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The indecomposable objects in the center of Deligne's category $\underline{\mathrm{Rep}}\, S_t$

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Abstract

We classify the indecomposable objects in the monoidal center of Deligne's interpolation category $\underline{\mathrm{Rep}}\, S_t$ by viewing $\underline{\mathrm{Rep}}\, S_t$ as a model-theoretic limit in rank and characteristic. We further prove that the center of $\underline{\mathrm{Rep}}\, S_t$ is semisimple if and only if t is not a non-negative integer. In addition, we identify the associated graded Grothendieck ring of this monoidal center with that of the graded sum of the centers of representation categories of finite symmetric groups with an induction product. We prove analogous statements for the abelian envelope.

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1 | INTRODUCTION

In the seminal paper [14], P. Deligne constructed symmetric tensor categories $\underline{\mathrm{Rep}}\, S_t$, where t can be any complex number, which interpolate the categories of representations over the symmetric groups S_d, $d \in \mathbb{Z}_{\geqslant 0}$. These categories, and their relatives for other series of groups, have proven interesting to the study of symmetric tensor categories, as well as to the study of stability phenomena in representation theory, for example, see [3, 8, 9, 19–21]. The category $\underline{\mathrm{Rep}}\, S_t$ can be constructed using the combinatorics of partitions (see Section 2.2) and has a universal property with respect to Frobenius algebra objects of dimension t in symmetric monoidal categories.

In addition to giving this combinatorial definition and universal property Deligne observed that for t transcendental $\underline{\mathrm{Rep}}\, S_t$ can be realized as an ultraproduct, a sort of model-theoretic limit,

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of the categories $\operatorname{Rep} S_d$, as $d \in \mathbb{Z}_{\geqslant 0}$ grows to infinity. This understanding was extended to all values of t by the second listed author in [26, Theorem 1.1] by viewing $\underline{\operatorname{Rep}} S_t$ as a limit of the categories $\underline{\operatorname{Rep}}_p S_d$, varying the rank d as well as the characteristic p. We recall the main ideas of this approach in Sections 2.3–2.4.

In this paper, we apply these ultrafilter techniques to prove several results on the monoidal centers $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$ of Deligne's categories. The *monoidal* or *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ [29, 33] of a monoidal category $\mathcal{C}$ is a universal construction of a braided monoidal category with a forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ and instrumental in the construction of quantum groups and solutions to the quantum Yang–Baxter equation, see, for example, [2, 30, 32].

In [24], the first and third listed authors started the investigation of $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$, showed that this is a ribbon category, and obtained invariants of framed links as an application. It was shown that the braided categories $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$ interpolate the braided categories $\mathcal{Z}(\operatorname{Rep} S_d)$ in the sense that $\mathcal{Z}(\operatorname{Rep} S_d)$ is the semisimplification of $\mathcal{Z}(\underline{\operatorname{Rep}} S_d)$ for $d \in \mathbb{Z}_{\geqslant 0}$. In the present paper, we answer several open structural questions about the categories $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$ including a classification of their indecomposable objects and computation of the (associated graded) Grothendieck rings.

We start the paper by providing some required background results on $\underline{\operatorname{Rep}} S_t$ in Section 2. Many of the results stated that there are known to experts but are sometimes not available in the literature. Thus, we have included proofs when appropriate.

A key statement we prove in Section 3.1 displays $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$ as a model-theoretic limit in rank and characteristic, similar to $\underline{\operatorname{Rep}} S_t$. For this, we define a bi-filtration on the center, with the filtration layer $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)^{\leqslant m,k}$ being the full subcategory on objects which are in the preimage of $(\underline{\operatorname{Rep}} S_t)^{\leqslant m,k}$ under the forgetful functor $\mathcal{Z}(\underline{\operatorname{Rep}} S_t) \to \underline{\operatorname{Rep}} S_t$. Here, $(\underline{\operatorname{Rep}} S_t)^{\leqslant m,k}$ consists of objects that are direct summands of objects of the form $X^{\otimes m_1} \oplus \ldots \oplus X^{\otimes m_k}$, with $m_i \leqslant m$, where X is the generating object of $\underline{\operatorname{Rep}} S_t$. With respect to this filtration, we show that $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)^{\leqslant k,m}$ is equivalent to the ultraproduct of the categories $\mathcal{Z}(\operatorname{Rep}_{\overline{\mathbb{F}}_{p_i}} S_{n_i})^{\leqslant k,m}$ with partially defined monoidal and additive structures, see Proposition 3.1. In particular, this enables us to solve the question of semisimplicity of $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$ raised in [24, Question 3.31]. Recall that $\underline{\operatorname{Rep}} S_t$ is semisimple if and only if $t \notin \mathbb{Z}_{\geqslant 0}$.

Theorem 3.3. *The category $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$ is semisimple if and only if $t \notin \mathbb{Z}_{\geqslant 0}$.*

Next, we construct a $\mathbb{C}$-linear functor

$$\underline{\operatorname{Ind}} : \mathcal{Z}(\operatorname{Rep} S_n) \boxtimes \underline{\operatorname{Rep}} S_{t-n} \longrightarrow \mathcal{Z}(\underline{\operatorname{Rep}} S_t). \tag{1.1}$$

for every $n \geqslant 0$ and $t \in \mathbb{C}$. This functor $\underline{\operatorname{Ind}}$ is a *separable Frobenius monoidal functor* compatible with the braidings, see Proposition 3.7. It enables us to classify the indecomposable objects in $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$. Let $n \geqslant 0$ be an integer, μ a singleton free partition of n, $Z(\mu)$ the centralizer of an element $\sigma \in S_n$ of cycle type μ, V an $Z(\mu)$-module, and U an object in $\underline{\operatorname{Rep}} S_{t-n}$. We denote the image of the object $\operatorname{Ind}_{Z(\mu)}^{S_n}(V) \boxtimes U$ under $\underline{\operatorname{Ind}}$ by $\underline{W}_{\mu,V,U}$. Up to isomorphism the object $\underline{W}_{\mu,V,U}$ does not depend on the choice of σ, and only depends on the isomorphism classes of U and V.

Theorem 3.9. *$\underline{W}_{\mu,V,U}$ is indecomposable if and only if V and U are, and the objects $\underline{W}_{\mu,V,U}$ for n, μ, V, U as above with V and U indecomposable form a complete list of all indecomposable objects in $\mathcal{Z}(\underline{\operatorname{Rep}} S_t)$ up to isomorphism.*

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The question of classifying the indecomposable objects in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ naturally emerged from the paper [24] but was independently raised by P. Etingof in his research statement. Moreover, in Corollary 3.14 we describe the blocks of the category $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ as

$$B_{\mu,V,B} = \{\underline{W}_{\mu,V,U} \mid U \in B\},$$

where B is a block of $\underline{\mathrm{Rep}}\, S_{t-n}$ as classified in [8], and the pair (μ, V) is as above (parametrizing blocks of $\mathcal{Z}(\mathrm{Rep}\, S_n)$ not induced from $\mathcal{Z}(\mathrm{Rep}\, S_m)$ with $m < n$).

We also classify the indecomposable and indecomposable projective objects in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$, for the *abelian envelope* $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ of $\underline{\mathrm{Rep}}\, S_d$, $d \in \mathbb{Z}_{\geqslant 0}$, as constructed in [9], see Section 3.7. Abelian envelopes and their general theory have been receiving an increasing amount of attention recently [4, 10, 19]. We show that $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ indeed satisfies the universal property of the abelian envelope of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$ in the sense of [4], see Corollary 3.24 and Appendix A.

We note that the first paper [24] on $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ already contained a general construction of objects via explicit idempotents and half-braidings using the combinatorial description of $\underline{\mathrm{Rep}}\, S_t$ by partitions. We identify the objects constructed in [24] in the image of $\underline{\mathrm{Ind}}$ in Section 3.6 and prove that these objects generate $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ as a Karoubian tensor category but are, in general, not indecomposable.

In Section 4, we address the question of describing the associated graded Grothendieck ring $\mathrm{gr}\, K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$ which was suggested by V. Ostrik. To this end, we introduce an induction tensor product structure on the direct sum of categories $\mathcal{Z}(\mathrm{Rep}\, S_n)$. Namely, we define in Section 4.1 the abelian monoidal category

$$\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0} := \bigoplus_{n \geqslant 0} \mathcal{Z}(\mathrm{Rep}\, S_n)$$

with the tensor product of an object V in $\mathcal{Z}(\mathrm{Rep}\, S_n)$ and W in $\mathcal{Z}(\mathrm{Rep}\, S_m)$ given by

$$V \odot W := \mathcal{Z}\,\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (V \boxtimes W) \in \mathcal{Z}(\mathrm{Rep}\, S_{n+m}).$$

Note that $\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0}$ is a tower of centers, *not* the center of a tower of representation categories. Here, $\mathcal{Z}\,\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (V \boxtimes W)$ is the usual induction of group representations with additional half-braiding defined in Proposition B.1. This induction product on the sum (or *tower*) of centers can be applied to other series of groups and may be of independent interest. More generally, in Appendix B we show that induction produces separable Frobenius monoidal functors

$$\mathcal{Z}(\mathrm{Rep}\, G) \longrightarrow \mathcal{Z}(\mathrm{Rep}\, H)$$

if $G \subseteq H$ is a subgroup.

From (1.1) we obtain an oplax monoidal functor

$$\underline{\mathrm{Ind}} : \mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0} \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t), \quad V \longmapsto \underline{\mathrm{Ind}}(V \boxtimes \mathbf{1}), \tag{1.2}$$

see Section 4.2, and a description of the associated graded of the *additive* Grothendieck ring $K_0^{\oplus}$.

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Theorem 4.4. *The functor* $\underline{\mathrm{Ind}}$ *from (1.2) induces an isomorphism of graded rings*

$$\operatorname{gr} K_0^{\oplus}(\underline{\mathrm{Ind}}) : K_0(\mathcal{Z}\operatorname{Rep} S_{\geqslant 0}) \xrightarrow{\sim} \operatorname{gr} K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)),$$

where the associated graded of $K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$ *is taken with respect to the filtration induced by the filtration* $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k}$ *of* $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$.

An analogous statement holds for the abelian envelope $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)$ if $t = d \in \mathbb{Z}_{\geqslant 0}$, see Theorem 4.5. Computations in $\mathcal{Z}\underline{\mathrm{Rep}}\, S_{\geqslant 0}$ — and hence in $K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_t)$ — can be carried out by computing induction of modules over centralizer groups of symmetric groups. Some sample computations are included in Section 4.4.

A particularly important class of tensor categories are *modular (fusion) categories* which find applications in topological field theory, see [41] and references therein. In particular, the center $\mathcal{Z}(\operatorname{Rep} G)$, for G a finite group and its cocycle twists appear in Dijkgraaf–Witten theory [16]. Modular categories and some of their applications have been generalized to non-semisimple finite tensor categories [5, 25]. In this generality, a *modular category* is a non-degenerate finite ribbon tensor category.

The categories $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, for t generic, and $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)$, for $d \in \mathbb{Z}_{\geqslant 0}$, are *infinite* analogs of modular categories. This interpretation follows from [24, Theorem 3.27] where $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ was shown to be a ribbon category and Section 3.8 where we prove that $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ and $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)$ are non-degenerate braided tensor categories. We note that these categories are also *factorizable* braided tensor categories by [22, Proposition 8.6.3]. In the finite case, non-degeneracy and factorizability of braided tensor categories are equivalent [39].

As $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ and $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)$ are infinite tensor categories, the concept of modular category and applications to topological field theory have not been developed for these categories. However, we note that these categories satisfy all conditions (beside finiteness) imposed on modular categories. For t generic, the category $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is, moreover, semisimple. Hence, the results on $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ of this paper and [24] give interesting infinite yet locally finite analogs of modular categories which are not equivalent to (co)modules over (quasi)-Hopf algebras.

2 | BACKGROUND

2.1 | Notation and conventions

In the following, $\Bbbk$ denotes a field and $\operatorname{Rep}_{\Bbbk} G$ the category of finite-dimensional G-representations over $\Bbbk$, with $\operatorname{Rep} G = \operatorname{Rep}_{\mathbb{C}} G$. Given a prime p, we abbreviate $\operatorname{Rep}_p G = \operatorname{Rep}_{\overline{\mathbb{F}}_p} G$.

The categories considered in this paper are, at the very least, $\Bbbk$-linear Karoubian rigid monoidal categories. The *Karoubian envelope* of a $\Bbbk$-linear category is the idempotent completion of the closure under finite direct sums, and a category is *Karoubian* if the inclusion into its Karoubian envelope gives an equivalence of categories. In general, the symbol $\mathcal{C} \boxtimes \mathcal{D}$ of two such categories $\mathcal{C}, \mathcal{D}$ denotes the external product as in [31, 35, Section 2.2] which is the Karoubian envelope of the naive $\Bbbk$-linear tensor product which has objects $X \boxtimes Y$, for $X \in \mathcal{C}, Y \in \mathcal{D}$. Note that in most cases studied, $\mathcal{C}$ is a finite semisimple (abelian) category, which implies that $\mathcal{C} \boxtimes \mathcal{D}$ is a finite direct sum of copies of $\mathcal{D}$. If, in addition, $\mathcal{D}$ is abelian, then $\mathcal{C} \boxtimes \mathcal{D}$ is abelian.

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For terminology on monoidal, braided and symmetric monoidal categories we follow [22]. In particular, an (abelian) tensor category is a $\Bbbk$-linear rigid monoidal category which is locally finite, abelian, with $\mathrm{End}(\mathbf{1}) = \Bbbk$, for the tensor unit $\mathbf{1}$. A *Karoubian tensor category* shall satisfy the same conditions with one exception: instead of being abelian, we require it only to be additive and idempotent-complete.

An important technical tool used in this paper is that of ultraproducts of categories (see, for example, [11]). For simplicity, we assume that the categories considered here are small so that objects and morphisms form sets. We replace representation-theoretic categories by equivalent small ones, noting that up to equivalence the ultraproduct will not depend on the choice of equivalent small categories. We assume throughout that $\mathcal{U}$ is a fixed (non-principal) *ultrafilter* on $\mathbb{N}$ and refer the reader to, for example, [37] for generalities on this concept. We may think of $\mathcal{U}$ as a collection of subsets of $\mathbb{N}$, each of which contains 'almost all' numbers.

Given an ultrafilter $\mathcal{U}$ and a collection of categories $(\mathcal{C}_i)_{i\in\mathbb{N}}$ we can defined their *ultraproduct* $\prod_{\mathcal{U}} \mathcal{C}_i$. Its objects are sequences $\prod_{\mathcal{U}} V_i$ of objects V_i of $\mathcal{C}_i$ defined for all i in a set belonging to the ultrafilter. Two such sequences are equal if they agree on all indices of some set belonging to $\mathcal{U}$. Similarly, morphisms are sequences $\prod_{\mathcal{U}} f_i : \prod_{\mathcal{U}} V_i \to \prod_{\mathcal{U}} W_i$ of morphisms $f_i : V_i \to W_i$ which are defined on a set belonging to the ultrafilter and identified when they agree on some set of the ultrafilter $\mathcal{U}$. We refer the reader to [11] or [26] for further explanations. If all categories $\mathcal{C}_i$ are $\Bbbk_i$-linear, then $\prod_{\mathcal{U}} \mathcal{C}_i$ is linear over the ultraproduct of fields $\prod_{\mathcal{U}} \Bbbk_i$. If all $\mathcal{C}_i$ are monoidal categories, then $\prod_{\mathcal{U}} \mathcal{C}_i$ is naturally a monoidal category.

We will make essential use of Łoś' theorem [37, Theorem 1.3.2] which allows us to transfer any first-order logical statement from the categories $\mathcal{C}_i$ to their ultraproduct $\prod_{\mathcal{U}} \mathcal{C}_i$. See also [28, Section 1] for examples on how this theorem is used. We further need Steinitz' theorem that states that an uncountable algebraically closed field is determined, up to isomorphism, by its characteristic and uncountable cardinality [40]. This theorem implies the existence of isomorphisms of the ultraproducts of fields $\prod_{\mathcal{U}} \mathbb{C}$, or $\prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i}$, for a sequence of primes with $\liminf_i p_i = \infty$, and the complex numbers, see, for example, [37, Chapter 1]. In the following, we consider equivalences of monoidal categories linear over an ultraproduct $\prod_{\mathcal{U}} \Bbbk_i$ of algebraically closed fields, which we regard as equivalences of $\mathbb{C}$-linear monoidal categories under an isomorphism of fields $\prod_{\mathcal{U}} \Bbbk_i \cong \mathbb{C}$ obtained from Steinitz' theorem.

2.2 | Deligne's categories as diagrammatic categories

In [14], Deligne constructs a class of Karoubian $\Bbbk$-linear symmetric monoidal categories $\underline{\mathrm{Rep}}\, S_t$ depending on a parameter t in the field $\Bbbk$ of characteristic zero. It is a well-known observation that every simple complex S_n-module appears as a direct summand of a tensor power $X_n^{\otimes k}$ of the n-dimensional permutation representation X_n of S_n for some $k \geqslant 1$. In other words, the category $\mathrm{Rep}\, S_n$ of finite-dimensional $\mathbb{C}S_n$-modules is the *Karoubian envelope* of the monoidal category generated by the single object X_n, cf. [14, §1.7–1.8]. The morphism spaces $\mathrm{Hom}(X_n^{\otimes k}, X_n^{\otimes l})$ are given by the S_n-invariants of $X_n^{\otimes(k+l)}$ and can be described combinatorially using the partition algebras $P_k(n)$, see, for example, [8, Section 2] for details. To define $\underline{\mathrm{Rep}}\, S_t$, n is now replaced by a general parameter $t \in \Bbbk$.

Deligne's category $\underline{\mathrm{Rep}}\, S_t$ can be constructed using a graphical calculus: it has a distinguished object X represented by a point and its tensor powers $X^{\otimes k}$ for $k \geqslant 0$ represented by k points. Morphisms between two such tensor powers $X^{\otimes k}$ and $X^{\otimes l}$ are represented using diagrams consisting of k upper points labeled $1, \dots, k$ and l lower points labeled $1', \dots, l'$, and an arbitrary number of

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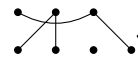
strings connecting points. Two such diagrams are considered equivalent, if the partitions of the points $1, \dots, k, 1', \dots, l'$ given by the connected components of each string diagram coincide. The morphism spaces between $X^{\otimes k}$ and $X^{\otimes l}$ are defined as the free $\Bbbk$-vector space spanned by the equivalence classes of such diagrams, so tensor products and compositions can be defined on diagrams and extended linearly. The following is a typical string diagram representing a morphism in $\underline{\mathrm{Rep}}\, S_t$:



The tensor product of two diagrams is given by stacking the diagrams horizontally. The composition of two diagrams π, μ is achieved by first stacking them vertically, and identifying the lower points of π with the upper points of μ. Then these identified points are removed from the string diagram, leaving only the upper points of π, the lower points of μ, and a number of strings. Each connected component of this string diagram which does not contain upper points of π or lower points of μ (such components may arise when removing the identified points) is removed and the resulting string diagram is multiplied by a factor t^{ℓ}, where $\ell \geqslant 0$ is the number of connected components removed in the process. The category $\underline{\mathrm{Rep}}\, S_t$ is now defined as the Karoubian envelope of the category defined on the objects $X^{\otimes k}$, for $k \geqslant 0$. We refer the reader to [14] and [8, Section 2] for details on the combinatorial construction of $\underline{\mathrm{Rep}}\, S_t$.

In the generic case, that is, if $t \notin \mathbb{Z}_{\geqslant 0}$, $\underline{\mathrm{Rep}}\, S_t$ is a semisimple symmetric tensor category. If $t = d$ is a non-negative integer, there is an essentially surjective full monoidal functor

$$\mathcal{F}_d : \underline{\mathrm{Rep}}\, S_d \longrightarrow \mathrm{Rep}\, S_d, \tag{2.1}$$

which maps X to the d-dimensional standard S_d-module X_d and, consequently, $X^{\otimes k}$ to $(X_d)^{\otimes k}$ for any $k \geqslant 0$. Choosing a basis $e_1, \dots, e_d$ in X_d provides a basis $(e_{\mathbf{i}} = e_{i_1} \otimes \cdots \otimes e_{i_k})_{i_1,\dots,i_k}$ indexed by tuples $\mathbf{i} = (i_1, \dots, i_k)$ for $\mathcal{F}_d(X^{\otimes k})$, for any $k \geqslant 0$, and the image $\mathcal{F}_d(\pi)$ of any diagram π with k upper and l lower points is the S_d-module morphism sending

$$e_{\mathbf{i}} \longmapsto \sum_{\mathbf{j}=(j_1,\dots,j_l)} f(\pi)^{\mathbf{i}}_{\mathbf{j}} e_{\mathbf{j}}, \tag{2.2}$$

where the coefficient $f(\pi)^{\mathbf{i}}_{\mathbf{j}}$ is 1 if the indices $i_1, \dots, i_k, j_1, \dots, j_l \in \{1, \dots, t\}$ induce a partition on the upper and lower points of the diagram π which refines the one given by the connected components of π, and 0 otherwise. The functors $\mathcal{F}_d$ allow us to view $\underline{\mathrm{Rep}}\, S_t$ as an interpolation category for the classical symmetric tensor categories $\mathrm{Rep}\, S_d$.

Further, we recall the (recursive) definition of the morphism x_π from [8, Equation (2.1)],

$$x_\pi = \pi - \sum_{\tau} x_\tau, \tag{2.3}$$

where the sum is taken over all partitions τ strictly coarser than π. Note that the set $\{x_\pi\}$, for all partitions of $\{1, \dots, k, 1', \dots, l'\}$, gives a basis for $\mathrm{Hom}_{\underline{\mathrm{Rep}}\, S_t}(X^{\otimes k}, X^{\otimes l})$. If, again, $t = d$, then $\mathcal{F}_d(x_\pi)$ is the S_d-module morphism

$$e_{\mathbf{i}} \longmapsto \sum_{\mathbf{j}=(j_1,\dots,j_l)} f'(x_\pi)^{\mathbf{i}}_{\mathbf{j}} e_{\mathbf{j}}, \tag{2.4}$$

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where now the coefficient $f'(x_\pi)^{\mathbf{i}}_{\mathbf{j}}$ is 1 if the indices $i_1, \dots, i_k, j_1, \dots, j_l \in \{1, \dots, t\}$ induce the same partition on the upper and lower points of the diagram π as the one given by the connected components of π, and 0 otherwise.

Indecomposable objects in $\underline{\mathrm{Rep}}\, S_t$ are classified by partitions, see [8, Section 3.1].

Theorem 2.1. *There is a bijection between partitions $\lambda \vdash n$, for $n \geqslant 0$, and indecomposable objects X_λ of $\underline{\mathrm{Rep}}\, S_t$. The object X_λ is a direct summand of $X^{\otimes n}$, but not of $X^{\otimes i}$ for $i < n$.*

For $t \notin \mathbb{Z}_{\geqslant 0}$ the objects X_λ behave uniformly in t. Their dimensions and character values are given by polynomials in t, and they can be cut out of $X^{\otimes n}$ by primitive idempotents which are $\mathbb{Q}(t)$-linear combinations of partition diagrams.

At non-negative integral values $t = d \in \mathbb{Z}_{\geqslant 0}$, some of the rational functions defining these primitive idempotents develop poles and the idempotents no longer exist. As such, some surviving idempotents which are generically not primitive become primitive at these special values, and the corresponding generically simple objects get 'glued together' in a sense.

As part of their analysis of blocks, Comes and Ostrik described a process of 'lifting' which takes an indecomposable object X_λ at $t = d$, and describes how it splits apart when we deform it to nearby semisimple values of t. The combinatorics of this process is completely described by Comes and Ostrik [8], but we will just use the following simplified version.

Theorem 2.2 ([8, Proposition 3.10, Proposition 3.12(a), Lemma 5.20]). *For every object $X \in \underline{\mathrm{Rep}}\, S_d$ there is an object $\mathrm{Lift}_d(X)$ in $\underline{\mathrm{Rep}}\, S_t$, uniquely defined up to isomorphism, for t in a formal neighborhood of d such that for all $X, Y \in \underline{\mathrm{Rep}}\, S_d$,*

(1) $\mathrm{Lift}_d(X \oplus Y) \cong \mathrm{Lift}_d(X) \oplus \mathrm{Lift}_d(Y)$;
(2) $\mathrm{Lift}_d(X \otimes Y) \cong \mathrm{Lift}_d(X) \otimes \mathrm{Lift}_d(Y)$;
(3) *for any indecomposable object $X_\lambda \in \underline{\mathrm{Rep}}\, S_d$, either*
   (a) $\mathrm{Lift}_d(X_\lambda)$ *remains indecomposable and is still labeled by X_λ*
   or
   (b) $\mathrm{Lift}_d(X_\lambda)$ *decomposes as $X_\lambda \oplus X_{\lambda'}$ for a partition λ' depending on d and λ satisfying $|\lambda| > |\lambda'|$.*

Moreover each $X_{\lambda'}$ arises as a summand of $\mathrm{Lift}_d(X_\lambda)$ for at most one partition λ with $|\lambda| > |\lambda'|$.

Remark 2.3. We believe that over the complex numbers one can check that this lifting operation can be defined for t in an analytic neighborhood of d, rather than just a formal one. For our purposes though this will not be necessary, so we will not pursue this direction further.

2.3 | Deligne's categories as limits in characteristic and rank

We now recall [26, Theorem 1.1] to display $\underline{\mathrm{Rep}}\, S_t$, for *any* $t \in \mathbb{C}$, as an ultraproduct of the representation categories $\mathrm{Rep}_p\, S_d$, cf. Section 2.1. The case for t transcendental was already contained in Deligne's seminal paper [14]. We are interested in three cases of this theorem to describe all cases of Deligne's category $\underline{\mathrm{Rep}}\, S_t$ up to equivalence of symmetric monoidal categories (under isomorphisms of fields $\mathbb{C} \to \mathbb{C}$ that send one transcendental parameter to the other).

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Definition 2.4. For any object X in a Karoubian or abelian tensor category, we denote by $\langle X \rangle$ the full Karoubian tensor subcategory generated by X.

Note that $\langle X \rangle$ is generated by duals, tensor products, direct sums, and direct summands involving X. In the situations we discuss, X will often be a self-dual object.

Theorem 2.5 ([14, 26, Theorem 1.1]). *In each case below, we specify an increasing sequence of positive integers* $\mathbf{t} = (t_i)_{i\in\mathbb{N}}$ *and a sequence of fields* $(\Bbbk_i)_{i\in\mathbb{N}}$. *We denote by* X *the object* $\prod_{\mathcal{U}} X_{t_i}$ *in* $\prod_{\mathcal{U}} \operatorname{Rep}_{\Bbbk_i} S_{t_i}$, *and by* t *the complex number corresponding to* $\prod_{\mathcal{U}} t_i$ *under the respective isomorphism of fields* $\prod_{\mathcal{U}} \Bbbk_i \cong \mathbb{C}$.

(1) *t transcendental: Consider a sequence* $\mathbf{t} = (t_i)_{i\in\mathbb{N}}$ *with* $\liminf_i t_i = \infty$ *and the ultraproduct* $\prod_{\mathcal{U}} \operatorname{Rep} S_{t_i}$. *Then, under an isomorphism of fields* $\prod_{\mathcal{U}} \mathbb{C} \cong \mathbb{C}$, $\langle X \rangle \subset \prod_{\mathcal{U}} \operatorname{Rep} S_{t_i}$ *is equivalent to* $\underline{\operatorname{Rep}}\, S_t$ *as a* $\mathbb{C}$*-linear symmetric monoidal category.*

(2) $t \in \overline{\mathbb{Q}} \setminus \mathbb{Z}_{\geqslant 0}$, *with minimal polynomial* m_t: *We can choose increasing sequences of positive integers* $\mathbf{t} = (t_i)_{i\in\mathbb{N}}$ *and primes* $\mathbf{p} = (p_i)_{i\in\mathbb{N}}$ *such that* $t_i < p_i$ *for any* $i \in \mathbb{N}$, *satisfying*

$$m_t(t_i) \equiv 0 \mod p_i, \qquad \forall i \in \mathbb{N}.$$

Then, under an isomorphism of fields $\prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i} \cong \mathbb{C}$, $\langle X \rangle \subset \prod_{\mathcal{U}} \operatorname{Rep}_{p_i} S_{t_i}$ *is equivalent to* $\underline{\operatorname{Rep}}\, S_t$ *as a* $\mathbb{C}$*-linear symmetric monoidal category.*

(3) $t = d \in \mathbb{Z}_{\geqslant 0}$: *Set* $t_i = p_i + d$ *for* p_i *the ith prime number. Then again, under an isomorphism of fields* $\prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i} \cong \mathbb{C}$, $\langle X \rangle \subset \prod_{\mathcal{U}} \operatorname{Rep}_{p_i} S_{p_i+d}$ *is equivalent to* $\underline{\operatorname{Rep}}\, S_d$ *as a* $\mathbb{C}$*-linear symmetric monoidal category.*

We remark that under the chosen isomorphisms of fields, the respective functors $\underline{\operatorname{Rep}}\, S_t \to \langle X \rangle$ are obtained by the universal property of $\underline{\operatorname{Rep}}\, S_t$ by sending the tensor generator of $\underline{\operatorname{Rep}}\, S_t$ to the object $X = \prod_{\mathcal{U}} X_{t_i}$ in the ultraproduct.

Since $\underline{\operatorname{Rep}}\, S_t$ is defined in terms of its generating object X, there is a natural exhaustive filtration $(\underline{\operatorname{Rep}}\, S_t)^{\leqslant k,m}$ encoding the complexity of objects in terms of this generator. Explicitly, $(\underline{\operatorname{Rep}}\, S_t)^{\leqslant k,m}$ is the full subcategory of objects which are direct summands of objects $X^{\otimes j_1} \oplus \cdots \oplus X^{\otimes j_\ell}$, where each j_i is at most k and the number of terms ℓ is at most m. We may similarly filter the categories $\operatorname{Rep}_{p_i} S_{n_i}$ in terms of the defining n_i-dimensional representation of S_{n_i}, analogously calling the filtered pieces $(\operatorname{Rep}_{p_i} S_{n_i})^{\leqslant k,m}$.

The advantage of these filtrations is that Theorem 2.5 tells us that $(\underline{\operatorname{Rep}}\, S_t)^{\leqslant k,m}$ is equivalent to the ultraproduct of $(\operatorname{Rep}_{p_i} S_{n_i})^{\leqslant k,m}$, with no need to further cut down the ultraproduct. These subcategories $(\underline{\operatorname{Rep}}\, S_t)^{\leqslant k,m}$ are not monoidal or even additive, as taking a tensor product or direct sum will possibly land you in a higher term of the filtration, but the ultraproduct does respect those products and sums that are defined.

Taking the union over all $m \in \mathbb{N}$ we obtain a coarser filtration $\mathcal{C}^{\leqslant k}$, for either $\mathcal{C} = \underline{\operatorname{Rep}}\, S_t$ or $\mathcal{C} = \operatorname{Rep}_{p_i} S_{n_i}$. These filtered pieces are additive and in fact abelian in the case where $t \notin \mathbb{Z}_{\geqslant 0}$. These are not monoidal subcategories but they satisfy the condition that if $V \in \mathcal{C}^{\leqslant k}$ and $W \in \mathcal{C}^{\leqslant k'}$, then $V \otimes W \in \mathcal{C}^{\leqslant k+k'}$, which makes the Grothendieck ring a filtered ring.

The descriptions of $\underline{\operatorname{Rep}}\, S_t$ can be used to transfer any first-order statement in the signature of symmetric monoidal categories with a distinguished object X from the classical (modular) representation theory to the limit, that is, the interpolation category $\underline{\operatorname{Rep}}\, S_t$ using Łoś' theorem. For more details and applications of this philosophy, see [26, 28].

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The combinatorial description of $\underline{\mathrm{Rep}}\, S_t$ recalled in Section 2.2 can be matched with the above characterization through ultrafilters in Theorem 2.5, giving an evaluation of the equivalence $\langle X \rangle \simeq \underline{\mathrm{Rep}}\, S_t$ on morphisms.

Given a partition π of $1, \dots, k, 1', \dots, l'$ viewed as a morphism $\pi : X^{\otimes k} \to X^{\otimes l}$, π corresponds to the ultraproduct $\prod_{\mathcal{U}} \pi_i$ described in the following. Using the notation $e_{\mathbf{i}}$ as in (2.2), $\pi_i : X_{t_i}^{\otimes k} \to X_{t_i}^{\otimes l}$ is defined by

$$\pi_i(e_{\mathbf{i}}) = \sum_{\mathbf{j}} f(\pi)^{\mathbf{i}}_{\mathbf{j}} e_{\mathbf{j}}, \tag{2.5}$$

for all k-tuples $\mathbf{i}$ and all l-tuples $\mathbf{j}$ of integers in $\{1, \dots, t_i\}$, where the coefficient $f(\pi)^{\mathbf{i}}_{\mathbf{j}}$ is the same as in (2.2), the complex case.

We may extend the assignment $\pi \mapsto \prod_{\mathcal{U}} \pi_i$ using the fixed isomorphism of fields $\mathbb{C} \xrightarrow{\sim} \prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i}$, $\alpha_i \mapsto \prod_{\mathcal{U}} \alpha_i$ by $\alpha\pi \mapsto \prod_{\mathcal{U}} \alpha_i \pi_i$, for $\alpha \in \mathbb{C}$. We note that $m \in \mathbb{Z}$ corresponds to $\prod_{\mathcal{U}} m$ under the isomorphism of fields, but for a general complex number α, α_i is only defined for almost all i.

2.4 | Representations of a fixed group in large characteristic

In addition to symmetric groups, we will often have an auxiliary finite group G for which we want to compare representations across different large characteristics.

Let G be a fixed finite group, and let X be a faithful representation of G defined over the integers (for example, the permutation representation). In a slight abuse of notation we will use X to denote the corresponding base changes to $\mathrm{Rep}\, G$ as well as to $\mathrm{Rep}_p\, G$. As X, in particular, labels an object in each of the categories $\mathrm{Rep}_{p_i}\, G$ for any sequence of primes $(p_i)_{i \in \mathbb{N}}$, we also have an object $\prod_{\mathcal{U}} X$ in the ultraproduct $\prod_{\mathcal{U}} \mathrm{Rep}_{p_i}\, G$, which we will also denote by X. The following result of Crumley realizes the characteristic zero representation theory of G as an ultraproduct of the representations of G in large characteristic.

Theorem 2.6 ([11, Section 9.5.1]). *Let $\mathbf{p} = (p_i)_{i \in \mathbb{N}}$ be an arbitrary increasing sequence of primes. Under an isomorphism of fields $\prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i} \cong \mathbb{C}$, there exists an equivalence of $\mathbb{C}$-linear symmetric monoidal categories between $\langle X \rangle \subset \prod_{\mathcal{U}} \mathrm{Rep}_{p_i}\, G$ and $\mathrm{Rep}\, G$.*

This result is nice in that it involves an equivalence of categories, and closely parallels the ultraproduct construction of Deligne categories above. However, in this case we can actually be very explicit about what happens at the level of objects, but first let us recall a bit about the representation theory of finite groups in large characteristic.

It was first observed by Dickson in 1902 [15] that if p does not divide $|G|$, then the representation theory of G over an algebraically closed field of characteristic p is 'the same' as over the complex numbers. Translated into a more modern set-up the following theorem explicitly describes this relationship. We refer to [38, Part III] for basic facts about the modular representation theory of finite groups.

Theorem 2.7. *Suppose that $\mathcal{O}$ is the ring of integers in a number field $\Bbbk$ such that every irreducible complex representation of G is defined over $\mathcal{O}$, p is a prime number not dividing $|G|$ or the*

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discriminant of $\mathcal{O}$, and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ lying above p. If V is an irreducible complex representation of G, choose an integral form $V_{\mathcal{O}}$ defined over $\mathcal{O}$ and define its reduction modulo p as $V_{\mathfrak{p}} := V_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p}$.

(1) *$V_{\mathfrak{p}}$ is an (absolutely) irreducible representation of G over $\mathcal{O}/\mathfrak{p}$.*
(2) *The isomorphism class of $V_{\mathfrak{p}}$ is independent of the choice of the integral form $V_{\mathcal{O}}$.*
(3) *If $\mathfrak{p}'$ is another prime lying above p, then there exists an automorphism σ of $\Bbbk$ sending $\mathfrak{p}$ to $\mathfrak{p}'$. The induced isomorphism $\tilde{\sigma} : \mathcal{O}/\mathfrak{p} \to \mathcal{O}/\mathfrak{p}'$ identifies $V_{\mathfrak{p}}$ with $V_{\mathfrak{p}'}$.*
(4) *Every irreducible representation of G over $\mathcal{O}/\mathfrak{p}$ arises this way.*
(5) *If V and W are two irreducible complex representations, then $V_{\mathfrak{p}} \cong W_{\mathfrak{p}}$ if and only if $V \cong W$.*

We are about ready to give an explicit description of what Crumley's equivalence does, but first let us recall basics about central characters. If $C \subset G$ is a conjugacy class, then the element $\sum_{g \in C} g \in Z(\mathbb{Z}G)$ defines an endomorphism of the identity functor in $\operatorname{Rep} G$ and $\operatorname{Rep}_p G$ for all primes p. Evaluating the trace of these endomorphisms on a representation V gives the central character of V. Over the complex numbers the central character of V is just a rescaling of the ordinary character, and in particular determines V up to isomorphism.

Fix $\mathcal{O}$ as in Theorem 2.7, and for each prime p_i as in Theorem 2.6 fix a prime ideal $\mathfrak{p}_i$ lying above p_i as well as an algebraic closure $\overline{\mathcal{O}/\mathfrak{p}_i}$ of $\mathcal{O}/\mathfrak{p}_i$. We may choose our isomorphism $\mathbb{C} \cong \prod_{\mathcal{U}} \overline{\mathcal{O}/\mathfrak{p}_i}$ such that on $\mathcal{O}$ it is the ultraproduct of the natural quotient maps $\mathcal{O} \to \mathcal{O}/\mathfrak{p}_i$.

Proposition 2.8. *Under the equivalence of categories defined in Theorem 2.6, the image of an irreducible representation $V \in \operatorname{Rep} G$ is isomorphic to $\prod_{\mathcal{U}} V_{\mathfrak{p}_i} \in \prod_{\mathcal{U}} \operatorname{Rep}_{\mathfrak{p}_i} G$.*

Proof. A priori we know that under this equivalence of categories $\prod_{\mathcal{U}} V_{p_i}$ gets identified with some irreducible complex representation V'; hence, it suffices to check that V and V' have the same central character. V is defined over $\mathcal{O}$, so therefore its central character is as well. Moreover, by construction the central character of $V_{\mathfrak{p}_i}$ is the reduction modulo $\mathfrak{p}_i$ of the central character of V. Since we chose our identification $\mathbb{C} \cong \prod_{\mathcal{U}} \overline{\mathcal{O}/\mathfrak{p}_i}$ to identify $\mathcal{O}$ with the product of its reductions modulo $\mathfrak{p}_i$, we see that indeed these characters agree. □

2.5 | The Grothendieck ring of $\underline{\operatorname{Rep}}\, S_t$

Given a $\Bbbk$-linear additive monoidal category $\mathcal{C}$, let $K_0^{\oplus}(\mathcal{C})$ denote its *additive Grothendieck ring*, that is, the quotient the free ring generated by all isomorphism classes of objects in $\mathcal{C}$ by the ideal of relations given through direct sums and tensor products. If we denote by $[Y]$ the symbol of an object Y of $\mathcal{C}$ inside of the ring $K_0^{\oplus}(\mathcal{C})$, these relations are

$$[Y_1] + [Y_2] = [Y_1 \oplus Y_2], \qquad [Y_1][Y_2] = [Y_1 \otimes Y_2]$$

for all $Y_1, Y_2 \in \mathcal{C}$.

The Grothendieck ring of $\underline{\operatorname{Rep}}\, S_t$ is a filtered ring: Recall the filtration on the category $\underline{\operatorname{Rep}}\, S_t$ from Section 2.3. Here, an object $Y \in \underline{\operatorname{Rep}}\, S_t$ is in $(\underline{\operatorname{Rep}}\, S_t)^{\leqslant k}$, for $k \geqslant 0$, if Y is isomorphic to a direct summand of a sum of objects $X^{\otimes l}$ with $l \leqslant k$, where X is the tensor generator of $\underline{\operatorname{Rep}}\, S_t$. This defines a filtration of the ring $K_0^{\oplus}(\underline{\operatorname{Rep}}\, S_t)$.

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Let us denote the irreducible complex S_n-module corresponding uniquely up to isomorphism to a partition $\lambda \vdash n$ by S^λ.

Lemma 2.9 ([14, Proposition 5.11], [8, Proposition 3.12], [27, Theorem 3.3]). *Sending* $[X_\lambda] \mapsto [S^\lambda]$ *induces an isomorphism* $\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_t) \cong \bigoplus_{n \geqslant 0} K_0(\mathrm{Rep}\, S_n)$*, where the product on the right-hand side is given by induction, or equivalently, by Littlewood–Richardson coefficients.*

As the ring on the right-hand side is a graded ring which does not depend on t, this result exhibits the Grothendieck ring of $\underline{\mathrm{Rep}}\, S_t$ as a filtered deformation of that ring.

2.6 | Induction and restriction between Deligne categories

The universal property of $\underline{\mathrm{Rep}}\, S_{t+k}$ gives a natural exact symmetric monoidal *restriction functor*

$$\underline{\mathrm{Res}}^{S_{t+k}}_{S_k \times S_t} : \underline{\mathrm{Rep}}\, S_{t+k} \longrightarrow \mathrm{Rep}\, S_k \boxtimes \underline{\mathrm{Rep}}\, S_t ,$$

which sends the defining object X_{t+k} to $X_k \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes X_t$. If $t = d$ is a non-negative integer, this descends to the ordinary restriction functor from $\mathrm{Rep}\, S_{d+k}$ to $\mathrm{Rep}\, S_k \boxtimes \mathrm{Rep}\, S_d \cong \mathrm{Rep}(S_k \times S_d)$, under the fiber functors $\mathcal{F}_{d+k}, \mathcal{F}_d$ to the semisimple categories from (2.1).

Etingof ([20], Section 2.3) considered *induction* and *co-induction* functors

$$\underline{\mathrm{Ind}}^{S_{t+k}}_{S_k \times S_t}, \widetilde{\underline{\mathrm{Ind}}}^{S_{t+k}}_{S_k \times S_t} : \mathrm{Rep}\, S_k \boxtimes \underline{\mathrm{Rep}}\, S_t \longrightarrow \underline{\mathrm{Rep}}\, S_{t+k} ,$$

defined a priori as the left and right adjoints to the restriction functor defined above. Etingof also considered restriction to a product of two Deligne categories, and in that case one needs to pass to an ind-completion in order to define these adjoints, but for our purposes we will only need the finite versions.

It was observed in [28] that if we think of $\underline{\mathrm{Rep}}\, S_t$ and $\mathrm{Rep}\, S_k$ as the model-theoretic limits of categories $\mathrm{Rep}_{p_i} S_{n_i}$ and $\mathrm{Rep}_{p_i} S_k$, respectively (in the sense described above), then these induction and co-induction functors are limits of the ordinary induction and co-induction functors

$$\mathrm{Ind}^{S_{t_i+k}}_{S_k \times S_{t_i}}, \widetilde{\mathrm{Ind}}^{S_{t_i+k}}_{S_k \times S_{t_i}} : \mathrm{Rep}_{p_i} S_k \boxtimes \mathrm{Rep}_{p_i} S_{t_i} \longrightarrow \mathrm{Rep}_{p_i} S_{k+t_i}$$

corresponding to the embedding of $S_k \times S_{t_i}$ into S_{k+t_i} for each t_i. Moreover, since induction and co-induction are naturally isomorphic for representations of finite groups (over any field), it follows that these Deligne category induction and co-induction functors are naturally isomorphic as well and we can view them as a single two-sided adjoint to the restriction functor which we will refer to just as induction.

These induction functors are also well behaved with respect to the filtrations $(\underline{\mathrm{Rep}}\, S_t)^{\leqslant m}$ defined earlier. In particular we have:

$$\underline{\mathrm{Ind}}^{S_{t+k}}_{S_k \times S_t} : \mathrm{Rep}\, S_k \boxtimes (\underline{\mathrm{Rep}}\, S_t)^{\leqslant m} \longrightarrow \underline{\mathrm{Rep}}\, S^{\leqslant m+k}_{t+k} , \tag{2.6}$$

moreover, this $m \to m+k$ shift in the filtration is optimal in a strong sense: If V is an object of $\mathrm{Rep}\, S_k \times (\underline{\mathrm{Rep}}\, S_t)^{\leqslant m}$ that does not lie in $\mathrm{Rep}\, S_k \times (\underline{\mathrm{Rep}}\, S_t)^{\leqslant m-1}$, then $\underline{\mathrm{Ind}}^{S_{t+k}}_{S_k \times S_t}(V)$ lies in $(\underline{\mathrm{Rep}}\, S_{t+k})^{\leqslant m+k}$ but not in $(\underline{\mathrm{Rep}}\, S_{t+k})^{\leqslant m+k-1}$.

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If $G \subset S_k$ is a subgroup, then we may further restrict from $\underline{\mathrm{Rep}}\, S_{t+k}$ to $\mathrm{Rep}\, G \boxtimes \underline{\mathrm{Rep}}\, S_t$. By the same reasoning as above, this also admits a two-sided adjoint induction functor

$$\underline{\mathrm{Ind}}_{G\times S_t}^{S_{t+k}} : \ \mathrm{Rep}\, G \boxtimes \underline{\mathrm{Rep}}\, S_t \longrightarrow \underline{\mathrm{Rep}}\, S_{t+k} \ .$$

We note that $\underline{\mathrm{Ind}}_{G\times S_t}^{S_{t+k}}$ is naturally isomorphic to $\underline{\mathrm{Ind}}_{S_k\times S_t}^{S_{t+k}} \circ (\mathrm{Ind}_G^{S_k} \boxtimes \mathrm{Id}_{\underline{\mathrm{Rep}}\, S_t})$, where we first perform ordinary induction from G to S_k and then perform Deligne category induction. In particular, it shifts up the filtration by the same amount.

Given a partition λ of size n we recall that S^λ denotes the corresponding irreducible representation of S_n. We will use X_λ to denote the corresponding indecomposable object of $\underline{\mathrm{Rep}}\, S_t$, see Section 2.1. Note that by Theorem 2.1, $X_\lambda \in (\underline{\mathrm{Rep}}\, S_t)^{\leqslant |\lambda|}$ but not in $(\underline{\mathrm{Rep}}\, S_t)^{\leqslant |\lambda|-1}$.

If $\lambda = (\lambda_1, \lambda_2, \dots \lambda_\ell)$ is a partition, then for n sufficiently large define the *padded partition* $\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \dots \lambda_\ell)$. In terms of Young diagrams, this padding operation just adds a long first row to λ to make it have n total boxes. The relevance for our purposes is that Comes and Ostrik showed that if $n - |\lambda| \geqslant \lambda_1$ (that is, if $\lambda[n]$ defines a Young diagram), then $X_\lambda \in \underline{\mathrm{Rep}}\, S_n$ gets mapped to $S^{\lambda[n]} \in \mathrm{Rep}\, S_n$ under the specialization functor $\mathcal{F}_n : \underline{\mathrm{Rep}}\, S_n \to \mathrm{Rep}\, S_n$ [8, Proposition 3.25].

For a partition λ let $\lambda - h.s.$ denote the set of partitions μ which can be obtained from λ by removing a horizontal strip, that is, by removing at most one box from each column. The following Deligne category Pieri rule gives us an upper triangularity property between the simple objects X_λ and certain easy to work with induced objects.

Lemma 2.10.

(1) *For $t \notin \mathbb{Z}_{\geqslant 0}$ or for $t \in \mathbb{Z}$ with $t \gg |\lambda|$*

$$\underline{\mathrm{Ind}}_{S_k\times S_t}^{S_{t+k}}(S^\lambda \boxtimes \mathbf{1}) = \bigoplus_{\mu\in\lambda-h.s.} X_\mu.$$

In particular, the right-hand side is X_λ plus terms X_μ with $|\mu| < |\lambda|$.

(2) *For $t \in \mathbb{Z}_{\geqslant 0}$ the object $\underline{\mathrm{Ind}}_{S_k\times S_t}^{S_{t+k}}(S^\lambda \boxtimes \mathbf{1})$ decomposes as a multiplicity-free direct sum of indecomposable objects X_μ, such that:*
   (a) *X_λ occurs with multiplicity one,*
   (b) *Each X_μ that appears has $\mu \in \lambda - h.s.$ (but not all such μ need appear).*

Proof. The ordinary Pieri rule for symmetric groups tells us that in characteristic 0 or $p > n + k$

$$\mathrm{Ind}_{S_k\times S_n}^{S_{n+k}}(S^\lambda \boxtimes \mathbf{1}) = \bigoplus_{\mu\in\lambda-h.s.} S^{\mu[n]}.$$

Part 1 follows since $S^{\mu[n]}$ corresponds to the object X_μ, under both the ultraproduct identification and under the quotient functor from $\underline{\mathrm{Rep}}\, S_n$ to $\mathrm{Rep}\, S_n$.

Now suppose $t = d \in \mathbb{Z}_{\geqslant 0}$. A priori one knows that $\mathrm{Ind}_{S_k\times S_d}^{S_{d+k}}(S^\lambda \boxtimes \mathbf{1})$ decomposes as a direct sum of indecomposable objects X_ν with some multiplicities c_ν. If we apply the Comes–Ostrik lifting operator Lift_d to this sum, we must obtain the answer from Part (1).

$$\mathrm{Lift}_d(\bigoplus_\nu c_\nu X_\nu) = \bigoplus_{\mu\in\lambda-h.s.} X_\mu.$$

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Comes and Ostrik's description of lifting (as summarized in Theorem 2.2) tells us that X_λ appears as a direct summand of $\mathrm{Lift}_d(X_\nu)$ for $\nu = \lambda$ and in that case, at most one other partition λ', with $|\lambda'| > |\lambda|$ appears as a summand giving $\mathrm{Lift}_d(X_{\lambda'}) = X_{\lambda'} \oplus X_\lambda$. Since λ is the largest partition appearing in $\lambda - h.s.$ and it appears with multiplicity one, we see that X_λ must occur at $t = d$ with multiplicity one, proving part (1).

To show part (2) we then inductively apply the same logic as above to the next largest partitions $\mu \in \lambda - h.s.$ Note that as we repeat this argument for some $\mu \in \lambda - h.s.$, the term X_μ might come from $\mathrm{Lift}_d(X_{\mu'})$ for some larger $\mu' \in \lambda - h.s.$ already accounted for, in which case X_μ does not appear in the induction at $t = d$. □

Note that this lemma gives us an alternative way of characterizing the indecomposable object X_λ as the unique direct summand of $\mathrm{Ind}_{S_k \times S_t}^{S_{t+k}}(S^\lambda \boxtimes \mathbf{1})$ not occurring as a summand of $\mathrm{Ind}_{S_j \times S_t}^{S_{t+j}}(S^\mu \boxtimes \mathbf{1})$ for any partition μ with $|\mu| < |\lambda|$.

Once one has established the Deligne category Pieri rule for induction, one can iteratively compute more general induced representations combinatorially. In particular, if we only keep track of the leading order terms, one obtains the following.

Corollary 2.11 See [27, Section 2.1].

$$\underline{\mathrm{Ind}}_{S_k \times S_t}^{S_{t+k}}(S^\lambda \boxtimes X_\mu) = \bigoplus_\nu c_{\lambda,\mu}^\nu X_\nu \oplus \{\textit{terms } X_\tau \textit{ with } |\tau| < |\lambda| + |\mu|\}$$

where $c_{\lambda,\mu}^\nu$ denotes a Littlewood–Richardson coefficient.

2.7 | The abelian envelope of Deligne's categories

An (abelian) multitensor category $\mathcal{D}$ is called an *abelian envelope* of a Karoubian rigid monoidal tensor category $\mathcal{C}$ if it contains $\mathcal{C}$ and for any multitensor category $\mathcal{D}'$, the category of tensor functors $\mathcal{D} \to \mathcal{D}'$ is equivalent to the category of faithful monoidal functors $\mathcal{C} \to \mathcal{D}'$ by restriction. If it exists, the abelian envelope is unique up to equivalence. We refer to [4] and [19, Section 9] for details on the abelian envelope and its universal property.

For $d \in \mathbb{Z}_{\geqslant 0}$, the abelian envelope $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ of $\underline{\mathrm{Rep}}\, S_d$ ([4, Example 2.45(1)]) has several explicit constructions. The first construction uses a symmetric monoidal functor $\underline{\mathrm{Rep}}\, S_d \to \underline{\mathrm{Rep}}\, S_{-1}$ and displays the abelian envelope as a category of representations over an algebraic group object internal to the semisimple category $\underline{\mathrm{Rep}}\, S_{-1}$. The second construction introduces a t-structure on complexes of objects in $\underline{\mathrm{Rep}}\, S_d$ and displays the abelian envelope as the heart of this t-structure — see [9] for details on these constructions.

A third construction, which we will employ, is given using ultrafilters. For this, we consider the ultraproduct of general modules over S_{p_i+d} in finite characteristic p_i (rather than just those in $\langle X_{p_i+d} \rangle$) in Theorem 2.5(c) and obtain the following modification of the ultraproduct description for the abelian envelope, which is obtained as the closure of $\langle X \rangle$ under subquotients.

Theorem 2.12 ([26, Theorem 1.1(b)]). *Set $t_i = p_i + d$ for p_i the ith prime number. Then there are equivalences of additive categories*

$$(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k} \cong \prod_{\mathcal{U}} (\mathrm{Rep}_{p_i}\, S_{p_i+d})^{\leqslant k},$$

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which provide, under an isomorphism of fields $\prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i} \cong \mathbb{C}$*, an equivalence of* $\mathbb{C}$*-linear symmetric tensor categories between* $\operatorname{colim}_{k \geqslant 0} \prod_{\mathcal{U}} (\operatorname{Rep}_{p_i} S_{p_i+d})^{\leqslant k}$ *and* $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$*.*

It follows from Theorem 2.12 (or can be deduced directly from the universal property of $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$) that the induction functor from Section 2.6 extends to the abelian envelope such that

$$\begin{tikzcd} \operatorname{Rep} S_k \boxtimes \underline{\operatorname{Rep}}\, S_{d-k} \arrow[r, "\underline{\operatorname{Ind}}^{S_d}_{S_k \times S_{d-k}}"] \arrow[d, hook] & \underline{\operatorname{Rep}}\, S_d \arrow[d, hook] \\ \operatorname{Rep} S_k \boxtimes \underline{\operatorname{Rep}}^{\mathrm{ab}} S_{d-k} \arrow[r, "\underline{\operatorname{Ind}}^{S_d}_{S_k \times S_{d-k}}"] & \underline{\operatorname{Rep}}^{\mathrm{ab}} S_d \end{tikzcd} \tag{2.7}$$

is a commutative diagram of $\mathbb{C}$-linear functors. This functor is again left and right adjoint to restriction and hence exact.

Comes and Ostrik showed ([9, Proposition 2.9, Corollary 4.6]) that each block of $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$ is equivalent to a block in the category of representations of quantum $SL(2)$, with the objects in $\underline{\operatorname{Rep}}\, S_d$ corresponding to tilting objects. As a consequence, $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$ itself forms a highest weight category with the indecomposable objects $X_\lambda \in \underline{\operatorname{Rep}}\, S_d$ as the indecomposable tilting objects. The simple objects D^λ of $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$ are again indexed by partitions, with D^λ appearing as a composition factor of X_λ with multiplicity one, and all other composition factors X_λ are of the form D^μ with μ satisfying $|\mu| < |\lambda|$.

Under the ultrafilter identification the indecomposable module X_λ corresponds to the so-called Young modules $Y(\lambda[p_i + d])$, which arise as direct summands of the permutation representation. The simple objects D^λ of $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$ correspond to ultraproducts of simple objects $D^{\lambda[p_i+d]}$. We note that while in general $\operatorname{Rep} S_n$ is not a highest weight category, the truncated categories $(\operatorname{Rep} S_n)^{\leqslant k}$ with k less than the characteristic p are highest weight with Specht modules as standard objects and Young modules as tilting objects. One can check that the ultraproduct of these highest weight structures defines a highest weight structure on $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$ agreeing with the one defined by Comes and Ostrik.

Induction does not preserve semisimplicity in general, so we cannot expect a version of Corollary 2.11 for induction of simple objects to hold as a direct sum decomposition in $\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d$. We can, however, relax it by passing to the Grothendieck ring and instead keeping track of composition multiplicities.

For an abelian category $\mathcal{A}$, the *(abelian) Grothendieck ring* $K_0(\mathcal{A})$ is the quotient of the additive Grothendieck ring $K_0^{\oplus}(\mathcal{A})$ from Section 2.5 by relations obtained from short exact sequences. If $\mathcal{A}$ is semisimple, then $K_0(\mathcal{A}) = K_0^{\oplus}(\mathcal{A})$.

Corollary 2.13. *In* $K_0(\underline{\operatorname{Rep}}^{\mathrm{ab}} S_d)$ *one has*

$$[\underline{\operatorname{Ind}}^{S_{d+k}}_{S_k \times S_d}(S^\lambda \boxtimes D^\mu)] = \sum_\nu c^\nu_{\lambda,\mu} [D^\nu] + \{\textit{terms } [D^\tau] \textit{ with } |\tau| < |\lambda| + |\mu|\}$$

where $c^\nu_{\lambda,\mu}$ *denotes a Littlewood–Richardson coefficient.*

Proof. This follows immediately from Corollary 2.11 by passing to the Grothendieck ring and then substituting $[X_\lambda] = [D^\lambda] + \{l.o.t\}$ everywhere and collecting all of the lower order terms to one side. □

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We have the following analog of Lemma 2.9 for the abelian Grothendieck ring of $\underline{\mathrm{Rep}}\, S_d$.

Lemma 2.14. *Sending $[D^\lambda] \mapsto [S^\lambda]$ induces an isomorphism of rings from $\mathrm{gr}\, K_0(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ to $\bigoplus_{n\geqslant 0} K_0(\mathrm{Rep}\, S_n)$, where the product on the right-hand side is given by induction, or equivalently, by Littlewood–Richardson coefficients.*

Proof. The proof will be via a chain of isomorphisms.

First note that the upper triangularity property $[X_\lambda] = [D^\lambda] + \{\text{l.o.t.}\}$ implies that these $[X_\lambda]$ form a basis for $K_0(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$, and therefore the inclusion $K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_d) \hookrightarrow K_0(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ is in fact an isomorphism. Moreover, this upper triangularity property implies that the assignment $[D^\lambda] \to [X_\lambda]$ defines an isomorphism of associated graded rings $\mathrm{gr}\, K_0(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d) \cong \mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_d)$.

Next, the first two properties of lifting in Theorem 2.2 say that the map $[X] \mapsto [\mathrm{Lift}_d(X)]$ defines a ring homomorphism $K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_d) \to K_0(\underline{\mathrm{Rep}}\, S_t)$ at a nearby transcendental value of t. The third property in Theorem 2.2 is again an upper triangularity property, which implies that this homomorphism is an isomorphism, but moreover that $[X_\lambda] \mapsto [X_\lambda]$ defines an isomorphism of associated graded rings $\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_d) \to \mathrm{gr}\, K_0(\underline{\mathrm{Rep}}\, S_t)$ at generic t.

Finally we use the isomorphism $\mathrm{gr}\, K_0(\underline{\mathrm{Rep}}\, S_t) \cong \bigoplus_{n\geqslant 0} K_0(\mathrm{Rep}\, S_n)$ of Lemma 2.9, which sends $[X_\lambda]$ to $[S^\lambda]$. Composing these three isomorphisms gives the desired result. □

2.8 | The projective objects in the abelian envelope

We now turn to describing the projective objects in $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ for later use. Key to understanding of the projectives will be the projective cover of the tensor unit.

Lemma 2.15. *Let $P \cong \prod_{\mathcal{U}} P_i$ be an object in $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$. Then P is projective in $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ if and only if there exists an integer $k_0 \geqslant 0$ such that for each integer $k \geqslant k_0$, almost all P_i are projective objects in $(\mathrm{Rep}_{p_i}\, S_{t_i})^{\leqslant k}$.*

Proof. By virtue of P being an object of $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$, there exists $k_0 \geqslant 0$ such that $P \in (\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k_0}$. In particular, P is contained in $P \in (\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k}$ for any $k \geqslant k_0$. Hence, for fixed k, almost all P_i are contained in $(\mathrm{Rep}_{p_i}\, S_{t_i})^{\leqslant k}$. Assume that P is projective. The property that P is projective is defined by the functor $\mathrm{Hom}(P, -)$ being right exact, as a functor on $(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k}$, which can be expressed as a first-order property. Thus, P_i is a projective object in $(\mathrm{Rep}_{p_i}\, S_{t_i})^{\leqslant k}$, for almost all i.

Conversely, again using Łoś' theorem, we see that if P_i is a projective object in $(\mathrm{Rep}_{p_i}\, S_{t_i})^{\leqslant k}$, for almost all i, then P is projective in $(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k}$. Since $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ is the filtered colimit of all of these categories, projectivity of P holds in the entire category $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$. □

Recall the indecomposable objects X_λ of $\underline{\mathrm{Rep}}\, S_t$, see Section 2.2 and consider the case $\lambda = (d+1)$.

Lemma 2.16. *The indecomposable object $X_{(d+1)} \in (\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant d+1}$ is the projective cover of the tensor unit* **1**.

Proof. Recall from Theorem 2.12 that $(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k}$ is equivalent to the ultraproduct of the categories $(\mathrm{Rep}_{p_i}\, S_{p_i+d})^{\leqslant k}$. Observe that $\underline{\mathrm{Ind}}$ is left adjoint of a functor-preserving epimorphisms and

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thus preserves projective objects; therefore, any object of the form $\mathrm{Ind}_{S_{d+1}\times S_{p_i-1}}^{S_{p_i+d}}(V \boxtimes W)$ is projective in $\mathrm{Rep}_{p_i} S_{p_i+d}$ as it is induced from a semisimple category where all objects are projective. In particular, observe that

$$P_i := \mathrm{Ind}_{S_{d+1}\times S_{p_i-1}}^{S_{p_i+d}}(\mathbf{1} \boxtimes \mathbf{1}) \in (\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)^{\leqslant d+1},$$

and that the surjective morphisms $P_i \to \overline{\mathbb{F}}_{p_i}$, $g \otimes (1 \boxtimes 1) \mapsto 1$ induce a surjective morphism $P \twoheadrightarrow \mathbf{1}$, for $P = \prod_{\mathcal{U}} P_i$.

By Lemma 2.10, $P \cong X_{(d+1)} \oplus$ (l.o.t). The indecomposables appearing as lower order terms are of the form $X_{(k)}$, with $k \leqslant d+1$. The combinatorial description of objects in the block of $X_{(d+1)}$ from [9, Theorem 2.6] implies that of the objects $X_{(k)}$, only $X_{(0)} = \mathbf{1}$ belongs to this block and $\mathrm{Hom}(P, Y) = 0$ for objects Y from other blocks. However, $\mathbf{1}$ is not projective as $\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d$ is not semisimple. Thus, $X_{(d+1)}$ is the projective cover of $\mathbf{1}$. □

Proposition 2.17. *Let $X \in (\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)^{\leqslant k}$. Then X is projective in $(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)^{\leqslant (k+d+1)}$ if and only if X is projective in $\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d$.*

Proof. Given an indecomposable object X in $(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)^{\leqslant k}$, we can use Lemma 2.16 to see that the projective cover of X is contained in $(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)^{\leqslant (k+d+1)}$. Indeed, $X \otimes X_{(d+1)}$ is a projective object in $(\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)^{\leqslant (k+d+1)}$ with an epimorphism to $X \otimes \mathbf{1} \cong X$. Now, X is projective in an additive subcategory $\mathcal{C}$ of $\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d$ if and only if its projective cover is contained in $\mathcal{C}$ and X is isomorphic to its projective cover. By the above observation and Lemma 2.15, this condition holds in $\mathcal{C} = \underline{\mathrm{Rep}}^{\mathrm{ab}} S_d$ if and only if it holds in $\mathcal{C} = (\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d)^{\leqslant (k+d+1)}$. □

The advantage of the above proposition is that projectivity of an object in $\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d$ can be checked using the components of an ultrafilter presentation, as well as the following observation.

Corollary 2.18. *All projective objects of $\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d$ are contained in the subcategory $\underline{\mathrm{Rep}}\, S_d$.*

Proof. This follows directly from [9, Remark 4.8] since for any object P of $\underline{\mathrm{Rep}}^{\mathrm{ab}} S_d$ there exists a surjective map $\bigoplus_{i=1}^{m} X^{\otimes n_i} \twoheadrightarrow P$. Hence, if P is projective, it is a direct summand of an object in $\underline{\mathrm{Rep}}\, S_d$. □

2.9 | Indecomposable Yetter–Drinfeld modules over S_n in arbitrary characteristic

In this section, let $\Bbbk$ be an algebraically closed field and G a finite group. We now turn to background results on the monoidal center $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ of the category $\mathrm{Rep}_{\Bbbk} G$. We employ the equivalent description of this braided tensor category as *Yetter–Drinfeld modules* [43, Definition 3.6], that is, G-graded G-representations $V = \bigoplus_{g\in G} V_g$ such that $g \cdot V_h = V_{ghg^{-1}}$, see, for example, [32, Proposition 7.1.6]. If $v \in V_g$, we write $|v| = g$ for the *degree* of v. Equivalently, $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ can be described as modules over the *Drinfeld double* $\mathrm{Drin}(G)$ of G [18]. We recall the following result found in [16] for $\Bbbk = \mathbb{C}$ and in [42, Corollary 2.3] for general characteristic.

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Theorem 2.19 (Dijkgraaf–Pasquier–Roche, Witherspoon). *Let G be a finite group and $\Bbbk$ be an algebraically closed field. A complete list of indecomposable (respectively, irreducible) objects in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ is given by modules of the form $W_{\sigma,V} = \mathrm{Ind}_Z^G(V)$ where σ is a representative of a conjugacy class of elements in G and V is an indecomposable (respectively, irreducible) module over the centralizer $Z = Z(\sigma)$ of σ in G.*

In fact, for each $\sigma \in G$, with $Z = Z(\sigma)$, we have functors

$$\mathrm{Ind}_Z^G : \ \mathrm{Rep}_{\Bbbk} Z \longrightarrow \mathcal{Z}(\mathrm{Rep}_{\Bbbk} G), \qquad V \longmapsto \mathrm{Ind}_Z^G(V).$$

The half-braiding on $W_{\sigma,V} := \mathrm{Ind}_Z^G(V)$ is given by $c_W((g \otimes v) \otimes w) = (g\sigma g^{-1}) \cdot w \otimes (g \otimes v)$, for any G-module W. The G-grading of the associated Yetter–Drinfeld module $\mathrm{Ind}_Z^G(V)$ here is given by $\delta(g \otimes v) = g\sigma g^{-1} \otimes v$. This grading does not depend on V, so it is clear that any morphism of $\Bbbk Z$-modules $f : V \to V'$ induces one in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$, namely, $\mathrm{Ind}_Z^G(f) : W_{\sigma,V} \to W_{\sigma,V'}$. The constructions are independent of the choice of a representative σ of a conjugacy class up to isomorphism.

The following lemma is easily seen from the description of $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ in terms of Yetter–Drinfeld modules over G.

Lemma 2.20. *The functors Ind_Z^G are full and faithful, that is, there are isomorphisms*

$$\mathrm{Hom}_{\mathrm{Rep}_{\Bbbk} Z}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)}(\mathrm{Ind}_Z^G V, \mathrm{Ind}_Z^G W),$$

for any $\Bbbk Z$-modules V and W.

If σ and τ are not conjugate, V is a $\Bbbk Z(\sigma)$-module, and W is a $\Bbbk Z(\tau)$-module, then

$$\mathrm{Hom}_{\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)}(\mathrm{Ind}_{Z(\sigma)}^G V, \mathrm{Ind}_{Z(\tau)}^G W) = \{0\}.$$

Hence, as an abelian category, $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ decomposes as a direct sum (as defined in [22, Section 1.3])

$$\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G) \cong \bigoplus_{[g] \in G^G} \mathrm{Rep}_{\Bbbk} Z(g),$$

where g ranges over a set of representatives of the conjugacy classes of G.

A direct consequence of this observation is the following: let W be a Yetter–Drinfeld G-module over any field $\Bbbk$, and let $W_{g,V}$ be the simple Yetter–Drinfeld G-module obtained from g in G and a simple module V over the centralizer $Z(g)$ of g in G. Then the multiplicity $[W : W_{g,V}]$ in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ equals the multiplicity $[W_g : V]$ in $\mathrm{Rep}_{\Bbbk} Z(g)$, where W_g is the homogeneous subspace of W of degree g.

Lemma 2.21. *$W_{\sigma,P}$ is projective in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ if and only if P is projective in $\mathrm{Rep}_{\Bbbk} Z(\sigma)$.*

Proof. For $\sigma \in G$, consider the regular $\Bbbk Z$-module and note that $\mathrm{Ind}_Z^G(\Bbbk Z) = \Bbbk G$. As an object in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$, this is the projective $\mathrm{Drin}(G)$-submodule $\mathrm{Drin}(G)\delta_\sigma = \Bbbk G \delta_\sigma$ of $\mathrm{Drin}(G)$, where $\{\delta_g\}_{g \in G}$ is the basis of $(\Bbbk G)^*$ dual to the basis $\{g\}_{g \in G}$. By functoriality of Ind_Z^G, P being a

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direct summand of $(\Bbbk Z)^{\oplus n}$, it readily follows that $W_{\sigma,P}$ is a direct summand of $\mathrm{Drin}(G)\delta_\sigma$ and thus projective.

Assume that $W := W_{\sigma,P}$ is a direct summand of a direct sum $R := ((\Bbbk G)^* \otimes \Bbbk G)^{\oplus m}$ of the regular module in $\mathcal{Z}(\mathrm{Rep}_\Bbbk G)$. Note that $R_\sigma = (\Bbbk G\delta_\sigma)^{\oplus m}$ and $W_\sigma = P$. Thus, we obtain that P is a direct summand of $(\Bbbk G\delta_\sigma)^{\oplus m}$ as a $\Bbbk Z$-module. Choosing a decomposition of G into right Z-cosets, we observe that as a $\Bbbk Z$-module, $\Bbbk G\delta_\sigma$ is simply a direct sum of copies of the regular module. Thus, P is projective. □

The Grothendieck ring of the monoidal center of representations over algebraically closed fields of arbitrary characteristic was studied in [42]. See also [17] for some concrete examples.

Corollary 2.22 ([42, Section 3]). *For G a finite group. There is an isomorphism of rings*

$$K_0(\mathcal{Z}(\mathrm{Rep}_\Bbbk G)) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_\sigma Z(\mathbb{C}Z(\sigma)),$$

where σ varies over a set of representatives of the conjugacy classes of G.

This corollary is proved in [42, p. 316] and uses a result of G. Lusztig.

3 | CLASSIFICATION OF INDECOMPOSABLE OBJECTS IN THE CENTER OF $\underline{\mathrm{Rep}}\, S_t$

3.1 | The center as a model-theoretic limit

Viewing $\underline{\mathrm{Rep}}\, S_t$ as a model-theoretic limit of categories $\mathrm{Rep}_{p_i} S_{t_i}$ as in Theorem 2.5 suggests that it may be possible to interpret the center $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ as a limit of the centers $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$. An object of the center of a monoidal category consists of the data of an object of the category along with a half-braiding. Therefore, the centers $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ and $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$ inherit filtrations $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$ and $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$ just by looking at the underlying object. Precisely, $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$ consists of those objects (V, c) where V lies in $(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$.

Proposition 3.1. *Recall the setup of Theorem 2.5(1)–(3). In all cases, the category $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$ is equivalent to the ultraproduct of the categories $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$ (or, $\mathcal{Z}(\mathrm{Rep}\, S_{t_i})^{\leqslant k,m}$) as $\mathbb{C}$-linear categories with partially defined monoidal and additive structures.*

Proof. One direction is fairly clear: If (Y_i, c_i) is an object of $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$ for each i, then we can take the ultraproduct of the underlying objects Y_i to obtain an object Y in $(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$. The fact that (Y_i, c_i) is an element of $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$ means that the half-braiding c_i is not just defined on $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$, but on all larger filtered pieces as well. Using Łoś' theorem, the ultraproduct (Y, c) becomes an object in $(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$ with half-braiding c_Z globally defined for all $Z \in \underline{\mathrm{Rep}}\, S_t$, hence lies in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$.

Going the other way requires a bit more work. If we start with an object (Y, c) in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$, we can always represent Y as an ultraproduct of objects Y_i in $(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$. However, a priori the half-braiding c need not come from an ultraproduct of half-braidings c_i globally on each $\mathrm{Rep}_{p_i} S_{t_i}$.

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Instead, initially we can just conclude that it gives a sequence of partially defined half-braidings c_i, each defined on some $(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k_i, m_i}$ with k_i and m_i tending to infinity as p_i does, but possibly not globally. This is essentially because the half-braiding is a natural transformation defined globally on the entire category, and the ultraproduct is only well behaved on these finite-filtered pieces. To correct for this we use Lemma 3.2, which shows that the data of a half-braiding are equivalent to some finite data that only involve objects and morphisms inside one of these finite-filtered pieces, namely, $(\underline{\mathrm{Rep}}\, S_t)^{\leqslant 2,2}$. Under the ultraproduct identification Y corresponds to a sequence of objects Y_i in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$ and c_X gives us a sequence of maps $c_{Y_i,X} : Y_i \otimes X \to X \otimes Y_i$ which satisfy the conditions of Lemma 3.2 for almost all i. Namely, as soon as $k_i, m_i \geqslant 2$. Hence for almost all i this corresponds to an object (Y_i, c_i) inside $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$, as desired. This completes the proof of Proposition 3.1. □

The following lemma specifies the finite datum required to obtain objects in the center of both interpolation categories and representation categories in finite characteristic.

Lemma 3.2. *Given an object Y in $\underline{\mathrm{Rep}}\, S_t$ or $\mathrm{Rep}_p S_d$, the datum of a half-braiding $c = \{c_V : Y \otimes V \to V \otimes Y \mid V \in \underline{\mathrm{Rep}}\, S_t\}$ is equivalent to the datum of a single morphism $c_X : Y \otimes X \to X \otimes Y$ such that*

$$c_X(\mathrm{Id}_Y \otimes \pi_*) = \mathrm{Id}_Y \otimes \pi_* \,, \quad (\pi^* \otimes \mathrm{Id}_Y) c_X = \pi^* \otimes \mathrm{Id}_Y \,, \tag{3.1}$$

$$c_{X\otimes X}(\mathrm{Id}_Y \otimes \pi_X) = (\pi_X \otimes \mathrm{Id}_Y) c_{X\otimes X} \quad c_{X\otimes X}(\mathrm{Id}_Y \otimes \pi_H) = (\pi_H \otimes \mathrm{Id}_Y) c_{X\otimes X} \,. \tag{3.2}$$

Here we denote

$\pi^* =$ $\in \mathrm{Hom}(X, \mathbf{1})$, $\pi_* =$ $\in \mathrm{Hom}(\mathbf{1}, X)$, $\pi_H =$, $\pi_X =$ $\in \mathrm{End}(X^{\otimes 2})$, (3.3)

and we note that

$$c_{X\otimes X} = (\mathrm{Id}_X \otimes c_X)(c_X \otimes \mathrm{Id}_X). \tag{3.4}$$

In the case of $\mathrm{Rep}_p S_d$, we use the morphisms $f_{\pi^*}, f_{\pi_*}, f_{\pi_H}, f_{\pi_X}$ instead of $\pi^*, \pi_*, \pi_H, \pi_X$.

Equivalently, (Y, c) being an object in the center is equivalent to c_X commuting with the structural maps of the Frobenius algebra structure of X.

A morphism $f : (Y, c) \to (Y', c')$ in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is equivalent to the datum of a morphism $f : Y \to Y'$ such that

$$c'_X(f \otimes \mathrm{Id}_X) = (\mathrm{Id}_X \otimes f) c_X. \tag{3.5}$$

Proof. See [24, Proposition 3.1] for details on the argument in the case of $\underline{\mathrm{Rep}}\, S_t$.

If $\mathrm{char}\, \Bbbk = p > d$, then $\mathrm{Rep}_p S_d$ is semisimple and the argument detailed in [24, Appendix A] proves the lemma since the Karoubian tensor subcategory $\langle X \rangle$ generated by X inside of $\mathrm{Rep}_p S_d$ is the entire category $\mathrm{Rep}_p S_d$.

In remains to consider the case where $p \leqslant d$ and $\mathrm{Rep}_p S_d$ is non-semisimple. In this case, the $\langle X_d \rangle$ contains all projective objects. Further, morphisms $X_d^{\otimes n} \to X_d^{\otimes m}$ are generated by $\pi_*, \pi^*, \pi_H, \pi_X$. Hence, Lemma A.1 and Corollary A.2 show that c is uniquely determined by its restriction to $\mathcal{A} = \langle X_d \rangle$. The half-braiding on $\langle X_d \rangle$, in turn, is determined by c_X using an argument as in [24, Proposition A.1]. □

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3.2 | Semisimplicity of the center

The techniques of viewing $\underline{\mathrm{Rep}}\, S_t$ as a model-theoretic limit via ultrafilters enable us to resolve a question raised in [24, Question 3.32] about semisimplicity of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$.

Theorem 3.3. *The category $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is semisimple if and only if $t \notin \mathbb{Z}_{\geqslant 0}$.*

Proof. It is clear that for $t \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is not semisimple since it contains the non-semisimple category $\underline{\mathrm{Rep}}\, S_t$ (with the standard symmetric braiding of $\underline{\mathrm{Rep}}\, S_t$ for all objects) as a full subcategory. So let us consider the remaining case where t is either transcendental or in $\overline{\mathbb{Q}} \setminus \mathbb{Z}_{\geqslant 0}$.

Since $\underline{\mathrm{Rep}}\, S_t$ is semisimple for these values of t, we know that $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is an abelian category with objects of finite length. Therefore, in order to prove that $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is semisimple, it suffices to check that every epimorphism in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ splits.

Suppose that $f : V \twoheadrightarrow W$ is an epimorphism in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$. There exist k and m such that V and W both lie in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$, so by Proposition 3.1 we can identify f with a sequence of morphisms $f_i : V_i \twoheadrightarrow W_i$ in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{n_i})^{\leqslant k,m}$ which are epimorphisms for almost every i. However, in these cases we have that $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{n_i})$ is a semisimple category, so for each value of i where this is an epimorphism we can find a monomorphism $g_i : W_i \hookrightarrow V_i$ in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{n_i})^{\leqslant k,m}$ such that $f_i \circ g_i$ is the identity, meaning that g_i defines a splitting of f_i.

Taking this sequence of monomorphisms g_i and applying Proposition 3.1 again we obtain a map $g : W \hookrightarrow V$ in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$. Since $f_i \circ g_i$ was the identity map for W_i almost always, $f \circ g$ must be the identity map for W. Hence g defines a splitting for f, and we see that every epimorphism in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ splits as desired. □

It was shown in [24, Corollary 3.40] that if $d \in \mathbb{Z}_{\geqslant 0}$, then $\mathcal{Z}(\mathrm{Rep}\, S_d)$ is the semisimplification of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$ in the sense of [23, Section 2.3].

3.3 | Induction functors to the center

In this section, we give a construction of objects in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ using induction functors defined on $\mathcal{Z}(\mathrm{Rep}\, S_n) \boxtimes \underline{\mathrm{Rep}}\, S_{t-n}$. For this, we fix $t \in \mathbb{C}$, $(t_i)_i$, and $(p_i)_i$ such that $\underline{\mathrm{Rep}}\, S_t \subset \prod_{\mathcal{U}} \mathrm{Rep}_{p_i} S_{t_i}$. Then it follows that $\underline{\mathrm{Rep}}\, S_{t-n} \subset \prod_{\mathcal{U}} \mathrm{Rep}_{p_i} S_{t_i - n}$ using that t_i grows to infinity, cf. Theorem 2.5, so that for almost all i, $t_i \geqslant n$. Note that in the transcendental case, we use $\mathrm{Rep}\, S_{t_i}$ for all i without explicitly mentioning this in the following statements, for which the proofs are easier in this case.

Definition 3.4. For any $n \geqslant 0$, μ a cycle type in S_n (that is, a partition of n), V a $\mathbb{C}$-representation of $Z = Z(\mu)$, the centralizer in S_n of an element of cycle type μ, and U an object in $\underline{\mathrm{Rep}}\, S_{t-n}$, we define

$$W := \underline{\mathrm{Ind}}_{Z \times S_{t-n}}^{S_t}(V \boxtimes U) \qquad \text{in } \underline{\mathrm{Rep}}\, S_t.$$

If we identify $U \cong \prod_{\mathcal{U}} U_i$ in the ultraproduct $\prod_{\mathcal{U}} \mathrm{Rep}_{p_i} S_{t_i}$ and, using Theorem 2.6, $V \cong \prod_{\mathcal{U}} V_i$ with $V_i \in \mathrm{Rep}_{p_i} Z$, we find that setting

$$W_i := \mathrm{Ind}_{Z_i}^{S_{t_i}} \circ \mathrm{Ind}_{Z \times S_{t_i - n}}^{Z_i}(V_i \boxtimes U_i) \qquad \text{in } \mathrm{Rep}_{p_i} S_{t_i},$$

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and $Z_i := Z_{S_{t_i}}(\sigma)$ whenever $t_i \geqslant n$ gives $W \cong \prod_{\mathcal{U}} W_i$ in the ultraproduct.

Fix an element $\sigma \in S_n$ of cycle type μ. Then, for all i such that $t_i \geqslant n$, we define c^i to be the Yetter–Drinfeld braiding of W_i in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i-n})$ for the grading determined by assigning degree σ to every element in $V_i \boxtimes U_i \subseteq W_i$. In other words, the structure of W_i as an object in the center is obtained using Theorem 2.19 from the Z_i-module $\mathrm{Ind}^{Z_i}_{Z\times S_{t_i-n}}(V_i \boxtimes U_i)$.

Lemma 3.5. *The morphism $c := \prod_{\mathcal{U}} c^i$ defines a half-braiding for W in $\underline{\mathrm{Rep}}\, S_t$. Up to isomorphism in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, c is independent of choice of the element $\sigma \in S_n$ of cycle type μ.*

Proof. By construction, $c^i_X : W_i \otimes X \to X \otimes W_i$ defines a morphism in $\mathrm{Rep}_{p_i} S_{t_i}$ for almost all i. Thus, we obtain a morphism $c_X = \prod_{\mathcal{U}} c^i_X : \prod_{\mathcal{U}} W_i \otimes X \to X \otimes \prod_{\mathcal{U}} W_i$, using that $\prod_{\mathcal{U}} X = X$. For almost all i, the morphism c_X satisfies the conditions of Lemma 3.2 and thus uniquely extends to give an object (W, c) in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$.

Now let $\tau \in S_n$ be conjugate to σ, that is, $\sigma = g\tau g^{-1}$ for some $g \in S_n$. Then $h \mapsto ghg^{-1}$ defines an isomorphism $\phi : Z(\tau) \to Z(\sigma)$. We regard V_i as a $\overline{\mathbb{F}}_{p_i}$-representation over $Z(\tau)$, denoted by V'_i, via restriction along ϕ. This way, we obtain isomorphisms

$$W'_i := \mathrm{Ind}^{S_{t_i}}_{Z(\tau)\times S_{t_i-n}}(V'_i \boxtimes U_i) \xrightarrow{\sim} W_i = \mathrm{Ind}^{S_{t_i}}_{Z(\sigma)\times S_{t_i-n}}(V_i \boxtimes U_i).$$

Now mapping $h \otimes (v \otimes u)$ to $hg^{-1} \otimes v \otimes u$ defines an isomorphism $(W'_i, (c')^i) \to (W_i, c^i)$ in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$. These induce isomorphisms in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ of the corresponding ultraproducts of objects. □

Definition 3.6. With $\mu \vdash n$, V a representation of $Z(\mu) \subseteq S_n$, and U an object in $\underline{\mathrm{Rep}}\, S_{t-n}$ as above we denote

$$\underline{W}_{\mu,V,U} := (W, c), \qquad \text{in } \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t).$$

The definition depends, by choice of $\mu \vdash n$, on n and we sometimes write $\underline{W}_{\mu\vdash n,V,U}$ to highlight this dependence.

We can make the constructions of Definition 3.6 functorial in the following way.

Proposition 3.7. *Let $n \geqslant 0$. There exists a $\mathbb{C}$-linear functor*

$$\underline{\mathrm{Ind}} : \mathcal{Z}(\mathrm{Rep}\, S_n) \boxtimes \underline{\mathrm{Rep}}\, S_{t-n} \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t),$$

sending the object $\mathrm{Ind}^{S_n}_{Z(\mu)}(V) \boxtimes U$ to $\underline{W}_{\mu,V,U}$. This functor $\underline{\mathrm{Ind}}$ is a separable Frobenius monoidal functor (see Appendix B) compatible with braidings in the sense that the diagram

$$\begin{tikzcd}
\underline{\mathrm{Ind}}((W \boxtimes U) \otimes (W' \boxtimes U')) \arrow[r, "\underline{\mathrm{Ind}}(\Psi_{W,W'} \boxtimes \Psi_{U,U'})"] \arrow[d, "\delta_{W\boxtimes U, W'\boxtimes U'}"'] & \underline{\mathrm{Ind}}((W' \boxtimes U') \otimes (W \boxtimes U)) \\
\underline{\mathrm{Ind}}(W \boxtimes U) \otimes \mathrm{Ind}(W' \boxtimes U') \arrow[r, "\Psi_{\underline{\mathrm{Ind}}(W\boxtimes U), \underline{\mathrm{Ind}}(W'\boxtimes U')}"] & \underline{\mathrm{Ind}}(W' \boxtimes U') \otimes \mathrm{Ind}(W \boxtimes U) \arrow[u, "\mu_{W\boxtimes U, W'\boxtimes U'}"']
\end{tikzcd} \tag{3.6}$$

commutes for any objects W, W' in $\mathcal{Z}(\mathrm{Rep}\, S_n)$, U, U' in $\underline{\mathrm{Rep}}\, S_{t-n}$.

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Proof. The composition $F \circ \underline{\mathrm{Ind}}$ with the forgetful functor is the functor $\underline{\mathrm{Ind}}^{S_t}_{S_n \times S_{t-n}}$ from Section 2.6. Thus, given an object (V, c) in $\mathcal{Z}(\mathrm{Rep}\, S_n)$ and $U \in \underline{\mathrm{Rep}}\, S_{t-n}$,

$$\underline{\mathrm{Ind}}((V, c) \boxtimes U) = \underline{\mathrm{Ind}}^{S_t}_{S_n \times S_{t-n}}(V \boxtimes U).$$

To define the half-braiding, we describe this functor using ultraproducts. Using Theorem 2.6, we fix an equivalence of symmetric monoidal categories between $\mathrm{Rep}\, S_n$ and the subcategory $\langle X \rangle$ of $\prod_{\mathcal{U}} \mathrm{Rep}_{p_i} S_n$, for the ultraproduct X of the standard representations. Under this equivalence, an object (V, c) of $\mathcal{Z}(\mathrm{Rep}\, S_n)$ corresponds to the pair $(\prod_{\mathcal{U}} V_i, \prod_{\mathcal{U}} c^i)$. By a similar argument as the one used in Proposition 3.1, the pairs (V_i, c^i) define objects in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_n)$ for almost all i.

Further, $U \cong \prod_{\mathcal{U}} U_i$ for some $U_i \in \mathrm{Rep}_{p_i} S_{t_i - n}$, which can be defined for almost all i. Since $\mathrm{Rep}_{p_i} S_{t_i - n}$ is symmetric monoidal, with braiding Ψ, there is a braided monoidal functor

$$\mathrm{Rep}_{p_i} S_{t_i - n} \longrightarrow \mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i - n}), \qquad U_i \mapsto (U_i, \Psi_{U_i, -}).$$

We now form the Deligne tensor product of this functor with the identity on $\mathcal{Z}(\mathrm{Rep}_{p_i} S_n)$, and compose with the braided equivalences of finite tensor categories (see, for example, [22])

$$\mathcal{Z}(\mathrm{Rep}_{p_i} S_n) \boxtimes \mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i - n}) \xrightarrow{\sim} \mathcal{Z}(\mathrm{Rep}_{p_i} S_n \boxtimes \mathrm{Rep}_{p_i} S_{t_i - n}) \simeq \mathcal{Z}(\mathrm{Rep}_{p_i}(S_n \times S_{t_i - n})).$$

The resulting functor is composed with the functor $\mathcal{Z}(\mathrm{Ind}^{S_{t_i}}_{S_n \times S_{t_i - n}})$ from Proposition B.1 to yields a functor

$$\mathrm{Ind}_i : \mathcal{Z}(\mathrm{Rep}_{p_i} S_n) \boxtimes \mathrm{Rep}_{p_i} S_{t_i - n} \longrightarrow \mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i}).$$

Passing to ultraproducts, Proposition 3.1 yields a functor

$$\underline{\mathrm{Ind}} : \mathcal{Z}(\mathrm{Rep}\, S_n) \boxtimes \underline{\mathrm{Rep}}\, S_{t-n} \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t).$$

By Proposition B.1, the functors Ind_i are separable Frobenius monoidal functors. Indeed, they are a composition of the Frobenius monoidal functor $\mathrm{Ind}^{S_{t_i}}_{S_n \times S_{t_i - n}}$ with strong monoidal functors (which are, in particular, separable Frobenius monoidal functors). Passing to ultraproducts induces natural transformations μ, δ, and the unit and counit maps η, ϵ for the functor $\underline{\mathrm{Ind}}$. The axioms of a lax and oplax monoidal structure, the Frobenius compatibilities Equation (B.3)–(B.4), as well as separability equation (B.5), and the braiding compatibility equation (B.6) are first-order expressions, since they are functional equalities (after fixing the needed tuples of objects to state these properties). Hence, by Łoś' theorem, $\underline{\mathrm{Ind}}$ inherits all of these properties. □

In particular, functoriality of $\underline{\mathrm{Ind}}$ implies that isomorphic $Z(\mu)$-representations V, V' and isomorphic objects U, U' in $\underline{\mathrm{Rep}}\, S_{t-n}$ yield isomorphic objects $\underline{W}_{\mu,V,U} \cong \underline{W}_{\mu,V',U'}$ in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$. From Corollary B.2, we obtain the following consequences of Proposition 3.7.

Corollary 3.8. *For any $n \geqslant 0$, we derive the following properties of the functor $\underline{\mathrm{Ind}}$.*

(1) $\underline{\mathrm{Ind}}$ *is exact and preserves duals.*
(2) $\underline{\mathrm{Ind}}$ *preserves Frobenius algebra objects.*
(3) *For any W, W' in $\mathcal{Z}(\mathrm{Rep}\, S_n)$ and U, U' in $\underline{\mathrm{Rep}}\, S_{t-n}$, the object $\underline{\mathrm{Ind}}((W \boxtimes U) \otimes (W' \boxtimes U'))$ is a direct summand of $\underline{\mathrm{Ind}}(W \boxtimes U) \otimes \underline{\mathrm{Ind}}(W' \boxtimes U')$.*

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Proof. This follows from the consequences collected in Corollary B.2 which are general properties of a separable Frobenius monoidal functor, see [13]. □

3.4 | Indecomposable objects of the center

We are now ready to classify all indecomposable and simple objects of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$. The following main theorem of this section settles [24, Question 3.32] by proving a classification of indecomposable objects in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$.

Theorem 3.9. *The objects $\underline{W}_{\mu,V,U}$ for μ, V, U as in Definition 3.6, for V a $Z(\mu)$-module, U in $\underline{\mathrm{Rep}}\, S_{t-n}$, and $\mu \vdash n$ singleton free are indecomposable if and only if V is irreducible and U is indecomposable. These objects provide a complete list of indecomposable objects in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ up to isomorphism.*

The condition that $\mu \vdash n$ is singleton free is equivalent to the condition that any σ of cycle type μ is a fixed-point free permutation of n.

Proof. Consider a triple μ, V, U as above. Write $Z = Z(\mu) = Z(\sigma)$ for $\sigma \in S_n$ of cycle type μ and assume that σ has no fixed points. Using Theorem 2.6, we identify $\mathrm{Rep}\, Z \simeq \prod_{\mathcal{U}} \mathrm{Rep}_{p_i} Z$, and fix an isomorphism $V \cong \prod_{\mathcal{U}} V_i$ for Z-modules over $\overline{\mathbb{F}}_{p_i}$. We also find an isomorphism $U \cong \prod_{\mathcal{U}} U_i$, where U_i is an object in $\mathrm{Rep}_{p_i} S_{t_i}$. As the sequence t_i is increasing, we may view $\sigma \in S_{t_i}$ for almost all i, and, using that $\sigma \in S_n$ is fixed-point free, $Z_{S_{t_i}}(\sigma) = Z \times S_{t_i - n}$. Using Łoś' theorem, $\mathrm{Ind}_{Z \times S_{t_i-n}}^{S_{t_i}}$ preserves indecomposable (respectively, irreducible) objects. Indeed, if V, U are indecomposable, then U_i is indecomposable for almost all i, and thus, the $Z \times S_{t_i-n}$-module $V_i \boxtimes U_i$ is indecomposable for almost all i. Here, we use that the category $\mathrm{Rep}_{p_i} Z$ is semisimple for sufficiently large i. Thus, any object in $\mathrm{Rep}_{p_i}(Z \times S_{t_i-n})$ is a direct sum of objects $V_i \boxtimes W_i$, where V_i is a simple Z-module and W_i is an object in $\mathrm{Rep}\, S_{t_i-n}$. Thus, for almost all i, $\mathrm{Ind}_{Z \times S_{t_i-n}}^{S_{t_i}}(V_i \boxtimes U_i)$ defines an indecomposable object in the center, and thus, $\underline{\mathrm{Ind}}_{Z \times S_{t-n}}^{S_t}(V \boxtimes U)$ is an indecomposable object in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$. Here, we use that an object being indecomposable (respectively, irreducible) can be expressed using first-order logic as in the proof of Theorem 3.3.

Conversely, let (W, c) be an indecomposable (or irreducible) object in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k,m}$. By Proposition 3.1, we know that (W, c) corresponds to an ultraproduct of objects (W_i, c^i) in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k,m}$. For almost all i, (W_i, c^i) is indecomposable (respectively, irreducible). Hence, we find elements $\sigma_i \in S_{t_i}$ and indecomposable (respectively, irreducible) $Z(\sigma_i)$-modules Y_i such that $W_i \cong \mathrm{Ind}_{Z(\sigma_i)}^{S_{t_i}}(Y_i)$. We may find $n_i \leqslant t_i$ such that (possibly after changing σ_i through conjugation), $\sigma_i \in S_{n_i}$ is fixed-point free. Then $Z(\sigma_i) \cong Z_i \times S_{t_i - n_i}$ for Z_i the centralizer of σ_i in S_{n_i}.

We observe that, since, for almost all i, (W_i, c^i) lies in $(\mathrm{Rep}_{p_i} S_{t_i})^{\leqslant k}$ of the coarser filtration — which is based on the maximum tensor power of X, $\mathrm{Ind}_{Z_i \times S_{t_i - n_i}}^{S_{n_i} \times S_{t_i - n_i}}(Y_i)$ lies in $\mathrm{Rep}_{p_i}(S_{n_i}) \boxtimes \mathrm{Rep}_{p_i}(S_{t_i})^{\leqslant k - n_i}$ (see Section 2.6) and in particular for almost all i, $n_i \leqslant k$. This provides an upper bound on n_i. Hence, by Łoś' theorem, there exists $n \in \mathbb{N}$ such that $n = n_i$ for almost all i. Hence, there are also only finitely many choices for the cycle type μ_i and one choice $\mu \vdash n$ is assumed for almost all i.

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We have seen above that (W_i, c^i) is indecomposable for almost all i and $W_i = \mathrm{Ind}_{Z\times S_{t_i-n}}^{S_{t_i}}(Y_i)$ for a fixed choice of n, $\sigma \in S_n$ and $Z = Z(\sigma)$. Hence, by Theorem 2.19 and Łoś' theorem, Y_i is indecomposable as an object in

$$\mathrm{Rep}_{p_i}(Z \times S_{t_i-n}) \simeq \mathrm{Rep}_{p_i} Z \boxtimes \mathrm{Rep}_{p_i} S_{t_i-n}$$

for almost all i. As the size of Z is fixed and the sequence p_i increases to infinity, for large enough i, $\mathrm{Rep}_{p_i} Z$ is semisimple. Thus, we find that the indecomposable object Y_i is isomorphic to an object $Y_i \cong V_i \boxtimes U_i$, where V_i is an indecomposable (and hence simple) Z-module and U_i is an indecomposable S_{t_i-n}-module.

Now, under the isomorphism $\prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i} \cong \mathbb{C}$, the ultraproduct $\prod_{\mathcal{U}} V_i$ corresponds to a Z-representation over $\mathbb{C}$ using Theorem 2.6. Further, by construction, $U := \prod_{\mathcal{U}} U_i$ defines an object in $\underline{\mathrm{Rep}}\, S_{t-n}$. The objects V, U are indecomposable (respectively, irreducible) again using Łoś' theorem. By construction, it follows that $W \cong W_{\sigma,V,U}$ as an object in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$. Thus, the objects $W_{\mu,V,U}$, with μ singleton free, V irreducible, and U indecomposable, provide a full list of indecomposable (respectively, irreducible) objects of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ as claimed. □

Remark 3.10. In the notation of Theorem 3.9, we may take $\mu = \varnothing \vdash 0$ and obtain the indecomposable objects of $\underline{\mathrm{Rep}}\, S_t$ embedded in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ using the symmetric half-braiding.

By Theorem 3.3, we have in particular classified the irreducible objects in the semisimple categories $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ for $t \notin \mathbb{Z}_{\geqslant 0}$ in Theorem 3.9.

Proposition 3.11. *Assume given two objects $\underline{W}_{\mu,V,U}$ and $\underline{W}_{\mu',V',U'}$, with singleton free $\mu \vdash n$, $\mu' \vdash n'$, V, V' irreducible as a $\mathbb{C}Z(\mu)$-module, respectively, $\mathbb{C}Z(\mu')$-module, U, U' indecomposable (respectively, irreducible) in $\underline{\mathrm{Rep}}\, S_{t-n}$, respectively, $\underline{\mathrm{Rep}}\, S_{t-n'}$. Then $\underline{W}_{\mu,V,U}$ and $\underline{W}_{\mu',V',U'}$ are isomorphic in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ if and only if $\mu = \mu'$, $V \cong V'$, and $U \cong U'$.*

Proof. We may use the ultraproduct description $(W, c) \cong \prod_{\mathcal{U}}(W_i, c^i)$ as in the proof of Theorem 3.9 which gives that $W = \underline{W}_{\mu,V,U}$ and $W' = \underline{W}_{\mu',V',U'}$ and almost all i, we find that $W_i \cong \mathrm{Ind}_{Z(\mu)\times S_{t_i-n}}^{S_{t_i}}(V_i \boxtimes U_i)$, for some indecomposable (respectively, irreducible) V_i, U_i; and similarly, $W_i' \cong \mathrm{Ind}_{Z(\mu')\times S_{t_i-n'}}^{S_{t_i}}(V_i' \boxtimes U_i')$. Now, if $W \cong W'$ as objects in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, then, for almost all i, $W_i \cong W_i'$ in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$. This yields, by Theorem 2.19, that $n = n'$, $\mu = \mu'$ and

$$\mathrm{Ind}_{Z(\mu)\times S_{t_i-n}}^{Z_i}(V_i \boxtimes U_i) \cong \mathrm{Ind}_{Z(\mu)\times S_{t_i-n}}^{Z_i}(V_i' \boxtimes U_i'),$$

for Z_i the centralizer of σ, of cycle type μ, in S_{t_i}. Using that μ is singleton free, we see that $Z_i = Z \times S_{t_i-n}$ and conclude that $V_i \cong V_i'$, $U_i \cong U_i'$ for almost all i. Thus, $\prod_{\mathcal{U}} V \cong \prod_{\mathcal{U}} V'$ which implies by Theorem 2.6 that $V \cong V'$. Further, $\prod_{\mathcal{U}} U_i \cong \prod_{\mathcal{U}} U'$ whence $U \cong U'$.

Conversely, if $U \cong U'$, $V \cong V'$, it was already established as a consequence of Proposition 3.7 that $\underline{W}_{\mu,V,U} \cong \underline{W}_{\mu',V',U'}$. □

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3.5 | The blocks of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$

We can now describe the blocks of the additive $\Bbbk$-linear category $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ based on the description of blocks of $\underline{\mathrm{Rep}}\, S_t$ from [8].

Lemma 3.12. *If $\mu \vdash n$ and $\nu \vdash m$ are singleton-free partitions, $V \in \mathcal{Z}(\mathrm{Rep}\, S_n)$, $V' \in \mathrm{Rep}\, S_m$, $U \in \underline{\mathrm{Rep}}\, S_{t-n}$, and $U' \in \underline{\mathrm{Rep}}\, S_{t-m}$. Then*

$$\mathrm{Hom}_{\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)}(\underline{W}_{\mu,V,U}, \underline{W}_{\nu,V',U'}) \cong \begin{cases} \mathrm{Hom}_{\mathrm{Rep}\, Z(\mu)}(V,V') \otimes \mathrm{Hom}_{\underline{\mathrm{Rep}}\, S_{t-n}}(U,U'), & \text{if } \mu = \nu, \\ \{0\}, & \text{if } \mu \neq \nu. \end{cases} \tag{3.7}$$

Proof. Assume that $\mu \neq \nu$. Using an ultraproduct description and Lemma 2.20 we see that there are no non-zero morphisms between any $\underline{W}_{\mu,V,U}$ and $\underline{W}_{\nu,V',U'}$. If $\mu = \nu$, we write $U \cong \prod_{\mathcal{U}} U_i$ and $U' \cong \prod_{\mathcal{U}} U_i'$, to see that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)}(\underline{W}_{\mu,V,U}, \underline{W}_{\nu,V',U'}) &\cong \prod_{\mathcal{U}} \mathrm{Hom}_{\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})}(\mathrm{Ind}_{Z\times S_{t_i-n}}^{S_{t_i}}(V_i \boxtimes U_i), \mathrm{Ind}_{Z\times S_{t_i-n}}^{S_{t_i}}(V_i' \boxtimes U_i')), \\ &\cong \prod_{\mathcal{U}} \mathrm{Hom}_{\mathrm{Rep}_{p_i} Z\times S_{t_i-n}}(V_i \boxtimes U_i, V_i' \boxtimes U_i') \\ &\cong \prod_{\mathcal{U}} \mathrm{Hom}_{\mathrm{Rep}_{p_i} Z}(V_i, V_i') \otimes \mathrm{Hom}_{\mathrm{Rep}\, S_{t_i-n}}(U_i, U_i') \\ &\cong \mathrm{Hom}_{\mathrm{Rep}\, Z}(V,V') \otimes \mathrm{Hom}_{\underline{\mathrm{Rep}}\, S_{t-n}}(U,U'), \end{aligned}$$

as desired. Here, we use Lemma 2.20 and the fact that, given that μ is singleton free, $Z(\mu) = Z \times S_{t_i-n} \leqslant S_{t_i}$, to obtain the third isomorphism. □

The above lemma enables us to decompose $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ as a direct sum of additive $\Bbbk$-linear categories.

Corollary 3.13. *Let $\mu \vdash n$ be singleton free. Then the $\Bbbk$-linear additive functor*

$$\underline{\mathrm{Ind}}: \ \mathrm{Rep}\, Z(\mu) \boxtimes \underline{\mathrm{Rep}}\, S_{t-n} \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$$

is fully faithful. As an additive $\Bbbk$-linear category, $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is equivalent to the direct sum of categories

$$\bigoplus_{n\geqslant 0, \mu\vdash n} \mathrm{Rep}\, Z(\mu) \boxtimes \underline{\mathrm{Rep}}\, S_{t-n}.$$

Proof. Lemma 3.12 implies that $\underline{\mathrm{Ind}}$ is fully faithful on objects of the form $U \boxtimes V$, for U, V indecomposable. Since the category $\mathrm{Rep}\, Z(\mu)$ is semisimple, this provides a full list of indecomposables of $\mathrm{Rep}\, Z(\mu) \boxtimes \underline{\mathrm{Rep}}\, S_{t-n}$. □

Note that the above decomposition is *not* a decomposition of tensor categories. We can now describe the blocks of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$. For this, denote by $\mathrm{Irrep}(Z)$ the set of isomorphism classes of irreducible representations of $\mathbb{C}Z$.

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Corollary 3.14. *The blocks in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ are parametrized by triples (μ, V, B), where $\mu \vdash n$ is a singleton-free partition of some integer $n \geqslant 0$ and $V \in \mathrm{Irrep}(Z(\mu))$ and B is a block of $\underline{\mathrm{Rep}}\, S_{t-n}$.*

Proof. Note that the categories $\mathrm{Rep}\, Z(\mu)$ are semisimple. Thus, each block contains a unique simple module V. Lemma 3.12 implies that $\underline{W}_{\mu,V,U}$ and $\underline{W}_{\mu,V',U'}$ are in the same block if and only if $\mu = \mu'$, $V \cong V'$, and U and U' are contained in the same block of $\underline{\mathrm{Rep}}\, S_{t-n}$. □

The blocks of $\underline{\mathrm{Rep}}\, S_t$ have been described combinatorially in [8, Section 5.1]. In particular they showed that for each $d \in \mathbb{Z}_{\geqslant 0}$ there are finitely many non-semisimple blocks in $\underline{\mathrm{Rep}}\, S_d$, all of which are equivalent to the unique non-semisimple block in $\underline{\mathrm{Rep}}\, S_0$. It follows immediately from the description above that the same holds for $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$. Corollary 3.14 implies that blocks of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ are parametrized by pairs of blocks B of $\underline{\mathrm{Rep}}\, S_{t-n}$ and (μ, V). The latter parametrize those blocks of $\mathcal{Z}(\mathrm{Rep}\, S_n)$ not induced from $\mathcal{Z}(\mathrm{Rep}\, S_m)$ with $m < n$.

3.6 | Comparison with the previous construction

In Section 3.4, we have constructed all objects in the center of $\underline{\mathrm{Rep}}\, S_t$. Hence, our construction should recover those central objects constructed in the previous paper [24]. In this section, we show how the objects constructed there are indeed special cases of the construction explained here. We also note that the old objects from [24, Definition 3.12] generate $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ as a Karoubian tensor category.

We recall the definition from [24] below. For this, we recall the embedding

$$x : \mathbb{C}S_n \hookrightarrow \mathrm{End}_{\underline{\mathrm{Rep}}\, S_t}(X^{\otimes n}), \quad g \mapsto x_g, \tag{3.8}$$

see, for example, [8, Equation (2.1)] for the definition of x_g. We write 1_n for the identity of S_n. Hence, we can embed $\mathbb{C}S_n \otimes M_k(\mathbb{C})$ into $\mathrm{End}_{\underline{\mathrm{Rep}}\, S_t}((X^{\otimes n})^{\oplus k})$.

Definition 3.15 (The central objects $\underline{W}^{\mathrm{old}}_{\mu,V}$). Consider $n, k \geqslant 0$, $\sigma \in S_n$ an element of cycle type $\mu \vdash n$, Z the centralizer of σ in S_n, V a k-dimensional representation of Z, and $e_V \in \mathbb{C}Z \otimes_{\mathbb{C}} M_k(\mathbb{C})$ an idempotent with image isomorphic to V. We denote by e the image of e_V under the above embedding (3.8), define $\mathrm{Im}\, e = (X^{\otimes n}, e)$ to be the subobject of $X^{\otimes n}$ defined by the idempotent e in $\underline{\mathrm{Rep}}\, S_t$ and set

$$d_1^{\sigma,V} := (e \otimes \mathrm{Id}_X)\left(1_{n+1} + \sum_{1 \leqslant i \leqslant n} E^i_{\sigma(i)} - E^i_i\right) \quad \in \mathrm{End}_{\underline{\mathrm{Rep}}\, S_t}(\mathrm{Im}\, e \otimes X)\,,$$

where E^i_j is defined as a partition of $n+1$ upper and lower points as follows:

$$E^i_j := \{\{k, k'\}\}_{1 \leqslant k \leqslant n} \setminus \{\{i, i'\}, \{j, j'\}\} \cup \{\{i, n+1\}, \{j', (n+1)'\}\}\,.$$

Let τ_n be the permutation $(1\ 2\ \ldots\ n+1)$ viewed as an endomorphism of $X^{\otimes(n+1)}$ in $\underline{\mathrm{Rep}}\, S_t$, that is, τ_n is the symmetric braiding $\Psi_{X^{\otimes n},X}$ of $\underline{\mathrm{Rep}}\, S_t$. It was shown in [24, Theorem 3.11] that $c_X := \tau_n d_1^{\sigma,V}$ determines an object $(\mathrm{Im}\, e, c)$ in the center of $\underline{\mathrm{Rep}}\, S_t$ via Lemma 3.2. It was also shown in [24, Proposition 3.22] that the isomorphism class of this central object depends only on the

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cycle type μ of σ and on the isomorphism class of V. Therefore, we will denote this central object considered in [24, Definition 3.12] by $\underline{W}^{\mathrm{old}}_{\mu,V}$.

Let us assume that μ has $r \geqslant 0$ singletons, so any permutation σ of cycle type μ has r fixed points. Let μ_0 be the partition of $n_0 := n - r$, we obtain from μ by removing all singletons. Then the centralizer Z of σ is isomorphic to $Z_0 \times S_r$, where Z_0 is the centralizer of an element of cycle type μ_0 in S_{n_0}, and the $\mathbb{C}Z$-module V decomposes as $V_0 \boxtimes U_0$ for a $\mathbb{C}Z_0$-module V_0 and a $\mathbb{C}S_r$-module U_0. We set $U_{\mu,V} := \underline{\mathrm{Ind}}^{S_{t-n_0}}_{S_r \times S_{t-n}} U_0 \boxtimes \mathbf{1}$. Note that, in particular, if $r = 0$, then $n_0 = n$, $Z_0 = Z$, $V_0 = V$, and $U_{\mu,V} = \mathbf{1}$.

Lemma 3.16. *If we interpret e_V as an idempotent e in $\underline{\mathrm{Rep}}\, S_t$ as in Definition 3.15, then*

$$\mathrm{Im}\, e \cong \underline{\mathrm{Ind}}^{S_t}_{Z_0 \times S_{t-n_0}} (V_0 \boxtimes U_{\mu,V}). \tag{3.9}$$

In particular, if μ has no singletons, that is, σ has no fixed points, then

$$\mathrm{Im}\, e \cong \mathrm{Ind}^{S_t}_{Z \times S_{t-n}} (V \boxtimes \mathbf{1}). \tag{3.10}$$

Proof. Again we use the ultraproduct concepts explained in Section 2.4 and Section 2.6: We choose an ultraproduct representation $\underline{\mathrm{Rep}}\, S_t \cong \langle X \rangle \subset \prod_{\mathcal{U}} \mathrm{Rep}_{p_i} S_{t_i}$ for $\underline{\mathrm{Rep}}\, S_t$, and similarly, we fix $\mathrm{Rep}\, S_n \cong \langle X \rangle \subset \prod_{\mathcal{U}} \mathrm{Rep}_{p_i} S_n$, for $n \geqslant 0$, cf. Theorem 2.6. For simplicity, write $\mathbb{F}_i := \overline{\mathbb{F}}_{p_i}$.

We prove the general case, Equation (3.9), first. Almost always, $t_i \geqslant n$, so let us assume this holds. Choose a finite field extension $\Bbbk$ of $\mathbb{Q}$ such that V_0, U_0 are defined over $\Bbbk$. Denote by $\mathcal{O}$ the ring of integers of $\Bbbk$ and for all i, choose a prime ideal $\mathfrak{p}_i \lhd \mathcal{O}$ containing p_i. Assume that i is sufficiently large i such that p_i does not divide the discriminant of $\Bbbk$ and $\mathcal{O}/\mathfrak{p}_i$ is $\mathbb{F}_{p_i^d}$ for some d, so in particular, $\mathcal{O}/\mathfrak{p}_i$ embeds into $\mathbb{F}_i$.

For the $\mathbb{C}S_r$ module U_0, by Proposition 2.8, we fix an ultraproduct presentation $U_0 \cong \prod_{\mathcal{U}} U_{0,i}$ where $U_{0,i}$ is the reduction of a chosen integral form of U_0 modulo $\mathfrak{p}_i$ viewed as an $\mathbb{F}_i S_{t_i}$-module. From Section 2.6 we know that the induction functor can be computed as an ultraproduct of the corresponding induction functors in finite characteristic. Thus, we find an ultraproduct presentation $U_{\mu,V} = \prod_{\mathcal{U}} U_i$, where

$$U_i = \mathrm{Ind}^{S_{t_i}-n_0}_{S_r \times S_{t_i-n}} (U_{0,i} \boxtimes \mathbb{F}_i).$$

Similarly, we choose an integral form of V_0 and write $V_{0,i}$ for the reduction of the integral form of V_0 modulo $\mathfrak{p}_i$ and observe that $V_0 \cong \prod_{\mathcal{U}} V_{0,i}$ by Proposition 2.8. Thus, the right-hand side of (3.9) is isomorphic to the ultraproduct of the $\mathbb{F}_i S_{t_i}$-modules

$$\begin{aligned}\mathrm{Ind}^{S_{t_i}}_{Z_0 \times S_{t_i-n_0}} (V_{0,i} \boxtimes U_i) &= \mathrm{Ind}^{S_{t_i}}_{Z_0 \times S_{t_i-n_0}} \left(V_{0,i} \boxtimes \mathrm{Ind}^{S_{t_i-n_0}}_{S_r \times S_{t_i-n}} (U_{0,i} \boxtimes \mathbb{F}_i) \right) \\ &\cong \mathrm{Ind}^{S_{t_i}}_{Z_0 \times S_r \times S_{t_i-n}} (V_{0,i} \boxtimes U_{0,i} \boxtimes \mathbb{F}_i).\end{aligned}$$

As we have chosen integral forms of V_0, U_0 over the ring of integers $\mathcal{O}$ of $\Bbbk$, we note that $(V_0 \boxtimes U_0) \otimes_{\mathcal{O}} \mathbb{C} \cong V$. Thus, we denote $V_i := (V_0 \boxtimes U_0) \otimes_{\mathcal{O}} \mathbb{F}_i$ and $V \cong \prod_{\mathcal{U}} V_i$. We may write, possibly

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replacing V with an isomorphic module,

$$e_V = \sum_{g\in Z} g \otimes_{\mathbb{C}} m_g,$$

with k-by-k-matrices $(m_g)_{g\in Z}$ whose entries lie in $\mathcal{O}$. It follows that

$$e = \sum_{g\in Z} x_g \otimes_{\mathbb{C}} m_g = \sum_{g\in Z} g x_{1_n} \otimes_{\mathbb{C}} m_g \in \operatorname{End}_{\underline{\operatorname{Rep}}\, S_t}((X^{\otimes n})^{\oplus k}).$$

We first observe that the image of the component $(x_{1_n})_i$ of the idempotent x_{1_n}, in the notation of (2.3), acting on the $\mathbb{F}_i S_{t_i}$-module $X_{t_i}^{\otimes n}$ is isomorphic to $\mathbb{F}_i S_{t_i}/S_{t_i-n}$ [24, Remark 2.5]. Denote by $m_{g,i}$ the $\mathbb{F}_i$-matrix obtained from m_g by reducing all entries modulo $\mathfrak{p}_i$. Now, the object $\operatorname{Im} e$ is isomorphic to the ultraproduct of the $\mathbb{F}_i S_{t_i}$-modules P_i, where P_i is the image of the idempotent

$$\sum_{g\in Z} g_i \otimes_{\mathbb{F}_i} m_{g,i} : \mathbb{F}_i S_{t_i}/S_{t_i-n} \otimes_{\mathbb{F}_i} \mathbb{F}_i^{\oplus k} \longrightarrow \mathbb{F}_i S_{t_i}/S_{t_i-n} \otimes_{\mathbb{F}_i} \mathbb{F}_i^{\oplus k}.$$

Here, g_i denotes the evaluation of the partition corresponding to g from (2.5). We note that g_i corresponds to the right multiplication action of the subgroup $\{1_{t_i-n}\} \times S_n$ on S_{t_i}/S_{t_i-n} [24, Remark 2.5]. Hence, it follows from a computation similar to [24, Proposition 3.8] that P_i is isomorphic to

$$\mathbb{F}_i S_{t_i}/S_{t_i-n} \otimes_Z V_i \cong \operatorname{Ind}_{Z\times S_{t_i-n}}^{S_{t_i}}(V_i \boxtimes \mathbb{F}_i) \cong \operatorname{Ind}_{Z_0\times S_r\times S_{t_i-n}}^{S_{t_i}}(V_{0,i} \boxtimes U_{0,i} \boxtimes \mathbb{F}_i).$$

Thus, for almost all i, the component in $\operatorname{Rep}_{p_i} S_{t_i}$ of the objects on the left and right sides in (3.9) are isomorphic, which proves the assertion.

It is clear from $V = V_0$ and $U_{\mu,V} = \mathbf{1}$ that (3.10) is a special case of (3.9). □

Proposition 3.17. *$\underline{W}_{\mu,V}^{\mathrm{old}}$ is isomorphic to $\underline{W}_{\mu_0,V_0,U}$ for $U = U_{\mu,V}$. In particular, if μ has no singletons, then $\underline{W}_{\mu,V}^{\mathrm{old}}$ is isomorphic to $\underline{W}_{\mu,V,\mathbf{1}}$.*

Proof. By Lemma 3.16, we know that the underlying objects are isomorphic. To see that the half-braidings coincide, it is enough to verify this for the half-braidings evaluated at the tensor-generating object.

With the same choices and symbols as in the proof of Lemma 3.16, let us pick a basis $e_1, \dots, e_{t_i}$ for X_{t_i}, the tensor generator in $\operatorname{Rep}_{p_i} S_{t_i}$. Then a basis of $X_{t_i}^{\otimes n}$ is given by the tensor products $e_{i_1} \otimes \cdots \otimes e_{i_n}$. Now the $\mathbb{F}_i S_{t_i}$-module map $(d_1^{\sigma,V})_i$ defined in (2.5), corresponding to $d_1^{\sigma,V}$, sends $(e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes v) \otimes e_j$ to

$$(e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes v) \otimes \begin{cases} e_{\phi\sigma\phi^{-1}(j)}, & \text{if } (i_1,\dots,i_n) \text{ are pairwise distinct and } j \in \{i_1,\dots,i_n\}, \\ e_j, & \text{if } (i_1,\dots,i_n) \text{ are pairwise distinct and } j \notin \{i_1,\dots,i_n\}. \\ 0, & \text{else,} \end{cases}$$

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for any $v \in V_i$ (the $\mathbb{F}_i$-reduction of the integral form of V), where $\phi : \{1, \dots, n\} \to \{1, \dots, t\}$ is the injective map defined by $\phi(k) = i_k$, for $k = 1, \dots, n$.

Under the isomorphism (3.9), evaluated on the ith part of the ultrafilter presentation used in the proof of Lemma 3.16, the map $(d_1^{\sigma,V})$ corresponds to the map

$$([g] \otimes v) \otimes e_j \longmapsto ([g] \otimes v) \otimes (g(1,\sigma)g^{-1} \cdot e_j)$$

for any $[g] \in S_{t_i}/S_{t_i-n}$, which is indeed the desired Yetter–Drinfeld braiding we expect from $\underline{W}_{\mu_0,V_0,U}$. This computation follows (similarly to [24, Proposition 3.36]) using that $[g]$ is identified with $e_{i_1} \otimes \cdots \otimes e_{i_n}$ if $g(k) = i_k$ under the isomorphisms of $(\mathrm{Im}\, x_{1_n})_i$ and $\mathbb{F}_i S_{t_i}/S_{t_i-n}$. □

We note that $U_{\mu,V}$ is generally (in particular, for $r \geqslant 1$) a decomposable object in $\underline{\mathrm{Rep}}\, S_{t-r}$ whose decomposition into indecomposable objects implies a decomposition for $\underline{W}^{\mathrm{old}}_{\mu,V}$. Hence, by Theorem 3.9, $\underline{W}^{\mathrm{old}}_{\mu,V}$ is typically not indecomposable.

We conclude our comparison with the results in [24] with the following observation, which answers [24, Question 3.31] in the affirmative.

Proposition 3.18. *The objects $\underline{W}^{\mathrm{old}}_{\mu,V}$ generate the entire center $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ as a Karoubian tensor category.*

Proof. By Theorem 3.9, it is enough to show that any object $\underline{W}_{\mu',V',U'}$ for μ' without singletons, V' irreducible, and U' indecomposable is isomorphic to a subobject of $\underline{W}^{\mathrm{old}}_{\mu,V}$ for some μ, V, with $r, n_0, \mu_0, V_0, U_0, U_{\mu,V}$ all defined as above. Note that by construction, $\underline{W}^{\mathrm{old}}_{\mu,V}$ contains $\underline{W}_{\mu_0,V_0,Y}$ for any subobject Y of $U_{\mu,V}$. Hence it is enough to show that, for any n_0, any indecomposable U' in $\underline{\mathrm{Rep}}\, S_{t-n_0}$ is isomorphic to a subobject of $\underline{\mathrm{Ind}}^{S_{t-n_0}}_{S_r \times S_{t-n_0-r}}(U_0 \boxtimes \mathbf{1})$ for some r, U_0, or equivalently, that any indecomposable U' in $\underline{\mathrm{Rep}}\, S_t$ is isomorphic to a subobject of $\underline{\mathrm{Ind}}^{S_t}_{S_r \times S_{t-r}}(U_0 \boxtimes \mathbf{1})$ for some r, U_0, for any t. But this is true by Lemma 2.10. □

3.7 | The center of $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$

Let $d \in \mathbb{Z}_{\geqslant 0}$ and consider the abelian envelope $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ of the category $\underline{\mathrm{Rep}}\, S_d$ of [9, 14]. There is a full and faithful functor of symmetric monoidal categories $\underline{\mathrm{Rep}}\, S_d \to \underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$.

Before determining the center of $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ we need some preliminary observations.

Proposition 3.19. *Let (Y, c) be an object in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$. Then the half-braiding c is uniquely determined by the morphism $c_X : Y \otimes X \to X \otimes Y$.*

Conversely, any pair (Y, c_X), where Y is an object in $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ and c_X is a morphism satisfying the finite list of conditions from Lemma 3.2 uniquely extends to give an object in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$.

Proof. The abelian category $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ has enough projectives. The projectives are direct summands of sums of tensor powers of the distinguished generating object X [9, Remark 4.8]. Hence, the projectives of $\mathcal{C} = \underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ are contained in $\mathcal{A} = \underline{\mathrm{Rep}}\, S_d$. Thus, we can use Lemma A.1 to conclude that c is uniquely determined by its restriction to $\underline{\mathrm{Rep}}\, S_d$. By a slight generalization of Lemma 3.2,

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allowing the object Y be in $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$ rather than $\underline{\mathrm{Rep}}\, S_d$, it follows that c is uniquely determined by a morphism c_X satisfying the conditions (3.1)–(3.2). □

The next corollary is a direct consequence of Proposition 3.19 and Lemma A.1.

Corollary 3.20. *The inclusion of* $\underline{\mathrm{Rep}}\, S_d$ *into its abelian envelope induces a full and faithful functor of braided monoidal categories*

$$\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d) \hookrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d).$$

To classify simple objects in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ we first note that $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ has the *Jordan–Hölder property* by virtue of being a locally finite abelian category [22, Theorem 1.5.4]. That is, any object in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ has a finite filtration by simple objects in this category. Moreover, the simple composition factors are unique up to order.

The induction functor from Proposition 3.7 extends to the abelian envelope such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{Z}(\mathrm{Rep}\, S_n) \boxtimes \underline{\mathrm{Rep}}\, S_{d-n} \arrow[r, "\underline{\mathrm{Ind}}"] \arrow[d, hook] & \mathcal{Z}(\underline{\mathrm{Rep}}\, S_d) \arrow[d, hook] \\
\mathcal{Z}(\mathrm{Rep}\, S_n) \boxtimes \underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_{d-n} \arrow[r, "\underline{\mathrm{Ind}}^{\mathrm{ab}}"] & \mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d),
\end{tikzcd} \tag{3.11}$$

where taking the abelian envelope of $\underline{\mathrm{Rep}}\, S_{d-n}$ is only required if $d - n \geqslant 0$. This follows, using the above Corollary 3.20, from right exactness of $\underline{\mathrm{Ind}}$ in the second tensor factor. Indeed, we may resolve a given object M in $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_{d-n}$ by $P_0 \xrightarrow{p} P_1 \to M \to 0$, where P_0, P_1 are projective objects of $\underline{\mathrm{Rep}}\, S_{d-n}$, and define $\underline{\mathrm{Ind}}(V \boxtimes M)$ to be the cokernel of the map $\underline{\mathrm{Ind}}(\mathrm{Id}_V \boxtimes p)$. Alternatively, we may also introduce the functor $\underline{\mathrm{Ind}}^{\mathrm{ab}}$ using ultraproducts, similarly to how $\underline{\mathrm{Ind}}$ was introduced in Proposition 3.7, working with objects from $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_{d-n}$ in the second tensor factor, cf. Theorem 2.12.

We can now extend Theorem 3.9 to the abelian envelope.

Corollary 3.21. *The objects* $\underline{W}_{\mu,V,U}$, *for* μ, V, U *as in Definition 3.6, where* V *is irreducible as a* $Z(\mu)$*-module,* U *is indecomposable (or irreducible, or indecomposable projective) in* $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_{d-n}$, *and* $\mu \vdash n$ *is singleton free, provide a full list of indecomposable (respectively, irreducible, or indecomposable projective) objects in* $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ *up to isomorphism.*

Proof. The classification of irreducible or indecomposable objects in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ is completely analogous to the proof of Theorem 3.9. The only difference is that we consider ultraproducts of general (indecomposable) modules in the categories $\mathcal{Z}(\mathrm{Rep}_{p_i}\, S_{t_i})$ rather than just objects for which the underlying S_{t_i}-modules are in $\langle X_{t_i} \rangle$. The projective objects among the indecomposables are identified in Proposition 3.22(2) below. □

Proposition 3.22. *Let* $d \in \mathbb{Z}_{\geqslant 0}$.

(1) *The object* $X_{(d+1)}$ *with symmetric half-braiding is the projective cover of* $\mathbf{1}$ *in* $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$.
(2) *An object* $\underline{W}_{\mu,V,U}$ *is projective if and only if* U *is projective.*
(3) *All projective objects of* $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ *are contained in the full subcategory* $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$.

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Proof. Note that the functor

$$I : \underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d \to \mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d), \qquad V \mapsto (V, \Psi_{V,-})$$

admits an ultrafilter description. It is given by the ultrafilter of the corresponding functors in characteristic p_i. This functor preserves projective objects (cf. Lemma 2.21) as an induction functor (for $\sigma = 1 \in S_0$). Thus I preserves projective objects and we can apply I to the projective cover $X_{(d+1)} \twoheadrightarrow \mathbf{1}$ from Lemma 2.16. This proves Part (1).

To prove Part (2), using Part (1), an object P in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k}$, P is projective in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ if and only if it is projective as an object of $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k+d+1}$ (cf. the proof of Proposition 2.17). Thus, for $P \cong \prod_{\mathcal{U}} P_i$ is projective if and only if for almost all i, P_i is projective. Since $P \cong \underline{W}_{\mu,V,U}$, we have that $P_i \cong \mathrm{Ind}_{Z\times S_{t_i-n}}^{S_{t_i}}(V_i \boxtimes U_i)$ and for almost all i. By Lemma 2.21, P_i is projective if and only if U_i is projective. Thus, almost all P_i are projective if and only if almost all U_i are projective, proving Part (2). Further, observe that for almost all i, P_i is in $\langle X_{t_i}\rangle$ and hence P is contained in the subcategory $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$. This proves Part (3). □

We can now collect a few consequences of the constructions of this section.

Corollary 3.23. *Any indecomposable object in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ is a quotient of an object in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$. In particular, the abelian category $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ has enough projectives.*

Proof. By the above Corollary 3.21, an indecomposable object in $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ is isomorphic to one of the form $\underline{W}_{\mu,V,U}$. We may choose a projective cover $P \twoheadrightarrow U$. Then $V \boxtimes P$ is projective in $\mathcal{Z}(\mathrm{Rep}\, S_n) \boxtimes \underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_{d-n}$ as the first tensor and is a semisimple category. By right exactness of $\underline{\mathrm{Ind}}$ in the second tensor factor, $\underline{W}_{\mu,V,U}$ arises as a quotient of the object $\underline{W}_{\mu,V,P}$, which lies in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$. This argument also shows that $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ has enough projectives. □

Recall the concept of the abelian envelope from [4], discussed in Section 2.7, to obtain the following result.

Corollary 3.24. *The category $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ is the abelian envelope of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$.*

Proof. The corollary follows from the more general results of Corollary A.4 since the functor $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d \to \mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ preserves projectives (as noted in the proof of Proposition 3.22). □

3.8 | Non-degeneracy of the centers

We conclude this section with a discussion on non-degeneracy of the center of Deligne's interpolation categories and their abelian envelopes in order to highlight the analogy with modular tensor categories.

We note that the categories $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, for t generic, and $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$, for $d \in \mathbb{Z}_{\geqslant 0}$, are tensor categories in the sense of [22]. That is, they are rigid $\Bbbk$-linear monoidal abelian categories, with bilinear tensor product, that are locally finite such that $\mathrm{End}(\mathbf{1}) = \Bbbk$. Recall that a braided tensor category $\mathcal{C}$ is *factorizable* if the canonoical functor $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \to \mathcal{Z}(\mathcal{C})$ gives an equivalence of

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braided monoidal categories. The centers $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ and $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ are factorizable using [22, Proposition 8.6.3].

Let $\mathcal{C}$ be a braided $\Bbbk$-linear monoidal category with braiding Ψ. The object V in $\mathcal{C}$ *centralizes* the object W of $\mathcal{C}$ if

$$\Psi_{W,V} \circ \Psi_{V,W} = \mathrm{Id}_{V \otimes W}\,.$$

The *Müger center* $\mathcal{C}'$ of $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ on those objects which centralize *all* objects of $\mathcal{C}$. Recall that $\mathcal{C}$ is called *non-degenerate* if the Müger center $\mathcal{C}'$ is generated by the tensor unit (that is, equivalent to $\mathrm{Vect}_{\Bbbk}$).

In the case of *finite* braided tensor categories, the concepts of factorizability and non-degeneracy are equivalent [39]. However, we saw that the categories $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ are only locally finite and have infinitely many simple objects, see Theorem 3.9 and Corollary 3.21. Nevertheless, we have the following result.

Proposition 3.25. *The braided tensor categories $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, for t generic, and $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$, for $d \in \mathbb{Z}_{\geqslant 0}$, are non-degenerate.*

Proof. By Theorem 3.9 and Corollary 3.21, we have a classification of indecomposable objects $W = \underline{W}_{\mu,U,V}$ in $\underline{\mathrm{Rep}}\, S_t$ and $\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d$, respectively. Assume that $\sigma \in S_n$ is an element of cycle type μ which is fixed-point free. Recall the ultrafilter description $W \cong \prod_{\mathcal{U}} W_i$, where $W_i = \mathrm{Ind}_{Z(\sigma)\times S_{t_i-n}}^{S_{t_i}} (U_i \boxtimes V_i)$.

First consider the case that $\sigma \neq 1$. Then for i large enough, $\sigma \notin Z(S_{t_i})$. Choose $\tau \in S_{t_i}$ which does not commute with σ and define $W' := \underline{W}_{\tau,\mathbf{1},\mathbf{1}}$. Then choose an ultraproduct representation $W' = \prod_{\mathcal{U}} W'_i$ and for any $j \geqslant i$ one computes that $\Psi_{W'_j,W_j} \circ \Psi_{W_j,W'_j} \neq \mathrm{Id}_{W_j \otimes W'_j}$. Indeed,

$$\begin{aligned}\Psi_{W'_j,W_j} \circ \Psi_{W_j,W'_j}((1 \otimes (u \boxtimes v)) \otimes (1 \otimes 1)) &= \Psi_{W,W'}((\sigma \otimes 1) \otimes (1 \otimes (u \boxtimes v))) \\ &= (\sigma\tau\sigma^{-1} \otimes (u \boxtimes v)) \otimes (\sigma \otimes 1)\end{aligned}$$

does not equal the identity as $\sigma\tau\sigma^{-1} \in Z(\sigma)$ if and only if $\tau \in Z(\sigma)$ (note that the tensor products in the induced modules are taken over the group algebras of subgroups, not over the base field).

If $\sigma = 1 \in S_0$, then without loss of generality $U = \mathbf{1}$ but assume that $V \neq \mathbf{1} \in \underline{\mathrm{Rep}}\, S_t$, as otherwise $W \cong \mathbf{1}$ as objects of the center. Let us fix an ultrafilter representation $V \cong \prod_{\mathcal{U}} V_i$. Then there exists an element $\tau \in S_{t_i}$ such that τ, viewed as an element of S_{t_j}, for $j \geqslant i$, acts non-trivially on V_j. Then, computing as above, the braiding again does not square to the identity. Indeed, for almost all j,

$$\begin{aligned}\Psi_{W'_j,W_j} \circ \Psi_{W_j,W'_j}((1 \otimes v) \otimes (1 \otimes 1)) &= (\sigma\tau\sigma^{-1} \otimes v) \otimes (\sigma \otimes 1) \\ &= (1 \otimes \tau v) \otimes (1 \otimes 1),\end{aligned}$$

for all $v \in \Bbbk \boxtimes V_j \cong V_j$. Passing to ultraproducts, as for almost all i, the braiding does not square to the identity, we conclude $\Psi_{W',W}\Psi_{W,W'} \neq \mathrm{Id}_{W \otimes W'}$ in both cases. Thus, W is not in the Müger center, which is hence trivial. □

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To summarize, $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ and $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$ are non-degenerate and factorizable braided tensor categories that possess a ribbon structure by [24, Theorem 3.28]. Thus, these categories can be seen as infinite (possibly non-semisimple) analogs of modular tensor categories (cf. [39] in the case of a finite tensor category).

4 | THE GROTHENDIECK RING OF THE CENTER OF $\underline{\mathrm{Rep}}\, S_t$

In this section, we show that the Grothendieck ring of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ is a filtered ring such that its associated graded ring can be identified with the Grothendieck ring of a braided monoidal category $\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0}$ defined as the sum of the categories $\mathcal{Z}(\mathrm{Rep}\, S_n)$, for all $n \geqslant 0$, with the induction product on centers from Appendix B. Thus, the Grothendieck ring can be described using a tower of centers of the symmetric groups S_n, varying n.

4.1 | The tower of centers of a sequence of groups

In this subsection, $\{G_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ can be any sequence of groups such that $G_0 = \{1\} \subseteq G_1 \subseteq G_2 \subseteq \ldots$, together with embeddings $\iota_{m,n} : G_m \times G_n \subseteq G_{m+n}$ such that $\iota_{m+n,r}(\iota_{m,n}, \mathrm{Id}) = \iota_{m,n+r}(\mathrm{Id}, \iota_{n,r})$ under the natural identifications of $(G_m \times G_n) \times G_r$ and $G_m \times (G_n \times G_r)$. In addition, we require that $G_n \times G_0 \to G_n$ and $G_0 \times G_n \to G_n$ are simply given by $(g, 1) \mapsto g$, respectively, $(1, g) \mapsto g$. We refer to such a sequence as a *tower of groups*. It induces a tower of algebras $\{\Bbbk G_n\}_{n\geqslant 0}$ as in [36]. Our main example of interest is the case where $G_n = S_n$, the tower of symmetric groups.

Note that, unlike the group algebras $\Bbbk G_n$, the Drinfeld doubles $\mathrm{Drin}(G_n)$, varying n, do *not* constitute a tower of algebras in the sense of [36]. Nevertheless, we can use Proposition B.1 to construct an external tensor product

$$\mathcal{Z}\,\mathrm{Ind}_{G_n\times G_m}^{G_{n+m}} : \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_n) \boxtimes \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_m) \cong \mathcal{Z}(\mathrm{Rep}_{\Bbbk}(G_n \times G_m)) \longrightarrow \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_{n+m}).$$

We denote

$$V \odot W := \mathrm{Ind}_{G_n\times G_m}^{G_{n+m}}(V \boxtimes W), \qquad V \in \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_n), W \in \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_m), \tag{4.1}$$

and define the additive $\Bbbk$-linear category

$$\mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_{\geqslant 0}) := \bigoplus_{n\geqslant 0} \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_n).$$

Its objects are formal direct sums $\bigoplus_n V_n$, where V_n is an object of $\mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_n)$ which is zero for all but finitely values of n. The morphism spaces in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_{\geqslant 0})$ are given by

$$\mathrm{Hom}_{\mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_{\geqslant 0})}\left(\bigoplus_n V_n, \bigoplus_n W_n\right) = \bigoplus_n \mathrm{Hom}_{\mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_n)}(V_n, W_n).$$

Lemma 4.1. *Given objects $V \in \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_n)$, $W \in \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_m)$, $U \in \mathcal{Z}(\mathrm{Rep}_{\Bbbk}\, G_k)$, there are natural isomorphisms*

$$\alpha_{V,W,U} : (V \odot W) \odot U \to V \odot (W \odot U) \tag{4.2}$$

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in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{n+m+k})$ *which satisfy the pentagon axiom of the associativity isomorphism of a monoidal category. In addition, there are coherent natural isomorphisms* $V \odot \mathbf{1} \cong V \cong \mathbf{1} \odot V$*, for* $\mathbf{1} \in \mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_0)$.

This way, $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{\geqslant 0})$ *obtains the structure of an abelian* $\Bbbk$*-linear symmetric monoidal category with biexact tensor product such that* $\mathrm{End}(\mathbf{1}) = \Bbbk$.

Proof. Given a fixed choice of adjunctions of $\mathrm{Ind}_{G_n \times G_m}^{G_{n+m}}$ and $\mathrm{Res}_{G_n \times G_m}^{G_{n+m}}$ we obtain that the two left adjoints $\mathrm{Ind}_{G_{n+m} \times G_k}^{G_{n+m+k}} \circ (\mathrm{Ind}_{G_n \times G_m}^{G_{n+m}} \boxtimes \mathrm{Id})$ and $\mathrm{Ind}_{G_n \times G_{m+k}}^{G_{n+m+k}} \circ (\mathrm{Id} \boxtimes \mathrm{Ind}_{G_m \times G_k}^{G_{m+k}})$ of the corresponding restriction $\mathrm{Res}_{G_n \times G_m \times G_k}^{G_{n+m+k}}$ are canonically naturally isomorphic by the Yoneda Lemma and hence necessarily coherent. Similarly, one proves the coherence of the natural isomorphisms

$$V \odot \mathbf{1} = \mathrm{Ind}_{G_n \times G_0}^{G_n}(V \boxtimes \mathbf{1}) \cong V \cong \mathrm{Ind}_{G_0 \times G_n}^{G_n}(\mathbf{1} \boxtimes V) \cong \mathbf{1} \odot V.$$

The category $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{\geqslant 0})$ is abelian and $\Bbbk$-linear as a direct sum of such categories. Note that $V \odot W$ is exact in both arguments as it is given by the composition of the exact functors of external tensor product [22, Section 1.11] and induction. Further,

$$\mathrm{End}_{\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{\geqslant 0})}(\mathbf{1}) \cong \mathrm{End}_{\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_0)}(\mathbf{1}) \cong \mathrm{End}_{\mathrm{Rep}_{\Bbbk} G_0}(\mathbf{1}) = \Bbbk.$$

The natural isomorphisms

$$V \odot W = \mathrm{Ind}_{G_n \times G_m}^{G_{n+m}}(V \boxtimes W) \xrightarrow{\sim} \mathrm{Ind}_{G_m \times G_n}^{G_{n+m}}(W \boxtimes V) = W \odot V, \quad g \otimes (v \boxtimes w) \mapsto g \otimes (w \boxtimes v),$$

square to the identity and give $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{\geqslant} 0)$ the structure of a symmetric monoidal category. □

In the terminology of [22, Definition 4.2.3], $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{\geqslant 0})$ is a *ring category*. Note that it does not have duals since there are no morphisms from $V \odot W$ to $\mathbf{1} \in \mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_0)$ as soon as either V or W are of degree at least 1. Moreover, Lemma 4.1 shows that the category $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{\geqslant 0})$ is an $\mathbb{N}$-graded monoidal category with respect to the grading given by

$$\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_{\geqslant 0})^k := \mathcal{Z}(\mathrm{Rep}_{\Bbbk} G_k).$$

Remark 4.2. In practice, the tensor product $\odot$ and the resulting product on the Grothendieck ring are computed as follows: Set $W = \mathrm{Ind}_{Z(\sigma)}^{G_n}(V)$ and $W' = \mathrm{Ind}_{Z(\tau)}^{G_m}(V')$ for $\sigma \in G_n$, $\tau \in G_m$ and V a $Z(\sigma)$-module, V' a $Z(\tau)$-module. Then, as an G_{n+m}-module,

$$W \odot W' \cong \mathrm{Ind}_{Z(\sigma) \times Z(\tau)}^{G_{n+m}}(V \boxtimes V').$$

The G_{n+m}-coaction for the element $g \otimes (v \otimes v') \in W \odot W'$ is given by

$$\delta(g \otimes (v \otimes v')) = g(\iota_{n,m}(\sigma, \tau))g^{-1} \otimes (v \otimes v').$$

Thus, one needs to decompose $\mathrm{Ind}_{Z(\sigma) \times Z(\tau)}^{Z}(V \boxtimes V')$ into indecomposable Z-modules, for $Z = Z(\iota_{n,m}(\sigma, \tau)) \subseteq G_{n+m}$, in order to compute $W \odot W'$.

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4.2 | An oplax tensor functor from $\mathcal{Z}(\mathrm{Rep}\, S_{\geqslant 0})$ to $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$

For the rest of this section, we restrict to the case of the symmetric groups $G_n = S_n$, $\Bbbk = \mathbb{C}$, and recall the functors

$$\underline{\mathrm{Ind}}\colon \mathcal{Z}(\mathrm{Rep}\, S_n) \boxtimes \underline{\mathrm{Rep}}\, S_{t-n} \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$$

from Proposition 3.7. We restrict to a functor

$$\underline{\mathrm{Ind}}\colon \mathcal{Z}(\mathrm{Rep}\, S_n) \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t), \qquad V \longmapsto \underline{\mathrm{Ind}}(W \boxtimes \mathbf{1}),$$

for the tensor unit $\mathbf{1}$ of $\underline{\mathrm{Rep}}\, S_{t-n}$ and $V \in \mathcal{Z}(\mathrm{Rep}\, S_n)$. These functors, varying $n \geqslant 0$, produce a functor

$$\underline{\mathrm{Ind}}\colon \mathcal{Z}(\mathrm{Rep}\, S_{\geqslant 0}) \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t). \tag{4.3}$$

We further recall the filtration $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k}$ induced by the filtration on $\underline{\mathrm{Rep}}\, S_t$ determined by the number of tensor powers of the generating object X, see Section 3.1.

Proposition 4.3. *There is a split injective natural transformation*

$$\tau_{V,W}\colon \underline{\mathrm{Ind}}(V \odot W) \to \underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W)$$

making $\underline{\mathrm{Ind}}\colon \mathcal{Z}(\mathrm{Rep}\, S_{\geqslant 0}) \to \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ *an oplax monoidal functor of braided* $\Bbbk$*-linear monoidal categories which is compatible with the respective filtrations.*

In particular, the oplax monoidal structure τ *is compatible with the braiding in the sense that the diagram*

$$\begin{tikzcd}
\underline{\mathrm{Ind}}(V \odot W) \arrow[r, "\underline{\mathrm{Ind}}(\Psi_{V,W})"] \arrow[d, hook, "\tau_{V,W}"] & \underline{\mathrm{Ind}}(W \odot V) \arrow[d, hook, "\tau_{W,V}"] \\
\underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W) \arrow[r, "\Psi_{\underline{\mathrm{Ind}}(V),\underline{\mathrm{Ind}}(W)}"] & \underline{\mathrm{Ind}}(W) \otimes \underline{\mathrm{Ind}}(V)
\end{tikzcd}$$

commutes. Here $\Psi_{V,W}$ *is the symmetric braiding on* $\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0}$.

Proof. Let $V \in \mathcal{Z}(\mathrm{Rep}\, S_n)$ and $W \in \mathcal{Z}(\mathrm{Rep}\, S_m)$ be objects. We know by (2.6) in Section 2.6 that $\underline{\mathrm{Ind}}(V) \in \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant n}$, $\underline{\mathrm{Ind}}(W) \in \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant m}$. Further, $V \odot W$ is an object in $\mathcal{Z}(\mathrm{Rep}\, S_{n+m})$ whence $\underline{\mathrm{Ind}}(V \odot W) \in \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant m+n}$ and hence $\underline{\mathrm{Ind}}$ is compatible with the filtrations (using the induced filtration from the grading on $\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0}$).

In order to construct τ and prove split injectivity, we use an ultrafilter description. Without loss of generality, consider i such that $t_i \geqslant n + m$ and write $\mathbb{F}_i := \overline{\mathbb{F}}_{p_i}$. We note that the trivial module $\mathbb{F}_i$ is a submodule and quotient of $\mathrm{Ind}_{S_{t_i-n-m} \times S_m}^{S_{t_i-n}}(\mathbb{F}_i \boxtimes \mathbb{F}_i)$ and $\mathrm{Ind}_{S_{t_i-n-m} \times S_n}^{S_{t_i-m}}(\mathbb{F}_i \boxtimes \mathbb{F}_i)$, respectively, in a canonical way (using the unit of the adjunction (Res, Ind), respectively, counit of the adjunction (Ind, Res) evaluated on the one-dimensional module). As induction is an exact functor, having a left and right adjoint, this implies that we have a surjective map in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$

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$$\begin{aligned}
&\mathrm{Ind}_{S_n\times S_m\times S_{t_i-n-m}}^{S_{t_i}}(V_i\boxtimes\mathbb{F}_i\boxtimes\mathbb{F}_i)\otimes\mathrm{Ind}_{S_n\times S_m\times S_{t_i-n-m}}^{S_{t_i}}(\mathbb{F}_i\boxtimes W_i\boxtimes\mathbb{F}_i)\\
&\cong\mathrm{Ind}_{S_n\times S_{t_i-n}}^{S_{t_i}}(V_i\boxtimes\mathrm{Ind}_{S_m\times S_{t_i-n-m}}^{S_{t_i-n}}(\mathbb{F}_i\boxtimes\mathbb{F}_i))\otimes\mathrm{Ind}_{S_m\times S_{t_i-m}}^{S_{t_i}}(W_i\boxtimes\mathrm{Ind}_{S_n\times S_{t_i-n-m}}^{S_{t_i-m}}(\mathbb{F}_i\boxtimes\mathbb{F}_i))\\
&\twoheadrightarrow\mathrm{Ind}_{S_n\times S_{t_i-n}}^{S_{t_i}}(V_i\boxtimes\mathbb{F}_i)\otimes\mathrm{Ind}_{S_m\times S_{t_i-m}}^{S_{t_i}}(W_i\boxtimes\mathbb{F}_i),
\end{aligned}$$

which displays the former as a quotient of the latter. These morphisms are natural in V_i, W_i by construction. We pre-compose with the natural transformation δ_{V_i,W_i} from Proposition B.1 to yield the natural transformation

$$\tau_{V_i,W_i}:\ \mathrm{Ind}_{S_n\times S_m\times S_{t_i-n-m}}^{S_{t_i}}(V_i\boxtimes W_i\boxtimes\mathbb{F}_i)\longrightarrow\mathrm{Ind}_{S_n\times S_{t_i-n}}^{S_{t_i}}(V_i\boxtimes\mathbb{F}_i)\otimes\mathrm{Ind}_{S_m\times S_{t_i-m}}^{S_{t_i}}(W_i\boxtimes\mathbb{F}_i).$$

We claim that τ_{V_i,W_i} is split injective. To see this, we note the equality

$$(S_n\times S_{t_i-n})\cap(S_m\times S_{t_i-m})=S_n\times S_m\times S_{t_i-n-m}$$

of subgroups of S_{t_i}, where S_n embeds as $S_{\{1,\dots,n\}}$ and S_m embeds as $S_{\{n+1,\dots,n+m\}}$. To show this, take an element $g=(g_n,h_n)=(g_m,h_m)$ in the intersection, with $g_n\in S_n$, $g_m\in S_m$, and $h_n\in S_{t_i-n}$, $h_m\in S_{t_i-m}$. We note that $g_n^{-1}g=(g_m,g_n^{-1}h_m)$ preserves the set $\{1,\dots,n\}$, which is fixed by g_m by assumption, and hence also preserved by $g_n^{-1}h_m\in S_{t_i-m}$. Thus, $g_n^{-1}h_m\in S_n\times S_m\times S_{t_i-n-m}$, and hence $g\in S_n\times S_m\times S_{t_i-n-m}$.

Next, we recall the elementary observation that for subgroups K,H of G, $g,h\in G$, the intersection of cosets $gH\cap hK$ is either empty or a coset $z(K\cap H)$ of $K\cap H$ in G, for any $z\in gH\cap hK$. We use this observation as follows. Fix coset decompositions

$$G=\bigsqcup_{\alpha}g_\alpha H,\qquad G=\bigsqcup_{\beta}h_\beta K,$$

to yield a coset decomposition

$$G=\bigsqcup_{(\alpha,\beta)}g_\alpha H\cap h_\beta K=\bigsqcup_{(\alpha,\beta)}\sigma_{\alpha,\beta}H\cap K,$$

over the pairs (α,β) for which the intersections of cosets are non-empty, and where

$$\sigma_{\alpha,\beta}=g_\alpha h_{\alpha,\beta}=h_\beta k_{\alpha,\beta}\in g_\alpha H\cap h_\beta K,$$

for some elements $h_{\alpha,\beta}\in H$, $k_{\alpha,\beta}\in K$. We set $H=S_n\times S_{t_i-n}$ and $K=S_m\times S_{t_i-m}$ viewed as subgroups of $G=S_{t_i}$.

Observe that a basis for $\mathrm{Ind}_{S_n\times S_m\times S_{t_i-n-m}}^{S_{t_i}}(V_i\boxtimes W_i\boxtimes\mathbb{F}_i)$ is given by the set

$$\{\sigma_{\alpha,\beta}\otimes(v_\gamma\boxtimes w_\delta\boxtimes 1)\}_{(\alpha,\beta,\gamma,\delta)},$$

where $\{v_\gamma\}_\gamma$ and $\{w_\delta\}_\delta$ are bases of V_i and W_i, respectively, and (α,β) with non-empty coset intersection as above. We consider the images of these basis elements under τ_{V_i,W_i}. Using the formula for δ_{V_i,W_i} in Proposition 3.7, we compute that

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$$\begin{aligned}\tau_{V_i,W_i}\big(\sigma_{\alpha,\beta}\otimes(v_\gamma\boxtimes w_\delta\boxtimes 1)\big) &= \big(\sigma_{\alpha,\beta}\otimes(v_\gamma\boxtimes 1)\big)\otimes\big(\sigma_{\alpha,\beta}\otimes(w_\delta\boxtimes 1)\big)\\ &= \big(g_\alpha h_{\alpha,\beta}\otimes(v_\gamma\boxtimes 1)\big)\otimes\big(h_\beta k_{\alpha,\beta}\otimes(w_\delta\boxtimes 1)\big)\\ &= \big(g_\alpha\otimes(v_\gamma\boxtimes 1)\big)\otimes\big(h_\beta\otimes(w_\delta\boxtimes 1)\big).\end{aligned}$$

Thus, the map τ_{V_i,W_i} sends the given basis to a subset of a basis for the target space $\mathrm{Ind}_{S_n\times S_{t_i-n}}^{S_{t_i}}(V_i\boxtimes\mathbb{F}_i)\otimes\mathrm{Ind}_{S_m\times S_{t_i-m}}^{S_{t_i}}(W_i\boxtimes\mathbb{F}_i)$. Hence, τ_{V_i,W_i} is injective.

Next, we claim that a complement of the image of τ_{V_i,W_i} is closed under the S_{t_i}-action. We have seen that the image of τ_{V_i,W_i} is spanned over $\mathbb{F}_i$ by those vectors

$$u_{\alpha,\beta,\gamma,\delta} := \big(g_\alpha\otimes(v_\gamma\boxtimes 1)\big)\otimes\big(h_\beta\otimes(w_\delta\boxtimes 1)\big)$$

for which $g_\alpha H\cap h_\beta K\neq\varnothing$, or equivalently, $g_\alpha^{-1}h_\beta\in H\cap K$. Those vectors $u_{\alpha,\beta,\gamma,\delta}$ for which $g_\alpha^{-1}h_\beta\notin H\cap K$ clearly span a G-invariant complement.

If $\{v_\gamma\}_\gamma$ and $\{w_\delta\}_\delta$ are chosen to be G-homogeneous bases, this complement is also a G-graded subspace, hence, a Yetter–Drinfeld submodule, showing that τ_{V_i,W_i} splits as a morphism in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$.

Next, we observe that τ_{V_i,W_i} is oplax monoidal. This is a consequence of δ_{V_i,W_i} being oplax monoidal mapping to a larger object in $\mathcal{Z}(\mathrm{Rep}_{p_i} S_{t_i})$ by Proposition 3.7 and naturality of $\underline{\mathrm{Ind}}(X\boxtimes Y)$ in $Y\in\underline{\mathrm{Rep}}\, S_{t-n}$.

Using (B.6) and naturality of the braiding of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, we see that this oplax monoidal structure is compatible with the braidings, using the symmetric braiding on $\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0}$.

By Łoś' theorem, we obtain an induced injective natural transformation $\tau_{V,W}$ as claimed. This structure makes $\underline{\mathrm{Ind}}$ an oplax monoidal functor compatible with braiding as claimed. □

4.3 | The Grothendieck ring of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$

Recall that

$$\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_t)\cong\bigoplus_{k\geqslant 0}K_0(\mathrm{Rep}\, S_n)$$

from Section 2.5, where the right-hand side is the ring of symmetric functions. To study the Grothendieck ring of the center $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, we first note that the direct sum of Grothendieck rings

$$K_0(\mathcal{Z}(\mathrm{Rep}\, S_{\geqslant 0}))=\bigoplus_{n\geqslant 0}K_0(\mathcal{Z}(\mathrm{Rep}\, S_n))$$

obtains the structure of a graded commutative algebra with the product

$$[V]\cdot[W] := [V\odot W]. \tag{4.4}$$

The unit is given by $[\mathbf{1}]$, for $\mathbf{1}\in\mathcal{Z}(\mathrm{Rep}\, S_0)$.

Theorem 4.4. *The functor* $\underline{\mathrm{Ind}}$ *from (4.3) induces an isomorphism of graded rings*

$$\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}}):\ K_0(\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0})\xrightarrow{\sim}\mathrm{gr}\, K_0^{\oplus}\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)),$$

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where the associated graded of $K_0^{\oplus}\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$ is taken with respect to the filtration induced by the filtration $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k}$ of categories.

Proof. The functor $\underline{\mathrm{Ind}}$ induces a morphism of abelian groups

$$\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}}) : K_0(\mathcal{Z}\,\mathrm{Rep}\, S_{\geqslant 0}) \longrightarrow \mathrm{gr}\, K_0(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)).$$

We first show $\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}})$ is an algebra map. Recall the split injective natural transformations

$$\tau_{V,W} : \underline{\mathrm{Ind}}(V \odot W) \to \underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W)$$

for $V \in \mathcal{Z}(\mathrm{Rep}\, S_n)$ and $W \in \mathcal{Z}(\mathrm{Rep}\, S_m)$ from Proposition 4.3 which gives an oplax monoidal structure. We use $\tau_{V,W}$ to show that

$$[\underline{\mathrm{Ind}}(V \odot W)] = [\underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W)]$$

in $\mathrm{gr}\, K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$. For this, we will first look at how these objects decompose inside of $\underline{\mathrm{Rep}}\, S_t$ which will help us identify the highest degree parts. To this end, we decompose

$$V = \bigoplus_i S^{\lambda_i}, \qquad W = \bigoplus_j S^{\mu_j},$$

as direct sums of simple modules in $\mathrm{Rep}\, S_n$ and $\mathrm{Rep}\, S_m$, respectively. By the Littlewood–Richardson rule, we have the decomposition

$$V \odot W = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}\Big(\bigoplus_i S^{\lambda_i} \boxtimes \bigoplus_j S^{\mu_j}\Big) = \bigoplus_{i,j,\nu} (S^{\nu})^{\oplus c^{\nu}_{\lambda_i,\mu_j}}$$

in $\mathrm{Rep}\, S_{n+m}$. By Lemma 2.10 we have, decomposing as elements in $K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_t)$,

$$[\underline{\mathrm{Ind}}(V)] = \sum_i [X_{\lambda_i}] + \{\text{l.o.t.}\}, \qquad [\underline{\mathrm{Ind}}(W)] = \sum_j [X_{\mu_j}] + \{\text{l.o.t.}\},$$

$$[\underline{\mathrm{Ind}}(V \odot W)] = \sum_{i,j,\nu} c^{\nu}_{\lambda_i,\mu_j} [X_\nu] + \{\text{l.o.t.}\},$$

where the lower order terms {l.o.t.} are in strictly lower filtration pieces $K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_t)^{\leqslant k}$. Further, we have by Lemma 2.9 that in $K_0^{\oplus}(\underline{\mathrm{Rep}}\, S_t)$,

$$[X_{\lambda_i}][X_{\mu_j}] = [X_{\lambda_i} \otimes X_{\mu_j}] = \sum_\nu c^{\nu}_{\lambda_i,\mu_j} [X_\nu] + \{\text{l.o.t.}\}.$$

Thus, all direct summands, as objects of $\underline{\mathrm{Rep}}\, S_t$, that have maximal degree $n+m$ in $\underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W)$, are in fact already contained in $\mathrm{Ind}(V \odot W)$ and hence are contained in the image of $\tau_{V,W}$ using split injectivity.

Now, we may decompose

$$\underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W) = \underline{\mathrm{Ind}}(V \odot W) \oplus Y,$$

in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, where Y is a complement of the image of $\tau_{V,W}$ (see Proposition 4.3). We want to identify all *leading* terms, that is, those that are contained in the leading filtration piece

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$K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))^{\leqslant n+m}$ but not contained in $K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))^{\leqslant n+m-1}$. By the Krull–Schmidt property of the category $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$, the decomposition in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ refines into a decomposition into indecomposables in $\underline{\mathrm{Rep}}\, S_t$. In such a refinement, any leading direct summand in $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ needs to contain at least a direct summand from $(\underline{\mathrm{Rep}}\, S_t)^{n+m}$ that is not contained in $(\underline{\mathrm{Rep}}\, S_t)^{n+m-1}$. However, all such summands occurring in $\underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W)$ are already contained in $\underline{\mathrm{Ind}}(V \odot W)$ by the above observations. This implies that

$$[\underline{\mathrm{Ind}}(V) \otimes \underline{\mathrm{Ind}}(W)] = [\underline{\mathrm{Ind}}(V \odot W)] + \{\text{l.o.t.}\}$$

in $K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$. Thus, the functor $\underline{\mathrm{Ind}}$ induces an algebra map to the associated graded Grothendieck ring as claimed.

It remains to show that $\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}})$ is an isomorphism. Indeed, using Proposition 3.18 we see that $\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}})$ is surjective, since by combining Theorem 3.9 and Proposition 3.11, we know that a $\mathbb{Z}$-basis for $K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$ is given by the objects $\underline{W}_{\mu,V,U}$ for V, U indecomposable and μ singleton free.

To prove injectivity of $\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}})$ we recall the notation from the proof of Proposition 3.18. Given a simple $Z(\mu)$-module V, with μ not necessarily singleton free, decompose $Z(\mu) \cong Z_0 \times S_r$ and $V \cong V_0 \boxtimes U_0$. Since we are working over $\mathbb{C}$ and V is simple, U_0 is simple as a $\mathbb{C}S_r$-module. Thus, by Lemma 2.10, $U_{\mu,V} = \underline{\mathrm{Ind}}_{S_r \times S_{t-n}}^{S_{t-n_0}}(U_0 \boxtimes \mathbf{1})$ contains a unique simple summand X of maximal filtration degree r. This implies that $\underline{W}_{\mu_0,V_0,X}$ is a subobject of $\underline{\mathrm{Ind}}(\mathrm{Ind}_{Z(\mu)}^{S_n} V) = \underline{W}_{\mu,V}^{\mathrm{old}}$, which has maximal filtration degree $n = n_0 + r$ by Corollary 2.11. By the same reasoning, decomposing $[\underline{W}_{\mu,V}^{\mathrm{old}}]$ in $K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$, all other summands are of strictly smaller filtration degree. Thus, $[\underline{\mathrm{Ind}}(\mathrm{Ind}_{Z(\mu)}^{S_n} V)] = [\underline{W}_{\mu_0,V_0,X}]$ in $\mathrm{gr}\, K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$. The datum (μ_0, V_0, X) uniquely determines (μ, V) since X determines r, U_0 and $V \cong V_0 \boxtimes U_0$. Thus, since the $[\underline{W}_{\mu_0,V_0,X}]$ are linearly independent in $\mathrm{gr}\, K_0^{\oplus}\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$, the map $\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}})$ is injective. □

The description of the additive Grothendieck ring of $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t)$ can be extended to the Grothendieck ring of the abelian envelope as follows.

Theorem 4.5. *For any $d \in \mathbb{Z}_{\geqslant 0}$, the functor $\underline{\mathrm{Ind}}$ from (4.3) induces an isomorphism of graded rings*

$$\mathrm{gr}\, K_0^{\oplus}(\underline{\mathrm{Ind}}) : K_0(\mathcal{Z}\, \mathrm{Rep}\, S_{\geqslant 0}) \xrightarrow{\sim} \mathrm{gr}\, K_0(\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)),$$

where the associated graded of $K_0(\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d))$ is taken with respect to the filtration induced by the filtration $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)^{\leqslant k}$ of categories.

Proof. First recall that the functor $\underline{\mathrm{Ind}}$, together with the split injective oplax monoidal structure τ from Proposition 4.3, extends to the abelian envelope as via composition

$$\underline{\mathrm{Ind}} : \mathcal{Z}\, \mathrm{Rep}\, S_{\geqslant 0} \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\, S_t) \hookrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d),$$

cf. the diagram (3.11).

The proof of Theorem 4.4 can be adapted to working with the abelian envelope using Corollary 2.13 instead of Lemma 2.9. Instead of the Krull–Schmidt property, we use the Jordan–Hölder property of the category $\mathcal{Z}(\underline{\mathrm{Rep}}^{\mathrm{ab}}\, S_d)$.

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For the proof in the case of the abelian envelope we also used that by Proposition 3.22(3), all projective objects are contained in the subcategory $\mathcal{Z}(\underline{\mathrm{Rep}}\, S_d)$. Thus, all simple objects occur as subquotients of the objects $\underline{\mathrm{Ind}}(V)$ by Proposition 3.18. This shows that $\mathrm{gr}\, K_0(\underline{\mathrm{Ind}})$ is an isomorphism in the case of the abelian envelope. □

4.4 | Some sample computation in the associated graded Grothendieck ring

As described in Remark 4.2, the induction product $W \odot W'$, for $W = \mathrm{Ind}_{Z(\sigma)}^{S_n}(V) \in \mathcal{Z}(\mathrm{Rep}\, S_n)$ and $W' = \mathrm{Ind}_{Z(\tau)}^{S_m}(V') \in \mathcal{Z}(\mathrm{Rep}\, S_m)$ can be computed by decomposing

$$\mathrm{Ind}_{Z(\sigma)\times Z(\tau)}^{Z(\sigma\times\tau)}(V \boxtimes V')$$

as a module over the centralizer $Z(\sigma \times \tau) \subset S_{n+m}$, where $\sigma \times \tau$ permutes $\{1, \dots, n\}$ using σ and $\{n+1, \dots, n+m\}$ using τ. With Theorem 4.4, this computes products in the associated graded Grothendieck ring $\mathrm{gr}\, K_0^{\oplus}(\mathcal{Z}(\underline{\mathrm{Rep}}\, S_t))$ or its abelian envelope. In this section, we consider a few examples. We start with the smallest non-trivial example.

Example 4.6. Let $\sigma = \tau = (12) \in S_2$. Then $\sigma \times \tau = (12)(34) \in S_4$ and $Z((12)(34))$ is the wreath product

$$\mathbb{Z}_2 \wr S_2 \cong \langle (12), (34), (13)(24) \rangle \subset S_4.$$

Using [7, Proposition 3.7], there are five simple modules of $\mathbb{Z}_2 \wr S_2$, one two-dimensional module V_2, and four one-dimensional modules $V^{\epsilon_1,\epsilon_2}$, where $\epsilon_i \in \{\pm 1\}$, and (12),(34) act via multiplication by ϵ_1, and (13)(24) acts via multiplication by ϵ_2.

We can choose V, V' to be either the trivial module $\Bbbk^{\mathrm{triv}}$ or the sign module $\Bbbk^{\mathrm{sign}}$ of $S_2 = Z(\sigma) = Z(\tau)$. The resulting module $V \odot V' = \mathrm{Ind}_{S_2\times S_2}^{\mathbb{Z}_2 \wr S_2}(V \boxtimes V')$ is two-dimensional, with a basis given by

$$v_1 = 1 \otimes (v \boxtimes v'), \quad v_2 = (13)(24) \otimes (v \boxtimes v'),$$

where v generates V and v' generates V'.

If $V = V' = \Bbbk^{\mathrm{triv}}$, then $V \odot V' \cong V^{+1,+1} \oplus V^{+1,-1}$ is a direct sum of the trivial and sign module over $\mathbb{Z}_2 \wr S_2$, splitting as $\Bbbk(v_1 + v_2) \oplus \Bbbk(v_1 - v_2)$. Similarly, if $V = V' = \Bbbk^{\mathrm{sign}}$, then $V \odot V' \cong V^{-1,+1} \oplus V^{-1,-1}$. If $V \neq V'$, then $V \odot V' \cong V' \odot V$ is the simple two-dimensional module V_2 over $\mathbb{Z}_2 \wr S_2$.

We include the product computations in a slightly more general situation involving cyclic permutations.

Example 4.7. Assume that σ and τ are k-cycles in S_k. Then their centralizers are cyclic groups isomorphic to $\mathbb{Z}_k$. Thus,

$$Z(\sigma \times \tau) = \mathbb{Z}_k \wr S_2 = \langle \sigma, \tau, \omega \rangle \subset S_{2k},$$

where conjugation by ω swaps σ and τ. Let $\Bbbk^{\zeta}$ denote the one-dimensional simple $\mathbb{Z}_k$-module where the generator acts through a root of unity $\zeta \in \Bbbk$. Again using, for example, [7, Proposition

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3.7], the simple modules of $\mathbb{Z}_k \wr S_2$ fall into two classes. First, two-dimensional simples

$$V_2^{\zeta_1,\zeta_2} = \mathrm{Ind}_{\mathbb{Z}_k\times\mathbb{Z}_k}^{\mathbb{Z}_k\wr S_2}\left(\Bbbk^{\zeta_1}\boxtimes\Bbbk^{\zeta_2}\right),$$

where ζ_1,ζ_2 are distinct kth roots of unity; second, one-dimensional simples $V^{\zeta,\epsilon}$, where both copies of $\mathbb{Z}_k$ act via multiplication by ζ and S_2 acts via $\Bbbk^{\mathrm{triv}}$ if $\epsilon = 1$ and via $\Bbbk^{\mathrm{sign}}$ if $\epsilon = -1$. By [7, Lemma 3.1], $V_2^{\zeta_1,\zeta_2} \cong V_2^{\zeta_2,\zeta_1}$.

We see that for all kth roots of unity ζ and $\zeta_1 \neq \zeta_2$,

$$\Bbbk^{\zeta}\odot\Bbbk^{\zeta} = \mathrm{Ind}_{\mathbb{Z}_k\times\mathbb{Z}_k}^{\mathbb{Z}_k\wr S_n}(\Bbbk^{\zeta}\boxtimes\Bbbk^{\zeta}) \cong V^{\zeta,+1}\oplus V^{\zeta,-1}, \qquad \Bbbk^{\zeta_1}\odot\Bbbk^{\zeta_2} \cong V_2^{\zeta_1,\zeta_2} \cong V_2^{\zeta_2,\zeta_1} \cong \Bbbk^{\zeta_2}\odot\Bbbk^{\zeta_1}.$$

If the cycle types of σ and τ have no common cycle length (including one-cycles), then $Z(\sigma\times\tau) = Z(\sigma)\times Z(\tau)$ and the induction product $\odot$ is simply given by the exterior tensor product $\boxtimes$.

In general, the representation theory of the products of wreath product groups that appear as centralizers in the symmetric groups is well understood and induction products may be computed using similar but more involved analogs of Littlewood–Richardson coefficients for wreath products.

APPENDIX A: THE ABELIAN ENVELOPE OF THE MONOIDAL CENTER

In this section, we collect general results about the monoidal center required in the core of the paper. Given a monoidal category $\mathcal{C}$, the center $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category [29, 33]. Objects in $\mathcal{Z}(\mathcal{C})$ are pairs (Y,c) where Y is an object of $\mathcal{C}$ and $c = \{c_V : Y\otimes V \to V\otimes Y\}_{V\in\mathcal{C}}$ is a natural isomorphism (in V), called a *half-braiding*, which satisfies the tensor compatibility

$$c_{V\otimes W} = (\mathrm{Id}_V\otimes c_W)(c_V\otimes\mathrm{Id}_W), \qquad \forall V,W\in\mathcal{C},$$

where the associativity isomorphism of $\mathcal{C}$ is omitted. A morphism $f : (Y,c)\to(Y',c')$ is a morphism $f : Y\to Y'$ in $\mathcal{C}$ such that

$$c'_V(f\otimes\mathrm{Id}_V) = (\mathrm{Id}_V\otimes f)c_V, \qquad \forall V\in\mathcal{C}.$$

For basic properties of the monoidal center, see, for example, [22, Section 7.13]. We employ the following extension property of the monoidal center.

Lemma A.1. *Let $\mathcal{C}$ be a locally finite abelian rigid monoidal category with enough projectives and $\mathcal{A}$ be an additive tensor subcategory of $\mathcal{C}$ containing all projective objects.*

Assume given a pair $(Y, c|_{\mathcal{A}})$ where Y is an object in $\mathcal{C}$ and

$$c|_{\mathcal{A}} : Y\otimes\mathrm{Id}_{\mathcal{A}} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{A}}\otimes Y$$

is a natural isomorphism which is tensor compatible in the sense that

$$\left(c|_{\mathcal{A}}\right)_{A\otimes A'} = \left(\mathrm{Id}_A\otimes\left(c|_{\mathcal{A}}\right)_{A'}\right)\left(\left(c|_{\mathcal{A}}\right)_A\otimes\mathrm{Id}_{A'}\right), \qquad \forall A,A'\in\mathcal{A}.$$

Then $c|_{\mathcal{A}}$ admits a unique extension to a half-braiding defining an object in $\mathcal{Z}(\mathcal{C})$.

In particular, the inclusion functor $\mathcal{A}\hookrightarrow\mathcal{C}$ extends to an inclusion functor $\mathcal{Z}(\mathcal{A})\hookrightarrow\mathcal{Z}(\mathcal{C})$.

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Proof. Assume given an object Y and

$$c|_{\mathcal{A}} : Y \otimes \mathrm{Id}_{\mathcal{A}} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{A}} \otimes Y$$

as in the statement of the lemma. We note that the functors of left and right tensoring with Y are exact (see, for example, [2, Proposition 2.1.8]). We can restrict c to a natural isomorphism of these functors on the full monoidal subcategory $\mathcal{P}$ on projective objects of $\mathcal{C}$, which is contained in $\mathcal{A}$ by assumption.

Now let M be an object of $\mathcal{C}$. Then we can find projective objects P_1, P_0 and an exact sequence

$$P_1 \to P_0 \to M \to 0.$$

We note that P_1, P_0 are both finite sums of indecomposable projectives by assumption of local finiteness of $\mathcal{C}$ and therefore the argument of [34, Theorem 5.4] can be adapted to extend the $c|_{\mathcal{A}}$ to the full subcategory of projective objects to all of $\mathcal{C}$ in a unique way to a natural isomorphism

$$c : Y \otimes \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}} \otimes Y.$$

It remains to check that c satisfies tensor compatibility and thus gives a half-braiding on all of $\mathcal{C}$. But this follows from exactness of the tensor product. In fact, for objects M, N of $\mathcal{C}$, given exact sequences

$$P_1 \to P_0 \to M \to 0, \qquad Q_1 \to Q_0 \to N \to 0,$$

with P_i, Q_i projective, exactness of $\otimes$ gives an exact sequence

$$P_1 \otimes Q_0 \oplus P_0 \otimes Q_1 \to P_0 \otimes Q_0 \to M \otimes N \to 0.$$

Now both $c = c_{M\otimes N}$ and $c = (c_M \otimes \mathrm{Id})(\mathrm{Id} \otimes c_N)$ make the following diagram commute:

$$\begin{tikzcd}
Y \otimes P_1 \otimes Y \otimes Q_0 \oplus Y \otimes P_0 \otimes Q_1 \arrow[r] \arrow[d, "\begin{pmatrix} c_{P_1\otimes Q_0} & 0 \\ 0 & c_{P_0\otimes Q_1} \end{pmatrix} = \begin{pmatrix} c_{P_1}\otimes \mathrm{Id} & 0 \\ 0 & c_{P_0}\otimes \mathrm{Id} \end{pmatrix}\begin{pmatrix} \mathrm{Id}\otimes c_{Q_0} & 0 \\ 0 & \mathrm{Id}\otimes c_{Q_1} \end{pmatrix}"] & Y \otimes P_0 \otimes Q_0 \arrow[r] \arrow[d, "c_{P_0\otimes Q_0} = (c_{P_0}\otimes \mathrm{Id})(\mathrm{Id}\otimes c_{Q_0})"] & Y \otimes M \otimes N \arrow[d, "c"] \\
P_1 \otimes Y \otimes Q_0 \otimes Y \oplus P_0 \otimes Q_1 \otimes Y \arrow[r] & P_0 \otimes Q_0 \otimes Y \arrow[r] & M \otimes N \otimes Y
\end{tikzcd}$$

Thus, these morphisms have to be equal by uniqueness of the morphism c. □

Corollary A.2. *Let X be a faithful representation in* $\mathrm{Rep}_p\, G$. *Assume given two objects* (Y, c), (Y, c') *in* $\mathcal{Z}(\mathrm{Rep}_p\, G)$. *Then* $c_X = c'_X$ *implies* $c = c'$.

Proof. Note that $\mathcal{A} = \langle X \rangle$ contains all projective objects [6, Theorem 1]. The half-braidings are determined by $c_{X\otimes n}$ which in turn is determined by c_X. □

In the following, let $\mathcal{A}$ be a Karoubian tensor category. We recall that an (abelian) multitensor category $\mathcal{A}^{\mathrm{ab}}$ with a fully faithful tensor functor $\iota : \mathcal{A} \to \mathcal{A}^{\mathrm{ab}}$ is called an *abelian envelope* ([4, Section 2.10]) of $\mathcal{A}$ if for any multitensor category $\mathcal{D}$, the category of tensor functors $\mathcal{A}^{\mathrm{ab}} \to \mathcal{D}$ is equivalent to the category of faithful monoidal functors $\mathcal{A} \to \mathcal{D}$ by restriction along ι. If it exists, the abelian envelope is unique up to equivalence. If the abelian envelope exists and has enough

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projectives, then a construction is given in [4]. A necessary and sufficient condition on $\mathcal{A}$ ('separated and complete') for $\mathcal{A}^{\mathrm{ab}}$ to have enough projectives is proven in *loc.cit.* In the following, we will identify $\mathcal{A}$ with a full tensor subcategory of $\mathcal{A}^{\mathrm{ab}}$ using ι.

Lemma A.3. *Assume that $\mathcal{A}^{\mathrm{ab}}$ is the abelian envelope of $\mathcal{A}$ and has enough projectives. Assume also that $\mathcal{A}$ admits a braiding c such that the functor $\mathcal{A}^{\mathrm{ab}} \to \mathcal{Z}(\mathcal{A}^{\mathrm{ab}}), X \mapsto (X, c_{X,-})$, preserves projectives. Then $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ has enough projectives and all its projectives lie in $\mathcal{Z}(\mathcal{A})$.*

Proof. We recall that the embedding of $\mathcal{A}$ into $\mathcal{A}^{\mathrm{ab}}$ is full on projectives by [4, Theorem 2.41, Theorem 2.42], so all projectives of $\mathcal{A}^{\mathrm{ab}}$ lie in $\mathcal{A}$.

As $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ is a multitensor category, having enough projectives is equivalent to the existence of a projective object in $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ with an epimorphism to the tensor unit in $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$.

The tensor unit of $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ is given by the tensor unit $\mathbf{1}$ of $\mathcal{A}^{\mathrm{ab}}$ together with the natural isomorphisms between the functors $\mathrm{Id} \otimes \mathbf{1} \cong \mathrm{Id} \cong \mathbf{1} \otimes \mathrm{Id}$. Since $\mathcal{A}^{\mathrm{ab}}$ has enough projectives, all of which lie in $\mathcal{A}$, there is a projective object $X \in \mathcal{A}^{\mathrm{ab}}$ which lies in $\mathcal{A}$ with an epimorphism to $\mathbf{1} \in \mathcal{A}^{\mathrm{ab}}$. Now $\mathcal{A}$ admits a braiding which yields a half-braiding c for X. Then (X, c) defines an object in the center of $\mathcal{A}$, hence by Lemma A.1, this is also an object in the center of $\mathcal{A}^{\mathrm{ab}}$. By our assumptions, (X, c) is a projective object in $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ with an epimorphisms to the tensor unit in $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$, so $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ has enough projectives.

Now consider any projective object (P, d) in $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$. Then $(P, d) \otimes (X, c)$ is a projective object in $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ with (P, d) as a quotient, hence, as a direct summand. However, $P \otimes X$ is a projective object in $\mathcal{A}^{\mathrm{ab}}$; hence, it lies in $\mathcal{A}$. This means that (P, d) appears as a direct summand in an object of $\mathcal{Z}(\mathcal{A})$; hence, it lies in $\mathcal{Z}(\mathcal{A})$, and we have shown that all projective objects of $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ lie in $\mathcal{Z}(\mathcal{A})$. □

Corollary A.4. *In the situation of Lemma A.3, $\mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ is the abelian envelope of $\mathcal{Z}(\mathcal{A})$.*

Proof. By [4, Theorem 2.42], the abelian envelope of a Karoubian tensor category $\mathcal{C}$, if it exists and has enough projectives, is given by any fully faithful monoidal functor $E : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a multitensor category with enough projectives; in this case, the abelian envelope is the abelian tensor subcategory generated by the image of E.

Set $\mathcal{D} := \mathcal{Z}(\mathcal{A}^{\mathrm{ab}})$ and let $E : \mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{A}^{\mathrm{ab}}) = \mathcal{D}$ be the functor induced by the inclusion $\mathcal{A} \to \mathcal{A}^{\mathrm{ab}}$ according to Lemma A.1. By Lemma A.3, $\mathcal{D}$ is a multitensor category with enough projectives, the projectives lying in $\mathcal{Z}(\mathcal{A})$. In particular, every object is a quotient of an object in $\mathcal{Z}(\mathcal{A})$ and the subcategory generated by the image of E is all of $\mathcal{D}$. □

APPENDIX B: SEPARABLE FROBENIUS MONOIDAL FUNCTORS AND THE MONOIDAL CENTER

In this section, we recall the definition of a (separable) Frobenius monoidal functor and show that induction functors of finite group representation display such a structure which extends to their monoidal centers.

A *Frobenius monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between two monoidal categories $\mathcal{C}$, $\mathcal{D}$ is a bilax monoidal functor, that is, comes with a lax monoidal structure (μ, η), and an oplax monoidal structure (δ, ϵ), where

$$\mu_{V,W} : F(V) \otimes F(W) \longrightarrow F(V \otimes W), \qquad \delta_{V,W} : F(V \otimes W) \longrightarrow F(V) \otimes F(W), \tag{B.1}$$

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$$\eta : \mathbf{1} \longrightarrow F(\mathbf{1}), \qquad \epsilon : F(\mathbf{1}) \longrightarrow \mathbf{1}, \tag{B.2}$$

for any objects V, W of $\mathcal{C}$, satisfying the additional compatibility conditions

$$\begin{tikzcd}
 & F(V) \otimes F(W) \otimes F(U) \arrow[dr, "\mu_{V,W} \otimes \mathrm{Id}_{F(U)}"] & \\
F(V) \otimes F(W \otimes U) \arrow[ur, "\mathrm{Id}_{F(V)} \otimes \delta_{W,U}"] \arrow[dr, "\mu_{V,W\otimes U}"'] & & F(V \otimes W) \otimes F(U), \\
 & F(V \otimes W \otimes U) \arrow[ur, "\delta_{V\otimes W,U}"'] &
\end{tikzcd} \tag{B.3}$$

$$\begin{tikzcd}
 & F(V) \otimes F(W) \otimes F(U) \arrow[dr, "\mathrm{Id}_{F(V)} \otimes \mu_{W,U}"] & \\
F(V \otimes W) \otimes F(U) \arrow[ur, "\delta_{V,W} \otimes \mathrm{Id}_{F(U)}"] \arrow[dr, "\mu_{V\otimes W,U}"'] & & F(V) \otimes F(W \otimes U). \\
 & F(V \otimes W \otimes U) \arrow[ur, "\delta_{V,W\otimes U}"'] &
\end{tikzcd} \tag{B.4}$$

For details on this definition, see, for example, [1, 13, Section 3.5]. A Frobenius monoidal functor is *separable* if, in addition, for any objects V, W of $\mathcal{C}$,

$$\mu_{V,W} \circ \delta_{V,W} = \mathrm{Id}_{F(V\otimes W)}\,. \tag{B.5}$$

Examples of Frobenius monoidal functors are obtained from the induction functor of representation categories of finite groups. We prove here that these Frobenius monoidal functors extend to the centers of the representation categories. This functor, without its Frobenius monoidal structure, already appeared in [12, Theorem 3.3.2]. In the following, $\Bbbk$ is any field.

Proposition B.1. *Let $\Bbbk$ be a field and $H \subseteq G$ be finite groups. Then the induction functor* Ind_H^G *induces a separable Frobenius monoidal functor*

$$\mathcal{Z}(\mathrm{Ind}_H^G) : \mathcal{Z}(\mathrm{Rep}_\Bbbk H) \longrightarrow \mathcal{Z}(\mathrm{Rep}_\Bbbk G).$$

The lax and oplax monoidal structures μ, δ are compatible with the braiding in the sense that the diagrams

$$\begin{tikzcd}
\mathrm{Ind}_H^G(V \otimes W) \arrow[r, "\mathrm{Ind}_H^G(\Psi_{V,W})"] \arrow[d, "\delta_{V,W}"'] & \mathrm{Ind}_H^G(W \otimes V) \arrow[d, "\delta_{W,V}"'] \\
\mathrm{Ind}_H^G(V) \otimes \mathrm{Ind}_H^G(W) \arrow[r, "\Psi_{\mathrm{Ind}_H^G(V),\mathrm{Ind}_H^G(W)}"] & \mathrm{Ind}_H^G(W) \otimes \mathrm{Ind}_H^G(V)
\end{tikzcd} \tag{B.6}$$

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$$\begin{tikzcd}
\mathrm{Ind}_H^G(V \otimes W) \arrow[r, "\mathrm{Ind}_H^G(\Psi_{V,W})"] & \mathrm{Ind}_H^G(W \otimes V) \\
\mathrm{Ind}_H^G(V) \otimes \mathrm{Ind}_H^G(W) \arrow[u, "\mu_{V,W}"] \arrow[r, "\Psi_{\mathrm{Ind}_H^G(V),\mathrm{Ind}_H^G(W)}"] & \mathrm{Ind}_H^G(W) \otimes \mathrm{Ind}_H^G(V) \arrow[u, "\mu_{W,V}"']
\end{tikzcd} \tag{B.7}$$

commute for any objects V, W in $\mathcal{C}$.

Proof. Given (V, c) in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} H)$, $\mathrm{Ind}_H^G(V) = \Bbbk G \otimes_{\Bbbk H} V$ can be equipped with the morphism

$$c'_X((g \otimes v) \otimes x) = (g|v|g^{-1} \cdot x) \otimes v,$$

for X any $\Bbbk G$-module and regarding the degree $|v| \in H \subseteq G$ as an element in G. One checks that c'_X defines a half-braiding. In fact, under the braided equivalence of $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ and the category of Yetter–Drinfeld modules over G, this half-braiding corresponds to the Yetter–Drinfeld module with coaction given by

$$\delta'(g \otimes v) = g|v|g^{-1} \otimes (g \otimes v).$$

This construction is clearly functorial with respect to morphisms in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} H)$.

The functor $\mathrm{Ind}_H^G : \mathrm{Rep}_{\Bbbk} H \to \mathrm{Rep}_{\Bbbk} G$ is both a left and right adjoint to the monoidal functor Res_H^G. Thus, Ind_H^G is both lax and oplax monoidal. Explicitly, the lax and oplax structures are given by

$$\begin{aligned}
\mu_{V,W} &: \mathrm{Ind}_H^G(V) \otimes \mathrm{Ind}_H^G(W) \longrightarrow \mathrm{Ind}_H^G(V \otimes W), & \eta &: \Bbbk \longrightarrow \mathrm{Ind}_H^G(\Bbbk), \\
(g \otimes v) \otimes (k \otimes w) &\xmapsto{\mu_{V,W}} \begin{cases} h \otimes (v \otimes g^{-1}kw), & \text{if } g^{-1}k \in H \\ 0, & \text{otherwise} \end{cases}, & 1 &\xmapsto{\eta} \sum_j g_j \otimes 1, \\
\delta_{V,W} &: \mathrm{Ind}_H^G(V \otimes W) \longrightarrow \mathrm{Ind}_H^G(V) \otimes \mathrm{Ind}_H^G(W), & \epsilon &: \mathrm{Ind}_H^G(\Bbbk) \longrightarrow \Bbbk, \\
g \otimes (v \otimes w) &\xmapsto{\delta_{V,W}} (g \otimes v) \otimes (g \otimes w), & g \otimes 1 &\xmapsto{\epsilon} 1,
\end{aligned}$$

where $\{g_j\}_{j \in J}$ is a set of representatives for the left cosets of H in G, that is, $H = \coprod_j g_j G$. We have to check that the natural transformations which determine the lax and oplax monoidal structure are compatible with the half-braidings defined above. It is easiest to check this using the formulation of objects in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$ as Yetter–Drinfeld modules and amounts to a straightforward computation.

Next, we check that the lax and oplax monoidal structures displayed above indeed satisfy Equations (B.3)–(B.4). For instance, Equation (B.3) follows from the terminal expressions in the following lines being equal:

$$\begin{aligned}
(g \otimes v) \otimes (k \otimes (w \otimes u)) &\mapsto (g \otimes v) \otimes (k \otimes w) \otimes (k \otimes u) \\
&\mapsto \begin{cases} (g \otimes (v \otimes g^{-1}kw)) \otimes (k \otimes u), & \text{if } g^{-1}k \in H, \\ 0, & \text{otherwise} \end{cases} \\
(g \otimes v) \otimes (k \otimes (w \otimes u)) &\mapsto \begin{cases} g \otimes (v \otimes g^{-1}kw \otimes g^{-1}ku), & \text{if } g^{-1}k \in H, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

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$$\mapsto \begin{cases} (g \otimes (v \otimes g^{-1}kw)) \otimes (g \otimes g^{-1}ku), & \text{if } g^{-1}k \in H, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we check compatibility with the braiding. The composition $\Psi_{\mathrm{Ind}_H^G(V),\mathrm{Ind}_H^G(W)} \circ \delta_{V,W}$ maps $g \otimes v \otimes w$ to

$$g|v|g^{-1}g \otimes w \otimes g \otimes v = g|v| \otimes w \otimes g \otimes v = g \otimes |v|w \otimes g \otimes v,$$

using that $|v| \in H$, where $\delta(v) = |v| \otimes v$ is the H-coaction on V. Hence, this composition equals $\delta_{W,V} \circ \mathrm{Ind}_H^G(\Psi_{V,W})$. This proves Equation (B.6). Equation (B.7) follows by noting that $\Psi_{V,W} \circ \mu_{V,W}(g \otimes v \otimes k \otimes w)$ is zero unless $g^{-1}k \in H$, in which case it evaluates to

$$g \otimes (|v|g^{-1}kw \otimes v) = g|v|g^{-1}k \otimes (w \otimes k^{-1}g|v|^{-1}v),$$

which is the image of $g \otimes v \otimes k \otimes w$ under the composition $\mu_{W,V} \circ \mathrm{Ind}_H^G(\Psi_{V,W})$. □

Corollary B.2. *Let $H \subset G$ be finite groups.*

(1) *The functor $\mathcal{Z}(\mathrm{Ind}_H^G)$ is exact and preserves duals.*
(2) *The functor $\mathcal{Z}(\mathrm{Ind}_H^G)$ preserves Frobenius algebra objects.*
(3) *The object $\mathrm{Ind}_H^G(V \otimes W)$ is naturally isomorphic to a direct summand of $\mathrm{Ind}_H^G(V) \otimes \mathrm{Ind}_H^G(W)$ in $\mathcal{Z}(\mathrm{Rep}_{\Bbbk} G)$.*

Proof. The functor $\mathcal{Z}(\mathrm{Ind}_H^G)$ is exact because it equals, on morphisms, the underlying functor Ind_H^G, which is both left and right adjoint to restriction. Further, a Frobenius monoidal functor preserves left and right duals and Frobenius algebras by [13]. Separability shows that the morphism $e_{V,W} = \delta_{V,W} \circ \mu_{V,W}$ is an idempotent, natural in V, W, that cuts out $\mathrm{Ind}_H^G(V \otimes W)$ as a direct summand of $\mathrm{Ind}_H^G(V) \otimes \mathrm{Ind}_H^G(W)$. □

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