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Multi-fuzzy sets and their correspondence to other sets

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Abstract. This paper discusses the properties of multi-fuzzy sets with product order and dictionary order on their membership functions. In particular the relationship between multi-fuzzy sets and other sets such as intuitionistic fuzzy sets, type-2 fuzzy sets, multisets and fuzzy multisets are studied.

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1. Introduction

The concept of a multi-fuzzy set [12, 13, 15, 16, 18] is an extension of a fuzzy set and Atanassov's intuitionistic fuzzy set. Multi-fuzzy sets are useful for handling problems with multi dimensional characterization properties. Applications of multi-fuzzy sets in image processing are discussed in [14]. Previous papers on multi-fuzzy sets concern set theoretical [11, 12, 13], topological [16] and algebraic [17] properties.

Our previous work on multi-fuzzy sets [19, 20] are based on the product order on their membership functions. This paper discusses the properties of multi-fuzzy sets with various order relations on their membership functions, in particular dictionary order and product order. It also studies the relationship between multi-fuzzy sets and intuitionistic fuzzy sets, type-2 fuzzy sets and fuzzy multisets.

2. Preliminaries

Throughout this paper X and Y stand for universal sets, I, J and K stand for indexing sets, $\{L_j : j \in J\}$ and $\{M_i : i \in I\}$ are families of lattices, unless it is stated otherwise and L^X stands for the set of all functions from X into L.

Let X be a nonempty crisp set and L be a complete lattice. An L-fuzzy set [5] on X is a mapping $A : X \to L$, and the family of all the L-fuzzy sets on X is just L^X consisting of all the mappings from X into L.

If $\{L_j : j \in J\}$ is a family of lattices, then the product $\prod_{j\in J} L_j$ is a lattice. If $x, y \in \prod_{j\in J} L_j$, then join $x \vee y$ and meet $x \wedge y$ of x, y are defined as: $(x \vee y)_j = x_j \vee y_j$ and $(x \wedge y)_j = x_j \wedge y_j,\ \forall x_j, y_j \in L_j,\ \forall j \in J$; or, equivalently, the product order $x \le y$ is defined as $x_j \le_j y_j, \forall j \in J$, where $\le$ and $\le_j$ are the order relations in $\prod_{j\in J} L_j$ and L_j respectively (Adopted from [24]).

Definition 2.1 ([22])**.** Let L and M be completely distributive lattices with order reversing involutions $' : M \to M$ and $' : L \to L$. A mapping $h : M \to L$ is called an order homomorphism, if it satisfies the conditions $h(0) = 0,\ h(\vee a_i) = \vee h(a_i)$ and $h^{-1}(b') = (h^{-1}(b))'$.

$h^{-1} : L \to M$ is defined by $\forall b \in L,\ h^{-1}(b) = \vee\{a \in M : h(a) \le b\}$. Order homomorphisms satisfy the following properties [22]: for every $a \in M$ and $p \in L$; $a \le h^{-1}(h(a)),\ h(h^{-1}(p)) \le p,\ h^{-1}(1_L) = 1_M,\ h^{-1}(0_L) = 0_M$ and $a \le h^{-1}(p) \Leftrightarrow h(a) \le p \Leftrightarrow h^{-1}(p') \le a'$. Both h and h^{-1} are order preserving and arbitrary join preserving maps. Moreover $h^{-1}(\wedge a_i) = \wedge h^{-1}(a_i)$.

2.1. Multi-fuzzy sets.

Definition 2.2 ([12, 16])**.** Let X be a nonempty set, J be an indexing set and $\{L_j : j \in J\}$ a family of partially ordered sets. A multi-fuzzy set A in X is a set :

$$A = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in X,\ \mu_j \in L_j^X,\ j \in J\}.$$

The function $\mu_A = (\mu_j)_{j\in J}$ is called the multi-membership function of the multi-fuzzy set A. $\prod_{j\in J} L_j$ is called the value domain. If $J = \{1, 2, ..., n\}$ (that is, $|J| = n$, a natural number), then n is called the dimension of A. Suppose that $L_j = [0, 1],\ \forall j \in J$ and dimension of multi-fuzzy sets in X is n, then $\mathbf{M^nFS}(X)$ denotes the set of all multi-fuzzy sets in X.

Let $\{L_j : j \in J\}$ be a family of partially ordered sets, $A = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in X,\ \mu_j \in L_j^X,\ j \in J\}$ and $B = \{\langle x, (\nu_j(x))_{j\in J}\rangle : x \in X,\ \nu_j \in L_j^X,\ j \in J\}$ be multi-fuzzy sets in X with product order on their membership functions. Then $A \sqsubseteq B$ if and only if $\mu_j(x) \le \nu_j(x)$, $\forall x \in X$ and $\forall j \in J$.

The equality, union and intersection(see [12]) of A and B are defined as:

- $A = B$ if and only if $\mu_j(x) = \nu_j(x)$, $\forall x \in X$ and $\forall j \in J$;
- $A \sqcup B = \{\langle x, (\mu_j(x) \vee \nu_j(x))_{j\in J}\rangle : x \in X\}$;
- $A \sqcap B = \{\langle x, (\mu_j(x) \wedge \nu_j(x))_{j\in J}\rangle : x \in X\}$.

Complement of A is $A' = \{\langle x, (\mu_j'(x))_{j\in J}\rangle : x \in X\}$, where μ_j' is the order reversing involution of μ_j.

If A, B, C are multi-fuzzy sets in X having same value domain with product order, then:

- $A \sqcup A = A,\ A \sqcap A = A$;
- $A \sqsubseteq A \sqcup B,\ \ B \sqsubseteq A \sqcup B,\ A \sqcap B \sqsubseteq A$ and $A \sqcap B \sqsubseteq B$;
- $A \sqsubseteq B$ if and only if $A \sqcup B = B$ if and only if $A \sqcap B = A$;

Definition 2.3 ([14, 15]). Let $\{L_j : j \in J\}$ be a family of complete lattices, $f : X \to Y$ and $h : \prod M_i \to \prod L_j$ be functions. The multi-fuzzy extension $F : \prod M_i^X \to \prod L_j^Y$ of f with respect to h is defined by

$$F(A)(y) = \bigvee_{y=f(x)} h(A(x)),\ A \in \prod M_i^X,\ y \in Y.$$

The lattice valued function $h : \prod M_i \to \prod L_j$ is called the bridge function of the multi-fuzzy extension of f. If $\{L_j : j \in J\}$ and $\{M_i : i \in I\}$ are families of completely distributive lattices and h^{-1} is the upper adjoint of h in Wang's [22] sense (see Definition 2.1), then the inverse $F^{-1} : \prod L_j^Y \to \prod M_i^X$ is defined by

$$F^{-1}(B)(x) = h^{-1}(B(f(x))),\ B \in \prod L_j^Y,\ x \in X;$$

3. Multi-fuzzy Sets and the Relationship to other Sets

This section discusses the relationship between multi-fuzzy sets and similar sets like intuitionistic fuzzy sets, type 2 fuzzy sets, Obtulowicz's general multi-fuzzy sets, multisets, fuzzy multisets, Syropoulo's multi-fuzzy sets, Blizard's multi-fuzzy sets and general sets.

3.1. **Multi-fuzzy Sets and Intuitionistic Fuzzy Sets.** An Atanassov's intuitionistic fuzzy set [1] on X is a set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : \mu_A(x) + \nu_A(x) \le 1, x \in X\}$, where $\mu_A(x) \in [0, 1]$ and $\nu_A(x) \in [0, 1]$ denote the membership degree and the non-membership degree of x in A respectively.

Consider multi-fuzzy sets A, B of X with value domain $L_1 \times L_2$, where $L_1 = L_2 = [0, 1]$. That is, $A = \{\langle x, \mu_1(x), \mu_2(x)\rangle : x \in X\} = (\mu_1, \mu_2)$ and $B = \{\langle x, \nu_1(x), \nu_2(x)\rangle : x \in X\} = (\nu_1, \nu_2)$ are multi-fuzzy sets of dimension 2. Define a partial order as follows $(\mu_1(x), \mu_2(x)) \le_M (\nu_1(x), \nu_2(x))$ if and only if $\mu_1(x) \le \nu_1(x)$ and $\mu_2(x) \ge \nu_2(x)$. If $\mu_1(x)$ and $\mu_2(x)$ are the grade membership and grade nonmembership values of x in A respectively and if $\mu_1(x) + \mu_2(x) \le 1,\ \forall x \in X$, then A is an intuitionistic fuzzy set. That is, every intuitionistic fuzzy set in X is a multi-fuzzy set in X of dimension 2.

If $A = \{\langle x, \mu_1(x), \mu_2(x)\rangle : x \in X\}$ and $B = \{\langle x, \nu_1(x), \nu_2(x)\rangle : x \in X\}$ are multi-fuzzy sets in X with the above order relation, then

- $A \sqsubseteq B$ if and only if $\mu_1(x) \le \nu_1(x)$ and $\mu_2(x) \ge \nu_2(x), \forall x \in X$, where $\le$ is the usual order relation and $\ge$ is its dual order in $[0, 1]$;
- $A \sqcup B = \{\langle x, \mu_1(x) \vee \nu_1(x), \mu_2(x) \wedge \nu_2(x)\rangle : x \in X\}$;
- $A \sqcap B = \{\langle x, \mu_1(x) \wedge \nu_1(x), \mu_2(x) \vee \nu_2(x)\rangle : x \in X\}$.

This shows that union and intersection defined in multi-fuzzy sets with the above order relation are the same as the union and intersection defined in intuitionistic fuzzy sets.

3.2. **Multi-fuzzy Sets and Type-2 Fuzzy Sets.** A type-2 fuzzy set is a fuzzy set having a membership function which itself is a fuzzy set (see page 17 of [6] and page 24 of‘[25]). Suppose $A = \{\langle x, \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \nu_B(x)\rangle : x \in X\}$ are multi-fuzzy sets of dimension 1 with value domain $L_1 = I^I$, where $I = [0, 1]$. Note that $\nu_A(x), \nu_B(x) \in I^I$, for each $x \in X$. Define a partial order as follows $\nu_A(x) \le_M \nu_B(x)$ if and only if $\nu_A(x) \subseteq \nu_B(x)$, where $\le_M$ and $\subseteq$are the order relations in multi-fuzzy sets and I^I respectively. $\subseteq$ is the fuzzy set inclusion operation in I^I. That is, type-2 fuzzy sets are multi-fuzzy sets.

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} = (\mu_A, \nu_A)$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\} = (\mu_B, \nu_B)$ be multi-fuzzy sets of dimension 2 with value domain $L_1 \times L_2$. Suppose $L_1 = [0, 1]$, $L_2 = I^I$, $I = [0, 1]$, and $\nu_A(x), \nu_B(x) \in I^I$, for each $x \in X$. Define the partial order $(\mu_A(x), \nu_A(x)) \le_M (\mu_B(x), \nu_B(x))$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \subseteq \nu_B(x)$. With respect to this order relation, A is a multi-fuzzy set and it is a generalization of type-2 fuzzy set.

3.3. **Multi-fuzzy Sets and Obtulowicz's General Multi-fuzzy Sets.** Obtulowicz's general multi-fuzzy sets [10] over a universal set X are functions $M : X \times \mathbb{N} \to I$ or equivalently, functions $\mathcal{M} : X \to I^{\mathbb{N}}$, where $\mathbb{N}$ is the set of all natural numbers and $I = [0, 1]$. The value $M(x, n)$ or $\mathcal{M}(x)(n)$ is the degree of certainty that n copies of an object $x \in X$ occur in a system or its part. In the general multi-fuzzy sets the order relations and operations are defined component wise.

Let $A = \{\langle x, \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \nu_B(x)\rangle : x \in X\}$ be multi-fuzzy sets of dimension 1 with value domain $L_1 = I^{\mathbb{N}}$, where $I = [0, 1]$, and $\nu_A(x), \nu_B(x) \in I^{\mathbb{N}}$ for each $x \in X$. Consider the partial order $\nu_A(x) \le_M \nu_B(x)$ if and only if $\nu_A(x) \subseteq \nu_B(x)$, where $\le_M$ and $\subseteq$are the order relations in multi-fuzzy sets and $I^{\mathbb{N}}$ respectively. Note that $\nu_A(x) = \mathcal{A}(x)(n)$ and $\nu_B(x) = \mathcal{B}(x)(n)$. Hence Obtulowicz's general multi-fuzzy sets are multi-fuzzy sets.

3.4. **Multi-fuzzy Sets and Multisets.** Let $\mathbb{W}$ be the set of nonnegative integers and let C_A be function the universal set X into $\mathbb{W}$. A multiset [2, 3] M over the set X is the set $A = \{(x, C_A(x)) : x \in X, C_A(x) > 0\}$. The value $C_A(x)$ is the number of copies of x occur in the multiset A. Let A and B be multisets over X. Then:

- $A \subseteq B$, if $C_A(x) \le C_B(x), \forall x \in X$;
- $A = B$, if $C_A(x) = C_B(x), \forall x \in X$;
- $C_{A\cup B}(x) = \max\{C_A(x), C_B(x)\}$;

- $C_{A\cap B}(x) = \min\{C_A(x), C_B(x)\}$.

Multisets are multi-fuzzy sets of dimension 1 with $L_1 = \mathbb{W}$ and $C_A(x) \leq_M C_B(x)$ equivalent to $C_A(x) \leq C_B(x)$, where $\leq_M$ and $\leq$ order relations in multi-fuzzy sets and multisets respectively. Miyamoto's [8, 9] above definition of inclusion, equality, union and intersection of multisets are the same as the definition of respective operations in multi-fuzzy sets of dimension 1 with value domain $L_1 = \mathbb{W}$.

3.5. **Multi-fuzzy Sets and Fuzzy Multisets.** Yager [23] proposed the notion of fuzzy bags and later Miyamoto [7, 8, 9] renamed them as fuzzy multisets. Let X be a nonempty set and $\mu_A^j(x) \in [0,1]$, for $j = 1, 2, ..., k$. A fuzzy multiset A in X is a set : $A = \{(x, \mu_A^1(x), \mu_A^2(x), ..., \mu_A^k(x)) : x \in X,\ \mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^k(x)\}$ (see [4]). In a fuzzy multiset an element of X may occur more than once with possibly the same or different membership values and fuzzy multiset membership value of $x \in X$ is a non-increasing sequence of fuzzy membership values of x. Usually we write the elements of X with nonzero membership values only. Appending any number of zeros at the right end of a finite sequence of the membership values of x will not make any difference to the occurrence of an element x. Let A and B be fuzzy multisets over X. Then:

- $A \subseteq B$ if and only if $\mu_A^j(x) \leq \mu_B^j(x),\ j = 1, 2, ..., k, \forall x \in X$;
- $A = B$ if and only if $\mu_A^j(x) = \mu_B^j(x),\ j = 1, 2, ..., k, \forall x \in X$;
- $\mu_{A\cup B}^j(x) = \max\{\mu_A^j(x), \mu_B^j(x)\},\ j = 1, 2, ..., k$;
- $\mu_{A\cap B}^j(x) = \min\{\mu_A^j(x), \mu_B^j(x)\},\ j = 1, 2, ..., k$.

A fuzzy multiset is a multi-fuzzy set with value domain $\prod L_j$ having the relation $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^k(x) \geq ...$ and $L_j = [0,1]$, for $j = 1, 2, ...$ If the order relation in the value domain of multi-fuzzy sets are product orders, then inclusion, union and intersection in fuzzy multisets and multi-fuzzy sets are identical.

3.6. **Multi-fuzzy Sets and Syropoulos's Multi-fuzzy Sets.** Syropoulos's multi-fuzzy sets [21] is a fuzzification of multisets. Let X be the universal set and $\mathbb{N}$ be the set of natural numbers. If $M : X \to \mathbb{N}$ characterizes a multiset M, then Syropoulos's multi-fuzzy set of M is characterized by a function $\mathcal{H} : X \to \mathbb{N} \times [0,1]$, such that if $M(x) = n$, then $\mathcal{H}(x) = (n, i)$, for every $x \in X$. In addition, the expression $\mathcal{H}(x) = (n, i)$ denotes the degree to which these n copies of x belongs to $\mathcal{H}$ is i. Union and intersection operations of Syropoulos's multi-fuzzy sets are defined as follows: Let $\mathcal{H}, \mathcal{G} : X \to \mathbb{N} \times [0,1]$ be two Syropoulos's multi-fuzzy sets, $\mathcal{H} \cup \mathcal{G}$ and $\mathcal{H} \cap \mathcal{G}$ be the union and intersection of $\mathcal{H}, \mathcal{G}$. The membership function can be defined as

$$(\mathcal{H} \cup \mathcal{G})(x) = (\max\{\mathcal{H}_m(x), \mathcal{G}_m(x)\}, \max\{\mathcal{H}_\mu(x), \mathcal{G}_\mu(x)\})$$

and

$$(\mathcal{H} \cap \mathcal{G})(x) = (\min\{\mathcal{H}_m(x), \mathcal{G}_m(x)\}, \min\{\mathcal{H}_\mu(x), \mathcal{G}_\mu(x)\}),$$

where $\mathcal{H}_\mu(x)$ and $\mathcal{G}_\mu(x)$ are the membership values of x in $\mathcal{H}$ and $\mathcal{G}$ respectively. Similarly $\mathcal{H}_m(x)$ and $\mathcal{G}_m(x)$ are the multiplicities of x in $\mathcal{H}$ and $\mathcal{G}$ respectively.

Syropoulos's multi-fuzzy sets are multi-fuzzy sets defined by the author with dimension 2, and $L_1 = L,\ L_2 = \mathbb{N}$. The order relation $\mathcal{H}(x) \leq_M \mathcal{G}(x)$ if and only if

$\mathcal{H}_\mu(x) \leq \mathcal{G}_\mu(x)$ and $\mathcal{H}_m(x) \leq \mathcal{G}_m(x)$, where $\leq_M$ and $\leq$ are order relations defined in multi-fuzzy sets and Syropoulos's multi-fuzzy sets respectively.

3.7. **Blizard's Multi-fuzzy Sets.** Blizard [2] proposed the notion of multi-fuzzy sets characterized by nonnegative and real valued membership functions. That is, membership function of a Blizard's multi-fuzzy set is $\mu(x) \in [0, \infty)$. He extended the value domain of membership functions into the set of all real numbers and called it a general set. Blizard's multi-fuzzy sets and general sets are our multi-fuzzy sets with dimension 1, and value domains $L_1 = [0, \infty)$ and $L_1 = \mathbb{R}$ respectively.

It is possible to define multi-fuzzy extensions of a crisp function $f : X \to Y$ with respect to bridge a function $h : \prod L_i \to \prod L_i$ in intuitionistic fuzzy sets, type-2 fuzzy sets, multisets, fuzzy multisets, Obtulowicz's general multi-fuzzy sets, Syropoulos's multi-fuzzy sets and Blizard's multi-fuzzy sets, since they are multi-fuzzy sets. If h is the identity function defined on $\prod L_i$, then the extension is Zadeh's extension. Using the bridge functions $h : \prod M_i \to \prod L_j$, we can extend a crisp function $f : X \to Y$ as a function from intuitionistic fuzzy sets (or any of the above sets) into type-2 fuzzy sets (or any of the above sets).

4. Different Order Relations on Membership Functions

This section discusses properties of multi-fuzzy sets with membership function having order relations other than product order.

Let $A = \{\langle x, \mu_1(x), \mu_2(x)\rangle : x \in X,\ \mu_1(x), \mu_2(x) \in [0, 1]\}$ and $B = \{\langle x, \nu_1(x), \nu_2(x)\rangle : x \in X,\ \nu_1(x), \nu_2(x) \in [0, 1]\}$ be multi-fuzzy sets in X of dimension 2 and value domain I^2 with dictionary order. That is, $A, B \in \mathbf{M^2FS}(X)$ with dictionary order in the value domain. Then

- $A = B$ if and only if $\mu_j(x) = \nu_j(x),\ j = 1, 2, \forall x \in X$;
- $A \subset B$ if and only if $\mu_1(x) < \nu_1(x)$ or if $\mu_1(x) = \nu_1(x)$ and $\mu_2(x) < \nu_2(x), \forall x \in X$;
- $A \sqcup B = \{\langle x, \mu_1(x) \vee \nu_1(x), \mu_2(x) \vee \nu_2(x)\rangle : x \in X\}$;
- $A \sqcap B = \{\langle x, \mu_1(x) \wedge \nu_1(x), \mu_2(x) \wedge \nu_2(x)\rangle : x \in X\}$.

In a similar manner we can define equality, set inclusion, union and intersection of $A, B \in \mathbf{M^nFS}(X)$ with dictionary order in the value domain.

If $A, B \in \mathbf{M^2FS}(X)$ with reverse dictionary order in the value domain, then $A \subset B$ if and only if $\mu_2(x) < \nu_2(x)$ or if $\mu_2(x) = \nu_2(x)$ and $\mu_1(x) < \nu_1(x), \forall x \in X$. Equality, union and intersection of $A, B \in \mathbf{M^2FS}(X)$ with reverse dictionary order in the value domain are similar to the respective relations of $A, B \in \mathbf{M^2FS}(X)$ with dictionary order in the value domain.
Let $A, B \in \mathbf{M^2FS}(X)$ with dictionary order in the value domain. If

- $A \sqcup^1 B = \{\langle x, \mu_1(x) \vee \nu_1(x), \mu_2(x)\rangle : x \in X\}$;
- $A \sqcup^2 B = \{\langle x, \mu_1(x) \vee \nu_1(x), \nu_2(x)\rangle : x \in X\}$;
- $A \sqcup^{min} B = \{\langle x, \mu_1(x) \vee \nu_1(x), \mu_2(x) \wedge \nu_2(x)\rangle : x \in X\}$;
- $A \sqcap^1 B = \{\langle x, \mu_1(x) \wedge \nu_1(x), \mu_2(x)\rangle : x \in X\}$;
- $A \sqcap^2 B = \{\langle x, \mu_1(x) \wedge \nu_1(x), \nu_2(x)\rangle : x \in X\}$;

- $A \sqcap^{max} B = \{\langle x, \mu_1(x) \wedge \nu_1(x), \mu_2(x) \vee \nu_2(x)\rangle : x \in X\}$.

then

- $A \sqcap B \sqsubseteq A \sqcup^{min} B \sqsubseteq A \sqcup^1 B \sqsubseteq A \sqcup B$;
- $A \sqcap B \sqsubseteq A \sqcup^{min} B \sqsubseteq A \sqcup^2 B \sqsubseteq A \sqcup B$;
- $A \sqcap B \sqsubseteq A \sqcup^1 B \sqsubseteq A \sqcup^{max} B \sqsubseteq A \sqcup B$;
- $A \sqsubseteq B \Rightarrow B \sqsubseteq A \sqcup B$ and $A \sqcap B \sqsubseteq A$. Note that $A \sqcup B$ need not be equal to B or $A \sqcap B$ need not be equal to A, (but equality holds if the dimension is 1). For example, consider the constant multi-fuzzy sets $A = (0.1, 0.3)$ and $B = (0.2, 0.2)$. Here $A \sqcup B = (0.2, 0.3)$ and $A \sqcap B = (0.1, 0.2)$. In general $\mathbf{M^2FS}(X)$ with dictionary order in the value domain need not be a lattice with respect to the operations union and intersection.

5. Conclusions

The relationships between multi-fuzzy sets and other sets like intuitionistic fuzzy sets, type-2 fuzzy sets, multisets, fuzzy multisets, Obtulowicz's general multi-fuzzy sets, Syropoulos's multi-fuzzy sets and Blizard's multi-fuzzy sets are important for the development of each of them. This paper showed that the above sets are multi-fuzzy sets. Finally we investigated some order relations in the value domain other than product order and showed that such order relations have significance in set theoretical study. In the future we can expect multi-fuzzy extensions of crisp functions on the above sets.

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