Interval Type-2 A-Intuitionistic Fuzzy logic for Regression Problems

Imo Eyoh, *Student Member, IEEE,* Robert John, *Senior Member, IEEE,* and Geert De Maere

***Abstract*—This paper presents an approach to prediction based on a new interval type-2 Atanassov-intuitionistic fuzzy logic system (IT2AIFLS) of Takagi-Sugeno-Kang (TSK) fuzzy inference with neural network learning capability. The gradient descent (GD) algorithm is used to adapt the parameters of the IT2AIFLS. The empirical comparison is made on the designed system using some benchmark regression problems - both artificial and real world datasets. Analyses of our results reveal that IT2AIFLS outperforms its type-1 variant, other type-1 fuzzy logic approaches and some type-2 fuzzy systems in the regression tasks. The reason for the improved performance of the proposed framework of IT2AIFLS is because of the introduction of non-membership functions and intuitionistic fuzzy indices into the classical IT2FLS model. This increases the level of fuzziness in the proposed IT2AIFLS framework, thus providing more accurate approximations than AIFLS, classical type-1 and interval type-2 fuzzy logic systems.**

***Index Terms*—Interval type-2 A-intuitionistic fuzzy logic system; Regression problems; Gradient descent algorithm.**

I. Introduction

Fuzzy set (FS) theory was introduced by Zadeh [1] as a generalisation of the classical notion of a set and has served as an indispensable mathematical tool for handling uncertainty and computing with words [2]. According to Zadeh [3],

> Fuzzy logic is a precise conceptual system of reasoning, deduction and computation in which the objects of discourse and analysis are, or are allowed to be, associated with imperfect information. Imperfect information is information which in one or more respects is imprecise, uncertain, incomplete, unreliable, vague or partially true.

Despite the extensive use of type-1 FS (T1FS) and its connotation of uncertainty, previous studies have shown that T1 fuzzy logic cannot directly handle the high level of uncertainty in many real world applications [4]. Zadeh [5] introduced type-2 fuzzy set (T2FS) which has the capacity to handle uncertainties that T1 struggles with because membership grades of T2FS are themselves fuzzy which give them the flexibility to adapt to uncertain environments. This flexibility provides a soft decision boundary and has a close resemblance to human decision making [6] such that classes of objects can have a gradual rather than abrupt transition from membership to non-membership. The hallmark of type-2 fuzzy logic systems therefore, is the ability to directly model uncertainties in data [4] such as noisy data and different word meanings.

Atanassov [7] extended the concept of Zadeh's fuzzy sets to intuitionistic fuzzy sets, hereafter referred to as AIFSs, which handle uncertainty by taking into account both the membership and non-membership degrees of an element x to a fuzzy set A together with extra degree of indeterminacy (hesitation). With AIFS, the fuzzy characteristic of "neither this or that" (neutral state) can be described, thus providing IFS the flexibility and the ability to capture more information than FS [8]. AIFSs are found to be useful for dealing with vagueness [9], [10]. Szmidt and Kacprzyk [11] state that AIFSs are useful in problem domains where the use of linguistic variable to describe the problem in terms of membership functions only seems too restrictive. According to Olej and Hajek [12], the representation of attributes by means of membership and non-membership functions provides a better way to express uncertainty. Castillo *et al.* [13] pointed out that the non-membership degrees or intuitionistic fuzzy indices enable the representation of imperfect knowledge and also allow adequate description of many real world problems. According to [14], when dealing with the problem of vagueness where there is insufficient information leading to an inability to satisfactorily specify the membership function, the AIFS theory becomes more suitable than fuzzy sets to deal with such problems. It is argued that AIFS is a tool for a more human consistent reasoning under imperfectly defined facts and imprecise knowledge [15].

Studies involving AIFSs have drawn much attention in recent times and have been successfully applied in different problem domains such as time series analysis [13], [16], threat assessment [17], prediction [12], [18], classification [19], control [20], [21], bankcruptcy forecasting [22], credit scoring [23] and e-learning to evaluate student knowledge of Mathematics in university courses [24]. As more number of neurons tend to slow down the learning process of a modular neural network (MNN), Sotirov *et al.* [25], proposed an A-intuitionistic fuzzy intercriteria analysis approach for reducing the number of neurons/parameters in an MNN thereby speeding up the learning process. However, most of these studies have focussed on type-1 Atanassov's intuitionistic fuzzy logic systems (AIFLSs). We believe that the use of type-1 FLS (classical or A-intuitionistic) in handling uncertainties in some areas may not be appropriate, especially in circumstances where the determination of exact membership function is difficult to pinpoint. Hence, similar to the notion of a classical T1FS, the type-1 AIFLS may not handle or minimize the plethora of uncertainties that are inherent in real world

Imo Eyoh and Robert John are with the Laboratory for Uncertainty in Data and Decision Making (LUCID) and Automated Scheduling, Optimisation and Planning (ASAP) Research Groups, University of Nottingham, UK.
e-mail: {ije, rij}@cs.nott.ac.uk
Geert De Maere is with the Automated Scheduling, Optimisation and Planning (ASAP) Research Group, University of Nottingham, UK.
e-mail: gdm@cs.nott.ac.uk

applications as their membership and non-membership degrees are exactly defined.

Perhaps the best way to describe this deficiency of FS and IFS is in the words of Gorzalczany [26] and Gerhre *et al.* [27]:

> "...it is not always possible for a membership function of the type $\mu : X \rightarrow [0, 1]$ to assign precisely one point from the interval $[0, 1]$ to each element $x \in X$ without loss of at least a part of information." [26].

Gorzalczany formalised the theory of IVFSs and studies have shown that IVFSs are equivalent to AIFSs [28], [29], [30], [7]. According to Gerhre *et al.* [27]

> "$\cdots$ But an increasingly prevalent view is that models based on [0,1] are inadequate. Many believe that assigning an exact number to an expert's opinion is too restrictive, and that the assignment of an interval of values is more realistic."

To tackle this problem, Atanassov and Gargov [31] extended the concept of IFS to interval valued AIFSs (IVAIFS) which are characterised by membership and non-membership functions and defined in the interval $[0, 1]$.

In the literature, IVFSs [26] are regarded as the special cases of IT2FSs [14], [32], [33], [34]. We argue that this will include both classical and A-intuitionistic IVFSs. Specifically, and more notably is the work of Bustince *et al* [33] which demonstrates indepth, a wider and general view of the relationship between IT2FSs and IVFSs. According to [33], IVFSs are only a special case of IT2FSs and as such both kinds of fuzzy sets should be treated differently. In their paper, four representations are defined for the primary membership functions of IT2FSs namely, as type-1 fuzzy sets, as interval-valued fuzzy sets, as multi-fuzzy sets and as multi-interval fuzzy sets. Thus, IT2FSs can easily be used to model other concepts, a capability not obtainable with IVFSs [33]. Similar to IT2FS and its representations, we argue that IT2AIFS can also be used in a more general perspective to represent concepts that are not possible with IVAIFSs, hence our adoption of IT2AIFS instead of IVAIFSs. It is useful to make this distinction in the context of this research as it serves to distinguish the much broader concept of IT2AIFS from the more specific concept of IVAIFS. Secondly, for IVAIFS, the general constraints is that the summation of the upper-bound membership and upper-bound non-membership degrees is less than or equal to 1. On the contrary, for IT2AIFS, the summation of the upper-bound membership and lower-bound non-membership is less than or equal to 1 and the summation of the lower-bound membership and upper-bound non-membership degrees is less than or equal to 1, i.e. for IT2AIFS, the constraints are: $0 \leq \overline{\mu}_{\tilde{A}^*}(x) + \underline{\nu}_{\tilde{A}^*}(x) \leq 1 \, and \; 0 \leq \underline{\mu}_{\tilde{A}^*}(x) + \overline{\nu}_{\tilde{A}^*}(x) \leq 1$ for all $x \in X$ [35]. This also presents IT2AIFS as a concepts different from IVAIFS. Consistent with previous studies on T2FLS [36], [37], we believe that the resulting interval type-2 A-intuitionistic fuzzy logic systems (IT2AIFLSs) whose degrees of membership and non membership are intervals are capable of providing better performance in some applications than the type-1 A-intuitionistic fuzzy logic systems (T1AIFLSs).

Some studies in the literature on applications involving T2FS and AIFS include, Nguyen *et al.* [35] who proposed a clustering approach using IT2 fuzzy C-mean (IT2FCM) and AIFS for the clustering of different types of images especially those corrupted with noise. Experimental results reveal improvement in the clustering quality of images using IT2FCM and AIFS compared to representative algorithms like FCM and IT2FCM. Nghiem *et al.* [38] also applied A-intuitionistic type-2 fuzzy set to image thresholding using Sugeno intuitionistic fuzzy generator. The authors in [38] claim that their proposed method exhibits higher thresholding quality with noisy images compared to typical algorithms such as image segmentation using type-1 fuzzy set and AIFS. Naim and Hagras [39], presented a hybrid approach where IT2 and AIFS are utilised in multi-criteria group decision making (MCGDM). The proposed system employs IT2FS to handle the linguistic uncertainty while utilising intuitionistic evaluation in the design of the non-membership function degrees. The authors applied the proposed method to the evaluation of postgraduate study involving 10 candidates. Analysis of results shows that variations in the group decision making using the proposed method of IT2FS and IF evaluation provided better agreement with the human experts decision than AIFS, FS and IT2 fuzzy systems. In Naim *et al.* [40], fuzzy logic-MCGDM (FL-MCGDM) is proposed for selecting appropriate and convenient lighting level for reading to meet each individual needs as this varies among users. The proposed hybrid system was developed using the concepts of IT2FS and the hesitation indices provided by the IFS. The membership function of the IT2FS for the left and right end-points were represented in intuitionistic values. Experimental evaluation revealed a significant correlation between the user's linguistic appraisal and the result provided by the proposed FL-MCGDM system. The authors concluded that the combination of T2FS and IFS provides FL-MCGDM with enhanced capability for decision making. Another FL-MCGDM is proposed in Naim and Hagras [41] for intelligent shared environment. The proposed model also utilises IT2FS and hesitation indices of AIFS in the design of the decision making model. In order to evaluate the effectiveness of the designed approach, the authors applied the model to an intelligent apartment and concluded that the results were consistent with the human decision as compared to classical fuzzy MCGDM.

Castillo *et al.* [42], proposed the concept of using IFS to represent IT2FS. The authors pointed out that this can be achieved with a suitable choice of interval function (g). As a follow up, in Castillo *et al.* [43] the authors discussed the use of IFS and its multidimensional (MDIFS) variant to interpret FSs, IT2FSs and generalised T2FSs (GT2FSs). Recently [44] proposed a new IT2FLS by introducing the non-membership function into IT2FS and incorporating the hesitation indices of AIFS into the FOUs of the proposed fuzzy set definitions, otherwise known as interval type-2 Atanassov intuitionstic fuzzy set (IT2AIFS). The proposed interval type-2 Atanassov-intuitionistic fuzzy logic system (IT2AIFLS) was applied to the prediction of non-linear systems. Evaluation of results reveal better performance of IT2AIFLS compared to some standard T2FLS and its type-1 counterpart. Cuong *et al.* [45]

have defined some operations for T2IFS and their properties and concluded that many applications will benefit from the use of such sets.

As discussed, FSs of type-1 are not able to directly model uncertainties, type-2 fuzzy sets (T2FSs) are therefore very appropriate for our purpose. The key advantage being that the membership functions of T2FSs are themselves fuzzy where the actual degree of membership is assumed to belong. That is, T2FSs has a greater capability to model imprecise and imperfect information as they are able to capture the uncertainties in their footprints of uncertainty (FOUs).

Type-2 fuzzy sets and systems can be classified into general type-2 (GT2) and interval type-2 (IT2). The GT2FSs are computationally intensive, difficult to use and understand [46] because the secondary membership grades of elements have different magnitudes. In recent years, research has focussed mostly on interval type-2 fuzzy sets which are quite practical with manageable computational intricacies since the secondary membership grades all take the value 1 [37]. The work in this research also adopt the principles of IT2FS.

The use of IT2FSs to model uncertainty in data cannot be over-emphasized as there exists abundance of applications involving IT2FLSs which employ at least one IT2FS in the rule base (see [47], [48], [49], [50], [51]). It is generally known that the membership functions of IT2FSs are themselves fuzzy which make them more versatile to handle uncertainty well. That is, IT2FSs are quite useful in cases where it is difficult to specify a single crisp numeric membership function value and where linguistic and numerical uncertainties abound, particularly in many real world applications. According to Wu [52], one of the reasons for the wide spread use of IT2FLSs is because the rule base is easy to design from expert knowledge and natural language which increases the robustness of the system. Also, IT2FLS are adaptive with the ability to model input-output relationships better than its type-1 counterpart. For a more detailed advantages of using IT2FLS, (see [52], [53]). Despite the advantages, the extensive use of IT2FLSs and their abilities to handle uncertainties in data better than their type-1 counterparts, they still make use of only the membership functions (upper and lower) to model these uncertainties where the non-membership is complementary to the membership (upper or lower). In a real life scenario, it is not always the case that the non-membership grade of an element to a set is complementary to the membership (upper or lower). There tend to be some extra degrees that represent evidence of neither belonging nor not-belonging (hesitation or neutral state) of an element to a set.

The traditional IT2FLS lack the mechanism of tackling this phenomenon. This research is an attempt to address this drawback by introducing Atanassov intuitionism (non-membership function and hesitation degrees) into IT2FS, leading to an enhanced FOUs definition for the proposed model - IT2AIFLs. With this approach, the evaluation becomes more precise and close to human reasoning than FLS and T2FLS. Thus, with the ability of IT2FSs to adequately capture the uncertainties in their FOUs and the ability of IFS to separately cater for the membership and non-membership grades of elements with extra degrees of hesitancy, we adopt the integration of these two concepts to design a new framework to uncertainty modeling - the so-called IT2AIFLS. The marriage of these two concepts - IFS and IT2FS - is able to provide a synergistic capability in dealing with imprecise and vague information. The proposed model utilises Takagi-Sugeno-Kang (TSK) fuzzy inferencing. In Lin *et al.* [54], the Takagi-Sugeno intuitionistic fuzzy systems are found to be universal approximators with arbitrarily high approximation accuracy.

The motivation for this study stem from the desire to extend the capability of IT2FSs. The novelty of this study is the possible integration of the extra notations of AIFS (non-membership function and hesitation degree) in IT2FS with the aim of designing a new IT2AIFLS framework where IT2AIFS are used to define linguistic concepts. In this study, the hesitancies of the experts are captured in the FOUs for both the membership and non-membership functions of the fuzzy sets through a process of scaling and shifting. This phenomenon is reflected as ripples along the bounds of the FOUs. The concept of taking into account the contributions of the non-membership functions and intuitionistic fuzzy indices in the partitioning of the input space is one of the advantages of this approach as they increase the fuzziness of the IT2AIFS. According to [55], "increased fuzziness in a description means increased ability to handle inexact information in a logically correct manner." Hence, we present the IT2AIFLS and the learning algorithms for the adaptation of its parameters based on gradient descent (GD) derivative-based method. Our aim is to apply the proposed framework to model uncertainty in data by considering both the membership and non-membership values of the fuzzy sets.

To the best knowledge of the authors, there is currently no work in the literature where IT2AIFS is applied in a fuzzy logic inference system for regression problems.

The rest of the paper is structured as follows: In Section II, AIFS, T2AIFS and IT2AIFS are defined. In Section III, IT2AIFLS is designed and in Section IV, parameter update rules are derived. We present our results in Sections V and VI, and conclude in Section VII.

II. Type-1 and Type-2 AIFSs

A. A-Intuitionistic Fuzzy Set (AIFS)

An ordinary FS A is specified by $A = \{(x, \mu_A(x)) \mid x \in X\}$ i.e. each set consists of elements and degrees of membership of the elements to the fuzzy set A. Intuitionistic FS proposed by Atanassov [7] is a generalisation of FS which consist of degree of membership and of non-membership given as: $A^* = \{(x, \mu_{A^*}(x), \nu_{A^*}(x)) \mid x \in X\}$ where $\mu_{A^*}(x)$ and $\nu_{A^*}(x)$ are element in [0,1] defined as degree of membership and non membership of element x to set A^* respectively with the constraint $0 \leq \mu_{A^*}(x) + \nu_{A^*}(x) \leq 1$. Atanassov also specified a hesitation degree, π, defined as 1 minus the sum of the degree of membership and non-membership of an element to a set i.e. $\pi_{A^*}(x) = 1 - (\mu_{A^*}(x) + \nu_{A^*}(x))$.

According to Ejegwa *et al.* [56], the degree of non-membership of an element in a fuzzy set may not always be 1 minus the degree of membership, that is, $(v(x) \neq 1 - \mu(x))$ because there may be some degree of hesitation of that element to the set. Thus the semantic representation of AIFS, A^* includes the degree of membership, degree of non-membership and the hesitation margin $\{(\mu_{A^*}(x), \nu_{A^*}(x), \pi_{A^*}(x)) \mid x \in X\}$ respectively. Given the background of AIFS, we now formally define an AIFS as follows:

Definition 1. *[7] Given a finite, non-empty set X, an AIFS A^* in X is an object having the form: $A^* = \{(x, \mu_{A^*}(x), \nu_{A^*}(x)) : x \in X)\}$, where the function $\mu_{A^*}(x)$: $X \to [0,1]$ defines the degree of membership and $\nu_{A^*}(x)$: $X \to [0,1]$ defines the degree of non-membership of element $x \in X$ and for every element $x \in X$, $0 \leq \mu_{A^*}(x) + \nu_{A^*}(x) \leq 1$.*

When $\nu_{A^*}(x) = 1 - \mu_{A^*}(x)$ for every $x \in X$, then the AIFS A^* collapses to ordinary fuzzy set A.

Thus, given an AIFS, the degree of hesitancy of x to A^* is given by:

$$\pi_{A^*}(x) = 1 - (\mu_{A^*}(x) + \nu_{A^*}(x)).$$

This is called the A-intuitionistic fuzzy (IF) index of x in A^*. For ordinary fuzzy set A, $\pi_A(x) = 0 \ \forall x \in X$.

B. Type-2 A-Intuitionistic Fuzzy Set (T2AIFS)

Definition 2. *A generalised T2AIFS denoted by $\tilde{A}^*$ is characterised by a type-2 membership function $\mu_{\tilde{A}^*}(x,u)$, and a type-2 non-membership function $\nu_{\tilde{A}^*}(x,u)$ [44], i.e.,*

$$\tilde{A}^* = \{(x,u), \mu_{\tilde{A}^*}(x,u), \nu_{\tilde{A}^*}(x,u) \mid \forall x \in X, \\ \forall u \in J_x^\mu \forall u \in J_x^\nu\}$$

in which $0 \leq \mu_{\tilde{A}^*}(x,u) \leq 1$ and $0 \leq \nu_{\tilde{A}^*}(x,u) \leq 1$ where $\forall u \in J_x^\mu$ *and* $\forall u \in J_x^\nu$ conform to the T1 constraint that $0 \leq \mu_{A^*}(x) + \nu_{A^*}(x) \leq 1$.

$$J_x^\mu = \left\{(x,u) : u \in \left[\underline{\mu}_{\tilde{A}^*}(x), \overline{\mu}_{\tilde{A}^*}(x)\right]\right\}$$
$$J_x^\nu = \{(x,u) : u \in [\underline{\nu}_{\tilde{A}^*}(x), \overline{\nu}_{\tilde{A}^*}(x)]\}$$

When there is no uncertainty, a type-1 AIFS is obtained.

Alternatively,

$$\int_{x \in X} \left[\int_{u \in J_x^\mu} \int_{u \in J_x^\nu} \{\mu_{\tilde{A}^*}(x,u), \nu_{\tilde{A}^*}(x,u)\}\right] / (x,u)$$

$$\sum_{x \in X} \left[\sum_{u \in J_x^\mu} \sum_{u \in J_x^\nu} \{\mu_{\tilde{A}^*}(x,u), \nu_{\tilde{A}^*}(x,u)\}\right] / (x,u)$$

where $\int$ is for continuous universe of discourse, and $\sum$ for discrete universe of discourse. When $\mu_{\tilde{A}^*}(x,u) = 1$, and $\nu_{\tilde{A}^*}(x,u) = 1$, a T2AIFS translates to an IT2AIFS (see Figure 2 and Equation (1)).

Definition 3. *[35] An IT2AIFS, $\tilde{A}^*$, is characterised by interval membership and non-membership functions defined as $\overline{\mu}_{\tilde{A}^*}(x)$, $\underline{\mu}_{\tilde{A}^*}(x)$ and $\overline{\nu}_{\tilde{A}^*}(x)$, $\underline{\nu}_{\tilde{A}^*}(x)$ respectively for all $x \in X$ with constraints: $0 \leq \overline{\mu}_{\tilde{A}^*}(x) + \underline{\nu}_{\tilde{A}^*}(x) \leq 1\, and\ 0 \leq \underline{\mu}_{\tilde{A}^*}(x) + \overline{\nu}_{\tilde{A}^*}(x) \leq 1$.*

In order to ensure that these constraints $0 \leq \overline{\mu}_{\tilde{A}^*}(x) + \underline{\nu}_{\tilde{A}^*}(x) \leq 1\, and\ 0 \leq \underline{\mu}_{\tilde{A}^*}(x) + \overline{\nu}_{\tilde{A}^*}(x) \leq 1$ are always satisfied, the maximum of the returned values for lower membership and lower non-membership and minimum of the values for upper membership and upper non-membership of the input uncertainty are obtained (see Equations (6) to (9)).
The two IF-indices used in this study are the IF-index of center and IF-index of variance. The IF-indices are $m-by-n$ matrix randomly generated in the interval [0,1], where m is the number of linguistic terms and n is the number of inputs. These IF-indices (hesitations) are then incorporated into the FOUs of the IT2AIFS. These indices were previously used in [23] and defined in this work as:

$$\pi_c(x) = max\,(0, (1 - (\mu_{\tilde{A}^*}(x) + \nu_{\tilde{A}^*}(x))))$$
$$\overline{\pi}_{var}(x) = max\,(0, (1 - (\overline{\mu}_{\tilde{A}^*}(x) + \underline{\nu}_{\tilde{A}^*}(x))))$$
$$\underline{\pi}_{var}(x) = max\left(0, \left(1 - \left(\underline{\mu}_{\tilde{A}^*}(x) + \overline{\nu}_{\tilde{A}^*}(x)\right)\right)\right)$$

such that: $0 \leq \pi_c(x) \leq 1 \ \ and \ \ 0 \leq \pi_{var}(x) \leq 1$.
The capability of taking the contribution of IF-index into account, aside from the non-membership degree, in the partitioning of the input space gives this approach a leverage over some conventional fuzzy approaches.
As defined above, an IT2AIFS $\tilde{A}^*$ is characterised by interval type-2 membership function, $\mu_{\tilde{A}^*}(x,u)$ and interval type-2 non-membership function, $\nu_{\tilde{A}^*}(x,u)$ for all $x \in X$ expressed as:

$$\begin{aligned} \tilde{A}^* &= \int_{x \epsilon X} \int_{u \epsilon J_x^\mu} \int_{u \epsilon J_x^\nu} 1/(x,u) \\ &= \int_{x \epsilon X} \left[\int_{u \epsilon J_x^\mu} \int_{u \epsilon J_x^\nu} 1/(u)\right] / x \end{aligned} \tag{1}$$

where x is the primary variable, and u is the secondary variable. The uncertainty about an IT2AIFS is completely described by the footprints of uncertainty (FOUs) as shown in Figure 2 and expressed as:

$$\begin{aligned} \bar{\mu}_{\tilde{A}}(x) &\equiv \overline{FOU_\mu(\tilde{A}^*)} \ \ \forall x \in X \\ \underline{\mu}_{\tilde{A}}(x) &\equiv \underline{FOU_\mu(\tilde{A}^*)} \ \ \forall x \in X \\ \bar{\nu}_{\tilde{A}}(x) &\equiv \overline{FOU_\nu(\tilde{A}^*)} \ \ \forall x \in X \\ \underline{\nu}_{\tilde{A}}(x) &\equiv \underline{FOU_\nu(\tilde{A}^*)} \ \ \forall x \in X \end{aligned} \tag{2}$$

Thus, two FOUs are defined namely: FOU_μ regarding the uncertainty of the membership function and FOU_ν defined with respect to the non-membership function of IT2AIFS $\tilde{A}^*$ (see Figure 2) as follows:

$$FOU_\mu\left(\tilde{A}^*\right) = \bigcup_{\forall x \in X} \left[\underline{\mu}_{\tilde{A}^*}(x), \bar{\mu}_{\tilde{A}^*}(x)\right] \tag{3}$$

$$FOU_\nu\left(\tilde{A}^*\right) = \bigcup_{\forall x \in X} [\underline{\nu}_{\tilde{A}^*}(x), \bar{\nu}_{\tilde{A}^*}(x)] \tag{4}$$

We have provided the background on which our fuzzy framework is based. We highlighted variants of fuzzy sets that have motivated this study. However, because of the associated complexities of the GT2FS as discussed above, we construct our A-intuitionistic fuzzy framework based on the notion of IT2FS.

III. Interval Type-2 A-Intuitionistic Fuzzy Logic System

The main purpose of this work is to introduce a new interval type-2 A-intuitionistic fuzzy logic system (IT2AIFLS) that utilises the membership and non-membership functions together with hesitancy index (HI) for regression problems. The proposed type-2 AIFLS (T2AIFLS) consists of the fuzzifier, rule base, fuzzy inference engine and output processing module which is similar to a T2FLS, but because of the intuitionism involved in the fuzzy set, the T2AIFLS modules are therefore referred to A-intuitionistic fuzzifier, A-intuitionistic rule base, A-intuitionistic fuzzy inference engine and A-intuitionistic output processing [17] as shown in Figure 1.

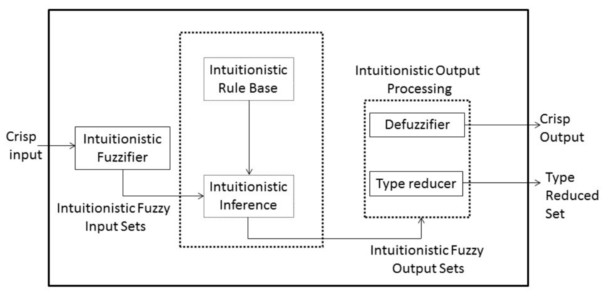

Fig. 1. Type-2 A-Intuitionistic Fuzzy Logic System

A. Fuzzification

There are two fuzzification procedures namely: singleton and non-singleton. For this study, our focus is on singleton fuzzification. We intend to use non-singleton fuzzification in a future study and compare their performances using the proposed model. The fuzzification process involves the mapping of a numeric input vector $x \in X$ into an IT2AIFS $\tilde{A}^*$ in X which activates the inference engine. For each crisp input $x \in X$, A-intuitionistic fuzzy values for membership and non-membership are generated. Here, interval singleton type-2 fuzzification is used to obtain membership and non-membership values.
For membership:

$$\mu_{\tilde{A}^*}(x) = \begin{cases} 1/1, & \text{if } x = x' \\ 1/0, & \text{if } x \neq x' \end{cases}$$

For non-membership:

$$\nu_{\tilde{A}^*}(x) = \begin{cases} 1/1, & \text{if } x = x' \\ 1/0, & \text{if } x \neq x' \end{cases}$$

The firing strength for membership and non-membership functions are intervals $[\underline{f^{\mu}}, \overline{f^{\mu}}]$ and $[\underline{f^{\nu}}, \overline{f^{\nu}}]$ respectively. A number of membership functions exist which are employed in the computation of type-2 fuzzy grades (fuzzification). These include triangular, trapezoidal, Gaussian, sigmoidal and others. In the literature, many applications benefit from the use of Gaussian functions for the design of FLSs because they have only two design parameters and thus help to reduce the computation time. In this study, the Gaussian is also adopted for the representation of both the membership and non-membership functions of the IT2AIFLS. According to Wu [57], "Gaussian IT2 FLCs are simpler in design because they are easier to represent and optimize, always continuous, and faster for small rulebases." For classical Gaussian IT2FLS, uncertainties can be associated to the standard deviation or mean of the fuzzy set. Mathematically, the classical Gaussian membership function is defined as follows:

$$\mu_{ik}(x_i) = exp\left(-\frac{(x_i - c_{ik})^2}{2\sigma_{ik}^2}\right) \tag{5}$$

where each membership function in the antecedents of the rule can be represented as an upper and lower membership functions with c and σ representing the center and standard deviation respectively assigned to the i_{th} input and k_{th} rule of the fuzzy system.

In this study, the classical Gaussian function is modified with the inclusion of hesitation indices. Thus, for IT2AIFS, A-intuitionistic Gaussian membership (Equations 6 and 7) and non-membership functions (Equations 8 and 9) with uncertain standard deviation are utilised which are defined as follows.

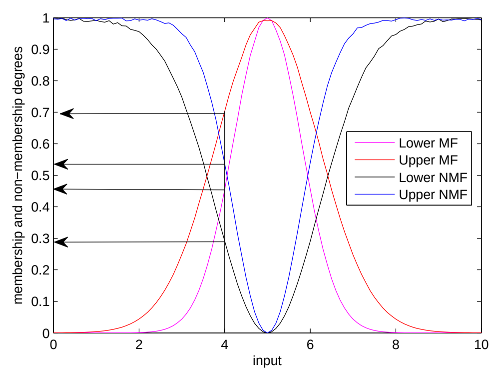

Fig. 2. An IT2 A-intuitionistic Gaussian membership and non-membership functions - IT2AIFS [44]

$$\overline{\mu_{ik}}(x_i) = min(1, exp\left(-\frac{(x_i - c_{ik})^2}{2\sigma_{2,ik}^2}\right) * (1 - \pi_{c,ik}(x_i))) \tag{6}$$

$$\underline{\mu_{ik}}(x_i) = max(0, exp\left(-\frac{(x_i - c_{ik})^2}{2\sigma_{1,ik}^2}\right) * (1 - \pi_{c,ik}(x_i))) \tag{7}$$

$$\overline{\nu_{ik}}(x_i) = min(1, (1 - \overline{\pi}_{var,ik}(x_i)) - \left[exp\left(-\frac{(x_i - c_{ik})^2}{2\sigma_{1,ik}^2}\right) * (1 - \pi_{c,ik}(x_i))\right]) \tag{8}$$

$$\underline{\nu_{ik}}(x_i) = \left(1 - \underline{\pi}_{var,ik}(x_i)\right) - \left[exp\left(-\frac{(x_i - c_{ik})^2}{2\underline{\sigma}^2_{2,ik}}\right) * (1 - \pi_{c,ik}(x_i))\right] \quad (9)$$

where $\pi_{c,ik}$ is the IF-index of center and $\pi_{var,ik}$ is the IF-index of variance [23][1]. The parameters $\sigma_{2,ik}$, $\sigma_{1,ik}$, $\pi_{c,ik}$, $\pi_{var,ik}$ and c are premise parameters. Shown in Figure 2 is an IT2 A-intuitionistic Gaussian membership and non-membership functions which characterise IT2AIFS. The FOU for the membership is bounded by lower membership and upper membership functions while the FOU of the non-membership is bounded by lower non-membership and upper non-membership functions respectively. The FOUs of the model are as shown in Figure 2. The bounds of the FOUs are somewhat wavy (ripples). A concept which incorporates the IF-indices (degrees of hesitancy) in the definitions of the FOUs of IT2AIFS. The scaling in Equations (6) and (7) captures the hesitation of the expert in the definition of the membership function FOU while Equations (8) and (9) include some shifting which captures the hesitation in the FOU of the non-membership function of the IT2AIFS. We have highlighted this concept in Figure 3. This representation satisfies the constraint in Definition 3. For instance, the membership and non-membership grades of $x = 4.0$ in Figure 2 are approximately $\{0.45,\ 0.69,\ 0.28,\ 0.54\}$, which satisfies the constraints: $0 \le \overline{\mu}_{\tilde{A}^*}(x) + \underline{\nu}_{\tilde{A}^*}(x) \le 1\, and\ 0 \le \underline{\mu}_{\tilde{A}^*}(x) + \overline{\nu}_{\tilde{A}^*}(x) \le 1$ as shown below following from Figure 2.

$$\begin{aligned}
\overline{\mu}_{\tilde{A}^*}(x) + \underline{\nu}_{\tilde{A}^*}(x) &= 0.69 + 0.28 \\
&= 0.97 \in [0, 1] \\
\pi_{\tilde{A}^*}(x) &= 1 - 0.97 \\
&= 0.03 \in [0, 1] \\
\underline{\mu}_{\tilde{A}^*}(x) + \overline{\nu}_{\tilde{A}^*}(x) &= 0.45 + 0.53 \\
&= 0.98 \in [0, 1] \\
\pi_{\tilde{A}^*}(x) &= 1 - 0.98 \\
&= 0.02 \in [0, 1]
\end{aligned}$$

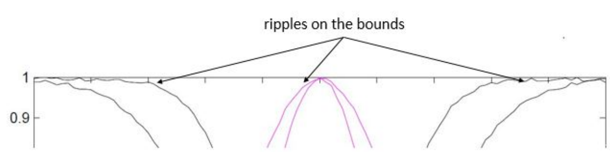

Fig. 3. Hesitancy of the expert as reflected on the FOUs of IT2AIFS

B. Rules

The rule representation of IT2AIFLS is similar to the classical IT2FLS, the only exception is that both membership and non-membership functions are involved in the inputs of the IT2AIFLS, that is, the fuzzy sets are A-intuitionistic fuzzy sets of type-2. The IF-THEN rule of an IT2AIFLS can thus be expressed as follows:

$$\begin{aligned}
R_k : IF\, x_1\, is\, \tilde{A}^*{}_{1k}\ \cdots and\, x_n\, is\, \tilde{A}^*{}_{nk} \\
THEN\, y_k\, is\, f\,(x_1, x_2, \cdots, x_n) \\
= w_{1k}x_1 + w_{2k}x_2 + \cdots + w_{nk}x_n + b_k \quad (10)
\end{aligned}$$

where $\tilde{A}^*{}_{1k}$,$\tilde{A}^*{}_{2k}$, $\cdots$,$\tilde{A}^*{}_{ik}$,$\cdots$,$\tilde{A}^*{}_{nk}$ are IT2AIFS and y_k is the output of the kth rule formed by linear combination of the input vector: $(x_1, x_2, \cdots, x_n)$. The above general rule for IT2AIFLS can be decomposed into both membership and non-membership functions as follows:
For membership function, the rule in Equation (10) translates to:

$$\begin{aligned}
R^{\mu}_k : IF\, x_1\, is\, \tilde{A}^*{}^{\mu}_{1k}\ \cdots and\, x_n\, is\, \tilde{A}^*{}^{\mu}_{nk} \\
THEN\, y^{\mu}_k = w^{\mu}_{1k}x_1 + w^{\mu}_{2k}x_2 + \cdots + w^{\mu}_{nk}x_n + b^{\mu}_k
\end{aligned}$$

For non-membership function, the rule becomes:

$$\begin{aligned}
R^{\nu}_k : IF\, x_1\, is\, \tilde{A}^*{}^{\nu}_{1k}\, and\ \cdots and\, x_n\, is\, \tilde{A}^*{}^{\nu}_{nk} \\
THEN\, y^{\nu}_k = w^{\nu}_{1k}x_1 + w^{\nu}_{2k}x_2 + \cdots + w^{\nu}_{nk}x_n + b^{\nu}_k
\end{aligned}$$

where y^{μ}_k and y^{ν}_k are the membership and non-membership outputs of the kth rule, $w's$ are the function parameters (coefficients of the independent variables) plus a constant term b known as the bias.

C. Inference

The inferencing approach adopted for this study is the Takagi-Sugeno-Kang fuzzy inferencing where the antecedent parts are IT2AIFS and the consequent parts are linear combinations of the inputs, otherwise known as A2-C0 model.

The learning of IT2AIFLS is similar to adaptive-neuro fuzzy inference system (ANFIS) [58] and T2-ANFIS [59] approaches. An IT2AIFLS structure with two inputs, three membership functions and nine rules is as shown in Figure 4. According to [22], the output of IFIS-TSK can be computed

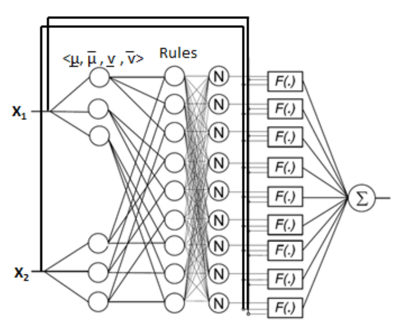

Fig. 4. An IT2AIFLS Structure - adapted from [60]

using two approaches: (i) by the combination of membership output, y^{μ}, and non-membership function output, y^{ν} and (ii) by direct defuzzification. In this study, the former approach is adopted and the final output of IT2AIFLS which is a closed form equation is defined as follows:

[1] Petr Hajek in an email conversation pointed out that *"IF-index of centre is used to express the hesitancy on the centre of the membership function while the IF-index of variance represents the hesitancy on the radius"* and these values are small numbers in the interval [0,1]

$$y = \frac{(1-\beta)\sum_{k=1}^{M}\left(\underline{f_k^{\mu}} + \overline{f_k^{\mu}}\right)y_k^{\mu}}{\sum_{k=1}^{M}\underline{f_k^{\mu}} + \sum_{k=1}^{M}\overline{f_k^{\mu}}} + \frac{\beta\sum_{k=1}^{M}\left(\underline{f_k^{\nu}} + \overline{f_k^{\nu}}\right)y_k^{\nu}}{\sum_{k=1}^{M}\underline{f_k^{\nu}} + \sum_{k=1}^{M}\overline{f_k^{\nu}}} \quad (11)$$

where $\underline{f}_k^{\mu}$, $\overline{f}_k^{\mu}$, $\underline{f}_k^{\nu}$ and $\overline{f}_k^{\nu}$ are the lower membership, upper membership, lower non-membership and upper non-membership firing strength respectively. This is a modification of a novel inference method proposed in [61] for IT2-TSK fuzzy system and motivated by the Nie-Tan [62] closed form type-reduction method for IT2FLS where iterations are not required in the computation of the defuzzified crisp value but depends only on the lower and upper bounds of the membership function footprint of uncertainty (FOU). As shown in equation (11), the final output of IT2AIFLS apart from also utilising the bounds of the membership function FOU, also utilises the upper and lower bounds of the non-membership function FOU with an additional design factor β [61] to weigh their contribution in the final output. In this study, the implication operator employed is the "prod" t-norm such that:

$$\underline{f_k^{\mu}}(x) = \underline{\mu}_{\tilde{A^*}_{1k}}(x_1) \cdot \underline{\mu}_{\tilde{A^*}_{2k}}(x_2) \cdot \cdots \cdot \underline{\mu}_{\tilde{A^*}_{nk}}(x_n)$$
$$\overline{f_k^{\mu}}(x) = \overline{\mu}_{\tilde{A^*}_{1k}}(x_1) \cdot \overline{\mu}_{\tilde{A^*}_{2k}}(x_2) \cdot \cdots \cdot \overline{\mu}_{\tilde{A^*}_{nk}}(x_n)$$
$$\underline{f_k^{\nu}}(x) = \underline{\nu}_{\tilde{A^*}_{1k}}(x_1) \cdot \underline{\nu}_{\tilde{A^*}_{2k}}(x_2) \cdot \cdots \cdot \underline{\nu}_{\tilde{A^*}_{nk}}(x_n)$$
$$\overline{f_k^{\nu}}(x) = \overline{\nu}_{\tilde{A^*}_{1k}}(x_1) \cdot \overline{\nu}_{\tilde{A^*}_{2k}}(x_2) \cdot \cdots \cdot \overline{\nu}_{\tilde{A^*}_{nk}}(x_n)$$

where $\cdot$ is the "prod" operator, y_k^{μ} and y_k^{ν} are the outputs of the kth rule for membership and non-membership functions respectively. The overall output of IT2AIFLS is a weighted average of each IF-THEN rule's output. The parameter β, $0 \leq \beta \leq 1$; specifies the weight of the membership and non-membership values in the final output. Apparently, if $\beta = 0$, the IT2AIFLS output is computed using only the membership function and if $\beta = 1$, then only the non-membership function contributes to the model's output. With the neural network learning ability, the parameters of the IT2AIFS are tuned as discussed in the next section.

IV. Parameter Update Rule

This study utilizes gradient descent algorithm (GDA) for the update of both the antecedent and the consequent parameters of the rules. The gradient-based algorithms are widely used in the literature for training FLSs and one of the important concepts involved in GD learning is the cost function. The cost function is a measure of the deviation of a particular solution from an optimal solution to the problem being solved. The GDA searches through the solution space to find a function that has the lowest possible cost (error). The cost function for a single output is defined as:

$$E = \frac{1}{2}\left(y^a - y\right)^2$$

where y^a is the actual output and y is the network output. The antecedent parameters are (c, σ_1^{μ}, σ_2^{μ}, σ_1^{ν}, σ_2^{ν}) while the consequent parameters are $(w^{\mu}, b^{\mu}, w^{\nu}, b^{\nu})$. For this study, σ_1^{μ} = σ_1^{ν} and σ_2^{μ} = σ_2^{ν}. The update rule for the generic parameter θ is as expressed in Equation (12):

$$\theta_{ik}(t+1) = \theta_{ik}(t) - \gamma\frac{\partial E}{\partial \theta_{ik}} \quad (12)$$

where γ is the learning rate (step size) that must be carefully chosen as a large value may lead to instability, and small value on the other hand may lead to a slow learning process. The IF-indices for this study are fixed i.e. they are not tuned. We intend to tune them in a future study. This will enable us to investigate the effect of tuning the IF-indices on the interval type-2 A-intuitionistic fuzzy sets separately. The parameter β in Equation (11) is initially specified and then tuned to allow for adaptive adjustment of the membership and non-membership function in the final output. A number of experiments are conducted using publicly available benchmark regression datasets in order to test the validity and efficiency of our proposed framework.

V. Experiments and Results

Next, we present our experimental analysis and discussion of simulation results. We demonstrate the effectiveness of IT2AIFLS on some regression problems. The IT2AIFLS utilises the IT2AIFS [44] which is represented in Figure 2 to as inputs to the fuzzy logic system. For a fair comparison of our proposed model with previous approaches, the experimental evaluations are conducted on the basis of same datasets (benchmark) that are publicly available, same numbers of input partitioning and same performance metrics (RMSE and NMSE). Each of the datasets are arranged as closely as possible to those reported previously in the literature. We measure the robustness of the approach by evaluation in the presence of some noise in the data such as the Friedman problem. For ease of comparison with previous works in the literature, we also adopt the root mean squared error (RMSE) as the performance criterion for all experiments. Using the test dataset to evaluate model performance gives an unbiased estimate [2] of the model error. The RMSE is computed as follows:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(y_i^a - y_i\right)^2}$$

where y^a is the actual output, y is the output of the model and N is the number of testing data points.

The number of parameters of the proposed framework for all datasets are $6n+2M(n+1)$, where n is the number of inputs, and M is the number of rules. For all experiments, we assumed that there are uncertainties in only the antecedent's part of each rule. The number of membership and non-membership functions for each input of the IT2AIFLS is arbitrarily set to 2 and the parameter β is initially set at 0.5 for all simulations. All experiments are carried out using $MATLAB^{©}$ 2015

[2] https://uk.mathworks.com

running on a 64-bit Intel core i3-4130 CPU@3.40GHz /8GB RAM configuration computer.

A. IT2AIFLS vs FIS, IFIS and IT2FLS on Regression Problems

This section compares the performance of IT2AIFLS with FIS and IFIS for both linear and non-linear regression problems. The regression datasets used for the analysis are Friedman, energy, stock and autoMPG6 which are obtained from [63]. We adopt the same computational protocol in [18] for Friedman, energy, stock and autoMPG6 dataset to aid comparison with FIS and IFIS.

1) Datasets Description:

TABLE I
DATASET CHARACTERISTICS

Dataset	No. of input	No. of samples	Trn set	Tst set
Friedman#1	5	1200	600	600
Friedman#2	5	1200	200	1000
Energy	6	365	183	182
Stock	9	950	475	475
AutoMPG6	5	392	196	196
Elect. volt. line	2	495	396	99
Elect.Maint	4	1059	847	212
Abalone	8	4177	3342	835

Friedman [64] The Friedman prediction problem is a synthetic dataset with the following data generation formula:

$$y = 10sin\left(\pi x_1 x_2\right) + 20\left(x_3 - 0.5\right)^2 + 10x_4 + 5x_5 + \hat{n} \quad (13)$$

where $x_i s$ are the input variables and $\hat{n}$ is white Gaussian noise with a mean of zero and standard deviation of 1. Each of the sample consists of five input variables x_1, x_2, x_3, x_4, x_5 independently and uniformly distributed over $[0,\ 1]$ and one target variable y generated using equation (13). For the Friedman dataset, 1200 data samples are randomly generated which are then split equally into 600 samples for training and 600 samples for testing, we shall refer to the dataset as Friedman#1. There are a total of 32 rules for Friedman dataset with 6(5) + 2*32(5+1) = 414 parameters.

Energy [63]: The daily electric energy problem involves the prediction of the daily average price of TkWhe electricity energy in Spain. The data set contains real values from 2003 about the daily consumption in Spain of energy from hydroelectric, nuclear electric, carbon, fuel, natural gas and other special sources of energy. There are a total of 365 data instances. For energy dataset, IT2AIFLS generated 64 rules with a total of 932 parameters.

Stock [63]: The stock dataset is a highly non-stationary dataset and consists of daily stock prices from January 1988 to October 1991 for ten aerospace companies. The task is to predict the price of the 10th company based on the prices of the other nine (9) companies. The dataset consists of 950 samples. Stock data is a high-dimensional dataset with a total number of 512 rules and 10294 parameters for the IT2AIFLS.

AutoMPG6 [63]: The task here is to predict the city-cycle fuel consumption in miles per gallon (mpg) in terms of 1 multi-valued discrete and 5 continuous attributes (where two multi-valued discrete attributes - Cylinders and Origin - from the original dataset are removed). For autoMPG6, 392 data samples are available for analysis. The total number of parameters for AutoMPG6 are 414 with 32 rules.

The analysis of the above datasets was previously conducted by Hajek and Olej [18] using type-1 A-intuitionistic fuzzy inference system. We extend this work by employing IT2AIFLS to the same datasets. For ease of comparison, the above datasets are arranged as closely as possible to those reported in [18]. The datasets (Friedman#1, energy, stock and autompg6) are randomly sampled 5 times and sequentially split into two equal parts as in Table I for each run with 500 training epochs. The results presented in Table II shows the average RMSE and standard deviation (SD) over 25 simulations for each dataset. The initial values of the consequent parts of the rule $(w\ \ and\ \ b)$ for membership and non-membership, are generated randomly from the interval $[0,\ 1]$ and updated using the above derived parameter update rule. The learning rate is chosen as 0.1. The RMSE defined as in Equation (V) is used as performance criterion. Table II shows the comparison of the RMSE on the test data of IT2AIFLS with FIS and IFIS (which also uses the design parameter β to weigh the membership and non-membership contributions to the final output). From Table II, IT2AIFLS outperforms both FIS and IFIS on the selected test samples. This is consistent with the reports in the literature that T2FLSs (IT2AIFLSs in this case) model uncertainty in certain applications better than T1FLSs. Our proposed model of IT2IFLS is also compared with classical IT2FLS, as shown in Table II, IT2IFLS performs better than the classical IT2FLS. Due to additive noise in the Friedman dataset, 30 Monte-Carlo simulations are also realised and the average RMSE and standard deviation for IT2AIFLS are 1.0865 and 0.058 respectively.

B. Friedman#2

This example studies the Friedman problem as reported in [65]. In this example, we perform experiments using Friedman to evaluate the performance of our model on non-fuzzy and fuzzy approaches, particularly with other T2 fuzzy approaches. For comparison purpose, we adopt the experimental set-up as reported in [65]. Similar to [65], 1400 samples are randomly generated using equation (13), 200 samples are used for training, 200 for validation while the remaining 1000 samples are used for testing (this we shall refer to test dataset 1) and this is repeated 20 times with the average RMSE and standard deviation reported in Table III. The learning rate $\alpha = 0.1$. The plot of the actual and predicted output is as shown in Figure 5). This problem was also analysed in [66] and [67]. While Carney and Cunningham [66] employed neural bootstrap aggregation (NBAG), benchmark and simple bagged ensemble; Lee *et al.* [67] on the other hand proposed a general regression neural

TABLE II
RMSE AND SD OF IT2AIFLS VS FIS/IFIS/IT2FLS ON REGRESSION PROBLEMS

Models	Friedman#1	Energy	Stock	AutoMPG6
FIS [18]	1.353 ± 0.026	7.443 ± 1.579	1.423± 0.227	3.702 ± 0.211
IFIS [18]	1.332 ± 0.032	4.776 ± 2.776	1.402 ± 0.219	3.684 ± 0.195
IT2FLS-TSK	1.095 ± 0.046	0.567 ± 0.125	0.750 ± 0.026	1.792 ± 0.048
IT2AIFLS	1.026 ± 0.011	0.558 ± 0.005	0.611 ± 0.006	1.700 ± 0.064

network with fuzzy adaptive resonance theory (GRNNFA) for the analysis of this first set of data. Similar to [65], we also study the performance of IT2AIFLS when the output of the nonlinear Friedman equation is noise free. In this second case, 1000 test samples are generated with $\hat{n} = 0$ (this we refer to test dataset 2). Similar to [65] we adopt self-constructing neural fuzzy inference network (SONFIN) and support vector based fuzzy model (SVR-FM) for type-1 comparison with our model. The parameters of SONFIN are learned using training-error minimisation through the combination of Kalman filtering and GDA. For type-2 systems, we adopt type-2 models such as type-2 FLS, self-evolving interval type-2 fuzzy neural network (SEIT2FNN) and interval type-2 fuzzy neural network with support vector regression (IT2FNN-SVR). T2FLS employs GDA for parameter learning referred to as T2FLS-G. SEIT2FNN is designed with structure learning and utilises rule-ordered Kalman filter together with GDA for parameter learning. SEIT2FNN has IT2FS in the antecedents trained with GDA with TSK interval type-1 sets in the consequent. Two flavors of IT2FNN-SVR are proposed in [65] namely IT2FNN-SVR(N) and IT2FNN-SVR(F). The difference between the two is in the representation of the input nodes. The former consists of input nodes with numerical values with interval output nodes while the latter consists of input nodes with fuzzy numbers and interval output nodes. SONFIN and SEIT2FNN are previous studies involving the first author in [65]. We compare our results with these models already reported in the literature as shown in Table III. The results in Table III indicate the RMSE and standard deviation for AIFLS, IT2AIFLS and similar works in the literature. It is shown that IT2AIFLS exhibits lower RMSE compared to its type-1 counterpart, the non-fuzzy, the two T1FLSs and the T2FLSs. For 30 Monte-Carlo realisations, the average RMSE and standard deviation for IT2AIFLS on Friedman#2 with additive noise are 1.5057 and 0.1022 respectively.

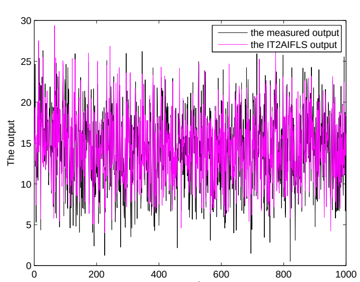

Fig. 5. Actual and predicted outputs of Friedman with Gaussian white noise

TABLE III
PERFORMANCE OF IT2AIFLS ON FRIEDMAN#2

Models	RMSE-tst1 (noisy)	Std-tst1	RMSE-tst2 (noise free)	Std-tst2
NBAG [66]	2.1218	-	-	-
Bench [66]	2.3178	-	-	-
Simple [66]	2.2244	-	-	-
GRNNFA [67]	2.136	-	-	-
SONFIN [68]	2.531	0.138	2.398	0.131
T2FLS-G [69]	2.597	0.137	2.479	0.145
SEIT2FNN [70]	1.941	0.170	1.598	0.216
IT2FNN-SVR(N) [65]	1.788	0.145	1.537	0.201
IT2FNN-SVR(F) [65]	1.597	0.120	1.291	0.151
IT2FLS-TSK	1.778	0.152	1.419	0.210
AIFLS	2.375	0.129	2.227	0.186
IT2AIFLS	1.494	0.111	1.116	0.104

C. The Electrical Engineering Distribution Problems

In [71], two problems involving electrical distribution in rural towns in Spain are proposed and have become real-world benchmark problems in fuzzy logic fields. The task here is to relate some characteristics of certain village with actual low voltage line it contains and also relate the maintenance cost of the network in certain towns with some of their characteristics.

1) Computing the Length of Low Voltage lines: The first problem proposed in [71] is to estimate the length of low voltage lines in rural towns using some available inputs. The dataset consist of 495 instances with actual values measured by a company. The dataset is divided into 396 samples for training set and 99 samples for testing set with each consisting of three attributes namely:

- Number of clients in population.
- Radius of i population in the sample.
- Line length, population i.

There are a total of 4 rules generated for low voltage line estimation with 36 number of parameters. The results presented in Table IV is averaged over ten simulations with 100 epochs and learning rate set to 0.1. It can be observed in Table IV that IT2AIFLS has superior performance. compared to the classical non-linear regression models, neural networks, the evolutionary approaches and other fuzzy approaches.

Experiment is conducted to ascertain the performance of IT2AIFLS with IT2FLS-TSK trained with the same number of design parameters using low voltage line length estimation

TABLE IV
LOW VOLTAGE LINE LENGTH ESTIMATION PROBLEM

Models	RMSE(tst)
Linear [71]	457.8821
Exponential [71]	443.8513
Second order Polynomial [71]	450.8126
Third-order polynomial [71]	450.5452
Three layer Perceptron [71]	408.7689
GA-P [71]	399.7962
Interval GA-P [71]	398.4181
WM Fuzzy model [71]	424.384
Mamdani Fuzzy model [71]	408.2511
TSK Fuzzy model [71]	385.3751
Gr + MF [72]	390.7979
Genetic Learning Process [73]	383.4866
HSLR(WM,3,5) [74]	409.04523
IT2FLS-SA [75]	606.84075
GT2FLS-sampling [75]	594.02365
GT2FLS-VSCTR [75]	590.90565
AIFLS	262.2775
IT2AIFLS	255.3325

TABLE V
PERFORMANCE COMPARISON OF IT2AIFLS WITH CLASSICAL IT2FLS ON VOLTAGE LENGTH ESTIMATION PROBLEM

Models	Parameter	RMSE	Run-time (s)
IT2FLS-TSK	24	260.7010	12.35
IT2AIFLS	24	260.1041	24.76

dataset. To achieve this, the same consequents are applied to both the membership and non-membership outputs of IT2AIFLS and this translate to 4 rules and 24 parameters for each model. The results in Table V is averaged over 10 simulation runs. As shown in Table V, with the same number of parameters defined for classical IT2FLS and intuitionistic IT2FLS, the accuracies of the two models tend to be quite close with IT2AIFLS performing slightly better than IT2FLS.

2) Computing the Maintenance Costs of Medium Voltage Lines: The second problem is to estimate the maintenance cost which are not based on real data. The dataset consists of 1059 samples with 5 attributes namely:

- Sum of the length of all street in the town.
- Total area of the town.
- Area that is occupied by buildings.
- Energy supply to the town.
- Maintenance costs of medium voltage line.

Similar to previous studies, the 1059 samples are divided into two sets: 847 samples for training and 212 samples for testing as reported in [71], [72], [73], [74] except [75] which used only 400 data instances for model evaluation (200 for training and 200 for testing). This, according to the authors, was because of the computational burden involved. Our model is executed for 100 epochs with learning rate set to 0.1. There are 16 rules generated for maintenance cost estimation with a total of 184 parameters. In order to relate the dependent variable (maintenance cost) with the independent variables, IT2AIFLS described above is applied to both the training and the test sets and results are compared with those reported in the literature and IT2FLS-TSK. Figure 6, shows the correlation between the actual and predicted output for electrical maintenance cost. This result is significant because it means that IT2AIFLS has a high descriptive capability and can be useful in modeling natural attributes of physical phenomenon. Table VI shows the performance of IT2AIFLS, classical IT2FLS-TSK and other models in the literature in terms of their RMSEs. The results in Table VI show a significant performance improvement of IT2AIFLS over IT2FLS-TSK and other works in the literature. For a fair comparison with IT2FLS-SA, GT2FLS-sampling and GT2FLS-VSCTR [75], we conducted a similar experiments as reported in [75] with the same computational set-up (i.e. 200 samples for training and 200 samples for testing). After 100 epochs of training, IT2AIFLS attains a RMSE of 123.6912 on the test dataset.

TABLE VI
MAINTENANCE COST ESTIMATION PROBLEM

Models	RMSE(tst)
Linear [71]	191.8828
Second order Polynomial [71]	212.9131
Three layer Perceptron [71]	181.9478
GA-P [71]	147.9324
Interval GA-P [71]	135.3699
WM Fuzzy model [71]	166.1776
Mamdani Fuzzy model [71]	150.3030
TSK Fuzzy model [71]	108.7934
Gr + MF [72]	102.049
Genetic Learning Process [73]	102.3034
HSLR(WM,3,5) [74]	154.3276
IT2FLS-SA [75]	353.99755*
GT2FLS-sampling [75]	424.3692*
GT2FLS-VSCTR [75]	317.43325*
IT2FLS-TSK	79.6075
AIFLS	61.1401
IT2AIFLS	53.7200

* used a subset of the dataset

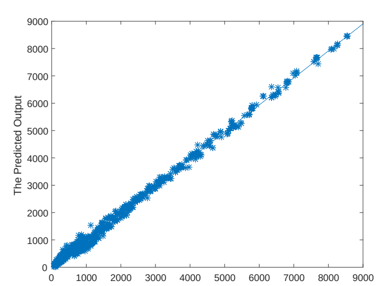

Fig. 6. Correlation analysis between the actual and predicted outputs for Electrical maintenance cost

VI. Complex High Dimensional Regression Problems

In this section, we analyse two high dimensional regression problems namely: abalone and house sales data.

- **Abalone dataset [63]**

The effectiveness of the proposed model is demonstrated using a real world high dimensional regression datasets namely the abalone dataset. The abalone dataset is a highly noisy dataset that contains physical measurements of abalone (large edible sea snails). The dataset consists of 4177 samples with 8 input attributes. The goal is to predict the age of abalone by counting the number of rings on the abalone through a microscope [63]. Similar to [76], [77], [78], [79], 5-fold cross validation is adopted where the dataset is randomly split into five folds with each set containing 20% of the dataset. For each run, four folds are used for training and one for testing. Each fold is executed 5 times and the average cross validation error for 25 trials is computed. Each trial was executed for 100 epochs with learning rate set to 0.1. For the abalone dataset, 256 rules are generated while 4656 parameters are tuned. The result of evaluation of the abalone dataset using IT2AIFLS is compared with IT2FLS-TSK, AIFLS and similar works in the literature. As shown in Table VII, IT2AIFLS exhibits MSE that is lower than other models in this problem domain.

TABLE VII
RESULTS COMPARISON OF IT2AIFLS FOR ABALONE DATASET WITH OTHER MODELS

Models	MSE(tst)	std MSE
TS-NSGA-II [79]	2.526	0.242
TS-NSGA-SPEA2$_{Acc}$ [79]	2.511	0.263
TS-NSGA-II$_A$ [79]	2.535	0.265
TS-NSGA-II$_U$ [79]	2.520	0.237
TS-NSGA-SPEA2 [79]	2.518	0.246
TS-NSGA-SPEA2$_{Acc^2}$ [79]	2.517	0.230
Multiobjective GFS [78]	2.423	0.173
FSMOGFS[77]	2.697	0.204
FSMOGFS^e [77]	2.708	0.216
FSMOGFS+TUN [77]	2.454	0.163
FSMOGFS^e +TUN^e [77]	2.509	0.184
ANFIS-SUB [76]	2.733	-
TSK-IRL [76]	2.642	-
Linear-LMS [76]	2.472	-
LEL-TSK [76]	2.412	-
METSK-HD^e[76]	2.392	-
IT2FLS-TSK	2.798	0.045
AIFLS	2.763	0.074
IT2AIFLS	1.042	0.034

In Table VIII, the final values of β for IT2AIFLS for the listed datasets are presented. The initial value for all experiments was chosen as 0.5.

TABLE VIII
TABLE SHOWING FINAL β VALUES FOR IT2AIFLS ON THE LISTED DATASETS

Fman#1	Energy	Stock	Auto MPG6	Elect. Volt	Elect. Maint	Abalone
0.22	0.64	0.9	0.27	0.41	0.37	0.58

- **House sales in King County, USA [80]**

The house sales dataset is one of the large-scale high dimensional regression problems obtained from [80]. The purpose of this analysis is to demonstrate the prediction performance between IT2AIFLS and classical IT2FLS. The house sales dataset consists of 18 features and 21,613 samples and the task is to predict the house price as closely as possible to the actual price. Figure 7 shows the house sales feature ranking. All the features below the mean ranking of 0.2 are regarded as negligible and a total of 15 input features are used in the analysis in order to reduce the computational burden of the system. The entire dataset is split into 70% training and 30% testing with 10 simulation runs and 100 epochs for each run.

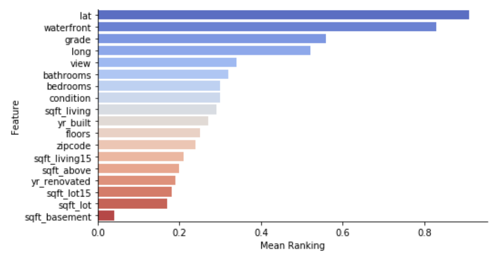

Fig. 7. Feature ranking of house sales data [81]

TABLE IX
COMPARISON OF IT2AIFLS WITH CLASSICAL IT2FLS USING LARGE AND HIGH DIMENSIONAL HOUSE SALES DATA

Models	RMSE (trn)	RMSE(tst)
IT2FLS-TSK	3.2348e-05	1.5337e-05
IT2AIFLS	2.9157e-05	1.4159e-05

Table IX shows the performance of IT2AIFLS (utilising membership and non-membership functions) compared with the classical IT2FLS with only the membership functions specification. As shown in the table, IT2AIFLS performs better than the classical IT2FLS with reduced RMSE on this problem domain. We conclude that the proposed model of IT2AIFLS is a more viable method for regression problems. The proposed framework incorporating IF-indices in the membership and non-membership functions tend to be more consistent with human or natural language description than the classical IT2FLS.

VII. Conclusion

In this study, an IT2AIFLS-TSK approach is applied to different regression problems. Consistent with previous studies in the literature on T2FLSs, the IT2AIFLS can accommodate more imprecision thereby modeling imperfect and imprecise knowledge better than T1FLSs. The key contribution in this design is the introduction of non-membership functions and Atanassov's IF-indices into IT2FLS (IT2AIFLS) which model the level of uncertainty of every input in each data set. The presence of the non-membership functions and IF-indices provides IT2AIFS more flexibility thus making it suitable for handling uncertainties even in complex situations. However, we intend to conduct more experiments in order to generalise our findings.

In the future, we intend to train IT2AIFLS using hybrid approaches involving combinations of derivative-based methods, derivative and non-derivative methods. We also intend to develop a non-singleton IT2AIFLS and apply AIFS to design a general type-2 FLS (GT2AIFLS). Although the GT2AIFLS is computationally more intensive, the results are more robust to the uncertainties inherent in many applications. For this study, only the Gaussian functions are utilised in the design of IT2AIFLS. We intend to apply other functions such as triangular and trapezoidal functions to design the proposed A-intuitionistic-based T2FLS and compare their performances. More importantly, in the future, we intend to carry out analysis of the characteristics of the proposed system in order to understand and interpret its performance in more technical terms. The associated shortcoming of the model proposed here is the high computational burden. As the dimension of the input increases, the parameters also increase (curse of dimensionality). It will be interesting to explore further ways of reducing the exponential growth of the parameters of IT2AIFLS.

Acknowledgement

The authors would like to thank Petr Hajek and Erdal Kayacan for finding time to respond to all queries on the model proposed here.

This research work was supported by the Government of Nigeria under the Tertiary Education Trust Fund (TETFund).

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] ——, "Fuzzy logic= computing with words," *Fuzzy Systems, IEEE Transactions on*, vol. 4, no. 2, pp. 103–111, 1996.

[3] ——, "Toward extended fuzzy logica first step," *Fuzzy sets and systems*, vol. 160, no. 21, pp. 3175–3181, 2009.

[4] J. M. Mendel, "Uncertain rule-based fuzzy logic system: introduction and new directions," 2001.

[5] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-i," *Information Sciences*, vol. 8, pp. 199–249, 1975.

[6] S. Mitra and S. K. Pal, "Fuzzy sets in pattern recognition and machine intelligence," *Fuzzy Sets and systems*, vol. 156, no. 3, pp. 381–386, 2005.

[7] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[8] D.-F. Li, *Decision and game theory in management with intuitionistic fuzzy sets*. Springer, 2014, vol. 308.

[9] L. Dengfeng and C. Chuntian, "New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions," *Pattern recognition letters*, vol. 23, no. 1, pp. 221–225, 2002.

[10] D.-F. Li, "Multiattribute decision making models and methods using intuitionistic fuzzy sets," *Journal of computer and System Sciences*, vol. 70, no. 1, pp. 73–85, 2005.

[11] E. Szmidt and J. Kacprzyk, "Intuitionistic fuzzy sets in some medical applications," in *Computational Intelligence. Theory and Applications*. Springer, 2001, pp. 148–151.

[12] V. Olej and P. Hájek, "If-inference systems design for prediction of ozone time series: the case of pardubice micro-region," in *Artificial Neural Networks–ICANN 2010*. Springer, 2010, pp. 1–11.

[13] O. Castillo, A. Alanis, M. Garcia, and H. Arias, "An intuitionistic fuzzy system for time series analysis in plant monitoring and diagnosis," *Applied Soft Computing*, vol. 7, no. 4, pp. 1227–1233, 2007.

[14] C.-M. Own, "Switching between type-2 fuzzy sets and intuitionistic fuzzy sets: an application in medical diagnosis," *Applied Intelligence*, vol. 31, pp. 283–291, 2009.

[15] E. Szmidt and J. Kacprzyk, "Medical diagnostic reasoning using a similarity measure for intuitionistic fuzzy sets," *Note on IFS*, vol. 10, no. 4, pp. 61–69, 2004.

[16] V. Olej and P. Hájek, "Comparison of fuzzy operators for if-inference systems of takagi-sugeno type in ozone prediction," in *Artificial Intelligence Applications and Innovations*. Springer, 2011, pp. 92–97.

[17] F. Yihong, L. Weimin, Z. Xiaoguang, and X. Xin, "Threat assessment based on adaptive intuitionistic fuzzy neural network," in *Fourth IEEE International Symposium on Computational Intelligence and Design (ISCID),*, vol. 1, 2011, pp. 262–265.

[18] P. Hájek and V. Olej, "Adaptive intuitionistic fuzzy inference systems of takagi-sugeno type for regression problems," in *Artificial Intelligence Applications and Innovations*. Springer, 2012, pp. 206–216.

[19] P. Intarapaiboon, "An application of intuitionistic fuzzy sets in text classification," in *Information Science, Electronics and Electrical Engineering (ISEEE), 2014 International Conference on*, vol. 1. IEEE, 2014, pp. 604–608.

[20] M. Akram, S. Shahzad, A. Butt, and A. Khaliq, "Intuitionistic fuzzy logic control for heater fans," *Mathematics in Computer Science*, vol. 7, no. 3, pp. 367–378, 2013.

[21] M. Akram, S. Habib, and I. Javed, "Intuitionistic fuzzy logic control for washing machines," *Indian Journal of Science and Technology*, vol. 7, no. 5, pp. 654–661, 2014.

[22] P. Hajek and V. Olej, "Defuzzification methods in intuitionistic fuzzy inference systems of takagi-sugeno type: The case of corporate bankruptcy prediction," in *11th IEEE International Conference on Fuzzy Systems and Knowledge Discovery (FSKD), 2014*, pp. 232–236.

[23] P. Hájek and V. Olej, "Intuitionistic fuzzy neural network: The case of credit scoring using text information," in *Engineering Applications of Neural Networks*. Springer, 2015, pp. 337–346.

[24] E. Sotirova, K. Atanassov, A. Shannon, T. Kim, M. Krawczak, P. Melo-Pinto, and B. Riečan, "Intuitionistic fuzzy evaluations for analysis of a student's knowledge of mathematics in university e-learning courses," in *Intelligent Systems (IS), 2016 IEEE 8th International Conference on*. IEEE, 2016, pp. 535–537.

[25] S. Sotirov, E. Sotirova, P. Melin, O. Castillo, and K. T. Atanassov, "Modular neural network preprocessing procedure with intuitionistic fuzzy intercriteria analysis method." in *FQAS*, 2015, pp. 175–186.

[26] M. B. Gorzałczany, "A method of inference in approximate reasoning based on interval-valued fuzzy sets," *Fuzzy sets and systems*, vol. 21, no. 1, pp. 1–17, 1987.

[27] M. Gehrke, C. Walker, and E. Walker, "Some comments on interval valued fuzzy sets!" *structure*, vol. 1, p. 2, 1996.

[28] G. Deschrijver and E. E. Kerre, "On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision," *Information Sciences*, vol. 177, no. 8, pp. 1860–1866, 2007.

[29] C. Cornelis, G. Deschrijver, and E. E. Kerre, "Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application," *International journal of approximate reasoning*, vol. 35, no. 1, pp. 55–95, 2004.

[30] H.-W. Liu and G.-J. Wang, "Multi-criteria decision-making methods based on intuitionistic fuzzy sets," *European Journal of Operational Research*, vol. 179, no. 1, pp. 220–233, 2007.

[31] K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets," *Fuzzy sets and systems*, vol. 31, no. 3, pp. 343–349, 1989.

[32] D. Dubois and H. Prade, "Interval-valued fuzzy sets, possibility theory and imprecise probability." in *EUSFLAT Conf.*, 2005, pp. 314–319.

[33] H. Bustince, J. Fernandez, H. Hagras, F. Herrera, M. Pagola, and E. Barrenechea, "Interval type-2 fuzzy sets are generalization of interval-valued fuzzy sets: toward a wider view on their relationship," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 5, pp. 1876–1882, 2015.

[34] H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera, and B. De Baets, "A historical account of types of fuzzy sets and their relationships," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 1, pp. 179–194, 2016.

[35] D. D. Nguyen, L. T. Ngo, and L. T. Pham, "Interval type-2 fuzzy c-means clustering using intuitionistic fuzzy sets," in *IEEE Third World Congress on Information and Communication Technologies (WICT)*, 2013, pp. 299–304.

[36] H. A. Hagras, "A hierarchical type-2 fuzzy logic control architecture for autonomous mobile robots," *IEEE Transactions on Fuzzy Systems*, vol. 12, no. 4, pp. 524–539, 2004.

[37] J. M. Mendel, R. I. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 808–821, 2006.

[38] T. Van Nghiem, D. D. Nguyen, and L. T. Ngo, "Intuitionistic type-2 fuzzy set approach to image thresholding," in *IEEE International Conference of Soft Computing and Pattern Recognition (SoCPaR)*, 2013, pp. 207–212.

[39] S. Naim and H. Hagras, "A hybrid approach for multi-criteria group decision making based on interval type-2 fuzzy logic and intuitionistic fuzzy evaluation," in *Fuzzy Systems (FUZZ-IEEE), 2012 IEEE International Conference on*. IEEE, 2012, pp. 1–8.

[40] S. Naim, H. Hagras, and A. Bilgin, "Employing an interval type-2 fuzzy logic and hesitation index in a multi criteria group decision making system for lighting level selection in an intelligent environment," in *Advances in Type-2 Fuzzy Logic Systems (T2FUZZ), 2013 IEEE Symposium on*. IEEE, 2013, pp. 1–8.

[41] S. Naim and H. Hagras, "A type 2-hesitation fuzzy logic based multi-criteria group decision making system for intelligent shared environments," *Soft Computing*, vol. 18, no. 7, pp. 1305–1319, 2014.

[42] O. Castillo, P. Melin, R. Tsvetkov, and K. Atanassov, "Short remark on interval type-2 fuzzy sets and intuitionistic fuzzy sets," in *Proceedings of the 18th International Conference on IFSs*, 2014, pp. 1–5.

[43] O. Castillo, P. Melin, R. Tsvetkov, and K. T. Atanassov, "Short remark on fuzzy sets, interval type-2 fuzzy sets, general type-2 fuzzy sets and intuitionistic fuzzy sets," in *Intelligent Systems' 2014*. Springer, 2015, pp. 183–190.

[44] I. Eyoh, R. John, and G. De Maere, "Interval type-2 intuitionistic fuzzy logic system for non-linear system prediction," in *2016 IEEE International Conference on Systems, Man and Cybernetics*, 2016, pp. 1063–1068.

[45] B. C. Cng, T. H. Anh, and B. D. Hi, "Some operations on type-2 intuitionistic fuzzy sets." *Journal of Computer Science and Cybernetics*, vol. 28, no. 3, pp. 274–283, 2012.

[46] J. M. Mendel and R. B. John, "Type-2 fuzzy sets made simple," *Fuzzy Systems, IEEE Transactions on*, vol. 10, no. 2, pp. 117–127, 2002.

[47] P. Melin and O. Castillo, "A review on type-2 fuzzy logic applications in clustering, classification and pattern recognition," *Applied soft computing*, vol. 21, pp. 568–577, 2014.

[48] O. Castillo and P. Melin, "A review on interval type-2 fuzzy logic applications in intelligent control," *Information Sciences*, vol. 279, pp. 615–631, 2014.

[49] E. Celik, M. Gul, N. Aydin, A. T. Gumus, and A. F. Guneri, "A comprehensive review of multi criteria decision making approaches based on interval type-2 fuzzy sets," *Knowledge-Based Systems*, vol. 85, pp. 329–341, 2015.

[50] T. Dereli, A. Baykasoglu, K. Altun, A. Durmusoglu, and I. B. Türksen, "Industrial applications of type-2 fuzzy sets and systems: A concise review," *Computers in Industry*, vol. 62, no. 2, pp. 125–137, 2011.

[51] H. Hagras and C. Wagner, "Towards the wide spread use of type-2 fuzzy logic systems in real world applications," *IEEE Computational Intelligence Magazine*, vol. 7, no. 3, pp. 14–24, 2012.

[52] D. Wu, "On the fundamental differences between interval type-2 and type-1 fuzzy logic controllers," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 5, pp. 832–848, 2012.

[53] H. Hagras, "Type-2 FLCs: a new generation of fuzzy controllers," *Computational Intelligence Magazine, IEEE*, vol. 2, no. 1, pp. 30–43, 2007.

[54] Y. Lin, X. Zhou, S. Gu, and S. Wang, "The takagi-sugeno intuitionistic fuzzy systems are universal approximators," in *Consumer Electronics, Communications and Networks (CECNet), 2012 2nd International Conference on*. IEEE, 2012, pp. 2214–2217.

[55] E. Hisdal, "The if then else statement and interval-valued fuzzy sets of higher type," *International Journal of Man-Machine Studies*, vol. 15, no. 4, pp. 385–455, 1981.

[56] P. Ejegwa, A. Akubo, and O. Joshua, "Intuitionistic fuzzy set and its application in career determination via normalized euclidean distance method," *European Scientific Journal*, vol. 10, no. 15, 2014.

[57] D. Wu, "Twelve considerations in choosing between gaussian and trapezoidal membership functions in interval type-2 fuzzy logic controllers," in *Fuzzy Systems (FUZZ-IEEE), 2012 IEEE International Conference on*. IEEE, 2012, pp. 1–8.

[58] J.-S. R. Jang, "ANFIS: Adaptive-network-based fuzzy inference system," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 23, no. 3, pp. 665–685, 1993.

[59] R. I. John and C. Czarnecki, "A type 2 adaptive fuzzy inferencing system," in *IEEE International Conference on Systems, Man, and Cybernetics, 1998.*, vol. 2, pp. 2068–2073.

[60] M. A. Khanesar, E. Kayacan, M. Teshnehlab, and O. Kaynak, "Analysis of the noise reduction property of type-2 fuzzy logic systems using a novel type-2 membership function," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 5, pp. 1395–1406, 2011.

[61] M. B. Begian, W. W. Melek, and J. M. Mendel, "Parametric design of stable type-2 tsk fuzzy systems," in *IEEE Annual Meeting of the North American Fuzzy Information Processing Society, (NAFIPS)*, 2008, pp. 1–6.

[62] M. Nie and W. W. Tan, "Towards an efficient type-reduction method for interval type-2 fuzzy logic systems," in *Fuzzy Systems, 2008. FUZZ-IEEE 2008.(IEEE World Congress on Computational Intelligence). IEEE International Conference on*. IEEE, 2008, pp. 1425–1432.

[63] http://www.keel.es/, accessed: 2016-06-23.

[64] J. H. Friedman, "Multivariate adaptive regression splines," *The annals of statistics*, pp. 1–67, 1991.

[65] C.-F. Juang, R.-B. Huang, and W.-Y. Cheng, "An interval type-2 fuzzy-neural network with support-vector regression for noisy regression problems," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 4, pp. 686–699, 2010.

[66] J. G. Carney and P. Cunningham, "Tuning diversity in bagged ensembles," *International Journal of Neural Systems*, vol. 10, no. 04, pp. 267–279, 2000.

[67] E. W. Lee, C. P. Lim, R. K. Yuen, and S. Lo, "A hybrid neural network model for noisy data regression," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 34, no. 2, pp. 951–960, 2004.

[68] C.-F. Juang and C.-T. Lin, "An online self-constructing neural fuzzy inference network and its applications," *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 1, pp. 12–32, 1998.

[69] J. M. Mendel, "Computing derivatives in interval type-2 fuzzy logic systems," *IEEE Transactions on Fuzzy Systems*, vol. 12, no. 1, pp. 84–98, 2004.

[70] C.-F. Juang and Y.-W. Tsao, "A self-evolving interval type-2 fuzzy neural network with online structure and parameter learning," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 6, pp. 1411–1424, 2008.

[71] O. Cordón, F. Herrera, and L. Sánchez, "Solving electrical distribution problems using hybrid evolutionary data analysis techniques," *Applied Intelligence*, vol. 10, no. 1, pp. 5–24, 1999.

[72] O. Cordón, F. Herrera, and P. Villar, "Generating the knowledge base of a fuzzy rule-based system by the genetic learning of the data base," *IEEE Transactions on fuzzy systems*, vol. 9, no. 4, pp. 667–674, 2001.

[73] O. Cordón, F. Herrera, L. Magdalena, and P. Villar, "A genetic learning process for the scaling factors, granularity and contexts of the fuzzy rule-based system data base," *Information Sciences*, vol. 136, no. 1, pp. 85–107, 2001.

[74] O. Cordón, F. Herrera, and I. Zwir, "Linguistic modeling by hierarchical systems of linguistic rules," *Fuzzy Systems, IEEE Transactions on*, vol. 10, no. 1, pp. 2–20, 2002.

[75] M. Almaraashi, R. John, A. Hopgood, and S. Ahmadi, "Learning of interval and general type-2 fuzzy logic systems using simulated annealing: Theory and practice," *Information Sciences*, vol. 360, pp. 21–42, 2016.

[76] M. J. Gacto, M. Galende, R. Alcalá, and F. Herrera, "Metsk-hd e: A multiobjective evolutionary algorithm to learn accurate tsk-fuzzy systems in high-dimensional and large-scale regression problems," *Information Sciences*, vol. 276, pp. 63–79, 2014.

[77] R. Alcalá, M. J. Gacto, and F. Herrera, "A fast and scalable multiobjective genetic fuzzy system for linguistic fuzzy modeling in high-dimensional regression problems," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 4, pp. 666–681, 2011.

[78] P. Pulkkinen and H. Koivisto, "A dynamically constrained multiobjective genetic fuzzy system for regression problems," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 1, pp. 161–177, 2010.

[79] M. J. Gacto, R. Alcalá, and F. Herrera, "Adaptation and application of multi-objective evolutionary algorithms for rule reduction and parameter tuning of fuzzy rule-based systems," *Soft Computing*, vol. 13, no. 5, pp. 419–436, 2009.

[80] https://www.kaggle.com/datasets, accessed: 2017-10-21.

[81] https://www.kaggle.com/arthurtok/feature-ranking\
\-rfe-random-forest-linear-models, accessed: 2017-10-21.