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Algebraic Quantum Field Theory on Spacetimes with Timelike Boundary

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Dedicated to Klaus Fredenhagen on the occasion of his 70th birthday.

Abstract. We analyze quantum field theories on spacetimes M with timelike boundary from a model-independent perspective. We construct an adjunction which describes a universal extension to the whole spacetime M of theories defined only on the interior int$M$. The unit of this adjunction is a natural isomorphism, which implies that our universal extension satisfies Kay's F-locality property. Our main result is the following characterization theorem: Every quantum field theory on M that is additive from the interior (i.e., generated by observables localized in the interior) admits a presentation by a quantum field theory on the interior int$M$ and an ideal of its universal extension that is trivial on the interior. We shall illustrate our constructions by applying them to the free Klein–Gordon field.

1. Introduction and Summary

Algebraic quantum field theory is a powerful and far developed framework to address model-independent aspects of quantum field theories on Minkowski spacetime [18] and more generally on globally hyperbolic spacetimes [7]. In addition to establishing the axiomatic foundations for quantum field theory, the algebraic approach has provided a variety of mathematically rigorous constructions of non-interacting models, see e.g., the reviews [1,3,4], and more interestingly also perturbatively interacting quantum field theories, see e.g., the recent monograph [26]. It is worth emphasizing that many of the techniques involved in such constructions, e.g., existence and uniqueness of Green's operators and the singular structure of propagators, crucially rely on the hypothesis that the spacetime is globally hyperbolic and has empty boundary.

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Even though globally hyperbolic spacetimes have plenty of applications to physics, there exist also important and interesting situations which require non-globally hyperbolic spacetimes, possibly with a non-trivial boundary. On the one hand, recent developments in high energy physics and string theory are strongly focused on anti-de Sitter spacetime, which is not globally hyperbolic and has a (conformal) timelike boundary. On the other hand, experimental setups for studying the Casimir effect confine quantum field theories between several metal plates (or other shapes), which may be modeled theoretically by introducing timelike boundaries to the system. This immediately prompts the question whether the rigorous framework of algebraic quantum field theory admits a generalization to cover such scenarios.

Most existing works on algebraic quantum field theory on spacetimes with a timelike boundary focus on the construction of concrete examples, such as the free Klein–Gordon field on simple classes of spacetimes. The basic strategy employed in such constructions is to analyze the initial value problem on a given spacetime with timelike boundary, which has to be supplemented by suitable boundary conditions. Different choices of boundary conditions lead to different Green's operators for the equation of motion, which is in sharp contrast to the well-known existence and uniqueness results on globally hyperbolic spacetimes with empty boundary. Recent works addressing this problem are [19,20,33], the latter extending the analysis of [31]. For specific choices of boundary conditions, there exist successful constructions of algebraic quantum field theories on spacetimes with timelike boundary, see e.g., [8,10–12]. The main message of these works is that the algebraic approach is versatile enough to account also for these models, although some key structures, such as for example the notion of Hadamard states [11,32], should be modified accordingly.

Unfortunately, model-independent results on algebraic quantum field theory on spacetimes with timelike boundary are more scarce. There are, however, some notable and very interesting works in this direction: On the one hand, Rehren's proposal for algebraic holography [25] initiated the rigorous study of quantum field theories on the anti-de Sitter spacetime. This has been further elaborated in [13] and extended to asymptotically AdS spacetimes in [27]. On the other hand, inspired by Fredenhagen's universal algebra [15–17], a very interesting construction and analysis of *global* algebras of observables on spacetimes with timelike boundaries has been performed in [30]. The most notable outcome is the existence of a relationship between maximal ideals of this algebra and boundary conditions, a result which has been of inspiration for this work.

In the present paper, we shall analyze quantum field theories on spacetimes with timelike boundary from a model-independent perspective. We are mainly interested in understanding and proving structural results for whole categories of quantum field theories, in contrast to focusing on particular theories. Such questions can be naturally addressed by using techniques from the recently developed operadic approach to algebraic quantum field theory [5]. Let us describe rather informally the basic idea of our construction and its implications: Given a spacetime M with timelike boundary, an algebraic

quantum field theory on M is a functor $\mathfrak{B} : \mathcal{R}_M \to \mathbf{Alg}$ assigning algebras of observables to suitable regions $U \subseteq M$ (possibly intersecting the boundary), which satisfies the causality and time-slice axioms. We denote by $\mathbf{QFT}(M)$ the category of algebraic quantum field theories on M. Denoting the full subcategory of regions in the interior of M by $\mathcal{R}_{\mathrm{int}\,M} \subseteq \mathcal{R}_M$, we may restrict any theory $\mathfrak{B} \in \mathbf{QFT}(M)$ to a theory $\mathrm{res}\,\mathfrak{B} \in \mathbf{QFT}(\mathrm{int}\,M)$ defined only on the interior regions. Notice that it is in practice much easier to analyze and construct theories on $\mathrm{int}\,M$ as opposed to theories on the whole spacetime M. This is because the former are postulated to be insensitive to the boundary by Kay's F-locality principle [22]. As a first result, we shall construct a left adjoint of the restriction functor $\mathrm{res} : \mathbf{QFT}(M) \to \mathbf{QFT}(\mathrm{int}\,M)$, which we call the universal extension functor $\mathrm{ext} : \mathbf{QFT}(\mathrm{int}\,M) \to \mathbf{QFT}(M)$. This means that given any theory $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\,M)$ that is defined only on the interior regions in M, we obtain a universal extension $\mathrm{ext}\,\mathfrak{A} \in \mathbf{QFT}(M)$ to all regions in M, including those that intersect the boundary. It is worth to emphasize that the adjective *universal* above refers to the categorical concept of universal properties. Below we explain in which sense ext is also "universal" in a more physical meaning of the word.

It is crucial to emphasize that our universal extension $\mathrm{ext}\,\mathfrak{A} \in \mathbf{QFT}(M)$ is always a bona fide algebraic quantum field theory in the sense that it satisfies the causality and time-slice axioms. This is granted by the operadic approach to algebraic quantum field theory of [5]. In particular, the $\mathrm{ext} \dashv \mathrm{res}$ adjunction investigated in the present paper is one concrete instance of a whole family of adjunctions between categories of algebraic quantum field theories that naturally arise within the theory of colored operads and algebras over them.

A far reaching implication of the above mentioned $\mathrm{ext} \dashv \mathrm{res}$ adjunction is a characterization theorem that we shall establish for quantum field theories on spacetimes with timelike boundary. Given any theory $\mathfrak{B} \in \mathbf{QFT}(M)$ on a spacetime M with timelike boundary, we can restrict and universally extend to obtain another such theory $\mathrm{ext}\,\mathrm{res}\,\mathfrak{B} \in \mathbf{QFT}(M)$. The adjunction also provides us with a natural comparison map between these theories, namely the counit $\epsilon_{\mathfrak{B}} : \mathrm{ext}\,\mathrm{res}\,\mathfrak{B} \to \mathfrak{B}$ of the adjunction. Our result in Theorem 5.6 and Corollary 5.7 is that $\epsilon_{\mathfrak{B}}$ induces an isomorphism $\mathrm{ext}\,\mathrm{res}\,\mathfrak{B}/\ker\epsilon_{\mathfrak{B}} \cong \mathfrak{B}$ of quantum field theories if and only if $\mathfrak{B}$ is *additive from the interior* as formalized in Definition 5.5. The latter condition axiomatises the heuristic idea that the theory $\mathfrak{B}$ has no degrees of freedom that are localized on the boundary of M, i.e., all its observables may be generated by observables supported in the interior of M. Notice that the results in Theorem 5.6 and Corollary 5.7 give the adjective *universal* also a physical meaning in the sense that the extensions are sufficiently large such that any additive theory can be recovered by a quotient. We strengthen this result in Theorem 5.10 by constructing an equivalence between the category of additive quantum field theories on M and a category of pairs $(\mathfrak{A}, \mathfrak{I})$ consisting of a theory $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\,M)$ on the interior and an ideal $\mathfrak{I} \subseteq \mathrm{ext}\,\mathfrak{A}$ of the universal extension that is trivial on the interior. More concretely, this means that every additive theory $\mathfrak{B} \in \mathbf{QFT}(M)$ may be naturally decomposed into two distinct pieces of data: (1) A theory

$\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\, M)$ on the interior, which is insensitive to the boundary as postulated by F-locality, and (2) an ideal $\mathfrak{I} \subseteq \mathrm{ext}\,\mathfrak{A}$ of its universal extension that is trivial on the interior, i.e., that is only sensitive to the boundary. Specific examples of such ideals arise from imposing boundary conditions. We shall illustrate this fact by using the free Klein–Gordon theory as an example. Thus, our results also provide a bridge between the ideas of [30] and the concrete constructions in [8,10–12].

The remainder of this paper is structured as follows: In Sect. 2, we recall some basic definitions and results about the causal structure of spacetimes with timelike boundaries, see also [9,29]. In Sect. 3, we provide a precise definition of the categories $\mathbf{QFT}(M)$ and $\mathbf{QFT}(\mathrm{int}\, M)$ by using the ideas of [5]. Our universal boundary extension is developed in Sect. 4, where we also provide an explicit model in terms of left Kan extension. Our main results on the characterization of additive quantum field theories on M are proven in Sect. 5. Section 6 illustrates our construction by focusing on the simple example of the free Klein–Gordon theory, where more explicit formulas can be developed. It is in this context that we provide examples of ideals implementing boundary conditions and relate to analytic results, e.g., [12]. We included "Appendix A" to state some basic definitions and results of category theory which will be used in our work.

2. Spacetimes with Timelike Boundary

We collect some basic facts about spacetimes with timelike boundary, following [29, Section 3.1] and [9, Section 2.2]. For a general introduction to Lorentzian geometry, we refer to [2,24], see also [1, Sections 1.3 and A.5] for a concise presentation.

We use the term *manifold with boundary* to refer to a Hausdorff, second countable, m-dimensional smooth manifold M with boundary, see e.g., [23]. This definition subsumes ordinary manifolds as manifolds with empty boundary $\partial M = \emptyset$. We denote by $\mathrm{int}\, M \subseteq M$ the submanifold without the boundary. Every open subset $U \subseteq M$ carries the structure of a manifold with (possibly empty) boundary and one has $\mathrm{int}\, U = U \cap \mathrm{int}\, M$.

Definition 2.1. A *Lorentzian manifold with boundary* is a manifold with boundary that is equipped with a Lorentzian metric.

Definition 2.2. Let M be a time-oriented Lorentzian manifold with boundary. The *Cauchy development* $D(S) \subseteq M$ of a subset $S \subseteq M$ is the set of points $p \in M$ such that every inextensible (piecewise smooth) future directed causal curve stemming from p meets S.

The following properties follow easily from the definition of Cauchy development.

Proposition 2.3. *Let $S, S' \subseteq M$ be subsets of a time-oriented Lorentzian manifold M with boundary. Then, the following holds true:*

(a) $S \subseteq S'$ *implies* $D(S) \subseteq D(S')$;

(b) $S \subseteq D(S) = D(D(S))$;
(c) $D(D(S) \cap D(S')) = D(S) \cap D(S')$.

We denote by $J_M^\pm(S) \subseteq M$ the *causal future/past* of a subset $S \subseteq M$, i.e., the set of points that can be reached by a future/past directed causal curve stemming from S. Furthermore, we denote by $I_M^\pm(S) \subseteq M$ the *chronological future/past* of a subset $S \subseteq M$, i.e., the set of points that can be reached by a future/past directed timelike curve stemming from S.

Definition 2.4. Let M be a time-oriented Lorentzian manifold with boundary. We say that a subset $S \subseteq M$ is *causally convex* in M if $J_M^+(S) \cap J_M^-(S) \subseteq S$. We say that two subsets $S, S' \subseteq M$ are *causally disjoint* in M if $(J_M^+(S) \cup J_M^-(S)) \cap S' = \emptyset$.

The following properties are simple consequences of these definitions.

Proposition 2.5. *Let $S, S' \subseteq M$ be two subsets of a time-oriented Lorentzian manifold M with boundary. Then, the following holds true:*

(a) *$D(S)$ and $D(S')$ are causally disjoint if and only if S and S' are causally disjoint;*
(b) *Suppose S and S' are causally disjoint. Then, the disjoint union $S \sqcup S' \subseteq M$ is causally convex if and only if both S and S' are causally convex.*

The following two definitions play an essential role in our work.

Definition 2.6. A *spacetime with timelike boundary* is an oriented and time-oriented Lorentzian manifold M with boundary, such that the pullback of the Lorentzian metric along the boundary inclusion $\partial M \hookrightarrow M$ defines a Lorentzian metric on the boundary ∂M.

Definition 2.7. Let M be a spacetime with timelike boundary.

(i) $\mathcal{R}_M$ denotes the category whose objects are causally convex open subsets $U \subseteq M$ and whose morphisms $i : U \to U'$ are inclusions $U \subseteq U' \subseteq M$. We call it the *category of regions* in M.
(ii) $\mathcal{C}_M \subseteq \operatorname{Mor} \mathcal{R}_M$ is the subset of *Cauchy morphisms* in $\mathcal{R}_M$, i.e., inclusions $i : U \to U'$ such that $D(U) = D(U')$.
(iii) $\mathcal{R}_{\operatorname{int} M} \subseteq \mathcal{R}_M$ is the full subcategory whose objects are contained in the interior $\operatorname{int} M$. We denote by $\mathcal{C}_{\operatorname{int} M} \subseteq \mathcal{C}_M$ the Cauchy morphisms between objects of $\mathcal{R}_{\operatorname{int} M}$.

Proposition 2.8. *Let M be a spacetime with timelike boundary. For each subset $S \subseteq M$ and each object $U \in \mathcal{R}_M$, i.e., a causally convex open subset $U \subseteq M$, the following holds true:*

(a) *$I_M^\pm(S)$ is the largest open subset of $J_M^\pm(S)$;*
(b) *$J_M^\pm(I_M^\pm(S)) = I_M^\pm(S) = I_M^\pm(J_M^\pm(S))$;*
(c) *$S \subseteq \operatorname{int} M$ implies $D(S) \subseteq \operatorname{int} M$;*
(d) *$D(U) \subseteq M$ is causally convex and open, i.e., $D(U) \in \mathcal{R}_M$.*

Proof. (a) and (b): These are standard results in the case of empty boundary, see e.g., [1,2,24]. The extension to spacetimes with non-empty timelike boundary can be found in [29, Section 3.1.1].

(c): We show that if $D(S)$ contains a boundary point, so does S: Suppose $p \in D(S)$ belongs to the boundary of M. By Definition 2.6, the boundary ∂M of M can be regarded as a time-oriented Lorentzian manifold with empty boundary; hence, we can consider a future directed inextensible causal curve γ in ∂M stemming from p. Since ∂M is a closed subset of M, γ must be inextensible also as a causal curve in M, hence γ meets S because it stems from $p \in D(S)$. Since γ lies in ∂M by construction, we conclude that S contains a boundary point of M.

(d): $D(U) \subseteq M$ is causally convex by the definition of Cauchy development and by the causal convexity of $U \subseteq M$. To check that $D(U) \subseteq M$ is open, we use quasi-limits as in [24, Definition 14.7 and Proposition 14.8]: First, observe that $I_M^{\pm}(U) \subseteq M$ is open by (a). Hence, it is the same to check whether a subset of $I := I_M^+(U) \cup I_M^-(U)$ is open in M or in I (with the induced topology). Indeed, $U \subseteq D(U) \subseteq I$ because U is open in M. From now on, in place of M, let us therefore consider I, equipped with the induced metric, orientation and time-orientation. By contradiction, assume that there exists $p \in D(U) \backslash U$ such that all of its neighborhoods intersect the complement of $D(U)$. Then, there exists a sequence $\{\alpha_n\}$ of inextensible causal curves in I never meeting U such that $\{\alpha_n(0)\}$ converges to p. We fix a convex cover of I refining the open cover $\{I_M^+(U), I_M^-(U)\}$. Relative to the fixed convex cover, the construction of quasi-limits allows us to obtain from $\{\alpha_n\}$ an inextensible causal curve λ through $p \in D(U)$. Hence, λ meets U, say in q. By the construction of a quasi-limit, q lies on a causal geodesic segment between p_k and p_{k+1}, two successive limit points for $\{\alpha_n\}$ contained in some element of the fixed convex cover. It follows that either p_k or p_{k+1} belongs to $J_I^+(U) \cap J_I^-(U)$, which is contained in U by causal convexity. Hence, we found a subsequence $\{\alpha_{n_j}\}$ of $\{\alpha_n\}$ and a sequence of parameters $\{s_j\}$ such that $\{\alpha_{n_j}(s_j)\}$ converges to a point of U (either p_k or p_{k+1}). By construction the sequence $\{\alpha_{n_j}(s_j)\}$ is contained in $I \backslash U$; however, its limit lies in U. This contradicts the hypothesis that U is open in I. $\qquad \square$

The causal structure of a spacetime M with timelike boundary can be affected by several pathologies, such as the presence of closed future directed causal curves. It is crucial to avoid these issues in order to obtain concrete examples of our constructions in Sect. 6. The following definition is due to [9, Section 2.2] and [29, Section 3.1.2].

Definition 2.9. A spacetime M with timelike boundary is called *globally hyperbolic* if the following two properties hold true:

(i) *Strong causality* Every open neighborhood of each point $p \in M$ contains a *causally convex* open neighborhood of p.

(ii) *Compact double-cones* $J_M^+(p) \cap J_M^-(q)$ is compact for all $p, q \in M$.

Remark 2.10. In the case of empty boundary, this definition agrees with the usual one in [1, 2, 24]. Simple examples of globally hyperbolic spacetimes with non-empty timelike boundary are the half space $\{x^{m-1} \geq 0\} \subseteq \mathbb{R}^m$, the spatial slab $\{0 \leq x^{m-1} \leq 1\} \subseteq \mathbb{R}^m$ and the cylinder $\{(x^1)^2 + \cdots + (x^{m-1})^2 \leq 1\} \subseteq \mathbb{R}^m$ in Minkowski spacetime $\mathbb{R}^m$, for $m \geq 2$, as well as all causally convex open subsets thereof.

The following results follow immediately from Definition 2.9 and Proposition 2.8.

Proposition 2.11. *Let M be a globally hyperbolic spacetime with timelike boundary.*

(a) *M admits a cover by causally convex open subsets.*
(b) *For each $U \in \mathcal{R}_M$, i.e., a causally convex open subset $U \subseteq M$, both U and $D(U)$ are globally hyperbolic spacetimes with (possibly empty) boundary when equipped with the metric, orientation and time-orientation induced by M. If moreover $U \subseteq \operatorname{int} M$ is contained in the interior, then both U and $D(U)$ have empty boundary.*

3. Categories of Algebraic Quantum Field Theories

Let M be a spacetime with timelike boundary. (In this section we do not have to assume that M is globally hyperbolic in the sense of Definition 2.9). Recall the category $\mathcal{R}_M$ of open and causally convex regions in M and the subset $\mathcal{C}_M$ of Cauchy morphisms (cf. Definition 2.7). Together with our notion of causal disjointness from Definition 2.4, these data provide the geometrical input for the traditional definition of algebraic quantum field theories on M.

Definition 3.1. An *algebraic quantum field theory* on M is a functor

$$\mathfrak{A} : \mathcal{R}_M \longrightarrow \mathbf{Alg} \tag{3.1}$$

with values in the category $\mathbf{Alg}$ of associative and unital $*$-algebras over $\mathbb{C}$, which satisfies the following properties:

(i) *Causality axiom* For all causally disjoint inclusions $i_1 : U_1 \to U \leftarrow U_2 : i_2$, the induced commutator

$$\big[\mathfrak{A}(i_1)(-), \mathfrak{A}(i_2)(-)\big]_{\mathfrak{A}(U)} : \mathfrak{A}(U_1) \otimes \mathfrak{A}(U_2) \longrightarrow \mathfrak{A}(U) \tag{3.2}$$

is zero.
(ii) *Time-slice axiom* For all Cauchy morphisms $(i : U \to U') \in \mathcal{C}_M$, the map

$$\mathfrak{A}(i) : \mathfrak{A}(U) \longrightarrow \mathfrak{A}(U') \tag{3.3}$$

is an $\mathbf{Alg}$-isomorphism.

We denote by $\mathbf{qft}(M) \subseteq \mathbf{Alg}^{\mathcal{R}_M}$ the full subcategory of the category of functors from $\mathcal{R}_M$ to $\mathbf{Alg}$ whose objects are all algebraic quantum field theories on M, i.e., functors fulfilling the causality and time-slice axioms. (Morphisms in this category are all natural transformations).

We shall now show that there exists an alternative, but equivalent, description of the category $\mathbf{qft}(M)$ which will be more convenient for the technical constructions in our paper. Following [5, Section 4.1], we observe that the time-slice axiom in Definition 3.1(ii) is equivalent to considering functors $\mathfrak{B} : \mathcal{R}_M \left[\mathcal{C}_M^{-1}\right] \to \mathbf{Alg}$ that are defined on the localization of the category $\mathcal{R}_M$ at the set of Cauchy morphisms $\mathcal{C}_M$. See Definition A.4 for the definition of localizations of categories. By abstract arguments as in [5, Section 4.6], one observes that the universal property of localizations implies that the category $\mathbf{qft}(M)$ is equivalent to the full subcategory of the functor category $\mathbf{Alg}^{\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]}$ whose objects are all functors $\mathfrak{B} : \mathcal{R}_M \left[\mathcal{C}_M^{-1}\right] \to \mathbf{Alg}$ that satisfy the causality axiom for the pushforward orthogonality relation on $\mathcal{R}_M \left[\mathcal{C}_M^{-1}\right]$. Loosely speaking, this means that the time-slice axiom in Definition 3.1(ii) can be hard-coded by working on the localized category $\mathcal{R}_M \left[\mathcal{C}_M^{-1}\right]$ instead of using the usual category $\mathcal{R}_M$ of regions in M.

The aim of the remainder of this section is to provide an explicit model for the localization functor $\mathcal{R}_M \to \mathcal{R}_M \left[\mathcal{C}_M^{-1}\right]$. With this model, it will become particularly easy to verify the equivalence between the two alternative descriptions of the category $\mathbf{qft}(M)$. Let us denote by

$$\mathcal{R}_M \left[\mathcal{C}_M^{-1}\right] \subseteq \mathcal{R}_M \tag{3.4a}$$

the full subcategory of $\mathcal{R}_M$ whose objects $V \subseteq M$ are stable under Cauchy development, i.e., $D(V) = V$ where $D(V) \subseteq M$ denotes the Cauchy development (cf. Definition 2.2). In the following, we shall always use letters like $U \subseteq M$ for generic regions in $\mathcal{R}_M$ and $V \subseteq M$ for regions that are stable under Cauchy development, i.e., objects in $\mathcal{R}_M \left[\mathcal{C}_M^{-1}\right]$. Recall from Definition 2.7 that each object $U \in \mathcal{R}_M$ is a causally convex open subset $U \subseteq M$, hence the Cauchy development $D(U) \subseteq M$ is a causally convex open subset by Proposition 2.8(d), which is stable under Cauchy development by Proposition 2.3(b). This shows that $D(U)$ is an object of $\mathcal{R}_M \left[\mathcal{C}_M^{-1}\right]$. Furthermore, a morphism $i : U \to U'$ in $\mathcal{R}_M$ is an inclusion $U \subseteq U'$, which induces an inclusion $D(U) \subseteq D(U')$ by Proposition 2.3(a) and hence a morphism $D(i) : D(U) \to D(U')$ in $\mathcal{R}_M \left[\mathcal{C}_M^{-1}\right]$. We define a functor

$$D : \mathcal{R}_M \longrightarrow \mathcal{R}_M \left[\mathcal{C}_M^{-1}\right] \tag{3.4b}$$

by setting on objects and morphisms

$$U \longmapsto D(U), \quad (i : U \to U') \longmapsto \left(D(i) : D(U) \to D(U')\right). \tag{3.4c}$$

Furthermore, let us write

$$I : \mathcal{R}_M \left[\mathcal{C}_M^{-1}\right] \longrightarrow \mathcal{R}_M \tag{3.5}$$

for the full subcategory embedding.

Lemma 3.2. *D and I form an adjunction (cf. Definition A.1)*

$$D : \mathcal{R}_M \rightleftarrows \mathcal{R}_M \left[\mathcal{C}_M^{-1}\right] : I \tag{3.6}$$

whose counit is a natural isomorphism (in fact, the identity), hence $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ is a full reflective subcategory of $\mathcal{R}_M$. Furthermore, the components of the unit are Cauchy morphisms.

Proof. For $U \in \mathcal{R}_M$, the U-component of the unit

$$\eta : \mathrm{id}_{\mathcal{R}_M} \longrightarrow I\,D \tag{3.7}$$

is given by the inclusion $U \subseteq D(U)$ of U into its Cauchy development, which is a Cauchy morphism, see Proposition 2.3(b) and Definition 2.7(ii). For $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$, the V-component of the counit

$$\epsilon : D\,I \longrightarrow \mathrm{id}_{\mathcal{R}_M[\mathcal{C}_M^{-1}]} \tag{3.8}$$

is given by the identity of the object $D(V) = V$. The triangle identities hold trivially. $\square$

Proposition 3.3. *The category* $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ *and the functor* $D : \mathcal{R}_M \to \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ *defined in* (3.4) *provide a model for the localization of* $\mathcal{R}_M$ *at* $\mathcal{C}_M$.

Proof. We have to check all the requirements listed in Definition A.4.

(a) By Definition 2.7, for each Cauchy morphism $i : U \to U'$ one has $D(U) = D(U')$ and hence $D(i) = \mathrm{id}_{D(U)}$ is an isomorphism in $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$.

(b) Let $F : \mathcal{R}_M \to \mathbf{D}$ be any functor to a category $\mathbf{D}$ that sends morphisms in $\mathcal{C}_M$ to $\mathbf{D}$-isomorphisms. Using Lemma 3.2, we define $F_W := F\,I : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathbf{D}$ and consider the natural transformation $F\eta : F \to F\,I\,D = F_W\,D$ obtained by the unit of the adjunction $D \dashv I$. Because all components of η are Cauchy morphisms (cf. Lemma 3.2), $F\eta$ is a natural isomorphism.

(c) Let $G, H : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathbf{D}$ be two functors. We have to show that the map

$$\mathrm{Hom}_{\mathbf{D}^{\mathcal{R}_M[\mathcal{C}_M^{-1}]}}\left(G, H\right) \longrightarrow \mathrm{Hom}_{\mathbf{D}^{\mathcal{R}_M}}\left(GD, HD\right) \tag{3.9}$$

is a bijection. Let us first prove injectivity: Let $\xi, \widetilde{\xi} : G \to H$ be two natural transformations such that $\xi D = \widetilde{\xi} D$. Using Lemma 3.2, we obtain commutative diagrams

$$
\begin{array}{ccc}
GDI & \xrightarrow{\xi DI} & HDI \\
{\scriptstyle G\epsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle H\epsilon} \\
G & \xrightarrow[\xi]{} & H
\end{array}
\qquad
\begin{array}{ccc}
GDI & \xrightarrow{\widetilde{\xi} DI} & HDI \\
{\scriptstyle G\epsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle H\epsilon} \\
G & \xrightarrow[\widetilde{\xi}]{} & H
\end{array}
\tag{3.10}
$$

where the vertical arrows are natural isomorphisms because the counit ϵ is an isomorphism. Recalling that by hypothesis $\xi D = \widetilde{\xi} D$, it follows that $\xi = \widetilde{\xi}$. Hence, the map (3.9) is injective.

It remains to prove that (3.9) is also surjective. Let $\chi : GD \to HD$ be any natural transformation. Using Lemma 3.2, we obtain a commutative diagram

$$
\begin{array}{ccc}
GD & \xrightarrow{\ \chi\ } & HD \\
{\scriptstyle GD\eta}\Big\downarrow & & \Big\downarrow{\scriptstyle HD\eta} \\
GDID & \xrightarrow[\ \chi ID\ ]{} & HDID
\end{array}
\tag{3.11}
$$

where the vertical arrows are natural isomorphisms because the components of the unit η are Cauchy morphisms and D assigns isomorphisms to them. Let us define a natural transformation $\xi : G \to H$ by the commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\ \xi\ } & H \\
{\scriptstyle G\epsilon^{-1}}\Big\downarrow & & \Big\uparrow{\scriptstyle H\epsilon} \\
GDI & \xrightarrow[\ \chi I\ ]{} & HDI
\end{array}
\tag{3.12}
$$

where we use that ϵ is a natural isomorphism (cf. Lemma 3.2). Combining the last two diagrams, one easily computes that $\xi D = \chi$ by using also the triangle identities of the adjunction $D \dashv I$. Hence, the map (3.9) is surjective. $\qquad\square$

We note that there exist two (a priori different) options to define an orthogonality relation on the localized category $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$, both of which are provided by [5, Lemma 4.29]: (1) The pullback orthogonality relation along the full subcategory embedding $I : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathcal{R}_M$ and (2) the pushforward orthogonality relation along the localization functor $D : \mathcal{R}_M \to \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$. In our present scenario, both constructions coincide and one concludes that two $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$-morphisms are orthogonal precisely when they are orthogonal in $\mathcal{R}_M$. Summing up, we obtain

Lemma 3.4. *We say that two morphisms in the full subcategory* $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \subseteq \mathcal{R}_M$ *are orthogonal precisely when they are orthogonal in* $\mathcal{R}_M$ *(i.e., causally disjoint, cf. Definition 2.4). Then, both functors* $D : \mathcal{R}_M \to \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ *and* $I : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathcal{R}_M$ *preserve (and also detect) the orthogonality relations.*

Proof. For I this holds trivially, while for D see Proposition 2.5(a). $\qquad\square$

With these preparations, we may now define our alternative description of the category of algebraic quantum field theories.

Definition 3.5. We denote by $\mathbf{QFT}(M) \subseteq \mathbf{Alg}^{\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]}$ the full subcategory whose objects are all functors $\mathfrak{B} : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathbf{Alg}$ that satisfy the following version of the *causality axiom*: For all causally disjoint inclusions $i_1 : V_1 \to V \leftarrow V_2 : i_2$ in $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$, i.e., V, V_1 and V_2 are stable under Cauchy development, the induced commutator

$$
\big[\mathfrak{B}(i_1)(-), \mathfrak{B}(i_2)(-)\big]_{\mathfrak{B}(V)} : \mathfrak{B}(V_1) \otimes \mathfrak{B}(V_2) \longrightarrow \mathfrak{B}(V)
\tag{3.13}
$$

is zero.

Theorem 3.6. *By pullback, the adjunction $D \dashv I$ of Lemma 3.2 induces an adjoint equivalence (cf. Definition A.2)*

$$I^* : \mathbf{qft}(M) \underset{\xrightarrow{\ \sim\ }}{\xleftarrow{\quad}} \mathbf{QFT}(M) : D^* . \tag{3.14}$$

In particular, the two categories $\mathbf{qft}(M)$ of Definition 3.1 and $\mathbf{QFT}(M)$ of Definition 3.5 are equivalent.

Proof. It is trivial to check that the adjunction $D : \mathcal{R}_M \rightleftarrows \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right] : I$ induces an adjunction

$$I^* : \mathbf{Alg}^{\mathcal{R}_M} \underset{\xleftarrow{\quad}}{\xrightarrow{\quad}} \mathbf{Alg}^{\mathcal{R}_M[\mathcal{C}_M^{-1}]} : D^* \tag{3.15}$$

between functor categories. Explicitly, the unit $\widetilde{\eta} : \mathrm{id}_{\mathbf{Alg}^{\mathcal{R}_M}} \to D^* I^*$ has components

$$\widetilde{\eta}_{\mathfrak{A}} := \mathfrak{A}\eta : \mathfrak{A} \longrightarrow D^*(I^*(\mathfrak{A})) = \mathfrak{A} I D, \tag{3.16}$$

where $\mathfrak{A} : \mathcal{R}_M \to \mathbf{Alg}$ is any functor and $\eta : \mathrm{id}_{\mathcal{R}_M} \to I D$ denotes the unit of $D \dashv I$. The counit $\widetilde{\epsilon} : I^* D^* \to \mathrm{id}_{\mathbf{Alg}^{\mathcal{R}_M[\mathcal{C}_M^{-1}]}}$ has components

$$\widetilde{\epsilon}_{\mathfrak{B}} := \mathfrak{B}\epsilon : I^*(D^*(\mathfrak{B})) = \mathfrak{B} D I \longrightarrow \mathfrak{B}, \tag{3.17}$$

where $\mathfrak{B} : \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right] \to \mathbf{Alg}$ is any functor and $\epsilon : D I \to \mathrm{id}_{\mathcal{R}_M[\mathcal{C}_M^{-1}]}$ denotes the counit of $D \dashv I$. The triangle identities for $I^* \dashv D^*$ follow directly from those of $D \dashv I$.

Next, we have to prove that this adjunction restricts to the claimed source and target categories in (3.14). Given $\mathfrak{A} \in \mathbf{qft}(M) \subseteq \mathbf{Alg}^{\mathcal{R}_M}$, the functor $I^*(\mathfrak{A}) = \mathfrak{A} I : \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right] \to \mathbf{Alg}$ satisfies the causality axiom of Definition 3.5 because of Lemma 3.4. Hence, $I^*(\mathfrak{A}) \in \mathbf{QFT}(M)$. Vice versa, given $\mathfrak{B} \in \mathbf{QFT}(M) \subseteq \mathbf{Alg}^{\mathcal{R}_M[\mathcal{C}_M^{-1}]}$, the functor $D^*(\mathfrak{B}) = \mathfrak{B} D : \mathcal{R}_M \to \mathbf{Alg}$ satisfies the causality axiom of Definition 3.5 because of Lemma 3.4 and the time-slice axiom of Definition 3.5 because D sends by construction morphisms in $\mathcal{C}_M$ to isomorphisms. Hence, $D^*(\mathfrak{B}) \in \mathbf{qft}(M)$.

Using Lemma 3.2, we obtain that the counit $\widetilde{\epsilon}$ of the restricted adjunction (3.14) is an isomorphism. Furthermore, all components of η are Cauchy morphisms, hence $\widetilde{\eta}_{\mathfrak{A}} = \mathfrak{A}\eta$ is an isomorphism for all $\mathfrak{A} \in \mathbf{qft}(M)$, i.e., the unit $\widetilde{\eta}$ is an isomorphism. This completes the proof that (3.14) is an adjoint equivalence. $\qquad\square$

Remark 3.7. Theorem 3.6 provides us with a constructive prescription of how to change between the two equivalent formulations of algebraic quantum field theories given in Definitions 3.1 and 3.5. Concretely, given any $\mathfrak{A} \in \mathbf{qft}(M)$, i.e., a functor $\mathfrak{A} : \mathcal{R}_M \to \mathbf{Alg}$ satisfying the causality and time-slice axioms as in Definition 3.1, the corresponding quantum field theory $I^*(\mathfrak{A}) \in \mathbf{QFT}(M)$ in the sense of Definition 3.5 reads as follows: It is the functor $I^*(\mathfrak{A}) = \mathfrak{A} I : \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right] \to \mathbf{Alg}$ on the category of regions $V \subseteq M$ that are stable under Cauchy development, which assigns to $V \subseteq M$ the algebra $\mathfrak{A}(V) \in \mathbf{Alg}$ and to an inclusion $i : V \to V'$ the algebra map

$\mathfrak{A}(i) : \mathfrak{A}(V) \to \mathfrak{A}(V')$. More interestingly, given $\mathfrak{B} \in \mathbf{QFT}(M)$, i.e., a functor $\mathfrak{B} : \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right] \to \mathbf{Alg}$ satisfying the causality axiom as in Definition 3.5, the corresponding quantum field theory $D^*(\mathfrak{B}) \in \mathbf{qft}(M)$ in the sense of Definition 3.1 reads as follows: It is the functor $D^*(\mathfrak{B}) = \mathfrak{B}\, D : \mathcal{R}_M \to \mathbf{Alg}$ defined on the category of (not necessarily Cauchy development stable) regions $U \subseteq M$, which assigns to $U \subseteq M$ the algebra $\mathfrak{B}(D(U))$ corresponding to the Cauchy development of U and to an inclusion $i : U \to U'$ the algebra map $\mathfrak{B}(D(i)) : \mathfrak{B}(D(U)) \to \mathfrak{B}(D(U'))$ associated with the inclusion $D(i) : D(U) \to D(U')$ of Cauchy developments.

Remark 3.8. It is straightforward to check that the results of this section still hold true when one replaces $\mathcal{R}_M$ with its full subcategory $\mathcal{R}_{\mathrm{int}\,M}$ of regions contained in the interior of M and $\mathcal{C}_M$ with $\mathcal{C}_{\mathrm{int}\,M}$ (cf. Definition 2.7). This follows from the observation that the Cauchy development of a subset of the interior of M is also contained in $\mathrm{int}\,M$, as shown in Proposition 2.8(c). We denote by

$$\mathbf{QFT}(\mathrm{int}\,M) \subseteq \mathbf{Alg}^{\mathcal{R}_{\mathrm{int}\,M}\left[\mathcal{C}_{\mathrm{int}\,M}^{-1}\right]} \tag{3.18}$$

the category of algebraic quantum field theories in the sense of Definition 3.5 on the interior regions of M. Concretely, an object $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\,M)$ is a functor $\mathfrak{A} : \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}] \to \mathbf{Alg}$ that satisfies the causality axiom of Definition 3.5 for causally disjoint interior regions.

4. Universal Boundary Extension

The goal of this section is to develop a universal construction to extend quantum field theories from the interior of a spacetime M with timelike boundary to the whole spacetime. (Again, we do not have to assume that M is globally hyperbolic in the sense of Definition 2.9). Loosely speaking, our extended quantum field theory will have the following pleasant properties: (1) It describes precisely those observables that are generated from the original theory on the interior, (2) it does not require a choice of boundary conditions, (3) specific choices of boundary conditions correspond to ideals of our extended quantum field theory. We also refer to Sect. 5 for more details on the properties (1) and (3).

The starting point for this construction is the full subcategory inclusion $\mathcal{R}_{\mathrm{int}\,M} \subseteq \mathcal{R}_M$ defined by selecting only the regions of $\mathcal{R}_M$ that lie in the interior of M (cf. Definition 2.7). We denote the corresponding embedding functor by

$$j : \mathcal{R}_{\mathrm{int}\,M} \longrightarrow \mathcal{R}_M \tag{4.1}$$

and notice that j preserves (and also detects) causally disjoint inclusions, i.e., j is a full orthogonal subcategory embedding in the terminology of [5]. Making use of Proposition 3.3, Lemma 3.2 and Remark 3.8, we define a functor $J : \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}] \to \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$ on the localized categories via the commutative

diagram

$$
\begin{array}{ccc}
\mathcal{R}_{\text{int } M}[\mathcal{C}^{-1}_{\text{int } M}] & \xrightarrow{\ J\ } & \mathcal{R}_M[\mathcal{C}^{-1}_M] \\
{\scriptstyle I}\Big\downarrow & & \Big\uparrow{\scriptstyle D} \\
\mathcal{R}_{\text{int } M} & \xrightarrow[\ j\ ]{} & \mathcal{R}_M
\end{array}
\tag{4.2a}
$$

Notice that J is simply an embedding functor, which acts on objects and morphisms as

$$
V \subseteq \text{int } M \longmapsto V \subseteq M, \quad (i : V \to V') \longmapsto (i : V \to V'). \tag{4.2b}
$$

From this explicit description, it is clear that J preserves (and also detects) causally disjoint inclusions, i.e., it is a full orthogonal subcategory embedding. The constructions in [5, Section 5.3] (see also [6] for details how to treat $*$-algebras) then imply that J induces an adjunction

$$
\text{ext} : \mathbf{QFT}(\text{int } M) \rightleftarrows \mathbf{QFT}(M) : \text{res} \tag{4.3}
$$

between the category of quantum field theories on the interior $\text{int } M$ (cf. Remark 3.8) and the category of quantum field theories on the whole spacetime M. The right adjoint $\text{res} := J^* : \mathbf{QFT}(M) \to \mathbf{QFT}(\text{int } M)$ is the pullback along $J : \mathcal{R}_{\text{int } M}[\mathcal{C}^{-1}_{\text{int } M}] \to \mathcal{R}_M[\mathcal{C}^{-1}_M]$, i.e., it restricts quantum field theories defined on M to the interior $\text{int } M$. The left adjoint $\text{ext} : \mathbf{QFT}(\text{int } M) \to \mathbf{QFT}(M)$ should be regarded as a universal extension functor which extends quantum field theories on the interior $\text{int } M$ to the whole spacetime M. The goal of this section is to analyze the properties of this extension functor and to develop an explicit model that allows us to do computations in the sections below.

An important structural result, whose physical relevance is explained in Remark 4.2 below, is the following proposition.

Proposition 4.1. *The unit*

$$
\eta : \text{id}_{\mathbf{QFT}(\text{int } M)} \longrightarrow \text{res ext} \tag{4.4}
$$

of the adjunction (4.3) *is a natural isomorphism.*

Proof. This is a direct consequence of the fact that the functor J given in (4.2) is a full orthogonal subcategory embedding and the general result in [5, Proposition 5.6]. $\qquad\square$

Remark 4.2. The physical interpretation of this result is as follows: Let $\mathfrak{A} \in \mathbf{QFT}(\text{int } M)$ be a quantum field theory defined only on the interior $\text{int } M$ of M and let $\mathfrak{B} := \text{ext } \mathfrak{A} \in \mathbf{QFT}(M)$ denote its universal extension to the whole spacetime M. The $\mathfrak{A}$-component

$$
\eta_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \text{res ext } \mathfrak{A} \tag{4.5}
$$

of the unit of the adjunction (4.3) allows us to compare $\mathfrak{A}$ with the restriction res $\mathfrak{B}$ of its extension $\mathfrak{B} = \text{ext } \mathfrak{A}$. Since $\eta_{\mathfrak{A}}$ is an isomorphism by Proposition 4.1, restricting the extension $\mathfrak{B}$ recovers our original theory $\mathfrak{A}$ up to

isomorphism. This allows us to interpret the left adjoint ext : $\mathbf{QFT}(\mathrm{int}\,M) \to$ $\mathbf{QFT}(M)$ as a genuine extension prescription. Notice that this also proves that the universal extension ext $\mathfrak{A} \in \mathbf{QFT}(M)$ of any theory $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\,M)$ on the interior satisfies F-locality [22].

We next address the question how to *compute* the extension functor ext : $\mathbf{QFT}(\mathrm{int}\,M) \to \mathbf{QFT}(M)$ explicitly. A crucial step toward reaching this goal is to notice that ext may be computed by a left Kan extension.

Proposition 4.3. *Consider the adjunction*

$$\mathrm{Lan}_J : \mathbf{Alg}^{\mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]} \underset{}{\overset{}{\rightleftarrows}} \mathbf{Alg}^{\mathcal{R}_M[\mathcal{C}_M^{-1}]} : J^*, \qquad (4.6)$$

corresponding to left Kan extension along the functor $J : \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}] \to \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$. Then, the restriction of Lan_J to $\mathbf{QFT}(\mathrm{int}\,M)$ induces a functor

$$\mathrm{Lan}_J : \mathbf{QFT}(\mathrm{int}\,M) \longrightarrow \mathbf{QFT}(M) \qquad (4.7)$$

that is left adjoint to the restriction functor res : $\mathbf{QFT}(M) \to \mathbf{QFT}(\mathrm{int}\,M)$ in (4.3). Due to uniqueness (up to unique natural isomorphism) of adjoint functors (cf. Proposition A.3), it follows that ext $\cong \mathrm{Lan}_J$, i.e., (4.7) is a model for the extension functor ext in (4.3).

Proof. A general version of this problem has been addressed in [5, Section 6]. Using in particular [5, Corollary 6.5], we observe that we can prove this proposition by showing that every object $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ is J-closed in the sense of [5, Definition 6.3]. In our present scenario, this amounts to proving that for all causally disjoint inclusions $i_1 : V_1 \to V \leftarrow V_2 : i_2$ with $V_1, V_2 \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ in the interior and $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ not necessarily in the interior, there exists a factorization of both i_1 and i_2 through a common interior region. Let us consider the Cauchy development $D(V_1 \sqcup V_2)$ of the disjoint union and the canonical inclusions $j_1 : V_1 \to D(V_1 \sqcup V_2) \leftarrow V_2 : j_2$. As we explain below, $D(V_1 \sqcup V_2) \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ is an interior region and j_1, j_2 provide the desired factorization: Since the open set $V_1 \sqcup V_2 \subseteq \mathrm{int}\,M$ is causally convex by Proposition 2.5(b), $D(V_1 \sqcup V_2)$ is causally convex, open and contained in the interior $\mathrm{int}\,M$ by Proposition 2.8(c–d). It is, moreover, stable under Cauchy development by Proposition 2.3(b), which also provides the inclusion $V_k \subseteq V_1 \sqcup V_2 \subseteq D(V_1 \sqcup V_2)$ inducing j_k, for $k = 1, 2$. Consider now the chain of inclusions $V_k \subseteq V_1 \sqcup V_2 \subseteq V$ corresponding to i_k, for $k = 1, 2$. From the stability under Cauchy development of V_1, V_2 and V, we obtain also the chain of inclusions $V_k \subseteq D(V_1 \sqcup V_2) \subseteq V$, for $k = 1, 2$, that exhibits the desired factorization

$$\begin{array}{ccc} & V & \\ {\scriptstyle i_1}\nearrow & {\scriptstyle \uparrow} & \nwarrow{\scriptstyle i_2} \\ & {\scriptstyle D(i_1 \sqcup i_2)}\Big\vert & \\ V_1 \underset{j_1}{\dashrightarrow} & D(V_1 \sqcup V_2) & \underset{j_2}{\dashleftarrow} V_2 \end{array} \qquad (4.8)$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We shall now briefly review a concrete model for left Kan extension along full subcategory embeddings that was developed in [5, Section 6]. This model is obtained by means of abstract operadic techniques, but it admits an intuitive graphical interpretation that we explain in Remark 4.4 below. It allows us to compute quite explicitly the extension $\operatorname{ext}\mathfrak{A} = \operatorname{Lan}_J\mathfrak{A} \in \mathbf{QFT}(M)$ of a quantum field theory $\mathfrak{A} \in \mathbf{QFT}(\operatorname{int}M)$ defined on the interior $\operatorname{int}M$ to the whole spacetime M. The functor $\operatorname{ext}\mathfrak{A} : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathbf{Alg}$ describing the extended quantum field theory reads as follows: To $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ it assigns a quotient algebra

$$\operatorname{ext}\mathfrak{A}(V) = \bigoplus_{\underline{i}:\underline{V}\to V} \mathfrak{A}(\underline{V}) \Big/ \sim \tag{4.9}$$

that we will describe now in detail. The direct sum (of vector spaces) in (4.9) runs over all tuples $\underline{i} : \underline{V} \to V$ of morphisms in $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$, i.e., $\underline{i} = (i_1 : V_1 \to V, \ldots, i_n : V_n \to V)$ for some $n \in \mathbb{Z}_{\geq 0}$, with the requirement that all sources $V_k \in \mathcal{R}_{\operatorname{int}M}[\mathcal{C}_{\operatorname{int}M}^{-1}]$ are interior regions. (Notice that the regions V_k are not assumed to be causally disjoint and that the empty tuple, i.e., $n = 0$, is also allowed). The vector space $\mathfrak{A}(\underline{V})$ is defined by the tensor product of vector spaces

$$\mathfrak{A}(\underline{V}) := \bigotimes_{k=1}^{|\underline{V}|} \mathfrak{A}(V_k), \tag{4.10}$$

where $|\underline{V}|$ is the length of the tuple. (For the empty tuple, we set $\mathfrak{A}(\emptyset) = \mathbb{C}$). This means that the (homogeneous) elements

$$(\underline{i}, \underline{a}) \in \bigoplus_{\underline{i}:\underline{V}\to V} \mathfrak{A}(\underline{V}) \tag{4.11}$$

are given by pairs consisting of a tuple of morphisms $\underline{i} : \underline{V} \to V$ (with all V_k in the interior) and an element $\underline{a} \in \mathfrak{A}(\underline{V})$ of the corresponding tensor product vector space (4.10). The product on (4.11) is given on homogeneous elements by

$$(\underline{i}, \underline{a})\,(\underline{i}', \underline{a}') := \big((\underline{i}, \underline{i}'), \underline{a} \otimes \underline{a}'\big), \tag{4.12}$$

where $(\underline{i}, \underline{i}') = (i_1, \ldots, i_n, i_1', \ldots, i_m')$ is the concatenation of tuples. The unit element in (4.11) is $\mathbb{1} := (\emptyset, 1)$, where $\emptyset$ is the empty tuple and $1 \in \mathbb{C}$, and the $*$-involution is defined by

$$\big((i_1, \ldots, i_n), a_1 \otimes \cdots \otimes a_n\big)^* := \big((i_n, \ldots, i_1), a_n^* \otimes \cdots \otimes a_1^*\big) \tag{4.13}$$

and $\mathbb{C}$-antilinear extension. Finally, the quotient in (4.9) is by the two-sided $*$-ideal of the algebra (4.11) that is generated by

$$\Big(\underline{i}(\underline{i}_1, \ldots, \underline{i}_n), \underline{a}_1 \otimes \cdots \otimes \underline{a}_n\Big) - \Big(\underline{i}, \mathfrak{A}(\underline{i}_1)(\underline{a}_1) \otimes \cdots \otimes \mathfrak{A}(\underline{i}_n)(\underline{a}_n)\Big) \in \bigoplus_{\underline{i}:\underline{V}\to V} \mathfrak{A}(\underline{V}), \tag{4.14}$$

for all tuples $\underline{i} : \underline{V} \to V$ of length $|\underline{V}| = n \geq 1$ (with all V_k in the interior), all tuples $\underline{i}_k : \underline{V}_k \to V_k$ of $\mathcal{R}_{\operatorname{int}M}[\mathcal{C}_{\operatorname{int}M}^{-1}]$-morphisms (possibly of length zero), for

$k = 1, \ldots, n$, and all $\underline{a}_k \in \mathfrak{A}(\underline{V}_k)$, for $k = 1, \ldots, n$. The tuple in the first term of (4.14) is defined by composition

$$\underline{i}(\underline{i}_1, \ldots, \underline{i}_n) := \left(i_1\, i_{11}, \ldots, i_1\, i_{1|\underline{V}_1|}, \ldots, i_n\, i_{n1}, \ldots, i_n\, i_{n|\underline{V}_n|} \right) \qquad (4.15a)$$

and the expressions $\mathfrak{A}(\underline{i})(\underline{a})$ in the second term are determined by

$$\mathfrak{A}(\underline{i}) : \mathfrak{A}(\underline{V}) \longrightarrow \mathfrak{A}(V), \quad a_1 \otimes \cdots \otimes a_n \longmapsto \mathfrak{A}(i_1)(a_1) \cdots \mathfrak{A}(i_n)(a_n), \quad (4.15b)$$

where multiplication in $\mathfrak{A}(V)$ is denoted by juxtaposition. To a morphism $i' : V \to V'$ in $\mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$, the functor $\mathrm{ext}\,\mathfrak{A} : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathbf{Alg}$ assigns the algebra map

$$\mathrm{ext}\,\mathfrak{A}(i') : \mathrm{ext}\,\mathfrak{A}(V) \longrightarrow \mathrm{ext}\,\mathfrak{A}(V'), \quad [\underline{i}, \underline{a}] \longmapsto [i'(\underline{i}), \underline{a}], \qquad (4.16)$$

where we used square brackets to indicate equivalence classes in (4.9).

Remark 4.4. The construction of the algebra $\mathrm{ext}\,\mathfrak{A}(V)$ above admits an intuitive graphical interpretation: We shall visualize the (homogeneous) elements $(\underline{i}, \underline{a})$ in (4.11) by decorated trees

$$(4.17)$$

where $a_k \in \mathfrak{A}(V_k)$ is an element of the algebra $\mathfrak{A}(V_k)$ associated to the interior region $V_k \subseteq V$, for all $k = 1, \ldots, n$. We interpret such a decorated tree as a formal product of the formal pushforward along $\underline{i} : \underline{V} \to V$ of the family of observables $a_k \in \mathfrak{A}(V_k)$. The product (4.12) is given by concatenation of the inputs of the individual decorated trees, i.e.,

$$(4.18)$$
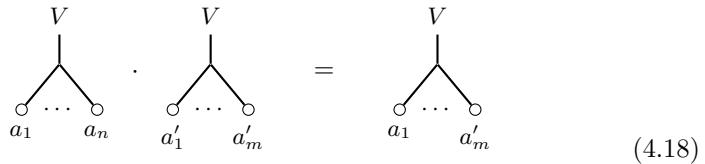
where the decorated tree on the right-hand side has $n + m$ inputs. The $*$-involution (4.13) may be visualized by reversing the input profile and applying $*$ to each element $a_k \in \mathfrak{A}(V_k)$. Finally, the $*$-ideal in (4.14) implements the following relations: Assume that $(\underline{i}, \underline{a})$ is such that the sub-family of embeddings $(i_k, i_{k+1}, \ldots, i_l) : (V_k, V_{k+1}, \ldots, V_l) \to V$ factorizes through some common interior region, say $V' \subseteq V$. Using the original functor $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\,M)$, we may form the product $\mathfrak{A}(i_k)(a_k) \cdots \mathfrak{A}(i_l)(a_l)$ in the algebra $\mathfrak{A}(V')$, which we

denote for simplicity by $a_k \cdots a_l \in \mathfrak{A}(V')$. We then have the relation

$$\sim \tag{4.19}$$

which we interpret as follows: Whenever $(i_k, i_{k+1}, \ldots, i_l) : (V_k, V_{k+1}, \ldots, V_l) \to V$ is a sub-family of embeddings that factorizes through a common interior region $V' \subseteq V$, then the formal product of the formal pushforward of observables is identified with the formal pushforward of the actual product of observables on V'.

5. Characterization of Boundary Quantum Field Theories

In the previous section, we established a universal construction that allows us to extend quantum field theories $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\, M)$ that are defined only on the interior $\mathrm{int}\, M$ of a spacetime M with timelike boundary to the whole spacetime. The extension $\mathrm{ext}\, \mathfrak{A} \in \mathbf{QFT}(M)$ is characterized abstractly by the $\mathrm{ext} \dashv \mathrm{res}$ adjunction in (4.3). We now shall reverse the question and ask which quantum field theories $\mathfrak{B} \in \mathbf{QFT}(M)$ on M admit a description in terms of (quotients of) our universal extensions.

Given any quantum field theory $\mathfrak{B} \in \mathbf{QFT}(M)$ on the whole spacetime M, we can use the right adjoint in (4.3) in order to restrict it to a theory $\mathrm{res}\, \mathfrak{B} \in \mathbf{QFT}(\mathrm{int}\, M)$ on the interior of M. Applying now the extension functor, we obtain another quantum field theory $\mathrm{ext}\, \mathrm{res}\, \mathfrak{B} \in \mathbf{QFT}(M)$ on the whole spacetime M, which we would like to compare to our original theory $\mathfrak{B} \in \mathbf{QFT}(M)$. A natural comparison map is given by the $\mathfrak{B}$-component of the counit $\epsilon : \mathrm{ext}\, \mathrm{res} \to \mathrm{id}_{\mathbf{QFT}(M)}$ of the adjunction (4.3), i.e., the canonical $\mathbf{QFT}(M)$-morphism

$$\epsilon_{\mathfrak{B}} \,:\, \mathrm{ext}\, \mathrm{res}\, \mathfrak{B} \longrightarrow \mathfrak{B}. \tag{5.1a}$$

Using our model for the extension functor given in (4.9) (and the formulas following this equation), the $\mathcal{R}_M\big[\mathcal{C}_M^{-1}\big] \ni V$-component of $\epsilon_{\mathfrak{B}}$ explicitly reads as

$$(\epsilon_{\mathfrak{B}})_V \,:\, \bigoplus_{\underline{i}:\underline{V}\to V} \mathfrak{B}(\underline{V})\Big/\!\sim \,\longrightarrow\, \mathfrak{B}(V), \quad [\underline{i}, \underline{b}] \longmapsto \mathfrak{B}(\underline{i})(\underline{b}). \tag{5.1b}$$

In order to establish positive comparison results, we have to introduce the concept of ideals of quantum field theories.

Definition 5.1. An *ideal* $\mathfrak{I} \subseteq \mathfrak{B}$ of a quantum field theory $\mathfrak{B} \in \mathbf{QFT}(M)$ is a functor $\mathfrak{I} : \mathcal{R}_M\big[\mathcal{C}_M^{-1}\big] \to \mathbf{Vec}$ to the category of complex vector spaces, which satisfies the following properties:

(i) For all $V \in \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$, $\mathfrak{I}(V) \subseteq \mathfrak{B}(V)$ is a two-sided $*$-ideal of the unital $*$-algebra $\mathfrak{B}(V)$.

(ii) For all $\mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$-morphisms $i : V \to V'$, the linear map $\mathfrak{I}(i) : \mathfrak{I}(V) \to \mathfrak{I}(V')$ is the restriction of $\mathfrak{B}(i) : \mathfrak{B}(V) \to \mathfrak{B}(V')$ to the two-sided $*$-ideals $\mathfrak{I}(V^{(\prime)}) \subseteq \mathfrak{B}(V^{(\prime)})$.

Lemma 5.2. *Let* $\mathfrak{B} \in \mathbf{QFT}(M)$ *and* $\mathfrak{I} \subseteq \mathfrak{B}$ *any ideal. Let us define* $\mathfrak{B}/\mathfrak{I}(V) := \mathfrak{B}(V)/\mathfrak{I}(V)$ *to be the quotient algebra, for all* $V \in \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$, *and* $\mathfrak{B}/\mathfrak{I}(i) : \mathfrak{B}/\mathfrak{I}(V) \to \mathfrak{B}/\mathfrak{I}(V')$ *to be the* $\mathbf{Alg}$*-morphism induced by* $\mathfrak{B}(i) : \mathfrak{B}(V) \to \mathfrak{B}(V')$, *for all* $\mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$*-morphisms* $i : V \to V'$. *Then* $\mathfrak{B}/\mathfrak{I} \in \mathbf{QFT}(M)$ *is a quantum field theory on* M *which we call the quotient of* $\mathfrak{B}$ *by* $\mathfrak{I}$.

Proof. The requirements listed in Definition 5.1 ensure that $\mathfrak{B}/\mathfrak{I} : \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right] \to \mathbf{Alg}$ is an $\mathbf{Alg}$-valued functor. It satisfies the causality axiom of Definition 3.5 because this property is inherited from $\mathfrak{B} \in \mathbf{QFT}(M)$ by taking quotients. $\qquad\square$

Lemma 5.3. *Let* $\kappa : \mathfrak{B} \to \mathfrak{B}'$ *be any* $\mathbf{QFT}(M)$*-morphism. Define the vector space* $\ker \kappa(V) := \ker \left(\kappa_V : \mathfrak{B}(V) \to \mathfrak{B}'(V) \right) \subseteq \mathfrak{B}(V)$, *for all* $V \in \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$, *and* $\ker \kappa(i) : \ker \kappa(V) \to \ker \kappa(V')$ *to be the linear map induced by* $\mathfrak{B}(i) : \mathfrak{B}(V) \to \mathfrak{B}(V')$, *for all* $\mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$*-morphisms* $i : V \to V'$. *Then* $\ker \kappa : \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right] \to \mathbf{Vec}$ *is an ideal of* $\mathfrak{B} \in \mathbf{QFT}(M)$, *which we call the kernel of* κ.

Proof. The fact that $\ker \kappa$ defines a functor follows from naturality of κ. Property (ii) in Definition 5.1 holds true by construction. Property (i) is a consequence of the fact that kernels of unital $*$-algebra morphisms $\kappa_V : \mathfrak{B}(V) \to \mathfrak{B}(V')$ are two-sided $*$-ideals. $\qquad\square$

Remark 5.4. Using the concept of ideals, we may canonically factorize (5.1) according to the diagram

$$
\begin{array}{ccc}
\text{ext res } \mathfrak{B} & \xrightarrow{\ \ \epsilon_{\mathfrak{B}}\ \ } & \mathfrak{B} \\
& \searrow_{\pi_{\mathfrak{B}}} \quad \nearrow_{\lambda_{\mathfrak{B}}} & \\
& \text{ext res } \mathfrak{B} \big/ \ker \epsilon_{\mathfrak{B}} &
\end{array}
\tag{5.2}
$$

where both the projection $\pi_{\mathfrak{B}}$ and the inclusion $\lambda_{\mathfrak{B}}$ are $\mathbf{QFT}(M)$-morphisms. $\qquad\square$

As a last ingredient for our comparison result, we have to introduce a suitable notion of additivity for quantum field theories on spacetimes with timelike boundary. We refer to [14, Definition 2.3] for a notion of additivity on globally hyperbolic spacetimes.

Definition 5.5. A quantum field theory $\mathfrak{B} \in \mathbf{QFT}(M)$ on a spacetime M with timelike boundary is called *additive (from the interior) at the object* $V \in \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$ if the algebra $\mathfrak{B}(V)$ is generated by the images of the $\mathbf{Alg}$-morphisms $\mathfrak{B}(i_{\mathrm{int}}) : \mathfrak{B}(V_{\mathrm{int}}) \to \mathfrak{B}(V)$, for all $\mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$-morphism $i_{\mathrm{int}} : V_{\mathrm{int}} \to V$ whose source $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M} [\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ is in the interior $\mathrm{int}\,M$ of M. We call $\mathfrak{B} \in \mathbf{QFT}(M)$ *additive (from the interior)* if it is additive at every object

$V \in \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$. The full subcategory of additive quantum field theories on M is denoted by $\mathbf{QFT}^{\mathrm{add}}(M) \subseteq \mathbf{QFT}(M)$.

We can now prove our first characterization theorem for boundary quantum field theories.

Theorem 5.6. *Let* $\mathfrak{B} \in \mathbf{QFT}(M)$ *be any quantum field theory on a (not necessarily globally hyperbolic) spacetime M with timelike boundary and let* $V \in \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$. *Then the following are equivalent:*

(1) *The V-component*

$$(\lambda_{\mathfrak{B}})_V \, : \, \mathrm{ext\,res}\,\mathfrak{B}(V) \big/ \ker \epsilon_{\mathfrak{B}}(V) \, \longrightarrow \, \mathfrak{B}(V) \tag{5.3}$$

of the canonical inclusion in (5.2) *is an* **Alg**-*isomorphism.*

(2) $\mathfrak{B}$ *is additive at the object* $V \in \mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$.

Proof. Let $i_{\mathrm{int}} : V_{\mathrm{int}} \to V$ be any $\mathcal{R}_M \left[\mathcal{C}_M^{-1} \right]$-morphism whose source $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ is in the interior $\mathrm{int}\,M$ of M. Using our model for the extension functor given in (4.9) (and the formulas following this equation), we obtain an **Alg**-morphism

$$[i_{\mathrm{int}}, -] \, : \, \mathfrak{B}(V_{\mathrm{int}}) \longrightarrow \mathrm{ext\,res}\,\mathfrak{B}(V), \quad b \longmapsto [i_{\mathrm{int}}, b]. \tag{5.4}$$

Composing this morphism with the V-component of $\epsilon_{\mathfrak{B}}$ given in (5.1), we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{ext\,res}\,\mathfrak{B}(V) & \xrightarrow{\quad(\epsilon_{\mathfrak{B}})_V\quad} & \mathfrak{B}(V) \\ & {\scriptstyle[i_{\mathrm{int}},-]}\nwarrow \quad \nearrow {\scriptstyle\mathfrak{B}(i_{\mathrm{int}})} & \\ & \mathfrak{B}(V_{\mathrm{int}}) & \end{array} \tag{5.5}$$

for all $i_{\mathrm{int}} : V_{\mathrm{int}} \to V$ with V_{int} in the interior.

Next we observe that the images of the **Alg**-morphisms (5.4), for all $i_{\mathrm{int}} : V_{\mathrm{int}} \to V$ with V_{int} in the interior, generate $\mathrm{ext\,res}\,\mathfrak{B}(V)$. Combining this property with (5.5), we conclude that $(\epsilon_{\mathfrak{B}})_V$ is a surjective map if and only if $\mathfrak{B}$ is additive at V. Hence, $(\lambda_{\mathfrak{B}})_V$ given by (5.2), which is injective by construction, is an **Alg**-isomorphism if and only if $\mathfrak{B}$ is additive at V. $\qquad \square$

Corollary 5.7. $\lambda_{\mathfrak{B}} \, : \, \mathrm{ext\,res}\,\mathfrak{B} \big/ \ker \epsilon_{\mathfrak{B}} \to \mathfrak{B}$ *given by* (5.2) *is a* $\mathbf{QFT}(M)$-*isomorphism if and only if* $\mathfrak{B} \in \mathbf{QFT}^{\mathrm{add}}(M) \subseteq \mathbf{QFT}(M)$ *is additive in the sense of Definition* 5.5.

We shall now refine this characterization theorem by showing that $\mathbf{QFT}^{\mathrm{add}}(M)$ is equivalent, as a category, to a category describing quantum field theories on the interior of M together with suitable ideals of their universal extensions. The precise definitions are as follows.

Definition 5.8. Let $\mathfrak{B} \in \mathbf{QFT}(M)$. An ideal $\mathfrak{I} \subseteq \mathfrak{B}$ is called *trivial on the interior* if its restriction to $\mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ is the functor assigning zero vector spaces, i.e., $\mathfrak{I}(V_{\mathrm{int}}) = 0$ for all $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ in the interior $\mathrm{int}\,M$ of M.

Definition 5.9. Let M be a spacetime with timelike boundary. We define the category $\mathbf{IQFT}(M)$ as follows:

- Objects are pairs $(\mathfrak{A}, \mathfrak{I})$ consisting of a quantum field theory $\mathfrak{A} \in \mathbf{QFT}(\operatorname{int} M)$ on the interior $\operatorname{int} M$ of M and an ideal $\mathfrak{I} \subseteq \operatorname{ext} \mathfrak{A}$ of its universal extension $\operatorname{ext} \mathfrak{A} \in \mathbf{QFT}(M)$ that is trivial on the interior.
- Morphisms $\kappa : (\mathfrak{A}, \mathfrak{I}) \to (\mathfrak{A}', \mathfrak{I}')$ are $\mathbf{QFT}(M)$-morphisms $\kappa : \operatorname{ext} \mathfrak{A} \to \operatorname{ext} \mathfrak{A}'$ between the universal extensions that preserve the ideals, i.e., κ restricts to a natural transformation from $\mathfrak{I} \subseteq \operatorname{ext} \mathfrak{A}$ to $\mathfrak{I}' \subseteq \operatorname{ext} \mathfrak{A}'$.

There exists an obvious functor

$$Q : \mathbf{IQFT}(M) \longrightarrow \mathbf{QFT}^{\mathrm{add}}(M), \tag{5.6a}$$

which assigns to $(\mathfrak{A}, \mathfrak{I}) \in \mathbf{IQFT}(M)$ the quotient

$$Q(\mathfrak{A}, \mathfrak{I}) := \operatorname{ext} \mathfrak{A} / \mathfrak{I} \in \mathbf{QFT}^{\mathrm{add}}(M). \tag{5.6b}$$

Notice that additivity of $\operatorname{ext} \mathfrak{A} / \mathfrak{I}$ follows from that of the universal extension $\operatorname{ext} \mathfrak{A}$ (cf. the arguments in the proof of Theorem 5.6) and the fact that quotients preserve the additivity property. There exists also a functor

$$S : \mathbf{QFT}^{\mathrm{add}}(M) \longrightarrow \mathbf{IQFT}(M), \tag{5.7a}$$

which 'extracts' from a quantum field theory on M the relevant ideal. Explicitly, it assigns to $\mathfrak{B} \in \mathbf{QFT}^{\mathrm{add}}(M)$ the pair

$$S\mathfrak{B} := \big(\operatorname{res} \mathfrak{B}, \ker \epsilon_{\mathfrak{B}}\big). \tag{5.7b}$$

Notice that the ideal $\ker \epsilon_{\mathfrak{B}} \subseteq \operatorname{ext} \operatorname{res} \mathfrak{B}$ is trivial on the interior: Applying the restriction functor (4.3) to $\epsilon_{\mathfrak{B}}$, we obtain a $\mathbf{QFT}(\operatorname{int} M)$-morphism

$$\operatorname{res} \epsilon_{\mathfrak{B}} : \operatorname{res} \operatorname{ext} \operatorname{res} \mathfrak{B} \longrightarrow \operatorname{res} \mathfrak{B}. \tag{5.8}$$

Proposition 4.1 together with the triangle identities for the adjunction $\operatorname{ext} \dashv \operatorname{res}$ in (4.3) then imply that (5.8) is an isomorphism with inverse given by $\eta_{\operatorname{res} \mathfrak{B}} : \operatorname{res} \mathfrak{B} \to \operatorname{res} \operatorname{ext} \operatorname{res} \mathfrak{B}$. In particular, $\operatorname{res} \epsilon_{\mathfrak{B}}$ has a trivial kernel and hence $\ker \epsilon_{\mathfrak{B}} \subseteq \operatorname{ext} \operatorname{res} \mathfrak{B}$ is trivial on the interior (cf. Definition 5.8). Our refined characterization theorem for boundary quantum field theories is as follows.

Theorem 5.10. *The functors Q and S defined in (5.6) and (5.7) exhibit an equivalence of categories*

$$\mathbf{QFT}^{\mathrm{add}}(M) \cong \mathbf{IQFT}(M). \tag{5.9}$$

Proof. We first consider the composition of functors $Q\, S : \mathbf{QFT}^{\mathrm{add}}(M) \to \mathbf{QFT}^{\mathrm{add}}(M)$. To $\mathfrak{B} \in \mathbf{QFT}^{\mathrm{add}}(M)$, it assigns

$$QS\mathfrak{B} = \operatorname{ext} \operatorname{res} \mathfrak{B} / \ker \epsilon_{\mathfrak{B}}. \tag{5.10}$$

The $\mathbf{QFT}(M)$-morphisms $\lambda_{\mathfrak{B}} : \operatorname{ext} \operatorname{res} \mathfrak{B} / \ker \epsilon_{\mathfrak{B}} \to \mathfrak{B}$ given by (5.2) define a natural transformation $\lambda : Q\, S \to \operatorname{id}_{\mathbf{QFT}^{\mathrm{add}}(M)}$, which is a natural isomorphism due to Corollary 5.7.

Let us now consider the composition of functors $SQ : \mathbf{IQFT}(M) \to \mathbf{IQFT}(M)$. To $(\mathfrak{A}, \mathfrak{I}) \in \mathbf{IQFT}(M)$, it assigns

$$SQ(\mathfrak{A}, \mathfrak{I}) = \Big(\mathrm{res}\,\big(\mathrm{ext}\,\mathfrak{A}/\mathfrak{I}\big), \ker \epsilon_{\mathrm{ext}\,\mathfrak{A}/\mathfrak{I}} \Big) = \Big(\mathrm{res}\,\mathrm{ext}\,\mathfrak{A}, \ker \epsilon_{\mathrm{ext}\,\mathfrak{A}/\mathfrak{I}} \Big), \quad (5.11)$$

where we also used that $\mathrm{res}\,\big(\mathrm{ext}\,\mathfrak{A}/\mathfrak{I}\big) = \mathrm{res}\,\mathrm{ext}\,\mathfrak{A}$ because $\mathfrak{I}$ is by hypothesis trivial on the interior. Using further the $\mathbf{QFT}(\mathrm{int}\,M)$-isomorphism $\eta_{\mathfrak{A}} : \mathfrak{A} \to \mathrm{res}\,\mathrm{ext}\,\mathfrak{A}$ from Proposition 4.1, we define a $\mathbf{QFT}(M)$-morphism $q_{(\mathfrak{A},\mathfrak{I})}$ via the diagram

$$(5.12)$$

Using the explicit expression for $\epsilon_{\mathrm{ext}\,\mathfrak{A}/\mathfrak{I}}$ given in (5.1) and the explicit expression for $\eta_{\mathfrak{A}}$ given by

$$(\eta_{\mathfrak{A}})_{V_{\mathrm{int}}} : \mathfrak{A}(V_{\mathrm{int}}) \longrightarrow \mathrm{res}\,\mathrm{ext}\,\mathfrak{A}(V_{\mathrm{int}}) = \bigoplus_{i:\underline{V}\to V_{\mathrm{int}}} \mathfrak{A}(\underline{V})\Big/\sim,$$
$$a \longmapsto [\mathrm{id}_{V_{\mathrm{int}}}, a], \quad (5.13)$$

for all $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}^{-1}_{\mathrm{int}\,M}]$, one computes from the diagram (5.12) that $q_{(\mathfrak{A},\mathfrak{I})}$ is the canonical projection $\pi : \mathrm{ext}\,\mathfrak{A} \to \mathrm{ext}\,\mathfrak{A}/\mathfrak{I}$. Hence, the $\mathbf{QFT}(M)$-isomorphisms $\mathrm{ext}\,\eta_{\mathfrak{A}} : \mathrm{ext}\,\mathfrak{A} \to \mathrm{ext}\,\mathrm{res}\,\mathrm{ext}\,\mathfrak{A}$ induce $\mathbf{IQFT}(M)$-isomorphisms

$$\mathrm{ext}\,\eta_{\mathfrak{A}} : (\mathfrak{A}, \mathfrak{I}) \longrightarrow SQ(\mathfrak{A}, \mathfrak{I}), \quad (5.14)$$

which are natural in $(\mathfrak{A}, \mathfrak{I})$, i.e., they define a natural isomorphism $\mathrm{ext}\,\eta : \mathrm{id}_{\mathbf{IQFT}(M)} \to SQ$. $\qquad \square$

Remark 5.11. The physical interpretation of this result is as follows: Every additive quantum field theory $\mathfrak{B} \in \mathbf{QFT}^{\mathrm{add}}(M)$ on a (not necessarily globally hyperbolic) spacetime M with timelike boundary admits an equivalent description in terms of a pair $(\mathfrak{A}, \mathfrak{I}) \in \mathbf{IQFT}(M)$. Notice that the roles of $\mathfrak{A}$ and $\mathfrak{I}$ are completely different: On the one hand, $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\,M)$ is a quantum field theory on the interior $\mathrm{int}\,M$ of M and as such it is independent of the detailed aspects of the boundary. On the other hand, $\mathfrak{I} \subseteq \mathrm{ext}\,\mathfrak{A}$ is an ideal of the universal extension of $\mathfrak{A}$ that is trivial on the interior, i.e., it only captures the physics that happens directly at the boundary. Examples of such ideals $\mathfrak{I}$ arise by imposing specific boundary conditions on the universal extension $\mathrm{ext}\,\mathfrak{A} \in \mathbf{QFT}(M)$, i.e., the quotient $\mathrm{ext}\,\mathfrak{A}/\mathfrak{I}$ describes a quantum field theory on M that satisfies specific boundary conditions encoded in $\mathfrak{I}$. We shall illustrate this assertion in Sect. 6 below using the explicit example given by the free Klein–Gordon field.

Let us also note that there is a reason why our universal extension captures only the class of additive quantum field theories on M. Recall that $\mathrm{ext}\,\mathfrak{A} \in \mathbf{QFT}(M)$ takes as an input a quantum field theory $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}\,M)$ on the interior $\mathrm{int}\,M$ of M. As a consequence, the extension $\mathrm{ext}\,\mathfrak{A}$ can only

have knowledge of the 'degrees of freedom' that are generated in some way out of the interior regions. Additive theories in the sense of Definition 5.5 are precisely the theories whose 'degrees of freedom' are generated out of those localized in the interior regions. □

6. Example: Free Klein–Gordon Theory

In order to illustrate and make more explicit our abstract constructions developed in the previous sections, we shall consider the simple example given by the free Klein–Gordon field. From now on M will be a *globally hyperbolic* spacetime with timelike boundary, see Definition 2.9. This assumption implies that all interior regions $\mathcal{R}_{\mathrm{int}\,M}$ are globally hyperbolic spacetimes with empty boundary, see Proposition 2.11. This allows us to use the standard techniques of [1, Section 3] on such regions.

6.1. Definition on $\mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$:

Let M be a globally hyperbolic spacetime with timelike boundary, see Definition 2.9. The free Klein–Gordon theory on $\mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ is given by the following standard construction, see e.g., [3,4] for expository reviews. On the interior $\mathrm{int}\,M$, we consider the Klein–Gordon operator

$$P := \Box + m^2 \ : \ C^\infty(\mathrm{int}\,M) \longrightarrow C^\infty(\mathrm{int}\,M) \ , \tag{6.1}$$

where $\Box$ is the d'Alembert operator and $m \geq 0$ is a mass parameter. When restricting P to regions $V \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$, we shall write

$$P_V \ : \ C^\infty(V) \longrightarrow C^\infty(V). \tag{6.2}$$

It follows from [1] that there exists a unique retarded/advanced Green's operator

$$G_V^\pm \ : \ C_{\mathrm{c}}^\infty(V) \longrightarrow C^\infty(V) \tag{6.3}$$

for P_V because every $V \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ is a globally hyperbolic spacetime with empty boundary, cf. Proposition 2.11.

The Klein–Gordon theory $\mathfrak{K} \in \mathbf{QFT}(\mathrm{int}\,M)$ is the functor $\mathfrak{K} : \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}] \to \mathbf{Alg}$ given by the following assignment: To any $V \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ it assigns the associative and unital $*$-algebra $\mathfrak{K}(V)$ that is freely generated by $\Phi_V(f)$, for all $f \in C_{\mathrm{c}}^\infty(V)$, modulo the two-sided $*$-ideal generated by the following relations:

- *Linearity* $\Phi_V(\alpha\,f + \beta\,g) = \alpha\,\Phi_V(f) + \beta\,\Phi_V(g)$, for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C_{\mathrm{c}}^\infty(V)$;
- *Hermiticity* $\Phi_V(f)^* = \Phi_V(f)$, for all $f \in C_{\mathrm{c}}^\infty(V)$;
- *Equation of motion* $\Phi_V(P_V f) = 0$, for all $f \in C_{\mathrm{c}}^\infty(V)$;
- *Canonical commutation relations (CCR)* $\Phi_V(f)\,\Phi_V(g) - \Phi_V(g)\,\Phi_V(f) = \mathrm{i}\,\tau_V(f,g)\,\mathbb{1}$, for all $f, g \in C_{\mathrm{c}}^\infty(V)$, where

$$\tau_V(f,g) := \int_V f\,G_V(g)\,\mathrm{vol}_V \tag{6.4}$$

with $G_V := G_V^+ - G_V^-$ the causal propagator and vol_V the canonical volume form on V. (Note that τ_V is antisymmetric, see e.g., [1, Lemma 4.3.5]).

To a morphism $i : V \to V'$ in $\mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$, the functor $\mathfrak{K} : \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}] \to \mathbf{Alg}$ assigns the algebra map that is specified on the generators by pushforward along i (which we shall suppress)

$$\mathfrak{K}(i) \,:\, \mathfrak{K}(V) \longrightarrow \mathfrak{K}(V'), \quad \Phi_V(f) \longmapsto \Phi_{V'}(f). \tag{6.5}$$

The naturality of τ (i.e., naturality of the causal propagator, cf. e.g., [1, Section 4.3]) entails that the assignment $\mathfrak{K}$ defines a quantum field theory in the sense of Definition 3.5.

6.2. Universal Extension

Using the techniques developed in Sect. 4, we may now extend the Klein–Gordon theory $\mathfrak{K} \in \mathbf{QFT}(\mathrm{int}\,M)$ from the interior $\mathrm{int}\,M$ to the whole spacetime M. In particular, using (4.9) (and the formulas following this equation), one could directly compute the universal extension $\mathrm{ext}\,\mathfrak{K} \in \mathbf{QFT}(M)$. The resulting expressions, however, can be considerably simplified. We therefore prefer to provide a more convenient model for the universal extension $\mathrm{ext}\,\mathfrak{K} \in \mathbf{QFT}(M)$ by adopting the following strategy: We first make an 'educated guess' for a theory $\mathfrak{K}^{\mathrm{ext}} \in \mathbf{QFT}(M)$ which we expect to be the universal extension of $\mathfrak{K} \in \mathbf{QFT}(\mathrm{int}\,M)$. (This was inspired by partially simplifying the direct computation of the universal extension). After this, we shall prove that $\mathfrak{K}^{\mathrm{ext}} \in \mathbf{QFT}(M)$ satisfies the universal property that characterizes $\mathrm{ext}\,\mathfrak{K} \in \mathbf{QFT}(M)$. Hence, there exists a (unique) isomorphism $\mathrm{ext}\,\mathfrak{K} \cong \mathfrak{K}^{\mathrm{ext}}$ in $\mathbf{QFT}(M)$, which means that our $\mathfrak{K}^{\mathrm{ext}} \in \mathbf{QFT}(M)$ is a model for the universal extension $\mathrm{ext}\,\mathfrak{K}$.

Let us define the functor $\mathfrak{K}^{\mathrm{ext}} : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathbf{Alg}$ by the following assignment: To any region $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$, which may intersect the boundary, we assign the associative and unital $*$-algebra $\mathfrak{K}^{\mathrm{ext}}(V)$ that is freely generated by $\Phi_V(f)$, for all $f \in C_{\mathrm{c}}^\infty(\mathrm{int}\,V)$ in the interior $\mathrm{int}\,V$ of V, modulo the two-sided ideal generated by the following relations:

- *Linearity* $\Phi_V(\alpha f + \beta g) = \alpha\,\Phi_V(f) + \beta\,\Phi_V(g)$, for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C_{\mathrm{c}}^\infty(\mathrm{int}\,V)$;
- *Hermiticity* $\Phi_V(f)^* = \Phi_V(f)$, for all $f \in C_{\mathrm{c}}^\infty(\mathrm{int}\,V)$;
- *Equation of motion* $\Phi_V(P_{\mathrm{int}\,V}f) = 0$, for all $f \in C_{\mathrm{c}}^\infty(\mathrm{int}\,V)$;
- *Partially-defined CCR* $\Phi_V(f)\,\Phi_V(g) - \Phi_V(g)\,\Phi_V(f) = \mathrm{i}\,\tau_{V_{\mathrm{int}}}(f, g)\,\mathbb{1}$, for all interior regions $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ with $V_{\mathrm{int}} \subseteq \mathrm{int}\,V$ and $f, g \in C_{\mathrm{c}}^\infty(\mathrm{int}\,V)$ with $\mathrm{supp}(f) \cup \mathrm{supp}(g) \subseteq V_{\mathrm{int}}$.

Remark 6.1. We note that our partially-defined CCR are consistent in the following sense: Consider $V_{\mathrm{int}}, V_{\mathrm{int}}' \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ with $V_{\mathrm{int}}^{(\prime)} \subseteq \mathrm{int}\,V$ and $f, g \in C_{\mathrm{c}}^\infty(\mathrm{int}\,V)$ with the property that $\mathrm{supp}(f) \cup \mathrm{supp}(g) \subseteq V_{\mathrm{int}}^{(\prime)}$. Using the partially-defined CCR for both V_{int} and V_{int}', we obtain the equality $\mathrm{i}\,\tau_{V_{\mathrm{int}}}(f, g)\,\mathbb{1} = \mathrm{i}\,\tau_{V_{\mathrm{int}}'}(f, g)\,\mathbb{1}$ in $\mathfrak{K}^{\mathrm{ext}}(V)$. To ensure that $\mathfrak{K}^{\mathrm{ext}}(V)$ is not the zero-algebra, we have to show that $\tau_{V_{\mathrm{int}}}(f, g) = \tau_{V_{\mathrm{int}}'}(f, g)$. This holds true due to the following argument: Consider the subset $V_{\mathrm{int}} \cap V_{\mathrm{int}}' \subseteq \mathrm{int}\,M$. This

is open, causally convex and by Proposition 2.3(c) also stable under Cauchy development, hence $V_{\text{int}} \cap V'_{\text{int}} \in \mathcal{R}_{\text{int}\,M}[\mathcal{C}^{-1}_{\text{int}\,M}]$. Furthermore, the inclusions $V_{\text{int}} \cap V'_{\text{int}} \to V^{(\prime)}_{\text{int}}$ are morphisms in $\mathcal{R}_{\text{int}\,M}[\mathcal{C}^{-1}_{\text{int}\,M}]$. It follows by construction that $\text{supp}(f) \cup \text{supp}(g) \subseteq V_{\text{int}} \cap V'_{\text{int}}$ and hence due to naturality of the τ's we obtain

$$\tau_{V_{\text{int}}}(f,g) = \tau_{V_{\text{int}} \cap V'_{\text{int}}}(f,g) = \tau_{V'_{\text{int}}}(f,g). \tag{6.6}$$

Hence, for any fixed pair $f, g \in C^\infty_{\text{c}}(\text{int}\,V)$, the partially-defined CCR are independent of the choice of V_{int} (if one exists).

To a morphism $i : V \to V'$ in $\mathcal{R}_M[\mathcal{C}^{-1}_M]$, the functor $\mathfrak{K}^{\text{ext}} : \mathcal{R}_M\left[\mathcal{C}^{-1}_M\right] \to \mathbf{Alg}$ assigns the algebra map that is specified on the generators by the pushforward along i (which we shall suppress)

$$\mathfrak{K}^{\text{ext}}(i) : \mathfrak{K}^{\text{ext}}(V) \longrightarrow \mathfrak{K}^{\text{ext}}(V'), \quad \Phi_V(f) \longmapsto \Phi_{V'}(f). \tag{6.7}$$

Compatibility of the map (6.7) with the relations in $\mathfrak{K}^{\text{ext}}$ is a straightforward check.

Recalling the embedding functor $J : \mathcal{R}_{\text{int}\,M}[\mathcal{C}^{-1}_{\text{int}\,M}] \to \mathcal{R}_M\left[\mathcal{C}^{-1}_M\right]$ given in (4.2), we observe that the diagram of functors

$$\begin{array}{ccc}
\mathcal{R}_{\text{int}\,M}[\mathcal{C}^{-1}_{\text{int}\,M}] & \xrightarrow{\quad\mathfrak{K}\quad} & \mathbf{Alg} \\
& {\scriptstyle J}\searrow \quad \Big\Downarrow{\scriptstyle\gamma} \quad \nearrow{\scriptstyle\mathfrak{K}^{\text{ext}}} & \\
& \mathcal{R}_M[\mathcal{C}^{-1}_M] &
\end{array} \tag{6.8}$$

commutes via the natural transformation $\gamma : \mathfrak{K} \to \mathfrak{K}^{\text{ext}}\,J$ with components specified on the generators by the identity maps

$$\gamma_{V_{\text{int}}} : \mathfrak{K}(V_{\text{int}}) \longrightarrow \mathfrak{K}^{\text{ext}}(V_{\text{int}}), \quad \Phi_{V_{\text{int}}}(f) \longmapsto \Phi_{V_{\text{int}}}(f), \tag{6.9}$$

for all $V_{\text{int}} \in \mathcal{R}_{\text{int}\,M}[\mathcal{C}^{-1}_{\text{int}\,M}]$. Notice that γ is a natural isomorphism because $\text{int}\,V_{\text{int}} = V_{\text{int}}$ and the partially-defined CCR on any interior region V_{int} coincides with the CCR.

Theorem 6.2. (6.8) *is a left Kan extension of* $\mathfrak{K} : \mathcal{R}_{\text{int}\,M}[\mathcal{C}^{-1}_{\text{int}\,M}] \to \mathbf{Alg}$ *along* $J : \mathcal{R}_{\text{int}\,M}[\mathcal{C}^{-1}_{\text{int}\,M}] \to \mathcal{R}_M\left[\mathcal{C}^{-1}_M\right]$. *As a consequence of uniqueness (up to unique natural isomorphism) of left Kan extensions and Proposition 4.3, it follows that* $\mathfrak{K}^{\text{ext}} \cong \text{ext}\,\mathfrak{K}$, *i.e.,* $\mathfrak{K}^{\text{ext}} \in \mathbf{QFT}(M)$ *is a model for our universal extension* $\text{ext}\,\mathfrak{K} \in \mathbf{QFT}(M)$ *of the Klein–Gordon theory* $\mathfrak{K} \in \mathbf{QFT}(\text{int}\,M)$.

Proof. We have to prove that (6.8) satisfies the universal property of left Kan extensions: Given any functor $\mathfrak{B} : \mathcal{R}_M\left[\mathcal{C}^{-1}_M\right] \to \mathbf{Alg}$ and natural transformation $\rho : \mathfrak{K} \to \mathfrak{B}\,J$, we have to show that there exists a unique natural transformation $\zeta : \mathfrak{K}^{\text{ext}} \to \mathfrak{B}$ such that the diagram

$$\begin{array}{ccc}
\mathfrak{K} & \xrightarrow{\quad\rho\quad} & \mathfrak{B}\,J \\
& {\scriptstyle\gamma}\searrow \quad \nearrow{\scriptstyle\zeta J} & \\
& \mathfrak{K}^{\text{ext}}\,J &
\end{array} \tag{6.10}$$

commutes. Because γ is a natural isomorphism, it immediately follows that ζJ is uniquely fixed by this diagram. Concretely, this means that the components $\zeta_{V_{\text{int}}}$ corresponding to interior regions $V_{\text{int}} \in \mathcal{R}_{\text{int } M}[\mathcal{C}^{-1}_{\text{int } M}]$ are uniquely fixed by

$$\zeta_{V_{\text{int}}} := \rho_{V_{\text{int}}} \, \gamma^{-1}_{V_{\text{int}}} \; : \; \mathfrak{K}^{\text{ext}}(V_{\text{int}}) \longrightarrow \mathfrak{B}(V_{\text{int}}). \tag{6.11}$$

It remains to determine the components

$$\zeta_V \; : \; \mathfrak{K}^{\text{ext}}(V) \longrightarrow \mathfrak{B}(V) \tag{6.12}$$

for generic regions $V \in \mathcal{R}_M[\mathcal{C}^{-1}_M]$. Consider any generator $\Phi_V(f)$ of $\mathfrak{K}^{\text{ext}}(V)$, where $f \in C^\infty_c(\text{int } V)$, and choose a finite cover $\{V_\alpha \subseteq \text{int } V\}$ of $\text{supp}(f)$ by interior regions $V_\alpha \in \mathcal{R}_{\text{int } M}[\mathcal{C}^{-1}_{\text{int } M}]$, together with a partition of unity $\{\chi_\alpha\}$ subordinate to this cover. (The existence of such a cover is guaranteed by the assumption that M is a globally hyperbolic spacetime with timelike boundary, see Proposition 2.11). We define

$$\zeta_V\big(\Phi_V(f)\big) := \sum_\alpha \mathfrak{B}(i_\alpha)\big(\zeta_{V_\alpha}\big(\Phi_{V_\alpha}(\chi_\alpha f)\big)\big), \tag{6.13}$$

where $i_\alpha : V_\alpha \to V$ is the inclusion. Our definition (6.13) is independent of the choice of cover and partition of unity: For any other $\{V'_\beta \subseteq \text{int } V\}$ and $\{\chi'_\beta\}$, we obtain

$$\sum_\beta \mathfrak{B}(i_\beta)\big(\zeta_{V'_\beta}\big(\Phi_{V'_\beta}(\chi'_\beta f)\big)\big) = \sum_{\alpha,\beta} \mathfrak{B}(i_{\alpha\beta})\big(\zeta_{V_\alpha \cap V'_\beta}\big(\Phi_{V_\alpha \cap V'_\beta}(\chi_\alpha \chi'_\beta f)\big)\big)$$

$$= \sum_\alpha \mathfrak{B}(i_\alpha)\big(\zeta_{V_\alpha}\big(\Phi_{V_\alpha}(\chi_\alpha f)\big)\big), \tag{6.14}$$

where $i_\beta : V'_\beta \to V$ and $i_{\alpha\beta} : V_\alpha \cap V'_\beta \to V$ are the inclusions. In particular, this implies that (6.13) coincides with (6.11) on interior regions $V = V_{\text{int}}$. (Hint: Choose the cover given by the single region V_{int} together with its partition of unity).

We have to check that (6.13) preserves the relations in $\mathfrak{K}^{\text{ext}}(V)$. Preservation of linearity and Hermiticity is obvious. The equation of motion relations are preserved because

$$\zeta_V\big(\Phi_V(P_{\text{int } V} f)\big) = \sum_\alpha \mathfrak{B}(i_\alpha)\big(\zeta_{V_\alpha}\big(\Phi_{V_\alpha}(\chi_\alpha P_{V_\alpha} f)\big)\big)$$

$$= \sum_{\alpha,\beta} \mathfrak{B}(i_{\alpha\beta})\big(\zeta_{V_\alpha \cap V_\beta}\big(\Phi_{V_\alpha \cap V_\beta}(\chi_\alpha P_{V_\alpha \cap V_\beta} \chi_\beta f)\big)\big)$$

$$= \sum_\beta \mathfrak{B}(i_\beta)\big(\zeta_{V_\beta}\big(\Phi_{V_\beta}(P_{V_\beta} \chi_\beta f)\big)\big) = 0. \tag{6.15}$$

Regarding the partially-defined CCR, let $V_{\text{int}} \in \mathcal{R}_{\text{int } M}[\mathcal{C}^{-1}_{\text{int } M}]$ with $V_{\text{int}} \subseteq \text{int } V$ and $f,g \in C^\infty_c(\text{int } V)$ with $\text{supp}(f) \cup \text{supp}(g) \subseteq V_{\text{int}}$. We may choose the cover given by the single region $i : V_{\text{int}} \to V$ together with its partition of

unity. We obtain for the commutator

$$\Big[\zeta_V\big(\Phi_V(f)\big), \zeta_V\big(\Phi_V(g)\big)\Big] = \Big[\mathfrak{B}(i)\big(\zeta_{V_{\mathrm{int}}}\big(\Phi_{V_{\mathrm{int}}}(f)\big)\big), \mathfrak{B}(i)\big(\zeta_{V_{\mathrm{int}}}\big(\Phi_{V_{\mathrm{int}}}(g)\big)\big)\Big]$$
$$= \mathrm{i}\,\tau_{V_{\mathrm{int}}}(f,g)\,\mathbb{1}, \tag{6.16}$$

which implies that the partially-defined CCR are preserved.

Naturality of the components (6.13) is easily verified. Uniqueness of the resulting natural transformation $\zeta : \mathfrak{K}^{\mathrm{ext}} \to \mathfrak{B}$ is a consequence of uniqueness of ζJ and of the fact that the **Alg**-morphisms $\mathfrak{K}^{\mathrm{ext}}(i_{\mathrm{int}}) : \mathfrak{K}^{\mathrm{ext}}(V_{\mathrm{int}}) \cong \mathfrak{K}(V_{\mathrm{int}}) \to \mathfrak{K}^{\mathrm{ext}}(V)$, for all interior regions $i_{\mathrm{int}} : V_{\mathrm{int}} \to V$, generate $\mathfrak{K}^{\mathrm{ext}}(V)$, for all $V \in \mathcal{R}_M\big[\mathcal{C}_M^{-1}\big]$. This completes the proof. $\qquad\square$

6.3. Ideals from Green's Operator Extensions

The Klein–Gordon theory $\mathfrak{K} \in \mathbf{QFT}(\mathrm{int}\,M)$ on the interior $\mathrm{int}\,M$ of the globally hyperbolic spacetime M with timelike boundary and its universal extension $\mathfrak{K}^{\mathrm{ext}} \in \mathbf{QFT}(M)$ depend on the *local* retarded and advanced Green's operators $G_{V_{\mathrm{int}}}^{\pm} : C_{\mathrm{c}}^{\infty}(V_{\mathrm{int}}) \to C^{\infty}(V_{\mathrm{int}})$ on all interior regions $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ in M. For constructing concrete examples of quantum field theories on globally hyperbolic spacetimes with timelike boundary as in [12], one typically imposes suitable boundary conditions for the field equation in order to obtain also *global* retarded and advanced Green's operators on M. Inspired by such examples, we shall now show that any choice of an *adjoint-related pair* (G^+, G^-) consisting of a retarded and an advanced Green's operator for P on M (see Definition 6.3 below) defines an ideal $\mathfrak{I}_{G^{\pm}} \subseteq \mathfrak{K}^{\mathrm{ext}} \in \mathbf{QFT}(M)$ that is trivial on the interior (cf. Definition 5.8). The corresponding quotient $\mathfrak{K}^{\mathrm{ext}}/\mathfrak{I}_{G^{\pm}} \in \mathbf{QFT}(M)$ then may be interpreted as the Klein–Gordon theory on M, subject to a specific choice of boundary conditions that is encoded in $G^{\pm}$.

Definition 6.3. A *retarded/advanced Green's operator* for the Klein–Gordon operator P on M is a linear map $G^{\pm} : C_{\mathrm{c}}^{\infty}(\mathrm{int}\,M) \to C^{\infty}(\mathrm{int}\,M)$ which satisfies the following properties, for all $f \in C_{\mathrm{c}}^{\infty}(\mathrm{int}\,M)$:

(i) $P\,G^{\pm}(f) = f$,
(ii) $G^{\pm}(Pf) = f$, and
(iii) $\mathrm{supp}(G^{\pm}(f)) \subseteq J_M^{\pm}(\mathrm{supp}(f))$.

A pair (G^+, G^-) consisting of a retarded and an advanced Green's operator for P on M is called *adjoint-related* if G^+ is the formal adjoint of G^-, i.e.,

$$\int_M G^+(f)\,g\,\mathrm{vol}_M = \int_M f\,G^-(g)\,\mathrm{vol}_M, \tag{6.17}$$

for all $f, g \in C_{\mathrm{c}}^{\infty}(\mathrm{int}\,M)$.

Remark 6.4. In contrast to the situation where M is a globally hyperbolic spacetime *with empty boundary* [1], existence, uniqueness and adjoint-relatedness of retarded/advanced Green's operators for the Klein–Gordon operator P is in general not to be expected on spacetimes with timelike boundary. Positive results seem to be more likely on globally hyperbolic spacetimes with

non-empty timelike boundary, although the general theory has not been developed yet to the best of our knowledge. Simple examples of adjoint-related pairs of Green's operators were constructed, e.g., in [12].

Given any region $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ in M, which may intersect the boundary, we use the canonical inclusion $i : V \to M$ to define local retarded/advanced Green's operators

$$
\begin{array}{ccc}
C_{\mathrm{c}}^\infty(\operatorname{int} V) & \xrightarrow{\;\;G_V^\pm\;\;} & C^\infty(\operatorname{int} V) \\
i_* \downarrow & & \uparrow i^* \\
C_{\mathrm{c}}^\infty(\operatorname{int} M) & \xrightarrow{\;\;G^\pm\;\;} & C^\infty(\operatorname{int} M)
\end{array}
\qquad (6.18)
$$

where i_* denotes the pushforward of compactly supported functions (i.e., extension by zero) and i^* the pullback of functions (i.e., restriction). Since $V \subseteq M$ is causally convex, it follows that $J_M^\pm(p) \cap V = J_V^\pm(p)$ for all $p \in V$. Therefore $G_V^\pm$ satisfies the axioms of a retarded/advanced Green's operator for P_V on V. (Here, we regard V as a globally hyperbolic spacetime with timelike boundary, see Proposition 2.11. $J_V^\pm(p)$ denotes the causal future/past of p in the spacetime V). In particular, for all interior regions $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$ in M, by combining Proposition 2.11 and [1, Corollary 3.4.3] we obtain that $G_{V_{\mathrm{int}}}^\pm$ as defined in (6.18) is the unique retarded/advanced Green's operator for the restricted Klein–Gordon operator $P_{V_{\mathrm{int}}}$ on the globally hyperbolic spacetime V_{int} with empty boundary.

Consider any adjoint-related pair (G^+, G^-) of Green's operator for P on M. For all $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$, we set $\mathfrak{I}_{G^\pm}(V) \subseteq \mathfrak{K}^{\mathrm{ext}}(V)$ to be the two-sided $*$-ideal generated by the following relations:

- $G^\pm$-CCR $\Phi_V(f)\,\Phi_V(g) - \Phi_V(g)\,\Phi_V(f) = \mathrm{i}\,\tau_V(f,g)\,\mathbb{1}$, for all $f, g \in C_{\mathrm{c}}^\infty(\operatorname{int} V)$, where

$$
\tau_V(f,g) := \int_V f\,G_V(g)\,\operatorname{vol}_V
\qquad (6.19)
$$

with $G_V := G_V^+ - G_V^-$ the causal propagator and vol_V the canonical volume form on V.

The fact that the pair (G^+, G^-) is adjoint-related (cf. Definition 6.3) implies that for all $V \in \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right]$ the causal propagator G_V is formally skew-adjoint, hence τ_V is antisymmetric.

Proposition 6.5. $\mathfrak{I}_{G^\pm} \subseteq \mathfrak{K}^{\mathrm{ext}}$ *is an ideal that is trivial on the interior (cf. Definition 5.8).*

Proof. Functoriality of $\mathfrak{I}_{G^\pm} : \mathcal{R}_M\left[\mathcal{C}_M^{-1}\right] \to \mathbf{Vec}$ is a consequence of (6.18), hence $\mathfrak{I}_{G^\pm} \subseteq \mathfrak{K}^{\mathrm{ext}}$ is an ideal in the sense of Definition 5.1. It is trivial on the interior because for all interior regions $V_{\mathrm{int}} \in \mathcal{R}_{\mathrm{int}\,M}[\mathcal{C}_{\mathrm{int}\,M}^{-1}]$, the Green's operators defined by (6.18) are the unique retarded/advanced Green's operators for $P_{V_{\mathrm{int}}}$ and hence the relations imposed by $\mathfrak{I}_{G^\pm}(V_{\mathrm{int}})$ automatically hold true in $\mathfrak{K}^{\mathrm{ext}}(V_{\mathrm{int}})$ on account of the (partially-defined) CCR. $\qquad \square$

Remark 6.6. We note that the results of this section still hold true if we slightly weaken the hypotheses of Definition 2.9 by assuming the strong causality and the compact double-cones property only for points in the interior int M of M. In fact, int M can still be covered by causally convex open subsets and any causally convex open subset $U \subseteq \text{int}\, M$ becomes a globally hyperbolic spacetime with empty boundary once equipped with the induced metric, orientation and time-orientation.

Example 6.7. Consider the sub-spacetime $M := \mathbb{R}^{m-1} \times [0, \pi] \subseteq \mathbb{R}^m$ of m-dimensional Minkowski spacetime, which has a timelike boundary $\partial M = \mathbb{R}^{m-1} \times \{0, \pi\}$. The constructions in [12] define an adjoint-related pair (G^+, G^-) of Green's operators for P on M that corresponds to Dirichlet boundary conditions. Using this as an input for our construction above, we obtain a quantum field theory $\mathfrak{K}^{\text{ext}}/\mathfrak{I}_{G^\pm} \in \mathbf{QFT}(M)$ that may be interpreted as the Klein–Gordon theory on M with Dirichlet boundary conditions. It is worth to emphasize that our theory in general *does not* coincide with the one constructed in [12]. To provide a simple argument, let us focus on the case of $m = 2$ dimensions, i.e., $M = \mathbb{R} \times [0, \pi]$, and compare our global algebra $\mathfrak{A}^{\text{BDS}}(M) := \mathfrak{K}^{\text{ext}}/\mathfrak{I}_{G^\pm}(M)$ with the global algebra $\mathfrak{A}^{\text{DNP}}(M)$ constructed in [12]. Both algebras are CCR-algebras, however the underlying symplectic vector spaces differ: The symplectic vector space underlying our global algebra $\mathfrak{A}^{\text{BDS}}(M)$ is $C_c^\infty(\text{int}\, M)/PC_c^\infty(\text{int}\, M)$ with the symplectic structure (6.19). Using that the spatial slices of $M = \mathbb{R} \times [0, \pi]$ are compact, we observe that the symplectic vector space underlying $\mathfrak{A}^{\text{DNP}}(M)$ is given by the space $\mathfrak{Sol}_{\text{Dir}}(M)$ of *all* solutions with Dirichlet boundary condition on M (equipped with the usual symplectic structure). The causal propagator defines a symplectic map $G : C_c^\infty(\text{int}\, M)/PC_c^\infty(\text{int}\, M) \to \mathfrak{Sol}_{\text{Dir}}(M)$, which however is not surjective for the following reason: Any $\varphi \in C_c^\infty(\text{int}\, M)$ has by definition compact support in the interior of M, hence the support of $G\varphi \in \mathfrak{Sol}_{\text{Dir}}(M)$ is schematically as follows

$$x = 0 \qquad\qquad x = \pi \qquad (6.20)$$
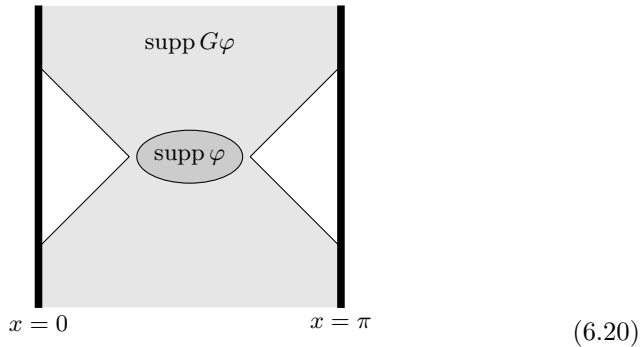
The usual mode functions $\Phi_k(t, x) = \cos(\sqrt{k^2 + m^2}\, t)\, \sin(kx) \in \mathfrak{Sol}_{\text{Dir}}(M)$, for $k \geq 1$, are clearly not of this form, hence $G : C_c^\infty(\text{int}\, M)/PC_c^\infty(\text{int}\, M) \to \mathfrak{Sol}_{\text{Dir}}(M)$ cannot be surjective. As a consequence, the models constructed in [12] are in general not additive from the interior and our construction $\mathfrak{K}^{\text{ext}}/\mathfrak{I}_{G^\pm}$ should be interpreted as the maximal additive subtheory of these examples.

It is interesting to note that there exists a case where both constructions coincide: Consider the sub-spacetime $M := \mathbb{R}^{m-1} \times [0, \infty) \subseteq \mathbb{R}^m$ of Minkowski spacetime with $m \geq 4$ even and take a massless real scalar field with Dirichlet boundary conditions. Using Huygens' principle and the support properties of the Green's operators, one may show that our algebra $\mathfrak{A}^{\mathrm{BDS}}(M)$ is isomorphic to the construction in [12].

Appendix A: Some Concepts from Category Theory

Adjunctions

This is a standard concept, which is treated in any category theory textbook, e.g., [28].

Definition A.1. An *adjunction* consists of a pair of functors

$$F : \mathbf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\rightleftarrows}} \mathbf{D} : G \qquad (A.1)$$

together with natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \to GF$ (called *unit*) and $\epsilon :$ $FG \to \mathrm{id}_{\mathbf{D}}$ (called *counit*) that satisfy the *triangle identities*

$$
\begin{array}{ccc}
F \xrightarrow{\ F\eta\ } FGF & \qquad & G \xrightarrow{\ \eta G\ } GFG \\
\quad\searrow_{\mathrm{id}_F} \quad \downarrow{\epsilon F} & & \quad\searrow_{\mathrm{id}_G} \quad \downarrow{G\epsilon} \\
F & & G
\end{array}
\qquad (A.2)
$$

We call F the *left adjoint* of G and G the *right adjoint* of F, and write $F \dashv G$.

Definition A.2. An *adjoint equivalence* is an adjunction

$$
F : \mathbf{C} \xrightleftharpoons[\ \sim\ ]{} \mathbf{D} : G \qquad (A.3)
$$

for which both the unit η and the counit ϵ are natural isomorphisms. Existence of an adjoint equivalence in particular implies that $\mathbf{C} \cong \mathbf{D}$ are equivalent as categories.

Proposition A.3. *If a functor $G : \mathbf{D} \to \mathbf{C}$ admits a left adjoint $F : \mathbf{C} \to \mathbf{D}$, then F is unique up to a unique natural isomorphism. Vice versa, if a functor $F : \mathbf{C} \to \mathbf{D}$ admits a right adjoint $G : \mathbf{D} \to \mathbf{C}$, then G is unique up to a unique natural isomorphism.*

Localizations

Localizations of categories are treated for example in [21, Section 7.1]. In our paper we restrict ourselves to small categories.

Definition A.4. Let $\mathbf{C}$ be a small category and $W \subseteq \mathrm{Mor}\,\mathbf{C}$ a subset of the set of morphisms. A *localization of $\mathbf{C}$ at W* is a small category $\mathbf{C}[W^{-1}]$ together with a functor $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ satisfying the following properties:

(a) For all $(f : c \to c') \in W$, $L(f) : L(c) \to L(c')$ is an isomorphism in $\mathbf{C}[W^{-1}]$.

(b) For any category $\mathbf{D}$ and any functor $F : \mathbf{C} \to \mathbf{D}$ that sends morphisms in W to isomorphisms in $\mathbf{D}$, there exists a functor $F_W : \mathbf{C}[W^{-1}] \to \mathbf{D}$ and a natural isomorphism $F \cong F_W L$.

(c) For all objects $G, H \in \mathbf{D}^{\mathbf{C}[W^{-1}]}$ in the functor category, the map

$$
\mathrm{Hom}_{\mathbf{D}^{\mathbf{C}[W^{-1}]}}(G, H) \longrightarrow \mathrm{Hom}_{\mathbf{D}^{\mathbf{C}}}(G\,L, H\,L) \qquad (A.4)
$$

is a bijection of Hom-sets.

Proposition A.5. *If $\mathbf{C}[W^{-1}]$ exists, it is unique up to equivalence of categories.*

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