

Annuitization and Asset Allocation with Borrowing Constraint

Jin Gi Kim, Bong-Gyu Jang, and Seyoung Park[†]

Department of Industrial and Management Engineering, POSTECH, Korea. Tel: +82-54-279-2864, Fax: +82-54-279-2870 E-mail: dgkimjg0203@postech.ac.kr

Department of Industrial and Management Engineering, POSTECH, Korea. Tel: +82-54-279-2372, Fax: +82-54-279-2870, E-mail: bonggyujang@postech.ac.kr

[†]*Corresponding Author: Nottingham University Business School, University of Nottingham, Jubilee Campus, Nottingham, NG8 1BB, UK, Tel: +44-7927-494518, E-mail: seyoung.park@nottingham.ac.uk*

Abstract

We generalize the result of Yaari (1965) on annuitization with borrowing constraint. We show that inability to borrow against future labor income has a significant influence on an individual's consumption and asset allocation strategies. We also show that there exists a certain threshold of wealth for annuitization. We find that the wealth threshold is lower in the presence of borrowing constraint than in its absence, implying the individual's earlier retirement.

Keywords: annuitization, wealth threshold, retirement, borrowing constraint

1. Introduction

The seminal work of [16] and [17] show that it is optimal to annuitize all of wealth under an uncertain lifetime without bequest motive. How does the presence of borrowing constraint against future labor income affect this result?

A wide range of studies analyze the implications of borrowing constraint for consumption and asset allocation while neglecting annuitization, whereas others focus on annuitization while neglecting borrowing constraint. On the one hand, [1], [6], [8], [10] investigate the effects of borrowing constraint on the optimal consumption and investment in the absence of annuitization. On the other hand, [5], [13], [15] investigate the result of [17] and Richard (1975) on annuitization in the more general setting, in the absence of borrowing constraint. This paper jointly models annuitization and borrowing constraint and analyzes quantitative interactions among annuitization, asset allocation, and borrowing constraint.

Allowing for an additional dimension of risk or a constraint in financial markets results in considerable challenges in solving the annuitization problem. It is well known that borrowing constraint against future labor income makes the derivations of optimal strategies much more tricky than in its absence ([7]). To our best knowledge, this is a first attempt to derive in closed-form the annuitization and asset allocation decisions with borrowing constraint.

We analyze the determinants of annuitization and asset allocation with borrowing constraint within the optimal consumption and investment framework of [11] and [12]. Following [15], we focus on DB pension plans that guarantee a defined life annuity, substituting for substantial portion of labor income. We consider a lifetime constant relative risk aversion (CRRA) utility function of consumption and leisure (which are nonseparable). We assume the simplest possible labor market setting, i.e., an individual receives a constant stream of labor income. We allow for the optimal stopping (endogenously determined) time for annuitization. One major departure from [15] is that we incorporate in the annuitization problem not only borrowing constraint, but also interestingly, a highly nonlinear hazard into death by introducing a geometric Brownian motion (GBM)-type time-varying mortality rate.

We derive in closed-form the optimal consumption, investment, and annuitization strategies with borrowing constraint. We show that inability to borrow against future labor income can significantly affect asset allocation. In particular, the borrowing constraint reduces consumption and investment in the stock market. Intuitively, the borrowing constraint reduces the value of labor income, and, thus, reduces available financial resources, leading to less consumption and stock investment. We also show that there exists a certain threshold of wealth for annuitization with the inability to borrow. We find that the wealth threshold is lower in the presence of borrowing constraint than in its absence. Consequently, individuals optimally enter retirement earlier than without borrowing constraint and annuitize all of their wealth at such an early retirement.

2. The Basic Model

Utility Function. Following [15], we assume that an individual has a utility function of the constant relative risk aversion (CRRA) type:

$$U(l_t, c_t) = \frac{1}{a} \frac{(l_t^{1-a} c_t^a)^{1-\gamma^*}}{1-\gamma^*}, \quad \gamma^* > 0,$$

where c_t is per-period consumption, l_t is leisure, $0 < a < 1$ is a weight for consumption, $\gamma \equiv 1 - \alpha(1 - \gamma^*)$ is the constant coefficient of relative risk aversion. Following [8], we employ a binomial choice of leisure with which the individual enjoys leisure $l_t = l_1$ while she is working and $l_t = \bar{l}$ ($\bar{l} > l_1$) after she retires. We assume that the individual receives a constant income stream of $I = w(\bar{l} - l_1) > 0$ while working. We normalize $l_1 = 1$.

Financial Market. We consider two tradable assets in the financial market: a riskless bond and a risky stock. The bond price B_t follows:

$$dB_t = rB_t dt,$$

where $r > 0$ is the risk-free interest rate. The stock price S_t follows a geometric Brownian motion (GBM)

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu > r$ is the expected rate of stock returns, $\sigma > 0$ is the volatility of stock returns, and W_t is a standard one-dimensional Brownian motion defined on a suitable probability space.

Annuity Market. We develop an analytically tractable framework to model and interpret the economics of a highly nonlinear hazard into death. Specifically, we introduce a GBM-type time-varying mortality rate ν_t as follows:

$$d\nu_t = \nu_t(-\nu dt + b d\tilde{W}_t), \quad \nu_0 = \nu > 0,$$

where $b > 0$ is the standard deviation of changes in mortality rate and \tilde{W}_t is a standard one-dimensional Brownian motion with a correlation $|\rho| < 1$, i.e., $dW_t \cdot d\tilde{W}_t = \rho dt$. In the limiting case of $b = 0$, the mortality rate reduces to a simple exponential distribution, i.e., $\nu_t = \nu e^{-\nu t}$. In an online appendix, we have tested whether the growth of mortality rates follows a normal distribution using the actual mortality data obtained from Human Mortality Database. According to our test, a hypothesis that the mortality rate growth follows a normal distribution cannot be rejected within 5% of p -value.

In the presence of the nonlinear hazard into death, the present value of annuity discounted

at the risk-free interest rate r is (for derivation, refer to an online appendix.):

$$E\left[\int_0^{\tau_M} e^{-rt} \$1 dt\right] = \frac{\$1}{r + \nu},$$

where τ_M is the time of death. Then the individual is expected to receive annuity income $\$X_t(r + \nu)$ annually until death when she enters annuitization with her wealth $\$X_t$.

Credit Market. Let X_t denote the individual's time- t wealth. Following [6], the individual is not allowed to access the credit market when using her labor income as a collateral due to market frictions such as informational asymmetry, agency conflicts, and limited enforcement. Thus, the individual is borrowing constrained, and the borrowing constraint can be formulated as follows:

$$X_t \geq 0, \text{ for } t \geq 0. \quad (1)$$

Annuitization Problem. The individual aims to maximize her expected discounted lifetime CRRA utility function by optimally controlling consumption and investment strategies, and the timing of annuitization. More specifically, the individual's value function is

$$\phi(x) = \max_{(c, \pi, \tau)} E\left[\int_0^{\tau \wedge \tau_M} e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\beta(\tau \wedge \tau_M)} \frac{\bar{l}^{\gamma-\gamma^*}}{1-\gamma} \frac{\left(X_{\tau \wedge \tau_M}(r + \nu)\right)^{1-\gamma}}{\beta + \nu} dt\right], \quad (2)$$

where π is the dollar amount invested in the stock market, τ is the optimal timing of annuitization, $\beta > 0$ is the subjective discount rate, $\gamma > 0$ is the constant coefficient of relative risk aversion, and X_τ is wealth at annuitization. The value function is subject to the following dynamic budget constraint:

$$dX_t = (rX_t - c_t + I)dt + \pi_t \sigma (dW_t + \theta dt), \quad X_0 = x \geq 0, \text{ for } 0 \leq t < \tau,$$

where θ represents the Sharpe ratio, $(\mu - r)/\sigma$. We only consider admissible consumption and investment strategies that satisfy the above dynamic budget constraint and the borrowing constraint given in (1).

We assume that annuitization must take place at retirement, i.e., the individual annuitizes

all of wealth at retirement. Then, the problem (2) can be regarded as a version of optimal retirement problem([3] and [4]). The individual, thus, consumes at a rate equal to $X_\tau(r + \nu)$ upon retirement, which is a fixed (real or nominal) continuous payout until death.

3. Optimal Annuitization

We show that there exists a certain threshold of wealth for annuitization with the inability to borrow against future labor income.

Theorem 3.1. *We define four constants α , α^* , η , and K as the following:*

$$\alpha \equiv \frac{\left(\beta + \nu - r + \frac{1}{2}(\theta + \rho b)^2\right) + \sqrt{\left(\beta + \nu - r + \frac{1}{2}(\theta + \rho b)^2\right)^2 + 2(\theta + \rho b)^2 r}}{(\theta + \rho b)^2} > 1,$$

$$\alpha^* \equiv \frac{\left(\beta + \nu - r + \frac{1}{2}(\theta + \rho b)^2\right) - \sqrt{\left(\beta + \nu - r + \frac{1}{2}(\theta + \rho b)^2\right)^2 + 2(\theta + \rho b)^2 r}}{(\theta + \rho b)^2} < 0,$$

$$\eta \equiv \frac{\gamma - 1}{\gamma} \left(r + \frac{(\theta + \rho b)^2}{2\gamma} \right) + \frac{\beta + \nu}{\gamma},$$

and

$$K \equiv \frac{\bar{l}^{\gamma - \gamma^*}}{\beta + \nu}.$$

We show that there exists a certain threshold of wealth, \bar{x} , over which it is optimal to enter retirement and annuitize all of wealth. The wealth threshold \bar{x} is obtained in closed-form:

$$\bar{x} = \left\{ \frac{\alpha}{r} - \frac{2C^{\alpha^*}}{(\theta + \rho b)^2} \left(1 + \frac{(\theta + \rho b)^2 \alpha}{2r} \right) \right\} \frac{K^{1/\gamma} I}{(r + \nu)} / \left[\left\{ K^{1/\gamma} \left(-\frac{\beta + \nu}{1 - \gamma} + \frac{r}{r + \nu} \right) \right. \right. \\ \left. \left. + \frac{\gamma}{1 - \gamma} + \frac{(\theta + \rho b)^2}{2\gamma\eta} - \frac{(\theta + \rho b)^2 \alpha^*}{2} \left(\frac{1}{\eta} - \frac{K^{1/\gamma}}{r + \nu} \right) \right\} \frac{2C^{\alpha^*}}{(\theta + \rho b)^2} + \left(\alpha - \frac{1}{\gamma} \right) \frac{C^{1/\gamma}}{\eta} \right],$$

where $C \in (0, 1)$ is a constant, which is determined more specifically in an online appendix.

Further, we obtain in closed-form the optimal consumption and investment strategies:

$$c_t^* = \eta \left(x + \frac{I}{r} \right) - \eta A \lambda(x)^{-\alpha} - \eta A^* \lambda(x)^{-\alpha^*}, \quad (3)$$

and

$$\pi_t^* = \frac{\theta + \rho b}{\gamma \sigma} \left(x + \frac{I}{r} \right) + \frac{\theta + \rho b}{\sigma} \left(\alpha - \frac{1}{\gamma} \right) A \lambda(x)^{-\alpha} + \frac{\theta + \rho b}{\sigma} \left(\alpha^* - \frac{1}{\gamma} \right) A^* \lambda(x)^{-\alpha^*}, \quad (4)$$

respectively, where A and A^* are positive constants, which are determined more specifically in an online appendix.

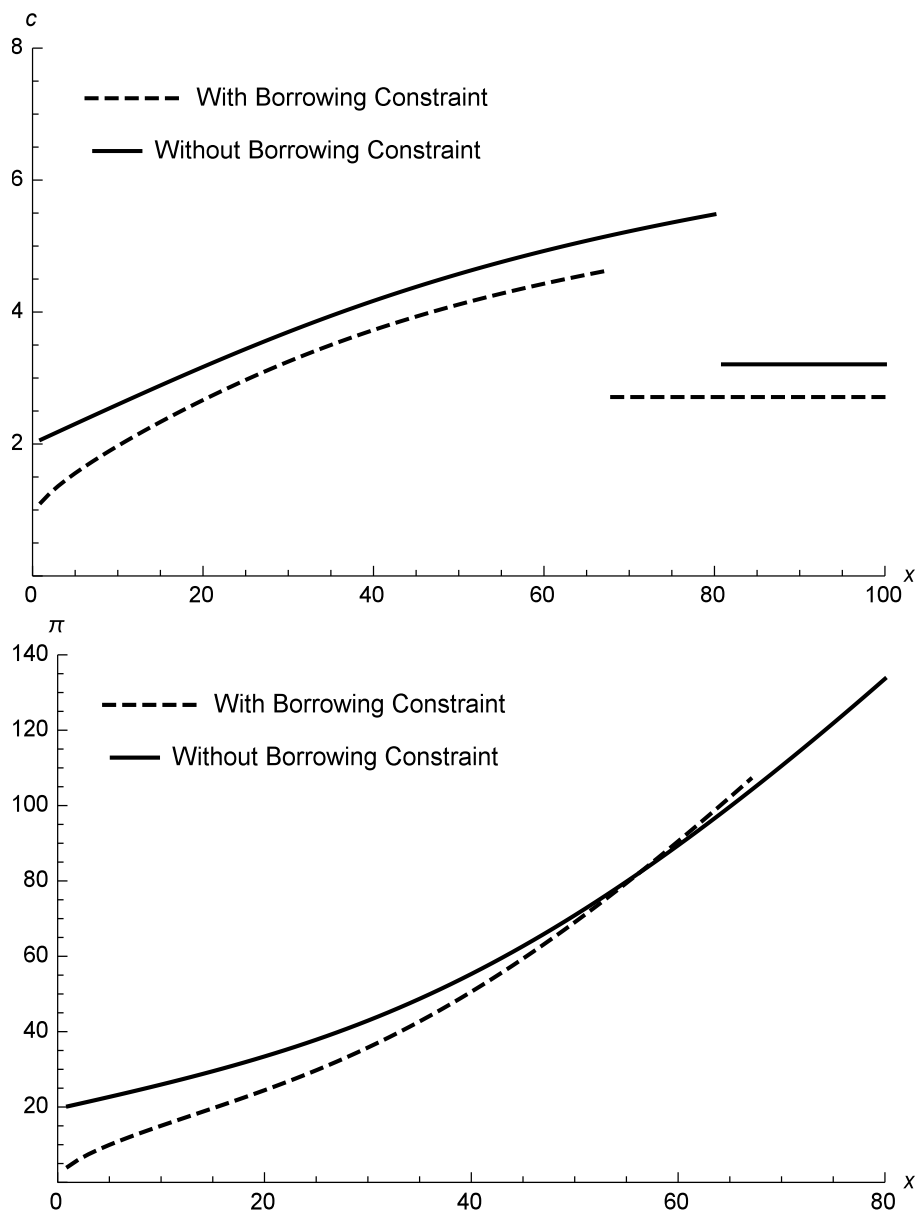


Figure 1: **Optimal consumption and stock investment.** The baseline parameter values are as follows ([8]): $r = 0.03$, $\mu = 0.1$, $\sigma = 0.2$, $\beta + \nu = 0.08$, $b = 0$, $\gamma = 3$, $K = 0.2$, $\rho = 0$, and $I = 1$.

Based on [16] and [17], we generalize the existing result of [8] on retirement with annuitization and the result of [15] on annuitization with borrowing constraint (for the solutions of the two existing results, refer to an online appendix). We show that inability to borrow against future labor income has a large impact on the optimal consumption and investment strategies. In particular, the borrowing constraint reduces consumption and investment in the stock market, as would be specified by the last negative terms involving A^* on the right hand side of (3) and (4) in Theorem 3.1. The negative effects of borrowing constraint on consumption and stock investment are illustrated in Figure 1, either. Intuitively, available financial resources for consumption and investment are expected to decrease due to the borrowing constraint that reduces the value of labor income.

We also show that there exists a certain threshold of wealth for annuitization with the inability to borrow, which is characterized by \bar{x} in Theorem 3.1. In Figure 1, the level of wealth where consumption jumps downward represents the wealth threshold for annuitization. We find that the wealth threshold is lower in the presence of borrowing constraint than in its absence. This is because with borrowing constraint, the value of labor income is lower than without borrowing constraint. Consequently, individuals are willing to retire earlier than without borrowing constraints and subsequently, annuitize all of their wealth at such an early retirement. Further, such an earlier retirement with borrowing constraints can make individuals more aggressive by investing more in the stock market as wealth approaches their retirement threshold. This is because the option value of retiring earlier becomes significantly more important and the investment motive becomes stronger. Indeed, the two lines for stock investment with and without borrowing constraint cross at high level of wealth (near retirement).

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