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ORIGINAL RESEARCH

"Greedy" demand adjustment in cooperative games

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Abstract

This paper studies a simple process of demand adjustment in cooperative games. In the process, a randomly chosen player makes the highest possible demand subject to the demands of other coalition members being satisfied. This process converges to the aspiration set; in convex games, this implies convergence to the core. We further introduce perturbations into the process, where players sometimes make a higher demand than feasible. These perturbations make the set of separating aspirations, i.e., demand vectors in which no player is indispensable in order for other players to achieve their demands, the one most resistant to mutations. We fully analyze this process for 3-player games. We further look at weighted majority games with two types of players. In these games, if the coalition of all small players is winning, the process converges to the unique separating aspiration; otherwise, there are many separating aspirations and the process reaches a neighbourhood of a separating aspiration.

Keywords Demand adjustment · Aspirations · Core · Stochastic stability

1 Introduction

In transferable utility cooperative games, we consider the following process. Suppose players currently have some demands. These demands can be interpreted as what they expect from the game. A player is randomly selected. This player is in a position to propose a coalition; but the coalition partners agree to form it only if their demands are satisfied. The player looks for a coalition that, after the demands of the coalition partners are subtracted from the coalition's worth, leaves the most to the player. The player then makes the demand equal to the residual.

The payoff vectors that allow each player to achieve such "maximal" demands in at least one coalition are called *aspirations* in Bennett (1983). (They are also called *semi-stable* demand vectors in Albers, 1979; Selten, 1981). Bennett et al. (1997) show that the process described in the previous paragraph converges to the set of aspiration payoff vectors.

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We analyze the implications of the process further. First, we show that in convex games the Bennett et al. (1997) result implies that the demand adjustment process converges to the core of the game.[1]

The set of aspirations is in general quite large, and there can be aspirations where some players demand very little. Cross (1967) argues that "scarce" players (players that are under-demanding and hence sought after as coalition partners) should be able to increase their demands. We formalize this argument by adding to the process the possibility of "mutations". With a small probability a player makes a demand different from the maximal feasible one; instead, the player (most likely) makes a higher demand. Since in the basic process players make the maximum feasible demands and the most likely mutations are to even higher demands, the process overall can be seen as "greedy". We show that separating aspirations (a subset of partnered aspirations, defined in Albers, 1979; Bennett, 1983), i.e., demand vectors in which no player is indispensable in order for other players to achieve their demands, are the ones most resistant to such upward mutations.

We fully analyze the process with mutations in 3-player superadditive games. In these games, if the core is non-empty, demand vectors that are in the core are stochastically stable (meaning that, as the mutation probability goes to zero, the process spends almost all of the time in the core). If the core is empty, the unique separating aspiration is stochastically stable. We then turn to weighted majority games.[2] In Montero and Possajennikov (2022), we showed that separating aspirations are stochastically stable in symmetric weighted majority games and in apex games. In this paper, we analyze weighted majority games with two types of players further.[3] In these games, if there are enough small players (i.e., if the coalition of all small players is winning), the process converges to the unique separating aspiration. On the other hand, if the coalition of all small players is losing, then there are many separating aspirations and the process reaches a neighborhood of a separating aspiration.

The paper contributes to the literature, reviewed in Newton (2018, Sect. 6), that applies evolutionary approaches to predicting outcomes in cooperative games. Agastya (1997) has a demand adjustment process in which players simultaneously make demands, and a coalition compatible with demands forms (with some probability, if several coalition structures are compatible). Using a myopic best response to incomplete memory samples, Agastya shows that in convex games the process converges to the core. Rozen (2013) allows the players, in addition to demands, to also name a list of potential coalition partners, obtaining the same result. With our process (without mutations), convergence to the core in convex games follows from the observation that in convex games the set of aspirations coincides with the core (Moldovanu & Winter, 1994).

As in Agastya (1997) and Rozen (2013), in our process the players need to know the (previous) demands of other players (and the characteristic function of the game) in order to find the best demand to make. In the demand adjustment process in Nax (2010, chapter 4), a player can increase or decrease the demand (although only by a small amount), depending on whether the player is in a coalition that satisfies the demand but without knowing the demands of others. The process predicts outcomes close to the core (but not exactly in the core, since players can temporarily increase demands with a non-zero probability) if the core

[1] Agastya (1997) and Rozen (2013) have this result for similar adjustment processes. We discuss the differences between their and our models in the discussion of related literature later in the introduction.

[2] (Weighted) majority games are a class of games in which some coalitions can "win" (have a positive worth) while others "lose" (have zero worth). Such games are often studied in political science and economics in the context of voting (for example, Felsenthal and Machover (1998), and, more recently, Kurz et al. (2023)).

[3] These games are often found in practice. For example, the distribution of party seats in the current German parliament (Bundestag) gives rise to such a game (see Example 2 in Sect. 5.2).

is non-empty. Using a variant of Nax's process, Issleib (2015) shows that in 3-player games an equitable allocation in the core is selected; this is related to the assumption in Nax (2010) that unsatisfied demands are more likely to be reduced the larger they are. Our process does not select within the core, but we are able to make predictions also when the core is empty, because demand configurations that are aspirations are stable in our process.

Other works that consider adaptive processes of demand (and coalition) adjustment also focus on games with a non-empty core. Arnold and Schwalbe (2002) obtain convergence to the core by allowing mutations only outside the core. Newton (2010, chapter 3; 2012) shows convergence to the core without the need for convexity by allowing for the possibility of jointly determined strategies. A joint adjustment of demands also allows convergence to the core in Nax (2019). Various models of mutations lead to selection (in the sense of stochastic stability) of some outcomes in the core in Agastya (1999), Newton (2012), and Sawa (2019). For matching problems and assignment games, convergence to the core (and possible selection within it) is obtained in Klaus et al. (2010), Nax and Pradelski (2015), Newton and Sawa (2015), and Klaus and Newton (2016). The one result for games with an empty core is in Nax (2019), where no demand configuration is stable in such games. In contrast, with our model of mutations we are able to obtain predictions also for some games with an empty core.

Our basic adjustment process contains an element of optimism. Because the player who is selected to adjust can also propose a coalition, this player only reduces the demand if no coalition can satisfy it. There is no consideration of the probability of a coalition satisfying it in the future. This feature allows us to maintain stability in games with an empty core. Similarly, there is a degree of optimism in our perturbed process, since mutations to a higher demand are more likely than mutations to a lower demand.

We see the process we discuss in the paper as one of the simplest and most natural processes of demand adjustment. While we think that the process has some behavioral plausibility, we do not claim that it describes human behavior literally. Rather, our process provides dynamic foundations for the cooperative solution concept of separating aspirations. The process allows us to make predictions for some classes of games, including for games with an empty core, for which the literature on dynamic processes in cooperative settings did not previously provide predictions.

2 Basic demand adjustment process

2.1 Demand adjustment in TU games

A transferable utility (TU) cooperative game in characteristic function form is given by (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ with $v(\varnothing) = 0$ is the characteristic function. We assume that the game is zero-normalized, $v(\{i\}) = 0$ for all i. A *coalition* $S \in 2^N$ is any subset of players. A *demand vector* $x \in \mathbb{R}^n$ is $x = (x_1, \ldots, x_n)$, with x_i being the demand of player i. Let the sum of demands of members of coalition S be $x(S) = \sum_{i \in S} x_i$.

Suppose that time is measured in discrete intervals $t = 0, 1, \ldots$. Suppose that at the beginning of time period t, a vector of demands is $x^{t-1} = (x_1^{t-1}, \ldots, x_n^{t-1})$ (we will also use x without the superscript if no confusion arises; at $t = 0$, x^0 is exogenously given). One player is randomly chosen; the only assumption on the probability of being chosen is that it is bounded away from 0 for each player. Let i be the chosen player. Player i knows the vector of demands x^{t-1}, and looks for a coalition that allows i to get the most, provided that the demands of the other coalition members are satisfied. This means that the player solves the problem

$$\max_{S:S\ni i}\{v(S) - x^{t-1}(S\backslash\{i\})\}. \tag{1}$$

Suppose that a certain coalition $Q \ni i$ solves the problem and y_i is the maximum value of the problem (that is, $y_i = v(Q) - x(Q\backslash\{i\})$). The player then sets the demand at y_i. Coalition Q forms (for period t) and all players in Q satisfy their demands.[4] The new demand vector at the end of period t is $x^t = (x_1^t, \ldots, x_n^t)$, with $x_i^t = y_i$ and $x_j^t = x_j^{t-1}$ for $j \neq i$. This new demand vector is then used in the next period. Coalition Q does not play a role in the next period; it is dissolved at the end of period t.

We introduced the above process in Montero and Possajennikov (2022), in which we analyzed it in specific classes of weighted majority games. The process is "greedy" in the sense that the player who is chosen in a given period sets the demand to the maximum payoff that this player can get, constrained only by the demands of other players. The player's decision rule is myopic, taking into account only the possibility to get the demand in the current period, but this is justified since only this player is able to change the demand in this period: no other player can.

2.2 Absorbing sets and aspirations

A *state* of the process defined above is a demand vector $x = (x_i)_{i\in N}$. The state can change from one period to the next as described above; we denote the set of all possible states as $\mathcal{S}$. Let $\Psi(x)$ denote the set of states that the process can move to (depending on which player is chosen) from a given state x in one step. For an arbitrary subset of states $\mathcal{A} \subset \mathcal{S}$, let $\Psi(\mathcal{A}) = \cup_{x\in\mathcal{A}}\Psi(x)$.

An *absorbing* set of states $\mathcal{A}$ is such set of states that the process cannot leave: $\Psi(\mathcal{A}) \subset \mathcal{A}$. The *minimal absorbing set* is an absorbing set that does not contain a strict subset which is absorbing. The union of all minimal absorbing sets is a *absorbing set solution*. The absorbing set solution is the set of states to which the process converges with probability 1; it contains all the states that the process will be in, or visit, in the long run.

A demand vector x is *maximal* if $x(S) \geq v(S)$ for all coalitions S, i.e., there is no coalition in which players can increase their demands while still satisfying the demands of the other members. Demand vector x is *feasible* if for each i, there exists $S \ni i$ such that $x(S) \leq v(S)$, i.e., every player can find a coalition that satisfies the demand of this player. Bennett (1983) defines

Definition 1 A demand vector x is an **aspiration** if x is maximal and feasible.

Such demand vectors are also called *semi-stable* (Albers, 1979; Selten, 1981). Given an aspiration x, the set of coalitions $GC(x) = \{S : x(S) = v(S)\}$ that can satisfy the demands of their members is called the *generating collection* of aspiration x.

Based on the more general adjustment process in Bennett et al. (1997), we show the following result in Montero and Possajennikov (2022):

Proposition 1 *The absorbing set solution for the basic process is the set of aspirations.*

[4] If there are several coalitions that solve (1), then any of them can be formed. Note that Q can be a singleton coalition $\{i\}$ if $\{i\}$ solves (1).

The intuition for the proof runs as follows. If a state (current demand vector) is an aspiration, then every player already makes the maximal demand possible: by maximality, there is no coalition in which a player can get a higher demand; by feasibility, each player has a coalition which is able to satisfy this player's demand, so the player does not need to lower the demand. Thus every aspiration is a minimal absorbing state. If the current state is not an aspiration, there are players that either can increase their demands, or have to lower them to find a feasible coalition. There is always a sequence of players, some lowering, some increasing demands such that the new demand vector is an aspiration. Since the probability of this sequence is non-zero, the process eventually gets to an aspiration, thus there are no other absorbing sets outside of the aspiration set.

2.3 Core convergence in convex games

For a cooperative TU game (N, v), the *core* $C(v)$ is the set of (demand) vectors that are maximal and feasible for the coalition $S = N$ of all players: $C(v) = \{x : x(S) \geq v(S)$ for all S and $x(N) = v(N)\}$. The core is one of the main solution concepts for cooperative games; however, it can be empty for some games. Obviously, demand vectors that are in the core are maximal; they are also feasible for all players since they are feasible for coalition N. Therefore, core allocations are also aspirations. The converse statement is not true, that is, there can be aspirations outside the core even if the core is nonempty, as the following "glove game" illustrates.

Example 1 (see Bennett, 1983) $N = \{1, 2, 3, 4, 5\}$, $R = \{1, 2\}$, $L = \{3, 4, 5\}$, $v(S) = \min(|S \cap R|, |S \cap L|)$. In particular, $v(\{1, j\}) = 1$ for $j = 3, 4, 5$, $v(\{2, j\}) = 1$ for $j = 3, 4, 5$ and $v(N) = 2$. The only point in the core is $(1, 1, 0, 0, 0)$, but any vector of the form $(a, a, 1 - a, 1 - a, 1 - a)$ with $0 \leq a \leq 1$ is an aspiration demand vector.

A *convex* game (Shapley, 1971) is a game in which the marginal contribution of each player is larger to a larger (in terms of set inclusion) coalition: for all i and all T, S such that $T \subset S \subset N \backslash \{i\}$, $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$. The core of a convex game is non-empty; indeed, it contains many outcomes. In particular, Shapley (1971) shows that the marginal contribution vectors are the vertices of the core, hence for each player i there are core outcomes that give this player his or her stand-alone value $v(\{i\})$.

Moldovanu and Winter (1994, Lemma 3.5) show that the set of semi-stable vectors (i.e. aspirations) coincides with the core in convex games. In particular, they show that in convex games an aspiration (that is, a demand vector that is maximal and feasible) has to be feasible for coalition N (intuitively, in convex games demands that are feasible for smaller coalitions are also feasible for larger coalitions). But then such an aspiration is in the core. Since, from the first paragraph of this subsection, demand vectors that are in the core are aspirations, it follows that the core and the set of aspirations coincide in convex games.

An immediate consequence of Proposition 1 for convex games (nevertheless, stated as a proposition rather than a corollary, to emphasize its importance) is that the "greedy" process converges to the core in these games.

Proposition 2 *In convex games, the basic process converges to the core with probability 1.*

This result relates to the results in Agastya (1997) and Rozen (2013), who show convergence to the core in convex games for similar processes. Intuitively, so long as coalitions that make demands of one player "maximal" while satisfying the demands of other players can form with a positive probability, processes similar to ours eventually hit the core and stay there. In our process, such "maximal" demand coalitions form with probability 1; in Agastya's paper, such "maximal" demand adjustments by only one player can happen with a positive probability because of incomplete sampling of past observations.[5]

In games that are not convex there can be aspirations that are not in the core (Example 1). In the next section we introduce certain perturbations in the process that can help to select among various aspirations.

3 Perturbed demand adjustment process

3.1 Introducing "greedy" perturbations

We now introduce certain perturbations of the process. To have a finite state space, we assume that the set of possible demands that players can make is a finite grid. To have the grid cover relevant allocations, we assume that all $v(S)$ are rational numbers. Taking m as the common denominator of these numbers, let $\delta = \frac{1}{lm}$, where $l > 0$ is a natural number. Let the grid be $\Gamma_\delta = \{k\delta : k \in \{0, 1, \ldots, K\}\}$, where $K = \frac{\max_S v(S)}{\delta}$. If $l \geq n$, then for any choices $x_j \in \Gamma_\delta$ of players in $S \backslash \{i\}$ with $\sum_{j \in S \backslash \{i\}} x_j < v(S)$, player i can choose demand $x_i \in \Gamma_\delta$ so that $x(S) = v(S)$. Therefore the set of demand vectors restricted to the grid contain (some) aspirations. The state space of the process is the finite space $\mathcal{S}$ of demand vectors on the grid. The process then is a finite Markov chain.

Let matrix M with elements m_{ab} describe the probability of moving from state a to state b in one period. Vector μ of size $|\mathcal{S}|$ (with $\sum \mu_i = 1$) is a *stationary distribution* for M if $\mu M = \mu$. Proposition 1 from the previous section implies that for the basic demand adjustment process we have considered, any probability distribution with the support on the set of aspirations on the grid is a stationary distribution. In particular, any degenerate distribution with all the mass on one particular aspiration is a stationary distribution.

As in Montero and Possajennikov (2022), we consider the following perturbation of the process from the previous section. With probability $1 - \varepsilon$, the choice of demand still follows from maximization problem (1). With probability ε, the adjusting player i chooses a demand differently; we refer to such an event as a "mutation". In particular, if the player experiences a mutation, then, with probability $1 - \varepsilon$, the demand is in the set $\{x_i^{t-1}, \ldots, \max_S v(S)\}$; with probability ε the demand is in the set $\{0, \ldots, x_i^{t-1}\}$. If ε is small, mutations in general are rare, but mutations to a higher demand are more likely than to a lower demand. This mutation model is based on *intentional* play in Naidu et al. (2010), and also has a flavor of "greediness": players hope to get a higher demand satisfied.

Let M^ε denote the transition matrix of the Markov chain of the process with mutation probability ε. The process is irreducible, since any demand vector can be obtained by a sequence of n mutations. Therefore, for $\varepsilon > 0$, the process has a unique stationary distribution μ^ε. Let $\mu^0 = \lim_{\varepsilon \to 0} \mu^\varepsilon$. States x that have a positive probability in μ^0 are *stochastically stable*: the process is much more likely to be in them as the mutation probability becomes arbitrarily small.

Stochastically stable sets are contained in the absorbing set solution of the process with $\varepsilon = 0$, which is the set of aspiration demand vectors. However, some aspirations are more easily disturbed than others with the kind of mutations that we consider.

[5] Since in Rozen (2013) players also name a list of potential coalition partners, the adjustment path to the core is more complicated. The part that is similar to setting "maximal" demands is the inclusion of one player into an already existing coalition, which in convex games leads to the core.

3.2 Perturbed process and separating aspirations

For an aspiration x, consider its generating collection $GC(x)$. Let $C_i(x) = \{S \in GC(x) : i \in S\}$ be the set of coalitions in $GC(x)$ that contain player i. Aspiration x is *partnered* if $C_i \subseteq C_j \Rightarrow C_j \subseteq C_i$ for all i, j (Bennett, 1983). In a partnered aspiration, for any pair of players i, j, either they are together in all their feasible coalitions, or, if player i has a feasible coalition without j, then so does j without i.

The latter property (that each player has a feasible coalition without any other particular player) is the one that is important for stochastic stability. In Montero and Possajennikov (2022) we consider the following definition:

Definition 2 Aspiration x is **separating** if $C_i \backslash C_j \neq \varnothing$ and $C_j \backslash C_i \neq \varnothing$ for all i, j.

(payoff vectors that are feasible for the grand coalition N with this property are called "completely separating" by Maschler and Peleg (1966), and "minimally partnered" by Reny et al. (2012)). In a separating aspiration, each player can find a coalition to satisfy his or her demand that does not contain any other particular player. Therefore mutations to a higher demand by any one player will not induce any other player to lower his or her demand.

With M^0 denoting the process without mutations ($\varepsilon = 0$), let $\mathcal{A} \subseteq \mathcal{S}$ be its absorbing set solution (from Proposition 1, it is the set of aspiration demand vectors in the game). For an arbitrary set $\mathcal{B} \subseteq \mathcal{S}$, let $\Psi^{\varepsilon}(\mathcal{B})$ denote the set of states that can arise from states in $\mathcal{B}$ with one most likely mutation: of one player to a higher demand. Let $\Psi^0_{\infty}(\mathcal{B})$ denote the set of states that the process without mutations can reach (in any number of steps) starting from a state in set $\mathcal{B}$.

Following (Nöldeke & Samuelson, 1993), we define

Definition 3 A set of states $\mathcal{B} \subseteq \mathcal{A}$ is called **minimal locally stable** if $\Psi^0_{\infty}(\Psi^{\varepsilon}(\mathcal{B})) \subseteq \mathcal{B}$, and there is no proper subset of $\mathcal{B}$ that has this property.

Starting from a state in a (minimal) locally stable set, one mutation can temporarily take the process out of it, but the process will converge back to it without further mutations.

Nöldeke and Samuelson (1993) show that the set of stochastically stable states is a subset of the set of states that are in minimal locally stable sets. In Montero and Possajennikov (2022, Lemma 3), we show that each separating aspiration is a minimal locally stable set. We also show that in general games there can be minimal locally stable sets different from separating aspirations. However, if a given game has no minimal locally stable sets other than the set of separating aspirations, then the set of separating aspirations contains all stochastically stable states.

4 Demand adjustment process in 3-player games

In this section, we apply the demand adjustment process (including perturbations) to 3-player superadditive games. Reordering players if necessary, a superadditive 3-player game is given by $v(\{12\}) = a \leq v(\{13\}) = b \leq v(\{23\}) = c \leq v(N)$.

The following lemma is well known (see, for example, Okada (2014), p. 965, Equation 4.19). We include its proof for completeness.

Lemma 1 *A 3-player superadditive game has a non-empty core if and only if* $a + b + c \leq 2v(N)$.

Proof A vector $x = (x_1, x_2, x_3)$ with $x_1 + x_2 + x_3 = v(N)$ is in the core if $x_1 + x_2 \geq a$, $x_1 + x_3 \geq b$ and $x_2 + x_3 \geq c$. Consider $x = (x_1, a - x_1, v(N) - a)$. It is in the core if $x_1 + v(N) - a \geq b$ and $v(N) - x_1 \geq c$, or if $a + b - v(N) \leq x_1 \leq v(N) - c$. If $a + b + c \leq 2v(N)$, then $a + b - v(N) \leq v(N) - c$, thus there exists x_1 satisfying the inequalities. Also, $x_1 + x_2 + x_3 = \frac{1}{2}(x_1 + x_2 + x_1 + x_3 + x_2 + x_3)$. If the core inequalities are satisfied, then $x_1 + x_2 + x_3 \geq \frac{1}{2}(a + b + c)$. However, if $2v(N) < a + b + c$, then no x with $x_1 + x_2 + x_3 = v(N)$ satisfies the core inequalities. □

We know that if an aspiration x has $GC(x) \ni N$, then x is in the core. The following lemma describes the generating collections that aspirations outside the core can have in 3-player superadditive games.

Lemma 2 *In a 3-player superadditive game, for an aspiration x outside the core either $GC(x) = \{\{ij\}, \{ik\}, \{jk\}\}$, or $GC(x) = \{\{ij\}, \{ik\}\}$, or $GC(x) = \{\{ij\}, \{ik\}, \{i\}\}$.*

Proof We divide the proof into two cases, depending on whether there is a singleton coalition in $GC(x)$. Suppose first that there are no singleton coalitions in $GC(x)$. Since all players need to have a feasible coalition and N is not feasible for x, then either $GC(x) = \{\{ij\}, \{ik\}, \{jk\}\}$ or $GC(x) = \{\{ij\}, \{ik\}\}$. Suppose now that $\{i\} \in GC(x)$. Suppose also that $\{j\} \in GC(x)$. By feasibility, either $\{k\} \in GC(x)$, or $\{ki\} \in GC(x)$, or $\{kj\} \in GC(x)$: in all cases, by superadditivity N is feasible, a contradiction. Hence, there is a most one singleton coalition in $GC(x)$. Thus, if $\{i\} \in GC(x)$, then $GC(x) = \{\{i\}, \{ij\}, \{ik\}\}$. □

The first result for our process in 3-player games is about the games with non-empty core.

Proposition 3 *If the core of a 3-player superadditive game is non-empty ($a+b+c \leq 2v(N)$), then all states in the core are stochastically stable, and there are no other stochastically stable states.*

Proof Consider aspiration x that is in the core, with $N \in GC(x)$. Suppose player i mutates to a higher demand $x_i + \Delta$. If $\{jk\} \in GC(x)$, then players j and k do not adjust their demand. Player i will lower the demand back to x_i; the process without mutations returns to x. Suppose now that $\{jk\} \notin GC(x)$, and suppose that player j is chosen to adjust. If N is one of the coalitions that solve the maximization problem (1), then another aspiration in the core is reached.

Suppose N is not a solution of problem (1). Since x is an aspiration, $x_i + x_j \geq v(\{ij\})$. Player j can form N, setting a demand equal to $x_j - \Delta_i$ or player j can form $\{ij\}$, getting $v(\{ij\}) - (x_i + \Delta_i)$. If player j strictly prefers $\{ij\}$ to N, it would be the case that $v(\{ij\}) - (x_i + \Delta_i) > x_j - \Delta_i$, contradicting $x_i + x_j \geq v(\{ij\})$. Therefore, j forms either $\{jk\}$ or $\{j\}$, setting demand $x_j - \Delta_j$, where $\Delta_j \geq 0$.

After the adjustment of player j, player i has no feasible coalition. In aspiration x, since N is feasible, $v(N) - x_j - x_k \geq v(\{ij\}) - x_j$. Therefore, $v(N) - (x_j - \Delta_j) - x_k \geq v(\{ij\}) - (x_j - \Delta_j)$. Therefore player i cannot strictly prefer $\{ij\}$ to N now. Also, in x, $v(N) - x_j - x_k \geq v(\{ik\}) - x_k$, thus $v(N) - (x_j - \Delta_j) - x_k \geq v(\{ik\}) - x_k$ and player i cannot strictly prefer $\{ik\}$ over N. Finally, in x, $v(N) - x_j - x_k \geq v(\{i\})$, thus $v(N) - (x_j - \Delta_j) - x_k \geq v(\{i\})$ and i cannot strictly prefer $\{i\}$ over N. Therefore i sets a demand for which N is a feasible coalition and the demand vector is in the core.

If j has formed $\{j\}$, then player k may need to adjust the demand, but, using an analogous reasoning to the one above, k adjusts to a demand that makes N feasible, thus to a core demand vector.

We have shown that, starting from an aspiration in the core, a mutation of one player to a higher demand leads to another aspiration in the core. Consider now aspiration x that is not in the core. By Lemma 2, such aspirations have $\{ij\}$ and $\{ik\}$ in $GC(x)$, meaning that $x_i + x_j = v(\{ij\})$ and $x_i + x_k = v(\{ik\})$. If also $\{jk\} \in GC(x)$, then $2x(N) = a + b + c$, and, since $a + b + c \leq 2v(N)$, then N is feasible, contradicting that x is not in the core. Therefore $\{jk\} \notin CG(x)$, i.e., $x_j + x_k > v(\{jk\})$.

Since $\{ij\}$ and $\{ik\}$ in $GC(x)$, $2x_i + x_j + x_k = v(\{ij\}) + v(\{ik\})$. Thus, $2x_i + v(\{jk\}) < v(\{ij\}) + v(\{ik\})$, and, therefore, $x_i < \frac{v(\{ij\})+v(\{ik\})-v(\{jk\})}{2}$.

Suppose i mutates to $\frac{v(\{ij\})+v(\{ik\})-v(\{jk\})}{2}$. Since $v(\{ij\}) + v(\{ik\}) + v(\{jk\}) \leq 2v(N)$, then $\frac{v(\{ij\})+v(\{ik\})+2v(\{jk\})-v(\{jk\})}{2} \leq v(N)$, and thus $\frac{v(\{ij\})+v(\{ik\})-v(\{jk\})}{2} \leq v(N) - v(\{jk\})$. Therefore, forming N is at least as good as forming $\{jk\}$ for any of the players j and k. Hence, either of them adjusts to a demand that makes N feasible and thus a demand vector in the core is reached.

The previous reasoning shows that the core is a locally stable set, and no aspiration outside the core is in a locally stable set. Consider any two aspirations in the core, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. If $x \neq y$, then there exists player i with $x_i < y_i$ and player k with $x_k > y_k$. Suppose that $x_j = y_j$. If player i mutates to y_i, and player k adjusts, aspiration y is reached. If $x_j \neq y_j$, the aspiration z reached after adjustment of player k (and possibly j) is not y but it is a core aspiration with $z_i = y_i$. One further mutation of the remaining "underdemanding" player would lead to y then.

Therefore, from any aspiration in the core another aspiration in the core can be reached by a sequence of most likely mutations, one at a time. Thus all core states are in the same locally stable set (component). From Proposition 1 in Nöldeke and Samuelson (1993), all aspirations in the core are stochastically stable. □

That the core is the unique minimal locally stable set is specific to 3-player games. For larger games, there are other locally stable sets: namely, separating aspirations (Lemma 3 in Montero and Possajennikov (2022)).

Example 1 (continued) *In the 5-player glove game, aspirations* $(a, a, 1-a, 1-a, 1-a)$ *are separating for any* $0 \leq a \leq 1$, *since each player can pair with at least two other players. Thus, any aspiration* $(a, a, 1-a, 1-a, 1-a)$ *is a (minimal) locally stable set, but only aspiration* $(1, 1, 0, 0, 0)$ *is in the core.*

Consider now 3-player games with an empty core. In these games, the result for our process is actually stronger.

Proposition 4 *If the core of a superadditive 3-player game is empty* ($a+b+c > 2v(N)$)*, then the aspiration* $x^* = (\frac{a+b-c}{2}, \frac{a+c-b}{2}, \frac{b+c-a}{2})$ *is the unique stochastically stable aspiration.*

Proof Consider $x^* = (\frac{a+b-c}{2}, \frac{a+c-b}{2}, \frac{b+c-a}{2})$. Its generating collection is $GC(x^*) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Therefore it is a separating aspiration and thus is a minimal locally stable set.

Consider aspiration x different from x^*. From Lemma 2, $GC(x)$ contains $\{ij\}$, $\{ik\}$ (and possibly $\{i\}$), i.e., players j and k depend on player i to get their demands. Since $\{jk\}$ is not feasible in x, $x_i < x_i^* = \frac{v(\{ij\})+v(\{ik\})-v(\{jk\})}{2}$ and $x_j > x_j^*$, $x_k > x_k^*$. Suppose that player i mutates to x_i^*. If subsequently players j and k are selected to adjust, they adjust their demand to the corresponding demands in x^*. Separating aspiration x^* is thus reached with one mutation.

Therefore, there are no other minimal locally stable sets in such games. Being the only aspiration in a (minimal) locally stable set, the separating aspiration x^* is the only stochastically stable one. □

We have shown in this section that the process allows selection of aspirations in all 3-player (superadditive) games, including games with an empty core. The next section considers the application of the process to another class of games with an empty core.

5 Demand adjustment in weighted majority games

5.1 Weighted majority games and separating aspirations

A TU game in characteristic function form is a *weighted majority game* if there are weights $(w_1, \ldots, w_n)$ of n players and a quota q such that $v(S) = 1$ if $\sum_{i \in S} w_i \geq q$ and $v(S) = 0$ otherwise. If $v(S) = 1$, then coalition S is considered winning; otherwise coalition S is losing. Coalition S is *minimal winning* if no $T \subsetneq S$ is winning. We assume that there are no null players, that is, each player belongs to at least one minimal winning coalition (and hence all weights must be strictly positive). We consider *constant-sum* games, in which $v(S) + v(N \backslash S) = 1$ for any S: if a coalition is winning, then its complement is losing, and vice versa.

A weighted majority game can be represented by the quota and the weights as $[q; w_1, \ldots, w_n]$. In general, there are many representations for the same game. A representation is homogeneous if for all minimal winning S, $\sum_{i \in S} w_i = q$. A weighted majority game is called *homogeneous* if there exists a homogeneous representation of it. If the game is moreover constant-sum, then the homogeneous representation is unique up to a multiplicative constant (Peleg, 1968).

If (N, v) is a constant-sum homogeneous game with homogeneous representation $[q; w_1, \ldots, w_n]$, then vector $x = \frac{1}{q}(w_1, \ldots, w_n)$ is a separating aspiration (we show this in Montero and Possajennikov, 2022; from Peleg 1968 it also follows that there exists a representation with integer weights thus x has rational coordinates). In general, there can be other separating aspirations in constant sum homogeneous games: consider the game with representation [4; 2, 2, 1, 1, 1]: any aspiration $x = (a, a, \frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2})$ with $\frac{1}{2} \leq a \leq 1$ is separating. In Montero and Possajennikov (2022) we show that in some classes of weighted majority games, namely symmetric games and apex games, the separating aspiration is unique and also the unique stochastically stable one in our "greedy" process. In the next section we analyze a more general class of weighted majority games.

5.2 Weighted majority games with two types of players

5.2.1 Symmetric and δ-symmetric aspirations

In a weighted majority game, two players i and j are said to be of the same *type* if substituting one by the other does not change the value of a coalition: $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N, i, j \notin S$. In a homogeneous representation of a weighted majority game players i, j of the same type have the same weight, $w_i = w_j$. In this section, we consider games with two types of players.[6] Such games have practical relevance as the following example illustrates.

Example 2 The 2021 German Federal Election resulted in the following distribution of seats: SPD 206, CDU/CSU 197, Grüne 118, FDP 92, AfD 83, DIE LINKE 39, SSW 1.

[6] If there is only one type of player in a game, then the game is symmetric. We analyzed symmetric majority games in Montero and Possajennikov (2022).

Assuming a majority of 369, this situation is equivalent to the weighted majority game [4; 2, 2, 1, 1, 1, 0, 0]. This is a game with two types of (non-null) players.

For a (constant-sum homogeneous) weighted majority game, the *minimal integer* representation has the smallest weight equal to 1. Accordingly, a constant-sum homogeneous weighted majority game with two types of players can be represented as $[q; a, \ldots, a, 1, \ldots, 1]$, with integer $a > 1$. Let p be the number of *large* players (with weight a) and $n - p$ the number of *small* players. Without loss of generality, the players are ordered in decreasing order of weights. We denote the large players as belonging to type t_a, and small players as belonging to type t_1 (we use t_w for a generic player type).

Let x be a vector of demands. Let $d_{ij}(x) = |x_i - x_j|$ be the absolute value of the difference in demands of players i and j. We denote $d_a(x) = \max_{i,j \in t_a} d_{ij}(x)$ the maximal difference in demands of large players, and $d_1(x) = \max_{i,j \in t_1} d_{ij}(x)$ the maximal difference in demands of small players. Further, let $d(x) = \max\{d_a(x), d_1(x)\}$. Aspiration x is *within-type symmetric*, or simply *symmetric*, if all players of the same type make the same demand, $d(x) = 0$.

We will show that only symmetric aspirations can be separating. A useful auxiliary result is the following:

Lemma 3 *Consider an aspiration x with $d(x) > 0$, i.e., $x_i < x_j$ for some players $i, j \in t_w$. Then for any coalition $S \in GC(x)$ such that $j \in S$, for any i with $x_i < x_j$, $i \in S$.*

Proof Suppose for a player i with $x_i < x_j$, $i \notin S$. Consider coalition $T = S \backslash \{j\} \cup \{i\}$. T has a lower sum of demands but, since players i and j are of the same type, the same $v(T) = v(S)$. This means that x is not maximal, contradicting that x is an aspiration. □

Lemma 4 *Consider an aspiration x. If x is separating, then $d(x) = 0$.*

Proof Consider aspiration x that has $d(x) > 0$ and let $x_i < x_j$ for players i, j of the same type. By Lemma 3, j does not have a feasible coalition without i, therefore x is not separating. □

Not all symmetric aspirations are separating (for example, aspiration (0, 1, 1, 1) in the apex game [3; 2, 1, 1, 1] is symmetric but not separating).

The lemma says that aspirations that are not symmetric cannot be separating, thus they are vulnerable to upward mutations of some players. However, sometimes symmetric aspirations cannot be easily reached by such mutations, as the following example shows.

Example 3 (see Montero and Possajennikov, 2022). Consider the game [8; 2, 2, 2, 2, 2, 2, 1, 1, 1], with nine players; players 1-6 have weight $a = 2$ and players 7-9 have weight 1.

Consider aspiration $x = (\frac{2}{8}, \frac{2}{8} + \delta, \ldots, \frac{2}{8} + \delta, \frac{1}{8} - \delta, \frac{1}{8} - \delta, \frac{1}{8} - \delta)$ in this game. With one mutation of player 1 (and adjustment of any of the players 2–6) the process would move to an aspiration which is a permutation of x (within type t_2), or, with adjustment of any of the players 7–9, to a permutation (within type t_1) of aspiration $y = (\frac{2}{8} + \delta, \frac{2}{8} + \delta, \ldots, \frac{2}{8} + \delta, \frac{1}{8} - 2\delta, \frac{1}{8} - \delta, \frac{1}{8} - \delta)$, but not to a symmetric aspiration.

Motivated by this example, we consider "nearly" symmetric aspirations. We call an aspiration x *δ-symmetric* if $d(x) \leq \delta$, where δ is the finite grid step size. (We will refer to symmetric aspirations as *0-symmetric*.)

The following proposition shows that the process can always reach the set of δ-symmetric aspirations with a sequence of most likely mutations, although, as we will see later, it is not always the case that the process stays within this set.

Proposition 5 *From any aspiration x, the process can reach a δ-symmetric aspiration with a sequence of mutations, one player at a time.*

Proof If $d(x) \leq \delta$, then x is already the required aspiration. Suppose thus that $d(x) > \delta$. Then either $d_a(x) > \delta$, or $d_1(x) > \delta$, or both.

Let t_w be the "winning" type, meaning that there exists a winning coalition S made exclusively of players of this type (there exists S such that $v(S) = 1$ and for all $i \in S$, $i \in t_w$). Let t_{-w} denote the other type (in a constant-sum game, one type is winning and the other is not: it cannot be that both are winning or both are losing). Let $x_{m,w} = \min_{k \in t_w} x_k$, $x_{M,w} = \max_{k \in t_w} x_k$, and $x_{m,-w} = \min_{k \in t_{-w}} x_k$, $x_{M,-w} = \max_{k \in t_{-w}} x_k$.

Case 1: At x there exists a feasible coalition U containing only players of type t_w (there exists U such that $x(U) = 1$ and for all $i \in U$, $i \in t_w$).

Case 1(a): Demands of players of type t_w are asymmetric ($d_w > 0$). Suppose $x_i = x_{m,w} > \frac{1}{|U|} - \delta$ and $x_j = x_{M,w}$. Then every player in U demands $\frac{1}{|U|}$ and thus $j \notin U$. By Lemma 3, any feasible coalition for j must contain all players in U, a contradiction. Therefore $x_{m,w} \leq \frac{1}{|U|} - \delta$. Suppose $x_{M,w} < \frac{1}{|U|} + \delta$. For U to be maximal, $i \notin U$. But a coalition $U \backslash \{j\} \cup \{i\}$ is then not maximal. Therefore $x_{M,w} \geq \frac{1}{|U|} + \delta$. Suppose player i mutates to $x_i + \delta$. Player j then has to adjust to $x_j - \delta$. If there are further players with $x_{m,w}$, they would not need to adjust downwards, and no player with $x_i \leq \frac{1}{|U|}$ would need to adjust downwards. In a new aspiration y, either $d_w(y) = d_w(x)$ but there are fewer players with $y_{m,w} = x_{m,w}$ and $y_{M,w} = x_{M,w}$, or $d_w(y) < d_w(x)$. Continuing, an aspiration z with $d_w(z) = 0$ can be reached.

Case 1(b): Demands of players of type t_w are symmetric but demands of players of type t_{-w} are asymmetric ($d_w = 0$ and $d_{-w} > \delta$). Suppose $x_i = x_{m,-w}$, $x_j = x_{M,-w}$. If player i mutates to $x_i + \delta$, player j would need to adjust to $x_j - \delta$. No player of type t_w needs to adjust; also no player of type t_{-w} with $x_i \leq x_{m,-w} + \delta$ would need to adjust downwards. Therefore in a new aspiration y, either $d_{-w}(y) = d_{-w}(x)$ but fewer players have $y_{m,-w} = x_{m,-w}$ and $y_{M,-w} = x_{M,-w}$, or $d_{-w}(y) < d_{-w}(x)$. Continuing, an aspiration z with $d_{-w}(z) = 0$ (and $d_w(z) = 0$ still) can be reached.

Case 2: Any coalition U of players of type t_w is infeasible (for any $U \subset t_w$, $\sum_{k \in U} x_k > v(U)$).

Case 2(a): Demands of players of type t_{-w} are asymmetric ($d_{-w} > 0$). Consider $i, j \in t_{-w}$ such that $x_i = x_{m,-w} < x_j = x_{M,-w}$. Consider any player $k \in t_w$ and consider any $U \in GC(x)$, $U \ni k$. Suppose $i \notin U$. If $j \in U$, then coalition $U \backslash \{j\} \cup \{i\}$ is not maximal; thus $j \notin U$ and j does not have a feasible coalition. Therefore, $i \in U$. Suppose i mutates to $x_i + \delta$. Player k would need to adjust to $x_k - \delta$. No other player with $x_{m,w}$ or $x_{m,w} + \delta$ would need to adjust. In a new aspiration y, either a coalition of players of type t_w will be feasible (Case 1), or $d_{-w}(y) = d_{-w}(x)$ (and $y_{M,-w} \leq x_{M,-w}$) but fewer players have $y_{m,-w} = x_{m,-w}$, or $d_{-w}(y) < d_{-w}(x)$. In the latter two cases, continuing, an aspiration z with $d_{-w}(z) = 0$ can be reached.

Case 2(b): Demands of players of type t_{-w} are symmetric but demands of players of type t_w are not δ-symmetric ($d_{-w} = 0$ and $d_w > \delta$). Consider $i, j \in t_w$ such that $x_i = x_{m,w} < x_j = x_{M,w}$. By Lemma 3, any coalition containing j also contains i. If player i mutates to $x_i + \delta$, player j would need to adjust to $x_j - \delta$. No player of type t_{-w} would need to adjust, thus d_{-w} stays at 0. In a new aspiration y, either a coalition of players of type w will be feasible (Case 1), or $d_w(y) = d_w(x)$ (and $y_{M,w} \leq x_{M,w}$) but fewer players have $y_{m,w} = x_{m,w}$, or $d_w(y) < d_w(x)$. In the latter two cases, continuing, an aspiration z with $d_w(z) = \delta$ can be reached. □

As we have seen in Example 3, Proposition 5 cannot be strengthened to convergence to 0-symmetric aspirations.

For further analysis, we consider the following division of the class of constant-sum weighted majority games with two types. Either there are enough large players to form a winning coalition ($pa \geq q$, for example, game [4; 2, 2, 1, 1, 1]), or there are enough small players to form a winning coalition ($n - p \geq q$, for example, game [3; 2, 1, 1, 1]). Since the game is constant-sum, it cannot be that both a coalition of only large players and a coalition of only small players are winning. We consider each of the subclasses in turn.

5.2.2 Apex-like games

Suppose that $n - p \geq q$, so that small players can form a winning coalition. Such games include apex games and other similar games, such as [5; 3, 1, 1, 1, 1, 1, 1] and [5; 2, 2, 1, 1, 1, 1, 1]. In this class of games, there is a unique separating aspiration as the following lemma shows.

Lemma 5 *If $n - p \geq q$, aspiration vector $z = (\frac{a}{q}, \ldots, \frac{a}{q}, \frac{1}{q}, \ldots, \frac{1}{q})$ is the unique separating aspiration.*

Proof Let x be a separating aspiration. We know from Lemma 4 that x must be symmetric, hence all players of the same type must make the same demand. Since $n - p \geq q$, there is a minimal winning coalition S comprised of q players.

Case 1: If $S \in GC(x)$, symmetry and maximality of x then imply that $x_i = \frac{1}{q}$ for all small players. It then follows that $x_i = \frac{a}{q}$ for all large players (otherwise either feasibility or maximality of x would be violated).

Case 2: If $S \notin GC(x)$, $x_i > \frac{1}{q}$ for all small players and no coalition consisting exclusively of small players is feasible. In order for small players to be able to obtain their demands, $x_j < \frac{a}{q}$ for all large players. Any feasible coalition $T \in GC(x)$ must then contain all large players; if there was a large player $k \notin T$, this player could replace a small players in T (T contains more than a small players since the set of all large players is losing) and the new coalition would have the same value but a lower total demand, contradicting maximality of x. Since any feasible coalition for a small player must contain all large players, x cannot be a separating aspiration. □

The next proposition shows that this unique separating aspiration is the unique stochastically stable one.

Proposition 6 *Consider a constant-sum homogeneous weighted majority game with representation $[q; a, \ldots, a, 1, \ldots, 1]$ and suppose $n - p \geq q$. Then $z = (\frac{a}{q}, \ldots, \frac{a}{q}, \frac{1}{q}, \ldots, \frac{1}{q})$ is the unique stochastically stable state of the process.*

Proof In the proof of Proposition 5 (Case 1), if there is a feasible coalition of only players of the winning type (in the class of apex-like games under consideration, this is type t_1), then a 0-symmetric aspiration can be reached, with still a feasible coalition of only players of type t_1. In this class of games, there is only one such aspiration, namely the aspiration z.

Consider therefore aspirations with any winning coalition of small players infeasible. In the proof of Proposition 5 (Case 2), an aspiration with $d_a = 0$ and $d_1 \leq \delta$ can be reached. Let this be aspiration $x = (b, \ldots, b, c, \ldots, c, c + \delta, \ldots, c + \delta)$. (There may be no players with demand $c + \delta$.)

In x, any coalition in $GC(x)$ contains all large players. Suppose player 1 mutates to $b+\delta$ and a player with $c+\delta$ (or, if there is no such player, a player with c) adjusts downwards. (Other small players may be also infeasible; they would also adjust downwards.) If there is a feasible coalition of small players, then Case 1 applies and the aspiration z can be reached with a sequence of mutations, one player at a time. If not, then we can apply Case 2(a) to get all large players demand $b+\delta$. Then all small players would lower demands to c or $c-\delta$. If a coalition of small players is not feasible, then we have an aspiration like x but with the minimal demand of large players larger than before. Mutations of one large player can be continued until large players demand $\frac{a}{q}$, and small players demand $\frac{1}{q}$, i.e., the separating aspiration. □

This result generalizes the result in Montero and Possajennikov (2022) for apex games, which are a subset of the games with two types. In apex games, there is only one large player, and the coalition of all small players is (minimal) winning. That in the process with mutations the unique separating aspiration is the stochastically stable one applies also for other games in this class, for example, [5; 2, 2, 1, 1, 1, 1, 1].

5.2.3 Games with large players winning

Consider now weighted majority games with $pa \geq q$, so that large players can form a winning coalition. Examples of such games are [4; 2, 2, 1, 1, 1] and [8; 2, 2, 2, 2, 2, 2, 1, 1, 1].

Let $u \leq p$ be the number of large players necessary for winning by themselves ($ua = q$). Then, winning coalitions consist of either u large players, or $u-1$ large players and a small players, or $u-2$ large players and $2a$ small players, etc, until $ra > n-p$. The maximal number r of small players that can be in a winning coalition is thus $r = a\left\lfloor \frac{n-p}{a} \right\rfloor$. The minimal number of large players that can be in a winning coalition is $s = \frac{q-r}{a}$. Separating aspirations in these games are $z = (\frac{b}{q}, \ldots, \frac{b}{q}, \frac{c}{q}, \ldots, \frac{c}{q})$, for all $0 \leq c \leq 1$ and $b = \frac{q-cr}{s}$ (then $a \leq b \leq \frac{aq}{ap-q+2}$). (For example, in game [4; 2, 2, 1, 1, 1], $a=2$, $q=4$, $p=2$, $n-p=3$, $r=2$, $s=1$ and $2 \leq b \leq 4$.)

If in an aspiration there is a feasible coalition consisting of players of only winning type t_a, then from the proof of Case 1 of Proposition 5 a 0-symmetric aspiration can be reached, still with a feasible coalition of players of winning type. There is only one such aspiration, namely the aspiration $(\frac{a}{q}, \ldots, \frac{a}{q}, \frac{1}{q}, \ldots, \frac{1}{q})$. However, if no coalition of players of type t_a is feasible, then a separating aspiration may not necessarily be reached with a sequence of upward mutations, one player at a time.

Example 4 Consider game [10; 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1], with $n=11$ players, $p=8$, $n-p=3$. Winning coalitions have either 5 large players, or 4 large players and 2 small players.

Consider aspiration $x^{(1)} = (\frac{a}{q}, \frac{a}{q}, \frac{a}{q}+\delta, \ldots, \frac{a}{q}+\delta, \frac{1}{q}-\delta, \frac{1}{q}-\delta, \frac{1}{q}-\delta)$. There are two players with demand $\frac{a}{q}$ that are in any feasible coalition.

Suppose player 1 mutates upward. Any of the other players would have to include one more player with demand $\frac{a}{q}+\delta$ into a feasible coalition thus a player lowers the demand by δ. If it is one of the players 3-8, then the process reaches a permutation of $x^{(1)}$. If it is player 2, then we have, as a result of the basic process, a permutation of $x^{(2)} = (\frac{a}{q}-\delta, \frac{a}{q}+\delta, \ldots, \frac{a}{q}+\delta, \frac{1}{q}-\delta, \frac{1}{q}-\delta, \frac{1}{q}-\delta)$. If it is one of the players 9-11, we have a permutation of $x^{(3)} = (\frac{a}{q}, \frac{a}{q}+\delta, \ldots, \frac{a}{q}+\delta, \frac{1}{q}-2\delta, \frac{1}{q}-\delta, \frac{1}{q}-\delta)$.

Consider aspiration $x^{(2)} = (\frac{a}{q} - \delta, \frac{a}{q} + \delta, \ldots, \frac{a}{q} + \delta, \frac{1}{q} - \delta, \frac{1}{q} - \delta, \frac{1}{q} - \delta)$. Any feasible coalition includes player 1. If player 1 mutates to $\frac{a}{q}$, then if one of players 2-8 adjusts, we are back to a permutation of $x^{(1)}$; if one of players 9-11 adjusts, we have a permutation of $x^{(3)}$. If player 1 mutates to $\frac{a}{q} + \delta$ or more, then if one of players 2-8 adjusts, we have a permutation of $x^{(2)}$. If one of players 9-11 adjusts, we have $x^{(4)} = (\frac{a}{q} + \delta, \ldots, \frac{a}{q} + \delta, \frac{1}{q} - 3\delta, \frac{1}{q} - \delta, \frac{1}{q} - \delta)$.

Consider aspiration $x^{(3)} = (\frac{a}{q}, \frac{a}{q} + \delta, \ldots, \frac{a}{q} + \delta, \frac{1}{q} - 2\delta, \frac{1}{q} - \delta, \frac{1}{q} - \delta)$. All feasible coalitions include player 1 and player 9. Suppose player 1 mutates. If one of players 2-8 adjusts, we have a permutation of $x^{(3)}$. If player 9 adjusts, we have a permutation of $x^{(4)}$. If one of players 10-11 adjusts, then so does the other, reaching separating aspiration $(\frac{a}{q} + \delta, \ldots, \frac{a}{q} + \delta, \frac{1}{q} - 2\delta, \frac{1}{q} - 2\delta, \frac{1}{q} - 2\delta)$. Suppose player 9 mutates. If player 1 adjusts, we have $x^{(2)}$. If one of players 2-8 adjusts, we have $x^{(1)}$. If one of players 10-11 adjusts, we have $x^{(3)}$.

Finally, consider aspiration $x^{(4)} = (\frac{a}{q} + \delta, \ldots, \frac{a}{q} + \delta, \frac{1}{q} - 3\delta, \frac{1}{q} - \delta, \frac{1}{q} - \delta)$. All feasible coalitions include player 9. Suppose player 9 mutates to $\frac{1}{q} - 2\delta$. If one of players 1-8 adjusts, we have $x^{(3)}$. If one of players 10-11 adjusts, then so does the other, reaching separating aspiration $(\frac{a}{q} + \delta, \ldots, \frac{a}{q} + \delta, \frac{1}{q} - 2\delta, \frac{1}{q} - 2\delta, \frac{1}{q} - 2\delta)$. Suppose player 9 mutates to $\frac{1}{q} - \delta$ or more. If one of players 1-8 adjusts, then we have $x^{(2)}$; if one of players 10-11 adjusts, we have $x^{(4)}$ as the result of the basic process.

Putting everything together, one-player mutations make the process stay within permutations of $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$, but there is a non-zero probability of reaching separating aspiration $(\frac{a}{q} + \delta, \ldots, \frac{a}{q} + \delta, \frac{1}{q} - 2\delta, \frac{1}{q} - 2\delta, \frac{1}{q} - 2\delta)$. Once the process is in a separating aspiration, no mutation of one player can disturb it. Therefore, (minimal) locally stable sets coincide with separating aspirations in this game.

Example 5 Consider game $[16; 2, \ldots, 2, 1 \ldots, 1]$, with $n = 18$ players, $p = 13, n - p = 5$. Winning coalitions have 8 large players, 7 large players and 2 small players, and 6 large players and 4 small players.

Suppose $b = \frac{2}{16}$ and $c = \frac{1}{16}$. Consider aspiration $x^{(1)} = (b, b, b + \delta, \ldots, b + \delta, c - \delta, \ldots, c - \delta)$, with coalitions of 6 large players and 4 small players in $GC(x^{(1)})$. All such coalitions contain players 1 and 2. If player 1 (or 2) mutates, the basic process of adjustment reaches a permutation of either $x^{(1)}$, or of $x^{(2)} = (b - \delta, b + \delta, \ldots, b + \delta, c - \delta, \ldots, c - \delta)$, or of $x^{(3)} = (b, b + \delta, \ldots, b + \delta, c - 2\delta, c - \delta, \ldots, c - \delta)$.

Consider $x^{(2)}$. All coalitions in $GC(x^{(2)})$ contain player 1. If player 1 mutates, we reach a permutation of either $x^{(1)}$, or of $x^{(2)}$, or of $x^{(3)}$, or of $x^{(4)} = (b + \delta, \ldots, b + \delta, c - 3\delta, c - \delta, \ldots, c - \delta)$.

Consider $x^{(3)}$. All feasible coalitions contain player 1 and player 14 (with demand $c - 2\delta$). If player 1 mutates, we reach a permutation of either $x^{(4)}$ or of $x^{(5)} = (b + \delta, \ldots, b + \delta, c - 2\delta, c - 2\delta, c - \delta, c - \delta, c - \delta)$. If player 14 mutates, we reach a permutation of either $x^{(1)}$, or of $x^{(2)}$, or of $x^{(3)}$.

Consider $x^{(4)}$. All coalitions in $GC(x^{(4)})$ contain player 14 with demand $c - 3\delta$. If player 14 mutates, we reach a permutation of either $x^{(2)}$, or of $x^{(3)}$, or of $x^{(4)}$, or of $x^{(5)}$ after an adjustment by the basic process.

Finally, consider $x^{(5)}$. All feasible coalitions contain player 14 with demand $c - 2\delta$. If player 14 mutates, the basic adjustment process reaches a permutation of either $x^{(2)}$, or of $x^{(3)}$, or of $x^{(4)}$, or of $x^{(5)}$.

The set of aspirations that are permutations of $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}$ is a (minimal) locally stable set in this game. Aspirations that are permutations of $x^{(4)}$ are 2δ away (by maximal coordinate-wise difference) from the separating aspiration $(b+\frac{2}{3}\delta, \ldots, b+\frac{2}{3}\delta, c-\delta, \ldots, c-\delta)$ and permutations of $x^{(2)}$ are 2δ away from $(b+\delta, \ldots, b+\delta, c-\frac{3}{2}\delta, \ldots, c-\frac{3}{2}\delta)$, while permutations of $x^{(1)}, x^{(3)}, x^{(5)}$ are less than 2δ away from a separating aspiration.

Example 5 illustrates that there may be locally stable sets that do not coincide with (indeed, do not even contain) separating aspirations. Nevertheless, Proposition 5 shows that a sequence of mutations, one at a time, can come close to a symmetric aspiration (within δ) in any two-type game. The following proposition describes how far away from a separating (and therefore symmetric) aspiration can such a sequence get.

Proposition 7 *Consider a constant-sum homogeneous weighted majority game with representation* $[q; a, \ldots, a, 1, \ldots, 1]$ *and suppose* $ap \geq q$. *Then, for any* δ, *there exists* k *such that all aspirations in locally stable sets are* $k\delta$*-close to separating aspirations. (In particular, one can choose* $k = s$.*)*

Proof From the proof of Proposition 5, either a 0-symmetric aspiration is reached (Case 1; it is then separating in the class of games under consideration), or an aspiration with $d_1 = 0$ and $d_a \leq \delta$ can be reached (Case 2(b)). Let this latter aspiration be $x^{(1)} = (b, \ldots, b, b+\delta, \ldots, b+\delta, c, \ldots, c)$. For it to be an aspiration (i.e. to have all coalitions maximal) $b \geq \frac{a}{q}$, $c < \frac{1}{q}$ and there is a feasible coalition with the maximal number $r = a\left\lfloor \frac{n-p}{a} \right\rfloor$ of small players and the minimal number $s = \frac{q-r}{a}$ of large players in it.

In $x^{(1)}$, all feasible coalitions contain all players with demand b, while for any other player j (with demand $b+\delta$ or c), for any other player i there exists S such that $i \in S$, $j \notin S$. If any of players like j mutates upwards, no other player would need to adjust and the process would return to $x^{(1)}$.

Suppose $s' < s$ players demand b. If each of them mutates in turn, and each time the same small player adjusts, then, like in Examples 4 and 5, an aspiration $x^{(k)} = (b+\delta, \ldots, b+\delta, c-s'\delta, c, \ldots, c)$ can be reached. As in the examples, of all the aspirations that can be reached by a sequence of one-player upward mutations, $x^{(k)}$ has the smallest demand $(c-s'\delta)$ of a small player.

If $s \leq r+1$ (as in Example 4), then a separating aspiration can be reached with a non-zero probability because those small players who demand more than others adjust, and all r small players demand $c-\delta$. In this case, aspirations in (minimal) locally stable sets are separating, in particular, they are 0-close to separating aspirations thus also k-close. If $s > r+1$ (as in Example 5), then there are not enough small players for all to demand $c-\delta$. Then a separating aspiration is never reached. But among aspirations in locally stable sets, the one furthest away from a separating aspiration is $x^{(k)}$, which is at most $s\delta$ away from $(b, \ldots, b, c, \ldots, c)$. □

Therefore, even if the process with mutations has locally stable sets that do not coincide with separating aspirations, these locally stable sets, for any given game, only have aspirations that are close to separating aspirations (especially for small δ).

On the other hand, for any fixed number k one can construct a game (with many players) such that locally stable sets contain aspirations that are more than $k\delta$ away from separating aspirations. In particular, to move $k\delta$ away from a separating aspiration, consider a game with an aspiration in which $r = a\left\lfloor \frac{n-p}{a} \right\rfloor$ small players demand δ less than in a separating aspiration, r large players demand δ more, and $k < r$ demand exactly what is in a separating aspiration. A minimal winning coalition S contains $k+r$ large players and r small players. This coalition has weight $\sum_{i \in S} w_i = a(k+r) + r = q$. The total weight of players is $\sum_{i \in N} w_i = 2q - 1 = 2a(k+r) + 2r - 1$. With $r+1$ small players, there are thus $\frac{2a(k+r)+r-2}{a}$ large players. For any r, a game with the total number of players larger than

$2(k+r)+\frac{r-2}{a}+r+1$ can therefore have locally stable sets that contain aspirations that are $k\delta$ away from separating ones.

6 Conclusion

This paper presented a simple "greedy" process of demand adjustment in cooperative games, in which players set maximal demands compatible with the demands of other players. This basic process converges to the set of aspirations; in convex games this means that it converges to the core.

We further extended the process by introducing "greedy" mutations, that is, mutations to higher demands. Separating aspirations then play an important role since they are the most resistant to such mutations. We analyzed this process in 3-player games and weighted majority games with two types of players.

For 3-player games, we derived complete results: either the core (when non-empty), or the unique separating aspiration (when the core is empty) are stochastically stable. For weighted majority games with two types of players, we found the following. In games in which the coalition of all small players is winning, there is a unique separating aspiration, which is stochastically stable. However, in games in which the coalition of large players is winning, the process does not necessarily converge to a separating aspiration. Instead, it will reach a neighborhood of such aspirations, where it remains. Hence, the simple process, augmented with appropriate mutations, can provide useful predictions for many games, also for those with an empty core.

Declarations

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