \mathcal{P}[W^{-1}] \rightarrow \mathcal{Q}$ and a multinatural isomorphism $F \cong F_W L$.
- (iii) For all multifunctors $G, H : \mathcal{P}[W^{-1}] \rightarrow \mathcal{Q}$, the whiskering map

$$(-)L : \text{mNat}(G, H) \longrightarrow \text{mNat}(GL, HL) \tag{2.17}$$

between the sets of 2-morphisms (i.e. multinatural transformations) is a bijection.

Remark 2.13 Similarly to Remark 2.10, the properties from Definition 2.12 can be rephrased more concisely as follows: For every colored operad \mathcal{Q} , the functor

$$(-)L : \text{Hom}_{\mathbf{Op}}(\mathcal{P}[W^{-1}], \mathcal{Q}) \longrightarrow \text{Hom}_{\mathbf{Op}}(\mathcal{P}, \mathcal{Q})^W \tag{2.18a}$$

is an equivalence of categories, where

$$\text{Hom}_{\mathbf{Op}}(\mathcal{P}, \mathcal{Q})^W \subseteq \text{Hom}_{\mathbf{Op}}(\mathcal{P}, \mathcal{Q}) \tag{2.18b}$$

denotes the full subcategory of multifunctors that send W to isomorphisms.

Proposition 2.14 *Suppose that $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}[W^{-1}]$ is a localization of orthogonal categories. Applying the 2-functor from Proposition 2.6 defines a multifunctor $\mathcal{O}_L : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathcal{O}_{\overline{\mathbf{C}}[W^{-1}]}$ that exhibits a localization of the colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$ at the set of 1-ary operations $W \subseteq \text{Mor}_1 \mathcal{O}_{\overline{\mathbf{C}}}$.*

Proof Because of 2-functoriality and uniqueness (up to equivalence) of localizations, we can pick without loss of generality a particular model for the underlying localized category $\mathbf{C}[W^{-1}]$. To allow for an effective use of the generators–relations description of the AQFT operads from Remark 2.5, we work with the Gabriel–Zisman model [23] in which $\mathbf{C}[W^{-1}]$ has the same objects as \mathbf{C} and its morphisms are given by equivalence classes of chains of zig-zags of \mathbf{C} -morphisms, where all reverse-pointing morphisms must be in W . In short, the category $\mathbf{C}[W^{-1}]$ is generated by all \mathbf{C} -morphisms $f : M \rightarrow N$ and formal inverses $w^{-1} : M \rightarrow N$ of all W -morphisms $w : N \rightarrow M$ modulo the equivalence relation described in [23]. The localization functor $L : \mathbf{C} \rightarrow \mathbf{C}[W^{-1}]$ then acts on objects and morphisms simply as $M \mapsto M$ and $(f : M \rightarrow N) \mapsto (f : M \rightarrow N)$.

We shall now explicitly verify that $\mathcal{O}_L : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathcal{O}_{\overline{\mathbf{C}}[W^{-1}]}$ satisfies the three properties from Definition 2.12 that characterize a localization of colored operads. Item (i) is obvious from the definition of \mathcal{O}_L in (2.6). For item (ii), consider any multifunctor $F : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathcal{Q}$ that sends operations in W to isomorphisms. We have to provide an extension of F along \mathcal{O}_L to a multifunctor $F_W : \mathcal{O}_{\overline{\mathbf{C}}[W^{-1}]} \rightarrow \mathcal{Q}$. Using the Gabriel–Zisman model for $\mathbf{C}[W^{-1}]$ and our generators–relations description from Remark 2.5, we can construct a strict extension (i.e. the multinatural isomorphism $F \cong F_W \mathcal{O}_L$ is the identity) by mapping the formal inverses of W -morphisms as follows

$$(w^{-1} : M \rightarrow N) \longmapsto (F(w)^{-1} : F(M) \rightarrow F(N)). \tag{2.19}$$

It is easy to check that this is compatible with the relations of $\mathcal{O}_{\overline{\mathbf{C}}[W^{-1}]}$, hence we obtain a multifunctor $F_W : \mathcal{O}_{\overline{\mathbf{C}}[W^{-1}]} \rightarrow \mathcal{Q}$ such that $F = F_W \mathcal{O}_L$.

For item (iii), injectivity is obvious and surjectivity is shown by the following argument: Given any multinatural transformation $\zeta : G \mathcal{O}_L \rightarrow H \mathcal{O}_L$ of multifunctors from $\mathcal{O}_{\overline{\mathbf{C}}}$ to \mathcal{Q} , we have to prove that the underlying components $\{\zeta_M : G(M) \rightarrow H(M)\}_{M \in \mathbf{C}}$ are multinatural on $\mathcal{O}_{\overline{\mathbf{C}}[W^{-1}]}$ too. Since multinaturality can be checked at the level of the generators, this amounts to showing that for each formal inverse $w^{-1} : M \rightarrow N$ the naturality diagram

$$\begin{array}{ccc} G(M) & \xrightarrow{G(w^{-1})} & G(N) \\ \zeta_M \downarrow & & \downarrow \zeta_N \\ H(M) & \xrightarrow{H(w^{-1})} & H(N) \end{array} \tag{2.20}$$

in \mathcal{Q} commutes. Since $G(w^{-1}) = G(w)^{-1}$ and $H(w^{-1}) = H(w)^{-1}$ in \mathcal{Q} , these naturality conditions are already enforced by the arity 1 generators of $\mathcal{O}_{\overline{\mathbf{C}}}$. \square

The relevance of this result for AQFT can be summarized as follows.

Corollary 2.15 *Let $\overline{\mathbf{C}}$ be an orthogonal category and $W \subseteq \text{Mor } \mathbf{C}$ a subset. Denote by*

$$\mathbf{AQFT}(\overline{\mathbf{C}})^W \subseteq \mathbf{AQFT}(\overline{\mathbf{C}}) \tag{2.21}$$

the full subcategory of AQFTs on $\overline{\mathbf{C}}$ that send W to isomorphisms in \mathbf{T} . In AQFT terminology, one may say that such theories satisfy the time-slice axiom for the set of morphisms W . Then the pullback functor

$$L^* := (-) \mathcal{O}_L : \mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}]) \longrightarrow \mathbf{AQFT}(\overline{\mathbf{C}})^W \tag{2.22}$$

associated to the localization $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}[W^{-1}]$ of orthogonal categories defines an equivalence between the category of AQFTs on $\overline{\mathbf{C}}[W^{-1}]$ and the category of AQFTs on $\overline{\mathbf{C}}$ that satisfy the time-slice axiom for W .

Proof By Proposition 2.14, the multifunctor $\mathcal{O}_L : \mathcal{O}_{\overline{\mathcal{C}}} \rightarrow \mathcal{O}_{\overline{\mathcal{C}}[W^{-1}]}$ is a localization of the colored operad $\mathcal{O}_{\overline{\mathcal{C}}}$ at W , hence the statement is a direct consequence of the universal property from Remark 2.13. \square

Remark 2.16 For a cocomplete closed symmetric monoidal category \mathbf{T} , Corollary 2.15 can be rephrased in the language of Proposition 2.8. This will be useful to understand our strategy and constructions in Sects. 3 and 4. The orthogonal localization functor $L : \overline{\mathcal{C}} \rightarrow \overline{\mathcal{C}}[W^{-1}]$ defines by Proposition 2.8 an adjunction

$$L_! : \mathbf{AQFT}(\overline{\mathcal{C}}) \rightleftarrows \mathbf{AQFT}(\overline{\mathcal{C}}[W^{-1}]) : L^*, \tag{2.23}$$

whose right adjoint is the pullback functor $L^* = (-)\mathcal{O}_L$ and whose left adjoint $L_!$ is given by operadic left Kan extension (see e.g. [9, Proposition 2.12] for an explicit colimit formula). Using this adjunction, we can associate to any $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathcal{C}})$ the AQFT $L^*L_!(\mathfrak{A}) \in \mathbf{AQFT}(\overline{\mathcal{C}})^W$ that satisfies the time-slice axiom, whether or not the original \mathfrak{A} satisfies this axiom, together with a comparison morphism $\eta_{\mathfrak{A}} : \mathfrak{A} \rightarrow L^*L_!(\mathfrak{A})$ given by the adjunction unit $\eta : \text{id} \rightarrow L^*L_!$. In general, the comparison morphism will *not* be an isomorphism, hence \mathfrak{A} and $L^*L_!(\mathfrak{A})$ define different AQFTs. The result in Corollary 2.15 implies that the adjunction (2.23) induces an adjoint equivalence

$$L_! : \mathbf{AQFT}(\overline{\mathcal{C}})^W \xrightarrow[\sim]{\simeq} \mathbf{AQFT}(\overline{\mathcal{C}}[W^{-1}]) : L^* \tag{2.24}$$

when we restrict to the full subcategory $\mathbf{AQFT}(\overline{\mathcal{C}})^W \subseteq \mathbf{AQFT}(\overline{\mathcal{C}})$ of AQFTs that satisfy the time-slice axiom. As a consequence, the comparison morphism $\eta_{\mathfrak{A}} : \mathfrak{A} \rightarrow L^*L_!(\mathfrak{A})$ is an isomorphism if and only if the theory $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathcal{C}})^W$ satisfies the time-slice axiom.

2.3 Model category structures

For the rest of this paper, we fix the target category

$$\mathbf{T} := \mathbf{Ch}_{\mathbb{K}} \tag{2.25}$$

to be the closed symmetric monoidal category of (possibly unbounded) chain complexes of vector spaces over a field \mathbb{K} of characteristic 0. By [29, Sections 2.3 and 4.2], $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ carries the structure of a *closed symmetric monoidal model category* in which the weak equivalences are the quasi-isomorphisms and the fibrations are the degree-wise surjective chain maps. As a consequence of Hinich’s results [25, 26], this model structure can be transferred to the category of $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs on an orthogonal category $\overline{\mathcal{C}}$. The following model categories appeared first in [10].

Theorem 2.17 *For each orthogonal category $\overline{\mathcal{C}}$, the associated category*

$$\mathbf{AQFT}(\overline{\mathcal{C}}) := \mathbf{Alg}_{\mathcal{O}_{\overline{\mathcal{C}}}}(\mathbf{Ch}_{\mathbb{K}}) := \mathbf{Hom}_{\mathbf{Op}}(\mathcal{O}_{\overline{\mathcal{C}}}, \mathbf{Ch}_{\mathbb{K}}) \tag{2.26}$$

of $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs carries a model structure in which a morphism $\zeta : \mathfrak{A} \rightarrow \mathfrak{B}$ is a weak equivalence (respectively, fibration) if each component $\zeta_M : \mathfrak{A}(M) \rightarrow \mathfrak{B}(M)$ is a quasi-isomorphism (respectively, degree-wise surjective). We call this the projective model structure on $\mathbf{AQFT}(\overline{\mathbf{C}})$.

Remark 2.18 Note that every object in $\mathbf{AQFT}(\overline{\mathbf{C}})$ is a fibrant object. We will frequently use this fact without further emphasis in our arguments below. This fact is not essential, but it simplifies some arguments.

An immediate but fundamental consequence of such model structures is given by the following result, which encodes numerous universal constructions among AQFTs.

Proposition 2.19 For each orthogonal functor $F : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{D}}$, the adjunction from Proposition 2.8, i.e. $F_{\downarrow} : \mathbf{AQFT}(\overline{\mathbf{C}}) \rightleftarrows \mathbf{AQFT}(\overline{\mathbf{D}}) : F^*$, is a Quillen adjunction for the projective model structures from Theorem 2.17.

Proof The right adjoint $F^* = (-)\mathcal{O}_F$ is a pullback functor, hence it preserves both the fibrations and the weak equivalences as these are defined component-wise. This in particular implies that F^* is a right Quillen functor. \square

The implementation of the time-slice axiom in this homotopical context is more subtle than in the case of an ordinary target category \mathbf{T} that we have discussed in Sect. 2.2. Demanding that a multifunctor $\mathfrak{A} : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathbf{Ch}_{\mathbb{K}}$ sends a subset $W \subseteq \text{Mor } \mathbf{C}$ to isomorphisms is neither realized in concrete examples (see e.g. the linear quantum gauge theories constructed in [4, 7]) nor is it compatible with weak equivalences: Indeed, given a weak equivalence $\zeta : \mathfrak{A} \rightarrow \mathfrak{B}$ in the model category $\mathbf{AQFT}(\overline{\mathbf{C}})$, it is in general *not* true that \mathfrak{A} satisfies this property if and only if \mathfrak{B} does. The homotopically correct generalization of the time-slice axiom, which is stable under weak equivalences of AQFTs, is given by replacing the concept of isomorphisms with that of quasi-isomorphisms.

Definition 2.20 Let $\overline{\mathbf{C}}$ be an orthogonal category and $W \subseteq \text{Mor } \mathbf{C}$ a subset. An object $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{C}})$ is said to satisfy the *homotopy time-slice axiom* for W if, as a multifunctor $\mathfrak{A} : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathbf{Ch}_{\mathbb{K}}$, it sends W to quasi-isomorphisms.

It is important to stress that this definition introduces the homotopy time-slice axiom only at the level of the objects in the model category $\mathbf{AQFT}(\overline{\mathbf{C}})$, but it does not define their morphisms and (higher) homotopies. Since model structures do not in general restrict in a sensible way to full subcategories, more work is required to obtain a model category that presents the homotopy theory of AQFTs satisfying the homotopy time-slice axiom. This issue has been addressed and solved in [13, Section 3.2], where two different but equivalent approaches have been identified. The first approach, which we shall sketch only briefly as it will not play an important role in our paper, uses the concept of *homotopical localization of operads*, generalizing Dwyer-Kan localization of simplicial categories [27], in order to invert (up to homotopy) the 1-ary operations W in the AQFT operad $\mathcal{O}_{\overline{\mathbf{C}}}$. Homotopical localizations can be determined in the model category \mathbf{sOp} of simplicial (i.e. \mathbf{sSet} -enriched) colored operads and they are

defined by the following homotopical generalization of the universal property from Definition 2.12 and Remark 2.13: A **sOp**-morphism $L^\infty : \mathcal{P} \rightarrow \mathcal{P}[W^{-1}]^\infty$ is called a homotopical localization of $\mathcal{P} \in \mathbf{sOp}$ at a subset W of 1-ary operations in \mathcal{P} if L^∞ sends W to equivalences and, for every $\mathcal{Q} \in \mathbf{sOp}$, the pullback map

$$(-) L^\infty : \text{Map}_{\mathbf{sOp}}(\mathcal{P}[W^{-1}]^\infty, \mathcal{Q}) \longrightarrow \text{Map}_{\mathbf{sOp}}(\mathcal{P}, \mathcal{Q})^{\text{ho}W} \tag{2.27}$$

is a weak homotopy equivalence, where $\text{Map}_{\mathbf{sOp}}(\mathcal{P}[W^{-1}]^\infty, \mathcal{Q}) \in \mathbf{sSet}$ is the mapping space in **sOp** and $\text{Map}_{\mathbf{sOp}}(\mathcal{P}, \mathcal{Q})^{\text{ho}W} \subseteq \text{Map}_{\mathbf{sOp}}(\mathcal{P}, \mathcal{Q})$ denotes the simplicial subset of maps that send W to equivalences. Homotopical localizations of operads can be computed by a homotopy pushout that is similar to the one for simplicial categories [27, Section 3] or by a generalization of Hammock localization to trees [3]. In our AQFT context, the universal property (2.27) implies that the homotopy time-slice axiom from Definition 2.20 can be enforced by considering algebras over the homotopically localized AQFT operad $\mathcal{O}_{\overline{\mathbf{C}}}[W^{-1}]^\infty \in \mathbf{sOp}$. Passing over **Ch_ℝ**-enriched colored operads by using the normalized chains functor $N_*(-, \mathbb{K}) : \mathbf{sSet} \rightarrow \mathbf{Ch}_{\mathbb{K}}$, we can define the model category

$$\mathbf{AQFT}(\overline{\mathbf{C}})^{\text{ho}W} := \mathbf{Alg}_{\mathcal{O}_{\overline{\mathbf{C}}}[W^{-1}]^\infty}(\mathbf{Ch}_{\mathbb{K}}) := \text{Hom}_{\mathbf{Op}_{\mathbf{Ch}_{\mathbb{K}}}}(N_*(\mathcal{O}_{\overline{\mathbf{C}}}[W^{-1}]^\infty, \mathbb{K}), \mathbf{Ch}_{\mathbb{K}}) \tag{2.28}$$

of **Ch_ℝ**-enriched multifunctors, endowed with the projective model structure from [25, 26]. This perspective on the homotopy time-slice axiom is conceptually very clear thanks to the universal property of homotopical localizations (2.27), but unfortunately it is difficult to use in practice because it is hard to compute homotopical localizations such as $\mathcal{O}_{\overline{\mathbf{C}}}[W^{-1}]^\infty$.

In order to prove our results in this paper, it will be convenient to use the equivalent approach from [13, Section 3.2] that enforces the homotopy time-slice axiom via a *left Bousfield localization*. (We refer the reader to [2] or to Hirschhorn’s book [28] for an introduction to Bousfield localizations of model categories and the related concepts of local objects and local equivalences.) The basic idea is to work with the same underlying category $\mathbf{AQFT}(\overline{\mathbf{C}})$, but to introduce a new model structure that has more weak equivalences and fewer fibrations than the projective one in Theorem 2.17. When designed correctly, this will lead to a new model category $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ that is Quillen equivalent to (2.28) and in which the homotopy time-slice axiom becomes a fibrancy condition. The construction of such a model structure is slightly technical and the relevant details can be found in [13] and [14, Section 6]. The main result may be summarized as follows.

Theorem 2.21 *Let $\overline{\mathbf{C}}$ be an orthogonal category and $W \subseteq \text{Mor } \mathbf{C}$ a subset. Denote by $\mathbf{AQFT}(\overline{\mathbf{C}})$ the projective AQFT model category from Theorem 2.17.*

- (a) *There exists a subset $\widehat{W} \subseteq \text{Mor } \mathbf{AQFT}(\overline{\mathbf{C}})$ (see Remark 2.22 below for an explicit description) such that an object $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{C}})$ is \widehat{W} -local if and only if it satisfies the homotopy time-slice axiom for $W \subseteq \text{Mor } \mathbf{C}$.*

- (b) The left Bousfield localization $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ of the projective AQFT model category at \widehat{W} from item (a) exists. The fibrant objects in this localized model structure are precisely the AQFTs satisfying the homotopy time-slice axiom for W .
- (c) The homotopical localization multifunctor $L^\infty : \mathcal{O}_{\overline{\mathbf{C}}} \rightarrow \mathcal{O}_{\overline{\mathbf{C}}}[W^{-1}]^\infty$ induces via pullback a Quillen equivalence

$$L^\infty : \mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}}) \xrightleftharpoons[\sim]{} \mathbf{AQFT}(\overline{\mathbf{C}})^{\text{ho}W} : L^{\infty*} \tag{2.29}$$

between $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ and the projective model category $\mathbf{AQFT}(\overline{\mathbf{C}})^{\text{ho}W}$ in (2.28).

Remark 2.22 The set \widehat{W} of AQFT morphisms is determined from the given set W of \mathbf{C} -morphisms by the following construction from [14, Definition 6.5 and Proposition 6.4]: Using the covariant Yoneda embedding

$$y(-) : \mathbf{C}^{\text{op}} \longrightarrow \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}), \quad M \longmapsto y(M) = \mathbf{C}(M, -) \otimes \mathbb{K}, \tag{2.30}$$

we can assign to every \mathbf{C} -morphism $f : M \rightarrow N$ a morphism $y(f) : y(N) \rightarrow y(M)$ of $\mathbf{Ch}_{\mathbb{K}}$ -valued functors on \mathbf{C} . We further can shift the homological degree by any integer $r \in \mathbb{Z}$ and consider $y(f)[r] : y(N)[r] \rightarrow y(M)[r]$. Since $i : \mathbf{C} \hookrightarrow \mathcal{O}_{\overline{\mathbf{C}}}$ embeds as the category of 1-ary operations in the AQFT operad, we obtain in analogy to Propositions 2.8 and 2.19 a Quillen adjunction $i_! : \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}) \rightleftarrows \mathbf{AQFT}(\overline{\mathbf{C}}) : i^*$. The subset $\widehat{W} \subseteq \text{Mor } \mathbf{AQFT}(\overline{\mathbf{C}})$ then consists of the morphisms

$$i_!(y(f)[r]) : i_!(y(N)[r]) \longrightarrow i_!(y(M)[r]), \tag{2.31}$$

where $(f : M \rightarrow N) \in W$ runs over all morphisms in W and $r \in \mathbb{Z}$ runs over all integers.

Remark 2.23 The reader might wonder about the physical interpretation of the left Bousfield localized model category $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ and its additional weak equivalences. In particular, one should *not* be surprised that every AQFT $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{C}})$, whether or not it satisfies the homotopy time-slice axiom, is weakly equivalent in $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ to an AQFT that satisfies the homotopy time-slice axiom, e.g. by taking a local fibrant replacement of \mathfrak{A} . We would like to stress that this behavior is not inconsistent and in fact necessary for the model category $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ to be Quillen equivalent to the model category $\mathbf{AQFT}(\overline{\mathbf{C}})^{\text{ho}W}$ in (2.28) that describes AQFTs satisfying the homotopy time-slice axiom through the concept of homotopical localizations, which is precisely the statement of Theorem 2.21(c). In other words, for the model category $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ to describe the homotopy theory of AQFTs satisfying the homotopy time-slice axiom, it necessarily must be true that every object in $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ is weakly equivalent to an AQFT that satisfies this axiom. To avoid any misconceptions about the additional weak equivalences in the model category $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$, let us recall the following standard result about left Bousfield localizations, see e.g. [28, Theorem 3.2.13]: Given any two fibrant objects $\mathfrak{A}, \mathfrak{B} \in \mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ in the localized model category, a morphism $\zeta : \mathfrak{A} \rightarrow \mathfrak{B}$ is a local weak equivalence if and only

if it is a projective weak equivalence in the sense of Theorem 2.17. Using Theorem 2.21(b) and translating this result to AQFT terminology, this means that both types of weak equivalences (local and projective) coincide whenever \mathfrak{A} and \mathfrak{B} satisfy the homotopy time-slice axiom. In particular, this implies that there are no additional weak equivalences between the objects that the model category $\mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}})$ is designed to describe.

The following result will be crucial for formulating and proving our strictification theorems for the homotopy time-slice axiom.

Proposition 2.24 *Suppose that $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}[W^{-1}]$ is a localization of orthogonal categories. The associated adjunction from Proposition 2.8 defines a Quillen adjunction*

$$L_! : \mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}}) \rightleftarrows \mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}]) : L^* \tag{2.32}$$

between the Bousfield localized model category $\mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}})$ from Theorem 2.21 and the projective model category $\mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}])$ of AQFTs on $\overline{\mathbf{C}}[W^{-1}]$ from Theorem 2.17.

Proof From the equivalent characterizations of Quillen adjunctions, see e.g. [28, Proposition 8.5.3], we find it most convenient to prove that $L_!$ preserves cofibrations and that L^* preserves fibrations. Recalling that the cofibrations in the left Bousfield localization $\mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}})$ agree by [28, Definition 3.3.1] with the cofibrations in the projective model category $\mathbf{AQFT}(\overline{\mathbf{C}})$, the first claim follows directly from Proposition 2.19.

To prove that L^* preserves fibrations, let us first observe that, for every $\mathfrak{B} \in \mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}])$, the AQFT $L^*(\mathfrak{B}) \in \mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}})$ satisfies the strict time-slice axiom and hence also the homotopy time-slice axiom. Recalling Theorem 2.21(b), this means that L^* maps to the fibrant objects in $\mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}})$. Combining this with Proposition 2.19, we obtain that L^* maps fibrations to projective fibrations between fibrant objects in $\mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}})$, which due to [28, Proposition 3.3.16] are also fibrations in the localized model structure. \square

Remark 2.25 Observe that the Quillen adjunction from Proposition 2.24 allows us to compare between the homotopy time-slice axiom from Definition 2.20 and the usual *strict* time-slice axiom that is formulated in terms of isomorphisms rather than quasi-isomorphisms. Indeed, by Theorem 2.21(c), the model category $\mathcal{L}_{\widehat{\mathfrak{W}}}\mathbf{AQFT}(\overline{\mathbf{C}})$ describes AQFTs that satisfy the homotopy time-slice axiom and, by Corollary 2.15, the model category $\mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}])$ describes AQFTs that satisfy the strict time-slice axiom.

3 Strictification theorem for reflective localizations

In this section we consider a special class of orthogonal localizations that are reflective in the sense of Definition 3.1. In this context we shall prove that the homotopy time-slice axiom is equivalent (in a suitable sense) to the strict time-slice axiom. The precise statement is our Theorem 3.6 below.

The following definition is an adaption of the concept of reflective localizations from ordinary category theory.

Definition 3.1 A localization of orthogonal categories $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}[W^{-1}]$ is called *reflective* if the orthogonal functor L admits a right adjoint $\iota : \overline{\mathbf{C}}[W^{-1}] \rightarrow \overline{\mathbf{C}}$ in the 2-category \mathbf{Cat}^\perp whose underlying functor $\iota : \mathbf{C}[W^{-1}] \rightarrow \mathbf{C}$ is fully faithful.

Remark 3.2 From this definition, it follows that the orthogonality relation $\perp_{\mathbf{C}[W^{-1}]}$ agrees with the pullback along ι of $\perp_{\mathbf{C}}$, i.e. $\perp_{\mathbf{C}[W^{-1}]} = \iota^*(\perp_{\mathbf{C}})$. The inclusion $\perp_{\mathbf{C}[W^{-1}]} \subseteq \iota^*(\perp_{\mathbf{C}})$ is a consequence of ι being an orthogonal functor and the inclusion $\perp_{\mathbf{C}[W^{-1}]} \supseteq \iota^*(\perp_{\mathbf{C}})$ is proven by using that ι is fully faithful. Indeed, given any $(g_1, g_2) \in \iota^*(\perp_{\mathbf{C}})$, we have by definition of the pullback orthogonality relation that $(\iota(g_1), \iota(g_2)) \in \perp_{\mathbf{C}}$, hence $(L\iota(g_1), L\iota(g_2)) \in \perp_{\mathbf{C}[W^{-1}]}$ because L is an orthogonal functor. Using that $\perp_{\mathbf{C}[W^{-1}]}$ is composition stable and that the adjunction counit $\epsilon : L\iota \rightarrow \text{id}_{\mathbf{C}[W^{-1}]}$ is a natural isomorphism, it follows that $(g_1, g_2) \in \perp_{\mathbf{C}[W^{-1}]}$.

The following result is useful to detect reflective localizations of orthogonal categories.

Proposition 3.3 *Suppose that $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{E}}$ is an orthogonal functor that admits a fully faithful right adjoint $j : \overline{\mathbf{E}} \rightarrow \overline{\mathbf{C}}$ in the 2-category \mathbf{Cat}^\perp . Then $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{E}}$ is a reflective localization of the orthogonal category $\overline{\mathbf{C}}$ at the set of morphisms $W := L^{-1}(\text{Iso } \overline{\mathbf{E}}) \subseteq \text{Mor } \overline{\mathbf{C}}$ that are sent via L to isomorphisms in $\overline{\mathbf{E}}$.*

Proof It is a basic exercise in category theory to prove that the underlying functor $L : \mathbf{C} \rightarrow \mathbf{E}$ is a localization of the category \mathbf{C} at $W = L^{-1}(\text{Iso } \mathbf{E}) \subseteq \text{Mor } \mathbf{C}$. The hypothesis that $L : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{E}}$ is an orthogonal functor implies that $L_*(\perp_{\mathbf{C}}) \subseteq \perp_{\mathbf{E}}$, so it remains to prove the inclusion $\perp_{\mathbf{E}} \subseteq L_*(\perp_{\mathbf{C}})$. Given any $g_1 \perp_{\mathbf{E}} g_2$, it follows that $j(g_1) \perp_{\mathbf{C}} j(g_2)$ and also $Lj(g_1) \perp_{\mathbf{E}} Lj(g_2)$ because j and L are orthogonal functors. Note that $(Lj(g_1), Lj(g_2)) \in L_*(\perp_{\mathbf{C}}) \subseteq \perp_{\mathbf{E}}$ defines an element of the pushforward orthogonality relation. Using that $L_*(\perp_{\mathbf{C}})$ is composition stable and that the adjunction counit $\epsilon : Lj \rightarrow \text{id}_{\mathbf{E}}$ is a natural isomorphism, it follows that $(g_1, g_2) \in L_*(\perp_{\mathbf{C}})$. \square

Example 3.4 Of particular interest for AQFT is the case where $\overline{\mathbf{C}}$ is some category of globally hyperbolic Lorentzian manifolds, with orthogonality relation determined by causal disjointness, and $W \subseteq \text{Mor } \mathbf{C}$ is the subset of all Cauchy morphisms. See e.g. [12, 18] and also [9, Section 3.5] for the relevant context and terminology. The following examples of reflective localizations have been established in the literature:

- In [5, Proposition 3.3], it is shown that the localization at all Cauchy morphisms of the orthogonal category $\overline{\mathcal{R}_M}$ of causally convex open subsets of a fixed globally hyperbolic Lorentzian manifold M (with or without time-like boundary) is reflective. This family of examples covers Haag–Kastler-type AQFTs.
- In [6, Section 3], it is shown that the localization at all Cauchy morphisms of the orthogonal category $\overline{\mathbf{CLoc}_2}$ of oriented and time-oriented connected globally hyperbolic conformal Lorentzian 2-manifolds is reflective. This example covers locally covariant conformal AQFTs, in the sense of [31], in two spacetime dimensions.

Another example is the localization of the orthogonal category $\overline{\mathbf{Loc}}_1$ of oriented and time-oriented connected globally hyperbolic Lorentzian 1-manifolds at all Cauchy morphisms. As this example has not been recorded in the literature yet, we include a proof in Appendix A. On the other hand, it is presently unclear to us whether or not the localization at all Cauchy morphisms of the orthogonal category $\overline{\mathbf{Loc}}_m$ of oriented and time-oriented connected globally hyperbolic Lorentzian m -manifolds is reflective for dimension $m \geq 2$.

Counterexample 3.5 This counterexample is inspired by topological field theories. Denote by $\mathbf{Disk}(\mathbb{R}^m)$ the category whose objects are open m -disks $U \subseteq \mathbb{R}^m$ and morphisms are subset inclusions. We endow this category with the disjointness orthogonality relation, i.e. $(U_1 \subseteq V) \perp (U_2 \subseteq V)$ if and only if $U_1 \cap U_2 = \emptyset$. Let us take W to be the isotopy equivalences, which are all morphisms in $\mathbf{Disk}(\mathbb{R}^m)$. The localization $L : \mathbf{Disk}(\mathbb{R}^m) \rightarrow \mathbf{Disk}(\mathbb{R}^m)[W^{-1}] = \{*\}$ of the underlying category is a singleton, i.e. a category with only one object $*$ and its identity morphism id_* . The induced orthogonality relation is given by $L_*(\perp) = \{(\text{id}_*, \text{id}_*)\}$, i.e. the identity id_* is orthogonal to itself. At the level of the underlying categories, this localization is reflective with right adjoint $\iota : \{*\} \rightarrow \mathbf{Disk}(\mathbb{R}^m)$ defined by $\iota(*) = \mathbb{R}^m$. However, the functor ι is *not* orthogonal because $\mathbb{R}^m \cap \mathbb{R}^m \neq \emptyset$ are not disjoint. This provides a simple example of an orthogonal localization $L : \overline{\mathbf{Disk}(\mathbb{R}^m)} \rightarrow \overline{\{*\}}$ that is not reflective in the sense of Definition 3.1, even though the underlying localization of categories is reflective.

Given any reflective localization of orthogonal categories $L : \overline{\mathbf{C}} \rightleftarrows \overline{\mathbf{C}}[W^{-1}] : \iota$, we obtain from the 2-functor (2.11) an adjunction

$$\iota^* : \mathbf{AQFT}(\overline{\mathbf{C}}) \rightleftarrows \mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}]) : L^* \tag{3.1}$$

between the associated AQFT categories. From the uniqueness (up to natural isomorphism) of adjoint functors, it then follows that ι^* is a model for the left Quillen functor $L_!$ in Proposition 2.24. This is the key observation to prove the following strictification theorem.

Theorem 3.6 *Let $L : \overline{\mathbf{C}} \rightleftarrows \overline{\mathbf{C}}[W^{-1}] : \iota$ be a reflective localization of orthogonal categories. Then the Quillen adjunction*

$$L_! = \iota^* : \mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}}) \rightleftarrows \mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}]) : L^* \tag{3.2}$$

from Proposition 2.24 is a Quillen equivalence.

Proof Let us fix any cofibrant replacement (Q, q) for the *projective* model structure on $\mathbf{AQFT}(\overline{\mathbf{C}})$, which by [28, Definition 3.3.1] determines a cofibrant replacement for the left Bousfield localized model category $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$. Let us also recall that every object in the projective model category $\mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}])$ is fibrant. We then may take $\mathbb{R}L^* := L^*$ and $\mathbb{L}\iota^* := \iota^* Q$ as models for the derived functors.

To show that the derived unit is a natural weak equivalence, it suffices to consider its components on those objects $\mathfrak{A} \in \mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$ that are both fibrant and cofibrant.

The question then reduces to proving that the associated components $\eta_{\mathfrak{A}} : \mathfrak{A} \rightarrow L^*i^*(\mathfrak{A})$ of the ordinary unit are weak equivalences in $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\overline{\mathbf{C}})$, or equivalently projective weak equivalences because both \mathfrak{A} and $L^*i^*(\mathfrak{A})$ are fibrant objects, see e.g. [28, Theorem 3.2.13]. Recalling that the adjunction (3.1) is obtained through the AQFT 2-functor (2.11), we have that $\eta_{\mathfrak{A}} = \mathfrak{A} \mathcal{O}_{\eta^\perp} : \mathfrak{A} \rightarrow L^*i^*(\mathfrak{A}) = \mathfrak{A} \mathcal{O}_{iL}$ where η^\perp is the unit for the adjunction $L : \overline{\mathbf{C}} \rightleftarrows \overline{\mathbf{C}}[W^{-1}] : \iota$ in \mathbf{Cat}^\perp . By Proposition 3.3, we may assume without loss of generality that $W = L^{-1}(\text{Iso } \mathbf{C}[W^{-1}])$, which combined with the triangle identities for the adjunction $L \dashv \iota$ implies that all components of η^\perp are morphisms in W . It then follows that $\eta_{\mathfrak{A}}$ is a projective weak equivalence because \mathfrak{A} satisfies the homotopy time-slice axiom as it is by hypothesis a fibrant object, see Theorem 2.21(b).

Concerning the derived counit, let us consider its component

$$i^* Q L^*(\mathfrak{B}) \xrightarrow{i^* q_{L^*(\mathfrak{B})}} i^* L^*(\mathfrak{B}) \xrightarrow{\epsilon_{\mathfrak{B}}} \mathfrak{B} \tag{3.3}$$

on an arbitrary object $\mathfrak{B} \in \mathbf{AQFT}(\overline{\mathbf{C}}[W^{-1}])$. By our choice of cofibrant replacement, $q_{L^*(\mathfrak{B})}$ is a projective weak equivalence and hence the first arrow is a weak equivalence since i^* preserves projective weak equivalences. It thus remains to show that the second arrow is a weak equivalence too. For this we recall once more that the adjunction (3.1) is obtained through the AQFT 2-functor (2.11), hence $\epsilon_{\mathfrak{B}} = \mathfrak{B} \mathcal{O}_{\epsilon^\perp} : \mathfrak{B} \mathcal{O}_{Li} = i^* L^*(\mathfrak{B}) \rightarrow \mathfrak{B}$. The claim then follows from the fact that the counit ϵ^\perp for the adjunction $L : \overline{\mathbf{C}} \rightleftarrows \overline{\mathbf{C}}[W^{-1}] : \iota$ in \mathbf{Cat}^\perp is a natural isomorphism since the right adjoint ι is fully faithful. \square

A direct consequence of Theorem 3.6 is that each $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{C}})$ that satisfies the homotopy time-slice axiom for W is *projectively* weakly equivalent to some $\mathfrak{A}_{\text{st}} \in \mathbf{AQFT}(\overline{\mathbf{C}})$ that satisfies the strict time-slice axiom for W . Such an object can be constructed very explicitly in the present case using only *underived* functors, i.e. there is no need to develop models for derived functors. This important and useful fact is summarized in the following

Corollary 3.7 *Let $\mathfrak{A} \in \mathbf{AQFT}(\overline{\mathbf{C}})$ be any AQFT that satisfies the homotopy time-slice axiom for W . Then the corresponding component $\eta_{\mathfrak{A}} : \mathfrak{A} \rightarrow L^*i^*(\mathfrak{A})$ of the underived unit is a projective weak equivalence, i.e. a weak equivalence in the usual projective model structure from Theorem 2.17. In particular, $\mathfrak{A}_{\text{st}} := L^*i^*(\mathfrak{A})$ defines an AQFT that satisfies the strict time-slice axiom and is projectively weakly equivalent to \mathfrak{A} .*

Proof It suffices to repeat the second paragraph of the proof of Theorem 3.6 (note that the cofibrancy assumption plays no role there). \square

Example 3.8 Recalling our list of reflective orthogonal localizations from Example 3.4 and Appendix A, we would like to emphasize that our Theorem 3.6, which strictifies the homotopy time-slice axiom, covers the following important cases: (i) Haag–Kastler-type AQFTs on a fixed globally hyperbolic Lorentzian manifold (with or without time-like boundary), (ii) locally covariant conformal AQFTs in two spacetime dimensions, and (iii) locally covariant AQFTs in one spacetime dimension.

4 Strictification criteria for empty orthogonality relations

In this section we consider the case of an orthogonal category $\overline{\mathbf{C}} = (\mathbf{C}, \emptyset)$ that carries an empty orthogonality relation and establish criteria under which there exists a strictification theorem for the associated homotopy time-slice axiom. This case is motivated by its relevance for describing the *relative Cauchy evolution* (RCE) for $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs, see [11] for earlier results that we will generalize in Example 4.4 below.

The main simplification that arises in the case of an empty orthogonality relation is that the category of $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs from Theorem 2.17 can be presented as the category

$$\mathbf{AQFT}(\mathbf{C}, \emptyset) \cong \mathbf{Fun}(\mathbf{C}, \mathbf{Alg}_{A_5}(\mathbf{Ch}_{\mathbb{K}})) \tag{4.1}$$

of functors from \mathbf{C} to associative and unital differential graded algebras. The projective model structure on $\mathbf{AQFT}(\mathbf{C}, \emptyset)$ from Theorem 2.17 then gets identified with the projective model structure on $\mathbf{Fun}(\mathbf{C}, \mathbf{Alg}_{A_5}(\mathbf{Ch}_{\mathbb{K}}))$.

Using Proposition 2.11, we obtain that the orthogonal localization $L : (\mathbf{C}, \emptyset) \rightarrow (\mathbf{C}[W^{-1}], \emptyset)$ of $\overline{\mathbf{C}} = (\mathbf{C}, \emptyset)$ at any set of morphisms $W \subseteq \text{Mor } \mathbf{C}$ is given by the usual categorical localization L and that the target category carries an empty orthogonality relation $L_*(\emptyset) = \emptyset$ too. The key tool that we will use in order to simplify the strictification problem of AQFTs is the following square of Quillen adjunctions

$$\begin{array}{ccc} \mathcal{L}_{\widetilde{W}}\mathbf{AQFT}(\mathbf{C}, \emptyset) & \begin{array}{c} \xleftarrow{L_{\dagger}} \\ \xrightarrow{L^*} \end{array} & \mathbf{AQFT}(\mathbf{C}[W^{-1}], \emptyset) \\ \begin{array}{c} \uparrow i_{\dagger} \\ \downarrow i^* \end{array} & & \begin{array}{c} \uparrow j_{\dagger} \\ \downarrow j^* \end{array} \\ \mathcal{L}_{\widetilde{W}}\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}) & \begin{array}{c} \xleftarrow{\text{Lan}_L} \\ \xrightarrow{L^*} \end{array} & \mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{Ch}_{\mathbb{K}}) \end{array} \tag{4.2}$$

The top horizontal adjunction is the one controlling the strictification problem for AQFTs, see Proposition 2.24. The bottom horizontal adjunction is its analogue for $\mathbf{Ch}_{\mathbb{K}}$ -valued functors, where Lan_L denotes the left Kan extension along the underlying functor $L : \mathbf{C} \rightarrow \mathbf{C}[W^{-1}]$. The left Bousfield localization $\mathcal{L}_{\widetilde{W}}\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$ is defined in analogy to Remark 2.22 with the set of maps \widetilde{W} given by $y(f)[r]$, for all $(f : M \rightarrow N) \in W$ and all integers $r \in \mathbb{Z}$, i.e. without applying the functor i_{\dagger} . The right adjoints i^* and j^* of the vertical adjunctions are the functors that forget the multiplications and units of an AQFT. Their left adjoints i_{\dagger} and j_{\dagger} are, in the present case of empty orthogonality relations (see (4.1)), simply the functors that take object-wise free algebras. For later use, we would like to record the following commutativity properties

$$i^* L^* = L^* j^*, \quad L_{\dagger} i_{\dagger} \cong j_{\dagger} \text{Lan}_L, \quad i_{\dagger} L^* = L^* j_{\dagger} \tag{4.3}$$

of the functors in the square of adjunctions (4.2). Note that the last property follows from the fact that $i_!$ and $j_!$ form object-wise free algebras, which commutes with pullback functors, hence it is special to the case of empty orthogonality relations.

The following result reduces the strictification problem for the homotopy time-slice axiom of AQFTs with empty orthogonality relation to a simpler, but still non-trivial, strictification problem for $\mathbf{Ch}_{\mathbb{K}}$ -valued functors.

Theorem 4.1 *Let $\overline{\mathbf{C}} = (\mathbf{C}, \emptyset)$ be an orthogonal category with empty orthogonality relation and $W \subseteq \text{Mor } \mathbf{C}$ a subset. Suppose that the Quillen adjunction*

$$\text{Lan}_L : \mathcal{L}_{\widehat{W}}\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}) \rightleftarrows \mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{Ch}_{\mathbb{K}}) : L^* \tag{4.4}$$

for $\mathbf{Ch}_{\mathbb{K}}$ -valued functors is a Quillen equivalence. Then the Quillen adjunction

$$L_! : \mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\mathbf{C}, \emptyset) \rightleftarrows \mathbf{AQFT}(\mathbf{C}[W^{-1}], \emptyset) : L^* \tag{4.5}$$

from Proposition 2.24 is a Quillen equivalence too.

Proof To describe the derived functors, we pick a cofibrant replacement (Q, q) for $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\mathbf{C}, \emptyset)$ and a cofibrant replacement (Z, z) for $\mathcal{L}_{\widehat{W}}\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$, such that q and z are natural projective weak equivalences. We have to prove the following two statements:

1. For all locally fibrant objects $\mathfrak{A} \in \mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\mathbf{C}, \emptyset)$, i.e. \mathfrak{A} satisfies the homotopy time-slice axiom, the derived unit $\eta_{Q(\mathfrak{A})} : Q(\mathfrak{A}) \rightarrow L^* L_! Q(\mathfrak{A})$ is a local weak equivalence, or equivalently a projective weak equivalence by [28, Theorem 3.2.13] because $Q(\mathfrak{A})$ and $L^* L_! Q(\mathfrak{A})$ are locally fibrant objects.
2. For all objects $\mathfrak{B} \in \mathbf{AQFT}(\mathbf{C}[W^{-1}], \emptyset)$, the derived counit

$$L_! Q L^*(\mathfrak{B}) \xrightarrow{L_! q_{L^*(\mathfrak{B})}} L_! L^*(\mathfrak{B}) \xrightarrow{\epsilon_{\mathfrak{B}}} \mathfrak{B} \tag{4.6}$$

is a projective weak equivalence.

Our proof strategy is to use suitable cotriple resolutions [22, Chapter 13.3] to reduce this problem to objects of the form $\mathfrak{A} = i_!(\mathfrak{F})$ and $\mathfrak{B} = j_!(\mathfrak{G})$, where $\mathfrak{F} \in \mathcal{L}_{\widehat{W}}\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$ is any locally fibrant object and $\mathfrak{G} \in \mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{Ch}_{\mathbb{K}})$ is any object. Leveraging the square of adjunctions (4.2), and in particular the commutativity properties (4.3), then allows us to deduce, from the hypothesis that (4.4) is a Quillen equivalence, that these components of the derived unit/counit are projective weak equivalences.

Our cotriple resolutions are determined by the vertical adjunctions in (4.2). Let us consider first the left vertical adjunction. The endofunctor $T := i_! i^*$ on $\mathcal{L}_{\widehat{W}}\mathbf{AQFT}(\mathbf{C}, \emptyset)$ defines a comonad with coproduct $i_! \eta^i i^* : T = i_! i^* \rightarrow i_! i^* i_! i^* = T^2$ given by the adjunction unit and counit $\epsilon^i : T = i_! i^* \rightarrow \text{id}$ given by the adjunction

count. This allows us to define a simplicial resolution

$$\text{Res}(\mathfrak{A}) := \left(T(\mathfrak{A}) \begin{matrix} \xleftarrow{\sim} \\ \xrightarrow{\sim} \end{matrix} T^2(\mathfrak{A}) \begin{matrix} \xleftarrow{\sim} \\ \xrightarrow{\sim} \\ \xleftarrow{\sim} \\ \xrightarrow{\sim} \end{matrix} \dots \right) \in \mathcal{L}_{\widehat{W}} \mathbf{AQFT}(\mathbf{C}, \emptyset)^{\Delta^{\text{op}}} \tag{4.7}$$

and an augmentation map $\text{Res}(\mathfrak{A}) \rightarrow \mathfrak{A}$. Since both $i_!$ and i^* act object-wise on the underlying category \mathbf{C} , so does Res . In fact, recalling (4.1), the resolution Res is simply an object-wise free resolution of dg-algebra valued functors. Using the normalized totalization functor $\text{Tot}^{\oplus} : \mathbf{Alg}_{\text{As}}(\mathbf{Ch}_{\mathbb{K}})^{\Delta^{\text{op}}} \rightarrow \mathbf{Alg}_{\text{As}}(\mathbf{Ch}_{\mathbb{K}})$, which is a homotopy colimit functor for simplicial diagrams of dg-algebras [24], the result in [22, Lemma 13.3.3] implies that

$$\text{Tot}^{\oplus} \text{Res}(\mathfrak{A}) \xrightarrow{\sim} \mathfrak{A} \tag{4.8}$$

is a projective weak equivalence. The same holds true for the right vertical adjunction in (4.2) by using instead of T the comonad $T' := j_! j^*$ on $\mathbf{AQFT}(\mathbf{C}[W^{-1}], \emptyset)$.

An important feature of these resolutions is that Tot^{\oplus} commutes (up to weak equivalence) with both the derived right adjoint L^* and the derived left adjoint $L_! Q$ of the top horizontal adjunction in (4.2). The former is a consequence of the fact that L^* is a pullback functor and Tot^{\oplus} acts by post-composition on functors, while the latter follows from the fact that derived left adjoints $L_! Q$ commute with homotopy colimits. Using also the explicit form of the simplicial resolution (4.7), this implies that the question of whether the derived unit $\eta_{Q(\mathfrak{A})} : Q(\mathfrak{A}) \rightarrow L^* L_! Q(\mathfrak{A})$ is a projective weak equivalence for all locally fibrant $\mathfrak{A} \in \mathcal{L}_{\widehat{W}} \mathbf{AQFT}(\mathbf{C}, \emptyset)$ can be reduced to objects of the form $\mathfrak{A} = T(\mathfrak{A}') = i_! i^*(\mathfrak{A}')$, for all locally fibrant $\mathfrak{A}' \in \mathcal{L}_{\widehat{W}} \mathbf{AQFT}(\mathbf{C}, \emptyset)$. Using that $i_!$ and i^* both preserve and detect local fibrancy (i.e. the homotopy time-slice axiom), one can equivalently consider objects of the form $\mathfrak{A} = i_!(\mathfrak{F})$, for all locally fibrant $\mathfrak{F} \in \mathcal{L}_{\widehat{W}} \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$. Applying the same arguments to the derived counit, one obtains that it is sufficient to consider its components on objects of the form $\mathfrak{B} = j_!(\mathfrak{G})$, for all $\mathfrak{G} \in \mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{Ch}_{\mathbb{K}})$.

Using now the commutativity properties (4.3), one obtains the commutative diagram

$$\begin{array}{ccccc} Q i_!(\mathfrak{F}) & \xleftarrow[\sim]{Q i_! z_{\mathfrak{F}}} & Q i_! Z(\mathfrak{F}) & \xrightarrow[\sim]{q_{i_! Z(\mathfrak{F})}} & i_! Z(\mathfrak{F}) & \xrightarrow{i_! \eta_{Z(\mathfrak{F})}^{\text{Fun}}} & \\ \eta_{Q i_!(\mathfrak{F})} \downarrow & & \eta_{Q i_! Z(\mathfrak{F})} \downarrow & & \eta_{i_! Z(\mathfrak{F})} \downarrow & & \\ L^* L_! Q i_!(\mathfrak{F}) & \xleftarrow[\sim]{L^* L_! Q i_! z_{\mathfrak{F}}} & L^* L_! Q i_! Z(\mathfrak{F}) & \xrightarrow[\sim]{L^* L_! q_{i_! Z(\mathfrak{F})}} & L^* L_! i_! Z(\mathfrak{F}) & \xrightarrow{\cong} & i_! L^* \text{Lan}_L Z(\mathfrak{F}) \end{array} \tag{4.9}$$

that relates the derived units of the top and the bottom horizontal adjunction in (4.2). Recall that (Z, z) denotes a projective cofibrant replacement for $\mathcal{L}_{\widehat{W}} \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$ and that projective weak equivalences are indicated by \sim . Since by hypothesis the derived unit $\eta_{Z(\mathfrak{F})}^{\text{Fun}}$ associated to $\text{Lan}_L \dashv L^*$ is a projective weak equivalence and since $i_!$ preserves projective weak equivalences, it follows that $\eta_{Q i_!(\mathfrak{F})}$ is a projective

weak equivalence too. Using again (4.3), one obtains the commutative diagram

$$\begin{array}{ccccc}
 L_! Q L^* j_!(\mathfrak{G}) & \xleftarrow{\cong} & L_! Q i_! L^*(\mathfrak{G}) & \xleftarrow{\sim} & L_! Q i_! Z L^*(\mathfrak{G}) & (4.10) \\
 \downarrow L_! q_{L^* j_!(\mathfrak{G})} & & \downarrow L_! q_{i_! L^*(\mathfrak{G})} & & \downarrow \sim L_! q_{i_! Z L^*(\mathfrak{G})} & \\
 L_! L^* j_!(\mathfrak{G}) & \xleftarrow{\cong} & L_! i_! L^*(\mathfrak{G}) & \xleftarrow{\sim} & L_! i_! Z L^*(\mathfrak{G}) & \\
 \downarrow \epsilon_{j_!(\mathfrak{G})} & & \downarrow \cong & & \downarrow \cong & \\
 j_!(\mathfrak{G}) & \xleftarrow{j_! \epsilon_{\mathfrak{G}}^{\text{Fun}}} & j_! \text{Lan}_L L^*(\mathfrak{G}) & \xleftarrow{j_! \text{Lan}_L z_{L^*(\mathfrak{G})}} & j_! \text{Lan}_L Z L^*(\mathfrak{G}) &
 \end{array}$$

that relates the derived counits of the top and the bottom horizontal adjunctions in (4.2). The bottom horizontal composition is the application of the functor $j_!$ to the derived counit associated to $\text{Lan}_L \dashv L^*$. Since by hypothesis the latter is a projective weak equivalence and since $j_!$ preserves projective weak equivalences, it follows that the left vertical composition is a projective weak equivalence too. This completes the proof. \square

Remark 4.2 In analogy to Corollary 3.7, a direct consequence of Theorem 4.1 is that each $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{C}, \emptyset)$ that satisfies the homotopy time-slice axiom for W is projectively weakly equivalent to some $\mathfrak{A}_{\text{st}} \in \mathbf{AQFT}(\mathbf{C}, \emptyset)$ that satisfies the strict time-slice axiom for W . However, the construction of such an object is more complicated in the present case because it requires derived functors: Fixing as in the proof of Theorem 4.1 any cofibrant replacement (Q, q) for the projective model structure on $\mathbf{AQFT}(\mathbf{C}, \emptyset)$ and recalling that every object in $\mathbf{AQFT}(\mathbf{C}[W^{-1}], \emptyset)$ is fibrant, we may take $\mathbb{R}L^* := L^*$ and $\mathbb{L}L_! := L_! Q$ as models for the derived functors. The component at $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{C}, \emptyset)$ of the derived unit of the Quillen adjunction $L_! \dashv L^*$ is then given by the zig-zag

$$\mathfrak{A} \xleftarrow{q_{\mathfrak{A}}} Q(\mathfrak{A}) \xrightarrow{\eta_{Q(\mathfrak{A})}} L^* L_! Q(\mathfrak{A}) . \tag{4.11}$$

The map $q_{\mathfrak{A}}$ is by construction a projective weak equivalence and, as explained in the proof of Theorem 4.1, so is the map $\eta_{Q(\mathfrak{A})}$. This implies that setting $\mathfrak{A}_{\text{st}} := L^* \mathbb{L}L_!(\mathfrak{A}) = L^* L_! Q(\mathfrak{A})$ defines an AQFT that satisfies the strict time-slice axiom and is equivalent to \mathfrak{A} via the zig-zag of projective weak equivalences (4.11). To obtain a computable model for \mathfrak{A}_{st} , one can use [22, Theorem 17.2.7] to describe, up to further projective weak equivalences,

$$\mathfrak{A}_{\text{st}} = L^* \mathbb{L}L_!(\mathfrak{A}) \simeq L^* \text{Tot}^{\oplus} B_{\Delta}(\mathcal{O}_{(\mathbf{C}[W^{-1}], \emptyset)}, \mathcal{O}_{(\mathbf{C}, \emptyset)}, \mathfrak{A}) \tag{4.12}$$

in terms of an operadic cotriple resolution, which can be worked out fairly explicitly using the definitions in [22, Chapter 13.3]. (Note that the normalized totalization functor Tot^{\oplus} is denoted by N_* in this book.)

Remark 4.3 A sufficient condition for the hypotheses of Theorem 4.1 to hold true is that the ordinary localization functor $L : \mathbf{C} \rightarrow \mathbf{C}[W^{-1}]$ exhibits $\mathbf{C}[W^{-1}]$ as an ∞ -categorical localization of \mathbf{C} at W . This is a direct consequence of Hinich’s rectification results [26, Theorem 4.1.1] for ∞ -functors. The question of whether the ordinary functor L is an ∞ -categorical localization can be addressed by using the explicit criteria established in [27, Key Lemma 1.3.6]. As a side-remark, we would like to note that reflective localizations of categories are ∞ -localizations, hence our strictification theorem for locally covariant AQFTs in one spacetime dimension (see Example 3.8) can be deduced alternatively from Theorem 4.1 and Appendix A.

Example 4.4 Given any globally hyperbolic Lorentzian manifold M with a sufficiently small and compactly supported metric perturbation h , one can form the RCE category

$$\mathbf{C} := \left(\begin{array}{ccc} & M_+ & \\ i_+ \swarrow & & \searrow j_+ \\ M & & M_h \\ i_- \swarrow & & \searrow j_- \\ & M_- & \end{array} \right) \tag{4.13}$$

which controls the relative Cauchy evolution indexed by the pair (M, h) . Here M_h denotes the perturbed spacetime and $M_{\pm} := M \setminus J^{\mp}(\text{supp}(h))$ is the subspacetime with the causal past/future of the support of h removed. The arrows in (4.13) are inclusions and, by construction, they are all Cauchy morphisms, i.e. $W = \text{Mor } \mathbf{C}$. It is shown in [11, Lemma 2.2] that the functor

$$\begin{aligned} L : \mathbf{C} &\longrightarrow \mathbf{B}\mathbb{Z}, \\ M, M_-, M_h, M_+ &\longmapsto *, \\ i_- &\longmapsto 1, \\ j_-, j_+, i_+ &\longmapsto 0, \end{aligned} \tag{4.14}$$

defines a localization of \mathbf{C} at W . Furthermore, the result in [11, Theorem 4.2] implies that (4.4) is a Quillen equivalence for the present example. (As a side-remark, we would like to note that similar ‘universal covering’ techniques for the RCE category as those initiated in [11, Equation (2.13)] can be used to check directly Hinich’s criteria [27, Key Lemma 1.3.6], which by Remark 4.3 leads to an independent proof of this theorem.) This means that the hypotheses of Theorem 4.1 hold true, hence we obtain a Quillen equivalence

$$L_! : \mathcal{L}_{\widehat{W}}\text{AQFT}(\mathbf{C}, \emptyset) \xrightleftharpoons{\sim} \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Alg}_{A_S}(\mathbf{Ch}_{\mathbb{K}})) : L^* . \tag{4.15}$$

This provides a generalization of the strictification theorem in [11, Theorem 5.4] from the case of linear homotopy AQFTs to all $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFTs on \mathbf{C} that satisfy the homotopy time-slice axiom. In particular, the strict \mathbb{Z} -action carried by the object $\mathbb{L}L_!(\mathfrak{A}) \in \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Alg}_{A_S}(\mathbf{Ch}_{\mathbb{K}}))$ provides a strict realization of the RCE for a $\mathbf{Ch}_{\mathbb{K}}$ -valued AQFT $\mathfrak{A} \in \text{AQFT}(\mathbf{C}, \emptyset)$ that satisfies the homotopy time-slice axiom.

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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A Localization of \mathbf{Loc}_1 at Cauchy morphisms

In this appendix we present a very explicit model for the localization at all Cauchy morphisms of the orthogonal category $\overline{\mathbf{Loc}}_1$ of connected¹ 1-dimensional spacetimes. The orthogonality relation in this case is empty $\perp_{\mathbf{Loc}_1} = \emptyset$, and so will be the orthogonality relation on the localized category. Up to equivalence of (orthogonal) categories, we can present \mathbf{Loc}_1 as the category whose objects are pairs (M, e) consisting of a 1-manifold $M \cong \mathbb{R}$ and a non-degenerate 1-form $e \in \Omega^1(M)$ (the vielbein, encoding the metric and time-orientation), and whose morphisms $f : (M, e) \rightarrow (M', e')$ are all embeddings $f : M \rightarrow M'$ such that $f^*(e') = e$. Note that each object $(M, e) \in \mathbf{Loc}_1$ embeds via a \mathbf{Loc}_1 -morphism into (\mathbb{R}, dt) , where by t we denote a (time) coordinate on \mathbb{R} . As a consequence, \mathbf{Loc}_1 is equivalent to its full subcategory $\mathbf{Loc}_1^{\text{skl}} \subseteq \mathbf{Loc}_1$ consisting of the objects $((a, b), dt) \in \mathbf{Loc}_1$ determined by open intervals $(a, b) \subseteq \mathbb{R}$, with $-\infty \leq a < b \leq +\infty$. (The superscript skl refers to the concept of ‘skeletal models’ as introduced in [6].) The morphisms are given by translations $f_\xi : ((a, b), dt) \rightarrow ((a', b'), dt)$, $t \mapsto t + \xi$, with $\xi \in \mathbb{R}$ satisfying the condition $a' - a \leq \xi \leq b' - b$ such that the interval (a, b) is mapped into the interval (a', b') .

¹ Restricting to connected manifolds simplifies the presentation of this appendix. Our proof below generalizes in a fairly obvious way to non-connected manifolds by treating each connected component separately.

Let us denote by $\mathbf{B}\mathbb{R}^\delta$ the category consisting of a single object, say $*$, with Hom-set the Abelian group $\mathbb{R}^\delta := \mathbb{R}$ (with no topology attached). The functor

$$\begin{aligned} j : \mathbf{B}\mathbb{R}^\delta &\longrightarrow \mathbf{Loc}_1^{\text{skl}}, \\ * &\longmapsto (\mathbb{R}, dt), \\ \xi \in \mathbb{R}^\delta &\longmapsto (f_\xi : (\mathbb{R}, dt) \rightarrow (\mathbb{R}, dt)) \end{aligned} \quad (\text{A.1})$$

is fully faithful and it admits a left adjoint given by

$$\begin{aligned} L : \mathbf{Loc}_1^{\text{skl}} &\longrightarrow \mathbf{B}\mathbb{R}^\delta, \\ ((a, b), dt) &\longmapsto *, \\ (f_\xi : ((a, b), dt) \rightarrow ((a', b'), dt)) &\longmapsto \xi \in \mathbb{R}^\delta. \end{aligned} \quad (\text{A.2})$$

For completeness, let us mention that the adjunction unit $\eta : \text{id} \rightarrow jL$ is given by the components $\eta_{(a,b)} := f_0 : ((a, b), dt) \rightarrow (\mathbb{R}, dt)$ and that the counit $\epsilon : Lj \rightarrow \text{id}$ is the identity natural isomorphism of $Lj = \text{id}$. Equipping all these categories with the empty orthogonality relation, we obtain from Proposition 3.3 that $L : \mathbf{Loc}_1^{\text{skl}} \rightarrow \overline{\mathbf{B}\mathbb{R}^\delta}$ is a reflective localization at all morphisms. (Note that every morphism in $\mathbf{Loc}_1^{\text{skl}}$ is Cauchy.) Choosing a quasi-inverse of the equivalence $\mathbf{Loc}_1^{\text{skl}} \xrightarrow{\sim} \overline{\mathbf{Loc}_1}$ of orthogonal categories, we obtain a localization $\overline{\mathbf{Loc}_1} \xrightarrow{\sim} \overline{\mathbf{Loc}_1^{\text{skl}}} \xrightarrow{L} \overline{\mathbf{B}\mathbb{R}^\delta}$ of $\overline{\mathbf{Loc}_1}$ at all (Cauchy) morphisms. Since equivalences in 2-categories can be upgraded to adjoint equivalences, the latter localization is reflective too.

References

1. Ayala, D., Francis, J.: Factorization homology of topological manifolds. *J. Topol.* **8**(4), 1045–1084 (2015). [arXiv:1206.5522](https://arxiv.org/abs/1206.5522) [math.AT]
2. Barwick, C.: On left and right model categories and left and right Bousfield localizations. *Homol. Homotopy Appl.* **12**(2), 245–320 (2010)
3. Basterra, M., Bobkova, I., Ponto, K., Tillmann, U., Yeakel, S.: Inverting operations in operads. *Topol. Appl.* **235**, 130–145 (2018). [arXiv:1611.00715](https://arxiv.org/abs/1611.00715) [math.CT]
4. Benini, M., Bruinsma, S., Schenkel, A.: Linear Yang–Mills theory as a homotopy AQFT. *Commun. Math. Phys.* **378**(1), 185–218 (2020). [arXiv:1906.00999](https://arxiv.org/abs/1906.00999) [math-ph]
5. Benini, M., Dappiaggi, C., Schenkel, A.: Algebraic quantum field theory on spacetimes with timelike boundary. *Annales Henri Poincaré* **19**(8), 2401–2433 (2018). [arXiv:1712.06686](https://arxiv.org/abs/1712.06686) [math-ph]
6. Benini, M., Giorgetti, L., Schenkel, A.: A skeletal model for 2d conformal AQFTs. *Commun. Math. Phys.* **395**(1), 269–298 (2022). [arXiv:2111.01837](https://arxiv.org/abs/2111.01837) [math-ph]
7. Benini, M., Musante, G., Schenkel, A.: Quantization of Lorentzian free BV theories: factorization algebra vs algebraic quantum field theory. [arXiv:2212.02546](https://arxiv.org/abs/2212.02546) [math-ph]
8. Benini, M., Schenkel, A.: Higher structures in algebraic quantum field theory. *Fortsch. Phys.* **67**(8–9), 1910015 (2019). [arXiv:1903.02878](https://arxiv.org/abs/1903.02878) [hep-th]
9. Benini, M., Schenkel, A., Woike, L.: Operads for algebraic quantum field theory. *Commun. Contemp. Math.* **23**(02), 2050007 (2021). [arXiv:1709.08657](https://arxiv.org/abs/1709.08657) [math-ph]
10. Benini, M., Schenkel, A., Woike, L.: Homotopy theory of algebraic quantum field theories. *Lett. Math. Phys.* **109**(7), 1487–1532 (2019). [arXiv:1805.08795](https://arxiv.org/abs/1805.08795) [math-ph]
11. Bruinsma, S., Fewster, C.J., Schenkel, A.: Relative Cauchy evolution for linear homotopy AQFTs. *Commun. Math. Phys.* **392**(2), 621–657 (2022). [arXiv:2108.10592](https://arxiv.org/abs/2108.10592) [math-ph]

12. Brunetti, R., Fredenhagen, K., Verch, R.: The generally covariant locality principle: a new paradigm for local quantum field theory. *Commun. Math. Phys.* **237**, 31–68 (2003). [arXiv:math-ph/0112041](#)
13. Carmona, V.: New model structures for algebraic quantum field theory. [arXiv:2107.14176](#) [math-ph]
14. Carmona, V., Flores, R., Muro, F.: A model structure for locally constant factorization algebras. [arXiv:2107.14174](#) [math.AT]
15. Costello, K., Gwilliam, O.: *Factorization Algebras in Quantum Field Theory*. New Mathematical Monographs, vol. 1. Cambridge University Press, Cambridge (2017)
16. Costello, K., Gwilliam, O.: *Factorization Algebras in Quantum Field Theory*. New Mathematical Monographs, vol. 2. Cambridge University Press, Cambridge (2021)
17. Elmendorf, A.D., Mandell, M.A.: Permutative categories, multicategories, and algebraic K -theory. *Algebr. Geom. Topol.* **9**, 2391–2441 (2009). [arXiv:0710.0082](#) [math.KT]
18. Fewster, C.J., Verch, R.: Algebraic quantum field theory in curved spacetimes. In: Brunetti, R., Dappiaggi, C., Fredenhagen, K., Yngvason, J. (eds.) *Advances in Algebraic Quantum Field Theory*. Springer, Heidelberg (2015). [arXiv:1504.00586](#) [math-ph]
19. Fewster, C.J., Verch, R.: Quantum fields and local measurements. *Commun. Math. Phys.* **378**(2), 851–889 (2020). [arXiv:1810.06512](#) [math-ph]
20. Fredenhagen, K., Rejzner, K.: Batalin–Vilkovisky formalism in the functional approach to classical field theory. *Commun. Math. Phys.* **314**, 93 (2012). [arXiv:1101.5112](#) [math-ph]
21. Fredenhagen, K., Rejzner, K.: Batalin–Vilkovisky formalism in perturbative algebraic quantum field theory. *Commun. Math. Phys.* **317**, 697 (2013). [arXiv:1110.5232](#) [math-ph]
22. Fresse, B.: *Modules Over Operads and Functors*. Lecture Notes in Mathematics, vol. 1967. Springer, Berlin (2009)
23. Gabriel, P., Zisman, M.: *Calculus of Fractions and Homotopy Theory*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* 35. Springer, New York (1967)
24. Harper, J.E.: Bar constructions and Quillen homology of modules over operads. *Algebr. Geom. Topol.* **10**(1), 87–136 (2010). [arXiv:0802.2311](#) [math.AT]
25. Hinich, V.: Homological algebra of homotopy algebras. *Commun. Algebra* **25**(10), 3291–3323 (1997). [arXiv:q-alg/9702015](#). Erratum: [arXiv:math/0309453](#) [math.QA]
26. Hinich, V.: Rectification of algebras and modules. *Doc. Math.* **20**, 879–926 (2015). [arXiv:1311.4130](#) [math.QA]
27. Hinich, V.: Dwyer–Kan localization revisited. *Homol. Homotopy Appl.* **18**, 27–48 (2016). [arXiv:1311.4128](#) [math.QA]
28. Hirschhorn, P.S.: *Model Categories and Their Localizations*. Mathematical Surveys and Monographs, vol. 99. Amer. Math. Soc., Providence (2003)
29. Hovey, M.: *Model Categories*. Mathematical Surveys and Monographs, vol. 63. Amer. Math. Soc., Providence (1999)
30. Lurie, J.: Higher algebra, draft available at <https://www.math.ias.edu/~lurie/papers/HA.pdf>
31. Pinamonti, N.: Conformal generally covariant quantum field theory: the scalar field and its Wick products. *Commun. Math. Phys.* **288**, 1117–1135 (2009). [arXiv:0806.0803](#) [math-ph]
32. Weiss, I.: Dendroidal sets, PhD thesis, Universiteit Utrecht (2007). Available at <https://dspace.library.uu.nl/handle/1874/22859>
33. Yau, D.: *Colored Operads*. Graduate Studies in Mathematics 170. American Mathematical Society, Providence (2016)

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