RESEARCH ARTICLE

The effect of Rényi entropy on Hawking radiation

Yang Liu[1,2]

Abstract

It is widely believed that Hawking radiation originates from excitations near the horizons of black holes (Hawking in Commun Math Phys 43:199, 1975, Giddings in Phys Lett B 754:39, 2016, Mathur in Class Quantum Gravity 26:224001, 2009). However, Giddings (Phys Lett B 754:39, 2016) proposed that the Hawking radiation spectrum that characterizes evaporating semi-classical black holes originates from a quantum "atmosphere", which extends beyond the horizon of a black hole. Although several research projects have been conducted in this field, they have not yet taken into account the effect of Rényi entropy. In the present article, we will therefore consider the effect of Rényi entropy on Hawking radiation power. We assume that if the effect of Rényi entropy is very small, we suggest that the Hawking radiation should originate from the quantum "atmosphere" which extends beyond the black hole's horizon for finite dimensions. That is, that Giddings' suggestion is the more likely of the above possibilities. However, for infinite dimensions, both suggestions are equally credible. We briefly consider the very large effect of Rényi entropy on Hawking radiation power as well. We find that if the effect of Rényi entropy is very large and ω/T_{BH} is very small, then the power spectral density S_R is proportional to the power spectral density S_{BH}.

Keywords Rényi entropy · Hawking radiation · Quantum atmosphere

Contents

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1 Introduction

Based on Hawking's original paper, the evaporation process of black holes is characterized by a non-unitary evolution of quantum fields in curved spacetimes [1]. More specifically, matter fields in a pure quantum state will collapse to form a black hole that eventually evaporates into a mixed thermal state [1]. It is widely believed that the semi-classical Hawking radiation spectra should be modified to ensure quantum unitarity since unitary temporal evolution is one of the fundamental principles of quantum mechanics [2,3].

It is generally believed that Hawking radiation spectra originate from quantum excitations near the horizon region, namely, $\Delta r = r - r_H \ll r_H$ [2,3]. Therefore, it was widely expected that modified Hawking radiation spectra should also be characterized by a relatively short length scale $\Delta r \ll r_H$, where r_H is the radius of the black hole's horizon [3].

However, Giddings suggested that the radiation spectra of a black hole originates from an effective quantum "atmosphere" which extends well beyond the horizon of the black hole [2]. By comparing the numerical results for the Hawking radiation power P_{BH} of an evaporating $(3 + 1)$-dimensional Schwarzschild black hole of horizon radius r_H with the familiar Stefan–Boltzmann radiation power $P_{BB} = \sigma A T^4$ of a $(3 + 1)$-dimensional flat space perfect blackbody emitter of radius r_A [4,5], Giddings concluded that the source of Hawking radiation was a quantum region which was located outside the black hole and that the effective radius r_A was described by the following relation

$$\Delta r = r_A - r_H \sim r_H. \tag{1.1}$$

According to Ref. [2], Eq. (1.1) is consistent with the existence of an effective emitting "atmosphere" which extends well beyond the horizon of a black hole, i.e., "atmosphere" is outside and far from the horizon of a black hole.

In considering the correctness of Giddings' conclusion, one interesting question that should be addressed is if the radiation spectra of a black hole in any $(D + 1)$-dimension originates from an effective quantum "atmosphere" which extends well beyond the horizon of the black hole? In order to answer this question, in Ref. [6], the author studied the Hawking radiation spectra of Schwarzschild black holes in a $(D + 1)$-dimension. Following Giddings's method, the author defined the effective radii $r_A(D)$ of the black-hole quantum atmospheres by equating the Hawking radiation

powers of the $(D+1)$-dimensional black holes to the corresponding Stefan–Boltzmann radiation powers of flat space perfect blackbody emitters [6]. The dimensionless radii r_A/r_H parameters, which characterize the effective black-hole quantum atmospheres, are functions which monotonically decrease as the number $(D + 1)$ increases [6]. In particular, in the large $D \gg 1$ regime, Ref. [6] suggests that radiating $(D + 1)$-dimensional Schwarzschild black holes are characterized by the relation $(r_A - r_H)/r_H \ll 1$.

Following Giddings' argument, other studies on black-hole quantum atmospheres have been conducted [7–9]. For example, in Ref. [7], Yan-Gang Miao and Zhen-Ming Xu investigated the Hawking radiation cascade from a five-dimensional charged black hole with a scalar field coupled to higher-order Euler densities in a conformally invariant manner. Myungseok Eune and Wontae Kim [8] showed that the outer temperature vanishes at the horizon and has a peak at a scale whose radial extent is set by the horizon radius which then decreases to the Hawking temperature at infinity. Ramit Dey, Stefano Liberati and Daniele Pranzetti [9] have shown that the Hawking quanta originate from a quantum atmosphere around the black hole with energy density and fluxes of particles peaking at about $4MG$, which is contrary to the popular belief that these quanta originate from ultra high energy excitations very close to the horizon. At present, the theory of a "black-hole quantum atmosphere" is a frontier area of research which is still far from reaching any final widely accepted conclusion. Results on this topic which are not directly related to the present article are not listed here for reasons of brevity.

One important element that appears to have received little attention in previous articles is the following. In 1902 Gibbs pointed out that, in a system where the partition function diverges, the standard Boltzmann–Gibbs theory cannot be applied, and large-scale gravitational systems are known to fall within this class [10]. Then Tsallis generalized standard statistical mechanics (which arises from the hypothesis of weak probabilistic correlations and their connection to ergodicity) to nonextensive one, which can be applied in all cases, and still possessing standard Boltzmann–Gibbs theory as a limit [10]. Therefore, the usual Boltzmann–Gibbs additive entropy must be generalized to the nonextensive, i.e., non-additive entropy (the entropy of the whole system is not necessarily the sum of the entropies of its sub-systems), which is named Tsallis entropy [10]. Non-extensive statistical mechanics does not only include one example, i.e., Tsallis entropy. Rényi entropy is another main example of such a non-extensive entropy. In this paper, we will therefore principally discuss the effect of Rényi entropy on Hawking radiation power, in particular on the effective radius of a Schwarzschild black hole quantum atmosphere. In Sect. 2, we introduce the basic concepts of Hawking radiation spectra of Schwarzschild black holes and Rényi entropy. In Sect. 3, we derive the effective radius of a black-hole quantum atmosphere taking Rényi entropy into consideration. In Sect. 4, numerical results of the effective radius of the black-hole quantum atmosphere are obtained. In Sect. 5, we briefly discuss when the effect of Rényi entropy is very large, how it impacts on Hawking radiation power. In Sect. 6, we discuss the results we have obtained.

2 Basics

2.1 The Hawking radiation spectra of (D+1)-dimensional Schwarzschild black holes

For $(D+1)$-dimensional Schwarzschild black holes, the semi-classical Hawking radiation power for one bosonic degree of freedom is described as [4,5,11,12]

$$P_{BH} = \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \sum_j \int_0^\infty \Gamma \frac{\omega^D}{\exp(\omega/T_{BH}) - 1} d\omega, \qquad (2.1)$$

Here j is the angular harmonic index of the emitted field modes and ω is the emitted frequency of field. The Bekenstein–Hawking temperature of the black hole is given by

$$T_{BH} = \frac{(D-2)}{4\pi r_{BH}}. \qquad (2.2)$$

We have set $G = c = k_B = \hbar = 1$. r_{BH} is the horizon radius of the black hole [6,13]. The greybody factors of the black-hole-field system composition are denoted as $\Gamma = \Gamma(\omega; j, D)$, which are dimensionless coefficients which quantify the interaction of the emitted fields with curved black-hole spacetime [4,5].

2.2 An area law prescription for Rényi entropy

One of the most remarkable discoveries in fundamental physics is that the entropy of a black hole has a value equal to a quarter of the horizon area in Planck units [3,14]:

$$S = \frac{Area(Horizon)}{4G_N}. \qquad (2.3)$$

Here G_N is the Newton's constant. Ryu and Takayanagi [14,15] generalized the relationship eq.(2.3) in the context of gauge/gravity duality. They proposed that the von Neumann entropy is determined by the area of a codimensional-2 minimal surface in the dual spacetime [14,15]:

$$S = \frac{Area(Minimal\ Surface)}{4G_N}. \qquad (2.4)$$

The above discussions of area laws eq.(2.3) and (2.4) are limited to the von Neumann entropy. However, according to Gibbs' argument [10], the Boltzmann–Gibbs (BG) theory cannot be applied in systems with divergence in the partition function, such as in a gravitational system [10]. Therefore, thermodynamic entropy of such non-standard systems cannot be described by an extensive entropy but must instead be generalized to a non-extensive entropy [10]. In Ref. [14], the author has generalized

the area laws to the case of Rényi entropy, in which a label index q is used and the entropy is defined in terms of the density matrix ρ of the entangling region as [14]:

$$S_q = \frac{1}{1-q} \ln Tr \rho^q. \tag{2.5}$$

When $q \to 1$, the von Neumann entropy $S = -Tr(\rho \ln \rho)$ is recovered. Rényi entropy contains richer physical information about the entanglement structure of a quantum state. The properties of Rényi entropy have been extensively studied using by numerical methods [16], in spin chains [17], in tensor networks [18], in free field theories [19], in two-dimensional CFTs [20] or higher [21], and in the context of gauge/gravity duality [22]. These results have also been generalized to charged [23] and supersymmetric cases [24].

The generalized area-law prescription for holographic Rényi entropy can be derived by applying the replica trick in the context of gauge/gravity duality [14,25]. Rényi entropy (2.5) of integer $q > 1$ is determined by the partition function of the QFT on a branched cover M_q, defined by taking q copies of the original Euclidean spacetime M_1 on which the QFT lives with a cut along the entangling region and gluing them along the cuts in a cyclic order. This can be written as [14,25]:

$$S_q = \frac{1}{1-q}(\ln Z[M_q] - q \ln Z[M_1]), \tag{2.6}$$

where $Z[M_1]$ and $Z[M_q]$ are the partition functions on the branched original and cover spacetime, respectively. For holographic QFTs, one can calculate $Z[M_q]$ by finding the dominant bulk solution B_q whose asymptotic boundary is M_q [14,25]. In the large N limit where the bulk physics is classical, we have [14]:

$$Z[M_q] = e^{-I_{bulk}[B_q]}, \tag{2.7}$$

where $I_{bulk}[B_q]$ denotes the on-shell action of the bulk solution B_q. If we assume the $\mathbb{Z}_q$ replica symmetry, then one can obtain that [14]:

$$S_q = \frac{q}{q-1}(I_{bulk}[\hat{B}_q] - I_{bulk}[\hat{B}_1]), \tag{2.8}$$

where $\hat{B}_q = B_q/\mathbb{Z}_q$ and the bulk action I_{bulk} is the Einstein action

$$I_{bulk} = -\frac{1}{16\pi G_N} \int d^{d+1}X \sqrt{G} R + I_{matter}. \tag{2.9}$$

Here X^μ, $G_{\mu\nu}$ and R are the coordinates, metric and Ricci scalar in the bulk [14]. As a result, one can obtain a generalized area law:

$$q^2 \partial_q \left(\frac{q-1}{q} S_q\right) = \frac{Area(Cosmic\ Brane_q)}{4G_N}, \tag{2.10}$$

where the cosmic brane is analogous to the Ryu-Takayanagi minimal surface [14]. The above arguments can be generalized to non-integer q [14]. More details can be found in Ref. [14].

2.3 Basic formulae for Rényi entropy

A long-standing problem of non-extensive thermodynamics is deriving a formulation which is compatible with the zeroth law of thermodynamics [26]. Based on the concept of composability, Abe showed [26,27] that the most general non-additive entropy composition rule which is compatible with a homogeneous equilibrium has the form

$$H_\lambda(S_{12}) = H_\lambda(S_1) + H_\lambda(S_2) + \lambda H_\lambda(S_1)H_\lambda(S_2), \tag{2.11}$$

where H_λ is a differentiable function of S and λ is a real constant parameter. S_1, S_2 and S_{12} are the entropies of the subsystems and the total system, respectively.

Biró and Ván [28] showed that, the most general entropy function compatible with the zeroth law for homogeneous systems has the following form:

$$L(S) = \frac{1}{\lambda} \ln[1 + \lambda H_\lambda(S)], \tag{2.12}$$

which is additive for composition, namely,

$$L(S_{12}) = L(S_1) + L(S_2). \tag{2.13}$$

The corresponding temperature is then

$$\frac{1}{T} = \frac{\partial L(E)}{\partial E}, \tag{2.14}$$

where we have assumed the principle of additivity holds for the energy composition [26].

For classical black holes, the Bekenstein–Hawking formula satisfies the Eq. (2.3), i.e.,

$$S_{12} = S_1 + S_2 + 2\sqrt{S_1}\sqrt{S_2}, \tag{2.15}$$

where $H_\lambda(S) = \sqrt{S}$ and $\lambda \to 0$ [26,27].

The Rényi entropy can be defined as

$$S_R = \frac{1}{1-q} \ln \sum_i p_i^q, \tag{2.16}$$

which is equivalent to the choices of $H_\lambda(S) = S$ and $\lambda = 1 - q$ in eq.(2.4) [26], if the original entropy functions satisfy the following condition

$$S_{12} = S_1 + S_2 + \lambda S_1 S_2, \tag{2.17}$$

where q is a real parameter called the non-extensivity parameter. It can be shown that the formal logarithm of Tsallis entropy

$$S_T = \frac{1}{1-q} \sum_i (p_i^q - p_i), \tag{2.18}$$

provides the Rényi entropy, namely,

$$S_R = L(S_T) = \frac{1}{1-q} \ln[1 + (1-q)S_T]. \tag{2.19}$$

When q approaches to 1 ($\lambda \to 0$), both Rényi and Tsallis entropy give the Shannon entropy $S = -\sum_i p_i \ln p_i$ [26].

3 The effective radius of a black-hole quantum atmosphere taking Rényi entropy into consideration

In this section, in order to determine the location of effective radius of a Schwarzchild black hole, we will derive the formulae for black hole thermodynamics considering the perturbations of Rényi entropy.

3.1 The Rényi temperature of a (D+1)-dimensional Schwarzschild black hole

Considering Eq. (2.2), $T_{BH} = M'(r_{BH})/S'(r_{BH}) = \frac{(D-2)}{4\pi r_{BH}}$, we have

$$S(r_{BH}) = \frac{4\pi}{(D-2)} \int r_{BH} M'(r_{BH}) dr_{BH}. \tag{3.1}$$

Based on Ref. [6], the horizon radius of a $(D+1)$-dimensional Schwarzschild black hole of mass M is given by

$$r_{BH} = [16\pi M/(D-1)\hat{A}_{D-1}]^{1/(D-2)}, \tag{3.2}$$

where $\hat{A}_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the generalized area of a unit $(D-1)$-sphere. Then we have

$$M'(r_{BH}) = \frac{(D-1)\pi^{D/2}}{8\pi \Gamma(D/2)} (D-2) r_{BH}^{D-3}. \tag{3.3}$$

Then the thermodynamic entropy [1] of $(D+1)$-dimensional Schwarzschild black hole is

$$S(r_{BH}) = \frac{4\pi}{(D-2)} \int r_{BH} M'(r_{BH}) dr_{BH} = \frac{\pi^{D/2}}{2\Gamma(D/2)} r_{BH}^{D-1}. \tag{3.4}$$

We have pointed out that the entropy of large-scale gravitational system cannot be described by Gibbs–Boltzmann statistical mechanics and should be replaced by Tsallis entropy [10]. Then if we regard the entropy of a black hole as being Tsallis entropy, namely, $S(r_{BH}) = S_T$ [26,29], then the Rényi entropy of a $(D + 1)$-dimensional Schwarzschild black hole is then

$$S_R = \frac{1}{\lambda} \ln(1 + \lambda S(r_{BH})) = \frac{1}{\lambda} \ln \left(1 + \lambda \frac{\pi^{D/2}}{2\Gamma(D/2)} r_{BH}^{D-1} \right), \tag{3.5}$$

where $\lambda = 1 - q$. We can therefore conclude that irrespective of the value of λ, the value of S_R will be positive. The derivative of Rényi entropy can then be obtained as

$$S_R' = \frac{1}{1 + \lambda \frac{\pi^{D/2}}{2\Gamma(D/2)} r_{BH}^{D-1}} \frac{\pi^{D/2}}{2\Gamma(D/2)} (D - 1) r_{BH}^{D-2}. \tag{3.6}$$

So that the Rényi temperature [2] of a $(D + 1)$-dimensional Schwarzschild black hole is

$$T_R = M'(r_{BH})/S_R' = \frac{(D-2)}{4\pi r_{BH}} + \lambda \frac{(D-2)\pi^{D/2}}{8\pi \Gamma(D/2)} r_{BH}^{D-2}. \tag{3.7}$$

When $D = 3$ and $\lambda \to 0$, the Rényi temperature becomes

$$T = \frac{1}{4\pi r_{BH}}, \tag{3.8}$$

which is the well-known result for a $(3 + 1)$-dimensional Schwarzschild black hole [6].

Equation (3.7) can be rewritten as $T_R = T_{BH}(1 + x)$, where $x = \lambda \frac{\pi^{D/2} r_{BH}^{D-1}}{2\Gamma(D/2)}$. In Sect. 3, we consider only the case of $|x| \ll 1$ and a modification to order x.

3.2 The effective radius of a black-hole quantum atmosphere taking Rényi entropy into account

On the one hand, for $(D + 1)$-dimensional Schwarzschild black holes, the semi-classical Hawking radiation power for one bosonic degree of freedom should be

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[1] The reason why we emphasize "thermodynamic" is that this formula is derived by thermodynamics absolutely without using the definitions of entropy in statistical mechanics.

[2] Here Rényi temperature refers to the black-hole temperature considering the effect of Rényi entropy.

modified as

$$P_R = \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \sum_j \int_0^\infty \Gamma \frac{\omega^D}{\exp(\omega/T_R)-1} d\omega, \tag{3.9}$$

where $T_R = T_{BH}(1+x)$ and $x = \lambda\frac{\pi^{D/2}r_{BH}^{D-1}}{2\Gamma(D/2)}$. Since $|x| \ll 1$, then $\frac{1}{\exp(\omega/T_R)-1}$ is approximately equal to

$$\frac{1}{\exp(\omega/T_R)-1} \approx \frac{1}{\exp(\omega/T_{BH})-1} + \frac{\frac{x\omega}{T_{BH}}\exp(\omega/T_{BH})}{[\exp(\omega/T_{BH})-1]^2}. \tag{3.10}$$

So that Eq. (3.9) can be written as

$$\begin{aligned}
P_R = &\frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \sum_j \int_0^\infty \Gamma \frac{\omega^D}{\exp(\omega/T_{BH})-1} d\omega \\
&+ \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \sum_j \int_0^\infty \Gamma\omega^D \frac{\frac{x\omega}{T_{BH}}\exp(\omega/T_{BH})}{[\exp(\omega/T_{BH})-1]^2} d\omega.
\end{aligned} \tag{3.11}$$

The first term in Eq. (3.11) is the original Hawking radiation power. The second term of Eq. (3.11) can be written as

$$-x\beta_{BH}\frac{\partial}{\partial\beta_{BH}}P_{BH}, \tag{3.12}$$

where $\beta_{BH} = 1/T_{BH}$. Then Eq. (3.11) is equal to

$$P_R = P_{BH} - x\beta_{BH}\frac{\partial}{\partial\beta_{BH}}P_{BH}. \tag{3.13}$$

On the other hand, the scalar radiation power of a spherically-symmetric blackbody (BB) at temperature T and having radius R in $(D+1)$ spacetime dimensions is given by the generalized Stefan–Boltzmann radiation law [6]

$$P_{BB} = \sigma A_{D-1}(R)T^{D+1}, \tag{3.14}$$

where σ is the Stefan–Boltzmann constant in $(D+1)$-dimensions:

$$\sigma = \frac{D\Gamma(D/2)\zeta(D+1)}{2\pi^{D/2+1}}, \tag{3.15}$$

and $A_{D-1}(R)$ is the surface area of an emitting body in $(D+1)$-dimensions:

$$A_{D-1}(R) = \frac{2\pi^{D/2}}{\Gamma(D/2)}R^{D-1}. \tag{3.16}$$

In this article, we suggest that the Stefan–Boltzmann law (3.14) still holds since it is a non-gravitational system which can be described by the standard statistical mechanics. However, it should be modified as,

$$P_{BB} = \sigma A_{D-1}(r) T_R^{D+1},\qquad(3.17)$$

where T_R and r are the Rényi temperature and radius in $(D+1)$-dimensions, respectively. The reason can be explained as the following. If we repeat the derivation of Stefan–Boltzmann's law [30], we can know that the only thermodynamic quantity is temperature T, while other physical quantities are mechanical quantities. Now that we have considered the effect of Rényi entropy and Rényi temperature, the only change in Stefan–Boltzmann's law is to change the ordinary temperature T into Rényi temperature T_R, and the other parts remain unchanged.

Giddings pointed out that the effective radius of a black-hole atmosphere can be determined by equating the radiation power P_{BB} of a flat space perfect blackbody emitter with the Hawking radiation power P_{BH} [2,6]. Therefore, following Ref. [2,6], we can define the effective radius r_A of the black-hole quantum atmosphere by the following relation:

$$P_{BH}(r_{BH}, T_{BH}) = P_{BB}(r_A, T_{BH}).\qquad(3.18)$$

In this article, the eq.(3.18) is modified to,

$$P_R(r_{BH}, T_R(T_{BH})) = P_{BB}(r_A, T_R(T_{BH})),\qquad(3.19)$$

where $T_R(T_{BH})$ means that T_R is a function of T_{BH}.
Since $T_R = T_{BH}(1+x)$ and $|x| \ll 1$, therefore

$$T_R^{D+1} = T_{BH}^{D+1}(1+x)^{D+1} \approx T_{BH}^{D+1}[1 + (D+1)x] = T_{BH}^{D+1} + (D+1)x T_{BH}^{D+1},$$
$$(3.20)$$

where we have used the binomial theorem in the second step.
Assuming now, that Eq. (3.18) still holds and considering Eqs. (3.13), (3.18), (3.19) as well as (3.20), we then have,

$$\sigma A_{D-1}(r_A)(D+1)x T_{BH}^{D+1} = -x\beta_{BH}\frac{\partial}{\partial \beta_{BH}} P_{BH}(r_{BH}, T_{BH}).\qquad(3.21)$$

From Eq. (3.21), we have,

$$T_{BH}^{D+1} = C_{D+1} P_{BH}(r_{BH}, T_{BH}),\qquad(3.22)$$

where C_{D+1} is a constant in $(D+1)$-dimensions. Considering Eqs. (3.14) and (3.18), we can therefore derive that,

$$C_{D+1} = 1/\sigma_D A_{D-1}(r_A),\qquad(3.23)$$

which means that our assumption is self-consistent.

Combining Eqs. (3.13), (3.17), (3.19) and (3.22), we can obtain the effective radius r_{AR} of a black-hole quantum atmosphere as being,

$$r_{AR} = \{\frac{\pi}{D\zeta(D+1)}(\frac{4\pi}{D-2})^{D+1}\bar{P}_{BH}\}^{\frac{1}{D-1}} \times r_{BH} \times \left\{\frac{\frac{D+1-x}{D+1}}{1+(D+1)x}\right\}^{\frac{1}{D-1}},$$

(3.24)

where we have defined the scaled Hawking radiation power of a black hole in $(D+1)$-dimensions [6] as,

$$\bar{P}_{BH} = P_{BH} \times r_{BH}^2.$$

(3.25)

If we define the effective radius r_A of the black-hole quantum atmosphere without taking into account the effect of Rényi entropy [6] as

$$r_A = \left\{\frac{\pi}{D\zeta(D+1)}\left(\frac{4\pi}{D-2}\right)^{D+1}\bar{P}_{BH}\right\}^{\frac{1}{D-1}} \times r_{BH},$$

(3.26)

and the entropy difference as,

$$F = \left\{\frac{\frac{D+1-x}{D+1}}{1+(D+1)x}\right\}^{\frac{1}{D-1}}.$$

(3.27)

When $x = 0$, then $F = 1$, and the effective radius r_{AR} of the black-hole quantum atmosphere, taking into account the effect of Rényi entropy, can be written as

$$r_{AR} = r_A F.$$

(3.28)

There are two reasons why we only expand $\frac{1}{\exp(\omega/T_R)-1}$ to order x. Firstly, if we expand the term to order x then the effective radius r_{AR} can be written as the product of the standard effective radius r_A and the difference F, which makes for an easier comparison of the results with that given by standard statistical mechanics. Secondly, standard statistical mechanics can describe the universe accurately, therefore, the effect of Rényi entropy can be neglected completely under normal circumstances. In fact, as we have pointed out in Sect. 2.3, when x is zero, the standard results are recovered.

4 Numerical results for the effective radius of a black-hole quantum atmosphere

Following Ref. [6], we can define the functional dependence $\bar{r}_{AR} = \bar{r}_{AR}(D)$ of the dimensionless ratio

$$\bar{r}_{AR} \equiv (r_{AR} - r_{BH})/r_{BH}, \tag{4.1}$$

which characterizes the effective quantum atmospheres of the radiating black holes in $(D + 1)$ dimensions. Considering Eq. (3.28), we can rewrite Eq. (4.1) as,

$$\bar{r}_{AR} = \tilde{r}_A F - 1, \tag{4.2}$$

where $\tilde{r}_A = r_A/r_{BH}$. Based on Hawking's and Giddings' arguments [2,6], there are two possible values of $\bar{r}_{AR}$ in $(D + 1)$ dimensions, namely, $\bar{r}_{AR} \sim O(0)$ or $\bar{r}_{AR} \sim O(1)$. We will calculate the numerical results for the two possibilities respectively.

For $\bar{r}_{AR} \sim O(0)$, if we set $\bar{r}_{AR} = 0$, we have,

$$x_D = \frac{(1 - \tilde{r}_A^{1-D})(D + 1)}{(D + 1)^2 \tilde{r}_A^{1-D} + 1}. \tag{4.3}$$

For $\bar{r}_{AR} \sim O(1)$, if we set $\bar{r}_{AR} = 1$, we have,

$$x_D = \frac{(1 - 2^{D-1}\tilde{r}_A^{1-D})(D + 1)}{2^{D-1}(D + 1)^2 \tilde{r}_A^{1-D} + 1}, \tag{4.4}$$

where the subscript D denotes the dimensionality of space.

4.1 The (3 +1)-dimensional case

The Hawking radiation power of scalar quanta from a $(3 + 1)$-dimensional Schwarzschild black hole is given by [6,31]

$$P_{BH}(D = 3) = 2.976 \times 10^{-4} \frac{1}{r_{BH}^2}. \tag{4.5}$$

Then we have $\tilde{r}_A = 2.679$ for the effective radius of the black-hole quantum atmosphere.

For $\bar{r}_{AR} \sim O(0)$, from Eq. (4.3), we have $x_3 = 1.0661$, which is $O(1)$ and inconsistent with the assumption in Sect. 3, namely, $|x| \ll 1$. We cannot adjust the value of $\bar{r}_{AR}$ to make $|x| \ll 1$. Therefore, for the $(3 + 1)$-dimensional case, Hawking's suggestion is less feasible.

For $\bar{r}_{AR} \sim O(1)$, from Eq. (4.4), we have $x_3 = 0.1786$. In fact, we can adjust the value of $\bar{r}_{AR}$ so that x can be very closed to zero. Therefore, for the $(3+1)$-dimensional case, Giddings' suggestion is more feasible.

4.2 The (D +1)-dimensional cases

In the previous subsection, we have seen that the effective quantum atmosphere for a $(3 + 1)$-dimensional Schwarzschild black hole is described by the relation

Table 1 The dimensionless radii $r_A(D) \equiv (r_A - r_{BH})/r_{BH}$ and $\tilde{r}_A = r_A/r_{BH}$ which characterize the effective quantum atmospheres of the radiating black holes in $(D+1)$ dimensions

D+1	5	6	7	8	9	10	11
$(r_A - r_{BH})/r_{BH}$	0.982	0.727	0.590	0.502	0.439	0.391	0.355
$\tilde{r}_A = r_A/r_{BH}$	1.982	1.727	1.590	1.502	1.439	1.391	1.355

Table 2 The values of x_D for the radiating black holes in $(D+1)$ dimensions using Hawking's suggestion

D+1	5	6	7	8	9	10	11
x_D	1.0349	1.0552	1.0841	1.1110	1.1302	1.1416	1.1612

Table 3 The values of x_D for the radiating black holes in $(D+1)$ dimensions using Giddings' suggestion

D+1	5	6	7	8	9	10	11
x_D	-5.152×10^{-3}	-0.0729	-0.0969	-0.1023	-0.0990	-0.0945	-0.0882

$$\bar{r}_{AR} \equiv (r_{AR} - r_{BH})/r_{BH}. \tag{4.6}$$

Based on the numerical results in $(3+1)$ dimensions, we can deduce, that Hawking's suggestion is less feasible than that of Giddings'.

We shall now calculate the values of the dimensionless radii $\bar{r}_{AR}$, which describe the effective quantum atmospheres of the radiating Schwarzschild black holes, and the value of x, which quantifies the effect of Rényi entropy in $(D+1)$-dimensions. In Ref. [6,32], the Hawking radiation powers of Schwarzschild black holes in $(D+1)$-dimensions have been obtained numerically. We list the results of the dimensionless radii $r_A(D)$ in Table 1 [6].

4.2.1 The Hawking's cases for intermediate D-values

For $\bar{r}_{AR} \sim O(0)$, from Eq. (4.3) and Table 1, we have $x_4 = 1.0349$. Similarly, we can obtain the values for other dimensions. The results have been listed in Table 2. Based on Table 2, we find that if $\bar{r}_{AR} \sim O(0)$, then the values of x_D cannot be $O(0)$, therefore we can conclude that for any dimensions, Hawking's suggestion does not appear to be feasible.

4.2.2 The Giddings' cases for intermediate D-values

For $\bar{r}_{AR} \sim O(1)$, from eq.(4.4) and Table 1, we have $x_4 = -5.152 \times 10^{-3}$. Similarly, we can obtain the values for other dimensions. The results have been listed in Table 3. Based on Table 3, we find that if $\bar{r}_{AR} \sim O(1)$, then the values of x_D can be very closed to zero and we can therefore conclude that for any dimensions, Giddings' suggestion appears to be feasible.

Combining this with the conclusions in Sect. 4.1 and 4.2.1, we can deduce that if we consider the effect of Rényi entropy, for any dimensions, Hawking's suggestions appear to be much less feasible that those of Gidding. Therefore, Hawking radiation most likely originates from a quantum atmosphere which extends beyond the horizons of black holes. We can also infer that the above conclusion is a general result and true independently of the value of λ in non-extensive statistical mechanics.

4.2.3 The large D regime

In this section, we will consider the case for in the large D regime to see whether the conclusions which we have obtained in Sects. 4.1, 4.2.1 and 4.2.2 are still established. The Hawking radiation spectrum for a Schwarzschild black holes in $(D + 1)$ dimensions is characterized by $\omega^D/(\exp(\omega/T_R) - 1)$ (see Eq. (3.9)). This frequency dependent function has a peak at [6]

$$\frac{\omega_{peak}}{T_R} = D + W(-De^{-D}),\tag{4.7}$$

where $W(x)$ is the Lambert function and T_R is the Rényi temperature of the black hole as given in Eq. (3.7). Inserting Eq. (3.7) into Eq. (4.7), we then have

$$\frac{\lambda_{peak}}{r_{BH}}(1 + x) \approx \frac{8\pi^2}{D^2}[1 + O(D^{-1})] \ll 1,\tag{4.8}$$

where λ_{peak} is the peak of wave length. Since $|x| \ll 1$, the modification term does not affect the above conclusion, namely, that in the large D regime the characteristic wavelength is much shorter than the horizon radius r_{BH} of a radiating Schwarzschild black hole, which is consistent with the result obtained without taking Rényi entropy into consideration.

References [6] and [32] have pointed out that the inequality (4.8) implies that the corresponding Hawking emission spectra of these higher-dimensional black holes are described extremely well by a short wavelength approximation. In particular, the effective radiating r_A of a black hole in the large D regime is determined by the high-energy (short wavelengths) absorptive radius of the black hole [6,12,32–38], namely,

$$\tilde{r}_A \equiv r_A/r_{BH} = (D/2)^{\frac{1}{D-2}}\sqrt{\frac{D}{D - 2}}.\tag{4.9}$$

In the large $D \gg 1$ regime, we find that ,

$$\tilde{r}_A \equiv r_A/r_{BH} = 1 + O\left(\frac{\ln D}{D}\right) \approx 1.\tag{4.10}$$

For $\bar{r}_{AR} \sim O(0)$, $F \approx 1$. Considering Eq. (4.3), we obtain that $(D + 1)x \approx 0$, which is the necessary condition for Hawking's arguments.

For $\bar{r}_{AR} \sim O(1)$, $F \approx 2$. Considering Eq. (4.4), we obtain that $\frac{1}{1+(D+1)x} \approx 2^{D-1}$. For large D, then $(D+1)x \approx -1$, which is the necessary condition for Giddings' arguments. We also find that $\lambda < 0$.

Since $D \gg 1$, we know that $|x|$ is extremely small for the two cases, so that the effect of Rényi entropy can be neglected completely. In addition, these two suggestions are both feasible.

5 Very large effect of Rényi entropy on Hawking radiation power

In this section, we will briefly mention the properties of Hawking radiation power in the situation where the effect of Rényi entropy is extremely large. Although this is a situation which does not exist in the world we have explored, it is nevertheless of interest.

On the one hand, since $T_R = T_{BH}(1 + x)$, if $|x| \gg 1$, then $T_R = xT_{BH}$. We then have

$$\exp(\omega/T_R) - 1 \approx \frac{\omega}{xT_{BH}}. \tag{5.1}$$

Then the power spectral density of P_R is

$$S_R = dP_R/d\omega = \frac{xT_{BH}}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \sum_j \Gamma(\omega; j, D)\omega^{D-1}, \tag{5.2}$$

where $\Gamma(\omega; j, D)$ are the greybody factors of the black-hole field system composition.

On the other hand, if $\omega/T_{BH} \ll 1$, then

$$\exp(\omega/T_{BH}) - 1 \approx \frac{\omega}{T_{BH}}. \tag{5.3}$$

The power spectral density of P_{BH} is

$$S_{BH} = dP_{BH}/d\omega = \frac{T_{BH}}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \sum_j \Gamma(\omega; j, D)\omega^{D-1}. \tag{5.4}$$

Therefore, if $x \gg 1$ and $\frac{\omega}{T_{BH}} \ll 1$, then we obtain

$$S_R = xS_{BH}. \tag{5.5}$$

The meaning of this relationship and the effect of Rényi entropy on Hawking radiation should be studied in further research.

6 Summary and discussion

It is widely believed that Hawking radiation originates from the excitations near the horizons of black holes. Therefore, most researchers assume that the modified Hawking radiation spectra should also be characterized by a relatively short length scale $\Delta r \equiv r_A - r_{BH} \ll r_{BH}$ [2,3]. One of the main arguments for the above assumption is based on Hawking's calculation, involving tracing back the modes all the way from future infinity to the past null infinity, through collapsing matter, so that one has a vacuum state near the horizon for a free-falling observer [9].

However, Giddings [2] has proposed that the Hawking radiation spectrum that characterizes evaporating semi-classical black holes originates from an quantum "atmosphere", which extends beyond the horizon of the black hole. Giddings has provided evidence that, for a Schwarzschild black hole in $(3 + 1)$ dimensions, the source of the Hawking radiation is a quantum region outside the black-hole horizon whose effective radius r_A is characterized by the relation $\Delta r \sim r_H$ [2]. Following Giddings' argument, other studies on black-hole quantum atmospheres have been conducted [7–9]. Reference [6] raised an interesting question, namely, whether Giddings' argument is of a general nature and is true for all radiating black holes irrespective of their dimensions? The results of Ref. [6] showed that at least in some cases, the effective radii of the black-hole quantum atmospheres are characterized by the relation $r_A \ll 1$. This conclusion implies that the Hawking radiation originates from quantum excitations very near the horizon of a black hole [6].

However, the following important consideration appears not to have been taken into account in these previous articles. Based on Gibbs' argument [10], the Boltzmann–Gibbs (BG) theory cannot be applied to systems with divergence in the partition function, such as in a gravitational system. Therefore, the thermodynamic entropy of such non-standard systems cannot be described merely by an extensive entropy but must instead be generalized to a non-extensive entropy of which Rényi entropy is one of the main examples.

This article therefore addressed the question in Ref. [6] concerning the effect of Rényi entropy. The general structure of the present article is similar to that of Ref. [6]. In Sect. 3, we derived the entropy and temperature of black holes taking Rényi entropy into consideration. The expressions are universally valid. Considering the fact that even if there indeed exists the effect of Rényi entropy, it must be very small since for now Gibbs–Boltzmann statistical mechanics can still describe physical phenomenon accurately. Therefore, we further obtained the expression for the effective radius of a quantum atmosphere including a modification term of order x (x is very small, i.e., $\lambda = 1 - q$ is very small and q approaches to 1).

In Sect. 4, we obtained numerical results for the effective radius of a black-hole quantum atmosphere in $(3 + 1)$ dimensions and in $(D + 1)$ dimensions. Based on our calculated results listed in Tables 2 and 3, we suggested that it is much more feasible that Hawking radiation originates from a quantum "atmosphere" which extends beyond the horizons of black holes having finite dimensions as suggested by Giddings. Furthermore, if we want to see whether this conclusion is still established, we found that for black holes having infinite dimensions, however, two suggestions are both feasible. In Sect. 5, we briefly considered also the very large effect of Rényi entropy

on Hawking radiation power. If the effect of Rényi entropy is very large and ω/T_{BH} is very small, we found that the power spectral density S_R is proportional to S_{BH}, that is, $S_R = x S_{BH}$, where x is defined as in Sect. 3.1 and includes the effect of Rényi entropy.

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