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Refined linearly anisotropic couple-stress elasticity

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Abstract

A recently developed, refined version of the conventional linear couple-stress theory of isotropic elasticity is extended to include the influence of anisotropic material effects. With this development, the implied refined theory (1) retains ability to determine the spherical part of the couple-stress and (2) is further furnished with constitutive ability to embrace modelling of linearly elastic solids that exhibit inherent polar material anisotropy of advanced levels that reach the class of locally monoclinic materials. This type of anisotropy embraces most of the structural material subclasses met in practice, such as those of general and special orthotropy, as well as the subclass of transverse isotropy. The thus obtained, enhanced version of the refined theory is furnished with ability to also handle structural analysis problems of polar fibrous composites reinforced by families of perfectly flexible fibres or, more generally, polar anisotropic solids possessing one or more material preference directions that do not possess bending resistance. A relevant example application considers and studies in detail the subclass of polar transverse isotropy caused by the presence of a single family of perfectly flexible fibres. By developing the relevant constitutive equation, and explicitly presenting it in a suitable matrix rather than indicial notation form, that application also exemplifies the way that the spherical part of the couple-stress is determined when the fibres are straight. It further enables this communication to initiate a discussion of further important issues stemming from (1) the positive definiteness of the full, polar form of the relevant strain energy function and (2) the lack of ellipticity of the final form attained by the governing differential equations.

Keywords

Anisotropic elasticity, constitutive equations, Cosserat theory, couple-stress theory, polar linear elasticity, polar transverse isotropy, refined couple-stress theory, weak discontinuity surfaces

1. Introduction

The indeterminacy of the conventional, Cosserat-type [1], couple-stress elasticity emerged in the open literature in the early 1960s [2,3] (see also [4, p. 124]). In terms of the terminology employed in Mindlin and Tiersten [2] and Koiter [3], this indeterminacy problem is described as a failure of the theory to determine the trace, m_{rr}, of the couple-stress tensor, $\mathbf{m}$, and has since become the subject of substantial debate and arguments. Up to a considerable extent, that problem also motivated the search for, and development of closely relevant models that avoid it, through the introduction and use of additional, micro-mechanics or higher-order deformation effects (e.g., [5–8]).

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In simple terms, this well-known problem stems from the fact that the conventional couple-stress theory [2,3] considers that m_{rr} is energetically reciprocal to a quantity which, regardless of deformation features or material properties, is always equal to zero. Namely, the divergence of the displacement spin-vector, $\mathbf{\Omega}$. That quantity, $\Omega_{i,i}$, is identically zero because is defined as the divergence of the curl of a vector, namely, the displacement vector, or, equivalently, as the divergence of the axial vector, $\mathbf{\Omega}$, of the corresponding (antisymmetric) rotation field.

A successful resolution of the outlined indeterminacy problem is most recently achieved [9,10] by refining the conventional model in a manner that considers m_{rr} energetically reciprocal to the gradient of a virtual spin-vector, $\mathbf{\Phi}$, that (1) is generally of the same order of magnitude with its actual counterpart, $\mathbf{\Omega}$, (2) its divergence is required to be non-zero ($\Phi_{i,i} \neq 0$), and (3) enables both the conventional [2,3] and the implied refined theoretical formulation [9,10] to account for the same amount of internally stored (elastic) energy. It has in fact be shown [9,10] that there exists an infinite number of such virtual vectors ($\mathbf{\Phi} \neq \mathbf{\Omega}$) that comply with all three of these requirements.

Since m_{rr} thus is considered energetically reciprocal to $\Phi_{i,i} \neq 0$, its action is successfully captured by an extra energy term that emerges in the strain energy function of the implied refined theoretical formulation. Proper consideration of that additional term then leads to the formation of an extra differential equation, whose solution enables determination of m_{rr} or, equivalently, of the spherical part of the couple-stress tensor. Most interestingly, the conventional theory [2,3] re-emerges, as a special case ($\mathbf{\Phi} \equiv \mathbf{\Omega}$), in the form of a singular theoretical model that is deprived the presence of the aforementioned extra energy term.

The implied refined formulation [9,10] is further capable to embrace the relevant hyperelasticity theory that models behaviour of elastic solids reinforced by fibres resistant in bending [11,12]. However, unlike the analysis detailed in Spencer and Soldatos [11], and Soldatos [12], where polar material response is specifically caused by fibre bending stiffness, Soldatos [9] has mainly been interested to establish connection with the pioneering developments detailed in Mindlin and Tiersten [2] and Koiter [3], where the anticipated polar material behaviour is regarded inherent in the selected material of interest, in the sense that its source is macroscopically unobservable and, therefore, unknown, or unimportant. The advances of the refined formulation presented in Soldatos [9] were thus principally confined within the bounds of linear polar material isotropy [2,3], although they also initiated a relevant discussion referring to polar transverse isotropy (see also section 6.2 of [10]).

This study embraces the viewpoint adopted in Mindlin and Tiersten [2], Koiter [3], and Soldatos [9], in the sense that it is still interested on linearly elastic materials that exhibit inherent polar material behaviour. However, it is predominantly interested to extend the analysis presented in Soldatos [9] for linear isotropic solids, by further considering cases of polar material anisotropy that is due to presence of one or more directions of material preference. It necessarily follows that, if such a preference material direction is felt representative of a single family of unidirectional fibres, the model considers that family as a perfectly flexible fibre phase of a polar elastic material. The class of locally monoclinic, polar, linearly elastic solids thus is regarded as possessing sufficient anisotropic generality for the present purposes.

Similar studies do exist in the literature [13,14], but their results are not compatible with the refined couple-stress theory of present interest. This is because Gourgiotis and Bigoni [13] focused on the anisotropic version of the conventional couple-stress theory [2,3], where the spherical couple-stress is indeterminate, while Ilkewicz et al. [14] were interested on corresponding analytical progress emerging in higher-order, micro-polar elasticity formalisms [5–8].

Under these considerations, a necessary recap of the main equations and features of the refined couple-stress theory is outlined in section 2. Section 3 follows with a proper introduction of the class of locally monoclinic anisotropic materials and derives, in indicial notation form, the relevant set of linear constitutive equations sought. The advanced anisotropy version of the refined couple-stress theory thus obtained is also furnished with ability to handle structural analysis problems of polar fibrous composites reinforced by one or more families of perfectly flexible fibres, including, for instance, the type of flexible cords first mentioned by Adkins and Rivlin [15] and Rivlin [16]. After suitable simplification of the obtained constitutive equations, such problems may include important cases of material anisotropy that is inferior to that of a locally monoclinic polar material, such as those represented by the material subclasses of general or special orthotropy and transverse isotropy.

In a relevant example application, section 4 continues and completes the relevant discussion initiated in section 4.2 of Soldatos [9] for the special case of local transverse isotropy. Particular attention is also paid to a further special case, where polar transverse isotropy is due to the presence of a single family of straight, perfectly flexible fibres. For that case, sections 5 and 6 study the consequences of the positive definiteness of the relevant strain energy function and the lack of ellipticity of the relevant governing equations, respectively. Section 7 summarises the main conclusions drawn in this study and highlights relevant directions of future expansion or relevant research.

2. Principal features and equations of the linear version of the refined couple-stress theory

The components of the traction and couple-traction vectors acting on any internal or external surface of a polar material are respectively as follows:

$$T_i^{(n)} = \sigma_{ji} n_j, \tag{1a}$$

$$L_i^{(n)} = m_{ji} n_j, \tag{1b}$$

where $\boldsymbol{\sigma}$ and $\mathbf{m}$ denote the stress and the couple-stress tensor, respectively, $\boldsymbol{n}$ denotes the outward unit normal of that surface, indices take the values 1, 2, and 3 in a suitable three-dimensional Cartesian co-ordinate framework, $\mathrm{O}x_i$, and the standard summation notation of repeated indices applies. As is shown by Koiter [3], only two of the three boundary conditions implied by equation (1b) can be set independently of the deformation when dealing with the external surface of a polar material.

2.1. Equilibrium

In the absence of body forces and body moments, and under the assumption that the couple-stress components are at least twice differentiable, the pair of standard equilibrium equations met in polar elasticity reduce into the following:

$$\sigma_{(ij),i} + \frac{1}{2}\varepsilon_{kji}\bar{m}_{\ell k,\ell i} = 0, \tag{2a}$$

$$\sigma_{[ij]} = \frac{1}{2}\varepsilon_{kji} m_{\ell k,\ell}, \tag{2b}$$

where partial differentiation is denoted by a comma, and the stress tensor has been subjected to the standard symmetric and antisymmetric parts decomposition:

$$\sigma_{ij} = \sigma_{(ij)} + \sigma_{[ij]}, \tag{3}$$

$\boldsymbol{\varepsilon}$ represents the alternating tensor, and:

$$\bar{m}_{\ell k} = m_{\ell k} - \frac{1}{3} m_{rr}\delta_{\ell k}, \tag{4}$$

stands for the deviatoric part of the couple-stress tensor.

Equation (2b) thus emerges as a constitutive equation that provides the antisymmetric part of the stress as soon as the couple-stress components are fully determined. Since equation (2a) then remains as the only equilibrium equation left, it makes evident that the spherical part of the couple-stress, m_{rr}, does not influence directly the state of equilibrium.

2.2. Kinematics

The standard kinematic quantities of non-polar linear elasticity still hold. Accordingly, the small strain and rotation tensors, as well the relevant spin-vector, are, respectively, defined as follows:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{5a}$$

$$\omega_{kj} = \frac{1}{2}\left(u_{k,j} - u_{j,k}\right) = \varepsilon_{ijk}\Omega_i, \tag{5b}$$

$$\Omega_i = \frac{1}{2}\varepsilon_{ijk}\omega_{kj} = \frac{1}{2}\varepsilon_{ijk}u_{k,j}, \tag{5c}$$

where **u** represents the displacement vector. Unlike its micro-rotation theory counterpart, which is represented by a non-symmetric tensorial quantity (e.g., [7,8]), the symmetric strain tensor ***e*** is here still considered adequate to capture the total strain observed during deformation.

Since the standard definition (5c) of the conventional spin-vector gives rise to the identity:

$$\Omega_{i,i} = 0, \tag{6}$$

an additional, auxiliary spin-vector, $\boldsymbol{\Phi}$, is further introduced. This is considered to be the axial vector of a corresponding antisymmetric tensor, $\boldsymbol{\varphi}$, in the sense that:

$$\Phi_i = \frac{1}{2}\varepsilon_{ijk}\varphi_{kj},\ \varphi_{kj} = \varepsilon_{ijk}\Phi_i, \tag{7}$$

and it also required to be such that:

$$\Phi_{i,i} \neq 0. \tag{8}$$

Moreover, its components are assumed related with those of the conventional spin, $\boldsymbol{\Omega}$, through the differential condition:

$$\left(m_{\ell i}\Phi_i\right)_{,\ell} = \left(m_{\ell i}\Omega_i\right)_{,\ell} = 2W^m, \tag{9}$$

where the energy quantity W^{m} will be identified in what follows.

As will be seen next (see equations (12) and (14)), satisfaction of equation (9) enables the outlined refined formulation to account for the same amount of elastic energy that is recorded by the conventional theory. It is also noted that, by virtue of equation (9), the vectors $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}$ are of the same order of magnitude and, by virtue of equation (8), the, thus, *macro*-vector $\boldsymbol{\Phi}$ is qualified to replace $\boldsymbol{\Omega}$ in the role of a spin-vector that is energetically reciprocal to the couple-traction vector $\mathbf{L}^{(\mathrm{n})}$.

Most importantly, on its own, the single equation (9) is evidently insufficient for unique determination of all three components of $\boldsymbol{\Phi}$. It follows that there exists a doubly infinite number of such unspecified and, therefore, virtual spin-vectors, $\boldsymbol{\Phi}$. Nevertheless, the choice $\boldsymbol{\Phi} = \boldsymbol{\Omega}$ is a trivial solution of the differential equation (9) and, hence, despite that it violates the additional condition (8), the actual spin-vector, $\boldsymbol{\Omega}$, may also be considered as an exceptional, singular member of the implied set of $\boldsymbol{\Phi}$-vectors.

2.3. Elastic energy considerations—determination of the spherical part of the couple-stress

The total energy that is stored within an arbitrary volume, V, of a polar linearly elastic material is:

$$E = \frac{1}{2}\int_S \left(T_i^{(n)}u_i + L_i^{(n)}\Phi_i\right)dS = \int_V W dV \geqslant 0, \tag{10}$$

where S denotes the surface that surrounds V, and dS and dV stand for the corresponding surface and volume element, respectively. Moreover, the polar version of the involved strain energy function, W, is anticipated quadratic in the gradients of the appearing displacement and spin-vectors, and the equality sign holds only in the absence of deformation.

Introduction of equation (1) into equation (10), followed by application of the divergence theorem and a relevant process detailed in Soldatos [9], requires from the quadratic strain energy function sought to attain the form:

$$W = W^e + W^{\Phi}, \tag{11}$$

where

$$W^e(e_{ij}) = \frac{1}{2}\sigma_{(ji)}e_{ij} \geqslant 0, \tag{12a}$$

$$W^{\Phi} = \frac{1}{2}\left\{[m_{\ell i}(\Phi_i - \Omega_i)]_{,\ell} + m_{\ell i}\Omega_{i,\ell}\right\} = \frac{1}{2}m_{\ell i}\Omega_{i,\ell} \geqslant 0, \tag{12b}$$

after equation (9) is also accounted for.

It is accordingly seen that, regardless of the type of material anisotropy involved, (1) the explicit form of W^e is identical to the standard strain energy function met in non-polar linear elasticity and (2) the value of the polar part of W, namely, W^{Φ}, comes out identical to its conventional counterpart:

$$W^{\Omega} = \frac{1}{2}m_{\ell i}\Omega_{i,\ell} = \frac{1}{2}\bar{m}_{\ell i}\Omega_{i,\ell}, \tag{13}$$

where use is also made of equations (4) and (6).

While equation (9) thus is inadequate for unique determination of the auxiliary vector $\mathbf{\Phi}$, its conjunction with equations (12) and (13) enables the following reduction:

$$W^{\Omega} = W^{\Phi} = \frac{1}{2}m_{\ell i}\Phi_{i,\ell} \geqslant 0. \tag{14}$$

The polar part of W thus is allowed to attain a form that is quadratic in the gradients of any member of the set of virtual spin-vectors $\mathbf{\Phi}$, including the actual spin-vector, $\mathbf{\Omega}$. Hence, a combination of equations (12a) and (14) underpins the search for a linear set of constitutive equations stemming from:

$$\sigma_{(ij)} = \frac{\partial W^e}{\partial e_{ij}}, \tag{15a}$$

$$m_{ji} = \frac{\partial W^{\Phi}}{\partial \Phi_{i,j}}. \tag{15b}$$

It, however, happens that, in the special case that $\mathbf{\Phi} = \mathbf{\Omega}$ where equation (13) holds, the identity equation (6) prevents the spherical part of the couple-stress, m_{rr}, to mark its contribution into the corresponding form of the strain energy function. Indeed, while equation (15a) still holds in that case, a combination of equation (15b) with equation (13) yields

$$\bar{m}_{ji} = \frac{\partial W^{\Omega}}{\partial \Omega_{i,j}}. \tag{16}$$

The conventional couple-stress theory thus emerges as a singular linear elasticity model which, for any well-posed boundary value problem, can lead to determination of the displacement, the symmetric stress, and the deviatoric couple-stress fields only. In doing so, it makes use of the total amount of elastic energy recorded by its refined counterpart, namely, equation (12), which, by virtue of equations (8) and (14), includes an extra term that accounts for the energy contribution of m_{rr}.

Since that indetermined spherical part of the couple-stress, m_{rr}, thus is not accounted for in the resulting antisymmetric part (2b) of the stress tensor, the value of the total strain energy equation (11) differs from that of its displacement gradient counterpart:

$$U(e_{ij}, \omega_{ij}) = \frac{1}{2}\sigma_{ji}u_{i,j} = \frac{1}{2}(\sigma_{(ji)} + \sigma_{[ji]})(e_{ij} + \omega_{ij}) = \frac{1}{2}(\sigma_{(ij)}e_{ij} + \sigma_{[ji]}\omega_{ij}) = W^e(e_{ij}) + W^{\omega}(\omega_{ij}). \tag{17}$$

This is because the here emerged rotation energy is found to be:

$$W^{\omega}(\omega_{ij}) = \frac{1}{2}\sigma_{[ji]}\omega_{ij} = \frac{1}{4}\varepsilon_{kij}m_{\ell k,\ell}\omega_{ij} = -\frac{1}{2}\Omega_k m_{\ell k,\ell} = \frac{1}{2}\Omega_{k,\ell}m_{\ell k} - \frac{1}{2}(\Omega_k m_{\ell k})_{,\ell}$$
$$= W^{\Omega} - \frac{1}{2}(\Omega_k m_{\ell k})_{,\ell} = W^{\Omega} - W^m = W^{\Phi} - W^m, \tag{18}$$

where use is also made of equations (2b), (5b), (9) and (14).

It thus becomes understood that the quantity W^m, introduced in equation (9), represents the part of the strain energy function of the refined formulation that is due to the action of the spherical part of the couple-stress, m_{rr}. That part of W^{Φ} can be identified only after the material symmetries and, therefore, the type of material anisotropy of the solid of interest is specified. Identification of W^m thus becomes a subject of particular importance in the subsequent section (see equation (25)).

Under these considerations, the already available information enables conversion of the second part of equation (9) into the following:

$$\Omega_\ell m_{rr,\ell} = 6W^m - 3(\bar{m}_{\ell i}\Omega_i)_{,\ell}, \tag{19}$$

where use is also made of equation (4). It will next be seen that, regardless of the involved type of material anisotropy, identification and proper consideration of W^m can always transform equation (19) into an extra partial differential equation (PDE) for m_{rr}. Solution of that PDE, subject to the remaining, unused part of the boundary conditions equation (1b), will thus enable determination of (1) the spherical part of the couple-stress, (2) the antisymmetric part of the stress, and, finally, (3) the total stress field.

3. Locally monoclinic materials

It is already seen that, regardless of the involved type of material anisotropy, the explicit form of W^e, appearing in equation (11), is identical to its well-known non-polar linear elasticity counterpart. Hence, no further attention is needed either in the development of explicit forms of the W^e-part of the strain energy function of an anisotropic, polar, linearly elastic material or in the development of corresponding constitutive equations referring to the symmetric part of the stress tensor.

Given the symmetries of the anisotropic polar material of interest, it is instead sufficient for this section to look for and develop explicit forms of (1) the polar parts, W^{Φ} and W^{Ω}, of the strain energy function and (2) the corresponding couple-stress constitutive equations stemming from equations (15b) and (16). Material anisotropy associated with the class of monoclinic solids thus is felt sufficiently general for the purposes of the present communication, as this embraces most subclasses of material anisotropy that is usually employed in structural materials, including structural fibre-reinforced composites.

Locally monoclinic elastic solids are accordingly characterised by the unit vectors, $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$, of two different material preference directions that are generally not mutually orthogonal, in the sense that:

$$a_i^{(1)}a_i^{(2)} = \cos\theta \neq 0. \tag{20}$$

The subclasses of local material orthotropy and local transverse isotropy are then obtained, as special cases, by considering that these directions are either orthogonal ($\cos\theta = 0$) or practically identical ($\mathbf{a}^{(1)} = \pm\mathbf{a}^{(2)}$), respectively.

It is re-emphasised although that, when dealing with fibre-reinforced materials, either of $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ may be selected to represent the direction of a family of perfectly flexible fibres. Unlike corresponding fibres that possess bending stiffness [10,11], such perfectly flexible fibres are expected to behave like flexible cords [15,16] that can resist extension, but not bending, twist, or compression that might be associated with relevant, macroscopically observed deformation modes of the polar material of interest.

3.1. Strain energy function

While the small strain tensor employed in equations (12a) and (15a) is symmetric, the spin gradient $\Phi_{i,j}$ appearing in the couple-stress part (15b) of the constitutive equations is a non-symmetric tensorial quantity. The search for appropriate forms of W^{Φ} that are invariant under rigid body rotation thus requires the split of $\Phi_{i,j}$ into its symmetric and antisymmetric parts:

$$\Phi_{(i,j)} = \frac{1}{2}\left(\Phi_{i,j} + \Phi_{j,i}\right), \Phi_{[i,j]} = \frac{1}{2}\left(\Phi_{i,j} - \Phi_{j,i}\right), \tag{21}$$

respectively.

Since W^{Φ} must be an isotropic invariant function of these quantities, its most general form must be expressed in terms of a corresponding complete set of relevant invariants that represent the symmetry group of polar monoclinic materials. Nevertheless, forms of W^{Φ} that are quadratic in their arguments are sufficient for the purposes of linear elasticity and, hence, third- and higher-order such invariants need not be included in a relevant list [17,18]. That list of invariants is then simplified and becomes:

$$\begin{aligned}
&I_1 = \Phi_{(i,i)} \equiv \Phi_{i,i}, I_2 = \Phi_{(i,j)}\Phi_{(j,i)}, I_3 = \Phi_{[i,j]}\Phi_{[j,i]}, I_4 = a_i^{(1)}\Phi_{(i,j)}a_j^{(1)}, I_5 = a_i^{(1)}\Phi_{(i,j)}\Phi_{(j,k)}a_k^{(1)},\\
&I_6 = a_i^{(1)}\Phi_{[i,j]}\Phi_{[j,k]}a_k^{(1)}, I_7 = a_i^{(1)}\Phi_{(i,j)}\Phi_{[j,k]}a_k^{(1)}, I_8 = a_i^{(2)}\Phi_{(i,j)}a_j^{(2)}, I_9 = a_i^{(2)}\Phi_{(i,j)}\Phi_{(j,k)}a_k^{(2)},\\
&I_{10} = a_i^{(2)}\Phi_{[i,j]}\Phi_{[j,k]}a_k^{(2)}, I_{11} = a_i^{(2)}\Phi_{(i,j)}\Phi_{[j,k]}a_k^{(2)}, I_{12} = a_i^{(1)}\Phi_{(i,j)}a_j^{(2)}, I_{13} = a_i^{(1)}\Phi_{(i,j)}\Phi_{(j,k)}a_k^{(2)},\\
&I_{14} = a_i^{(1)}\Phi_{[i,j]}a_j^{(2)}, I_{15} = a_i^{(1)}\Phi_{[i,j]}\Phi_{[j,k]}a_k^{(2)}, I_{16} = a_i^{(1)}\Phi_{(i,j)}\Phi_{[j,k]}a_k^{(2)},\\
&I_{17} = a_i^{(1)}\left(\Phi_{(i,j)}\Phi_{[j,k]} - \Phi_{[i,j]}\Phi_{(j,k)}\right)a_k^{(2)}, I_{18} = a_i^{(1)}a_i^{(2)} = \cos\theta.
\end{aligned} \tag{22}$$

The most general quadratic form sought for the polar part of the strain energy function thus is as follows:

$$\begin{aligned}
W^{\Phi}\left(\Phi_{i,j}\right) = \frac{1}{2}\{&\eta_0 I_1\left[I_1 + 2\hat{\eta}_0^{(1)}I_4 + 2\hat{\eta}_0^{(2)}I_8 + 2\left(\hat{\eta}_0^{(3)}I_{12} + \hat{\eta}_0^{(4)}I_{14}\right)I_{18}\right] + \eta_1 I_2 + \eta_2 I_3 +\\
&\eta_3 I_4\left[I_4 + 2\hat{\eta}_3^{(1)}I_8 + 2\left(\hat{\eta}_3^{(2)}I_{12} + \hat{\eta}_3^{(3)}I_{14}\right)I_{18}\right] + \eta_4 I_5 + \eta_5 I_6 + 2\eta_6 I_7 +\\
&\eta_7 I_8\left[I_8 + 2\left(\hat{\eta}_7^{(1)}I_{12} + \hat{\eta}_7^{(2)}I_{14}\right)I_{18}\right] + \eta_8 I_9 + \eta_9 I_{10} + 2\eta_{10}I_{11} +\\
&\eta_{11}I_{12}\left(I_{12} + 2\hat{\eta}_{11}^{(1)}I_{14}\right) + \eta_{12}I_{13}I_{18} + \eta_{13}I_{14}^2 + \left(\eta_{14}I_{15} + 2\eta_{15}I_{16} + 2\eta_{16}I_{17}\right)I_{18}\},
\end{aligned} \tag{23}$$

and involves a total of 27 independent elastic moduli having dimensions of force. Seventeen of those moduli are represented by the symbols η_0 to η_{16}. For notational convenience, each of the remaining 10, additional elastic moduli is noted as the product of one of the four moduli η_α (α = 0, 3, 7, 11) with some suitably noted non-dimensional parameter of the type $\hat{\eta}_\alpha^{(\beta)}$.

It is also noted that the deformation invariants I_{12} to I_{17} are all odd in the components of both unit vectors $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$. Hence, wherever necessary in equation (23), each of these invariants is multiplied by the non-deformation invariant I_{18} which is also odd in both $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$. While W^{Φ} thus remains quadratic in the spin gradients, its expression is not affected by the sense of either direction of material preference, as is usually considered true in the case of structural fibre-reinforced materials.

In the singular case of the conventional couple-stress theory ($\mathbf{\Phi} = \mathbf{\Omega}$), validity of equation (6) implies that $I_1 = 0$ and, hence, equation (23) reduces to:

$$\begin{aligned}
W^{\Omega}\left(\Omega_{i,j}\right) = \frac{1}{2}\{&\eta_1 I_2 + \eta_2 I_3 + \eta_3 I_4\left[I_4 + 2\hat{\eta}_3^{(1)}I_8 + 2\left(\hat{\eta}_3^{(2)}I_{12} + \hat{\eta}_3^{(3)}I_{14}\right)I_{18}\right] + \eta_4 I_5 + \eta_5 I_6 +\\
&2\eta_6 I_7 + \eta_7 I_8\left[I_8 + 2\left(\hat{\eta}_7^{(1)}I_{12} + \hat{\eta}_7^{(2)}I_{14}\right)I_{18}\right] + \eta_8 I_9 + \eta_9 I_{10} + 2\eta_{10}I_{11} +\\
&\eta_{11}I_{12}\left(I_{12} + 2\hat{\eta}_{11}^{(1)}I_{14}\right) + \eta_{12}I_{13}I_{18} + \eta_{13}I_{14}^2 + \left(\eta_{14}I_{15} + 2\eta_{15}I_{16} + 2\eta_{16}I_{17}\right)I_{18}\}.
\end{aligned} \tag{24}$$

A comparison of this expression with equation (23) makes it then understood that the energy contribution of the spherical couple-stress is marked in W^{Φ} through incorporation of the terms that are missing in equation (24). Namely, the terms multiplied by the elastic moduli η_0, $\eta_0\hat{\eta}_0^{(1)}$, $\eta_0\hat{\eta}_0^{(2)}$, $\eta_0\hat{\eta}_0^{(3)}$, and $\eta_0\hat{\eta}_0^{(4)}$.

It follows that the extra energy contribution, associated in equation (19) with the action of the spherical part of the couple-stress, is:

$$W^m = \frac{1}{2}\eta_0 I_1\left[I_1 + 2\hat{\eta}_0^{(1)}I_4 + 2\hat{\eta}_0^{(2)}I_8 + 2\left(\hat{\eta}_0^{(3)}I_{12} + \hat{\eta}_0^{(4)}I_{14}\right)I_{18}\right]. \tag{25}$$

With the use of equation (22), the extra differential equation (19) is then enabled to attain the more specific form:

$$\Omega_\ell m_{rr,\ell} = 3\eta_0 \Phi_{r,r}\left[\Phi_{k,k} + 2\hat{\eta}_0^{(1)} a_r^{(1)} \Phi_{(r,k)} a_k^{(1)} + 2\hat{\eta}_0^{(2)} a_r^{(2)} \Phi_{(r,k)} a_k^{(2)} + 2\left(\hat{\eta}_0^{(3)} a_r^{(1)} \Phi_{(r,k)} a_k^{(2)} + \hat{\eta}_0^{(4)} a_r^{(1)} \Phi_{[r,k]} a_k^{(2)}\right)\cos\theta\right] - 3\left(\bar{m}_{\ell i}\Omega_i\right)_{,\ell}. \tag{26}$$

3.2. Constitutive equations

Connection of equation (23) with equation (15b), followed by the process detailed in Appendix 1, yields the following couple-stress constitutive equation:

$$\begin{aligned} m_{ji} = {} & \eta_0 \Phi_{k,k}\left(\delta_{ij} + \vartheta_{ij}^{(0)}\right) + \eta_1 \Phi_{(i,j)} + \eta_2 \Phi_{[j,i]} + \left(\eta_0 \hat{\eta}_0^{(1)} \delta_{ij} + \vartheta_{ij}^{(3)}\right) I_4 + a_k^{(1)}\left(\eta_4 \Phi_{(j,k)} + \eta_5 \Phi_{[j,k]}\right) a_i^{(1)} + \\ & \eta_6 a_k^{(1)}\left(\Phi_{(k,i)} a_j^{(1)} - \Phi_{[k,j]} a_i^{(1)}\right) + \left(\eta_0 \hat{\eta}_0^{(2)} \delta_{ij} + \vartheta_{ij}^{(7)}\right) I_8 + a_k^{(2)}\left(\eta_8 \Phi_{(j,k)} + \eta_9 \Phi_{[j,k]}\right) a_i^{(2)} + \\ & \eta_{10} a_k^{(2)}\left(\Phi_{(k,i)} a_j^{(2)} - \Phi_{[k,j]} a_i^{(2)}\right) + \left(\vartheta_{ij}^{(11)} + \eta_0 \hat{\eta}_0^{(3)} I_{18} \delta_{ij}\right) I_{12} + \left(\vartheta_{ij}^{(13)} + \eta_0 \hat{\eta}_0^{(4)} I_{18} \delta_{ij}\right) I_{14} + \\ & I_{18} a_k^{(2)}\left[(\eta_{12} - \eta_{16})\Phi_{(j,k)} + (\eta_{14} + \eta_{15} + \eta_{16})\Phi_{[j,k]}\right] a_i^{(1)} + I_{18} a_k^{(1)}\left[(\eta_{15} + \eta_{16})\Phi_{(k,i)} - \eta_{16} \Phi_{[k,i]}\right] a_j^{(2)}, \end{aligned} \tag{27}$$

where considerable brevity is achieved with the use of the following tensorial quantities:

$$\begin{aligned} \vartheta_{ij}^{(0)} &= \hat{\eta}_0^{(1)} a_i^{(1)} a_j^{(1)} + \hat{\eta}_0^{(2)} a_i^{(2)} a_j^{(2)} + \left(\hat{\eta}_0^{(3)} + \hat{\eta}_0^{(4)}\right) I_{18} a_i^{(1)} a_j^{(2)}, \\ \vartheta_{ij}^{(3)} &= \eta_3\left[a_i^{(1)} a_j^{(1)} + \hat{\eta}_3^{(1)} a_i^{(2)} a_j^{(2)} + \left(\hat{\eta}_3^{(2)} + \hat{\eta}_3^{(3)}\right) I_{18} a_i^{(1)} a_j^{(2)}\right], \\ \vartheta_{ij}^{(7)} &= \eta_7\left[a_i^{(2)} a_j^{(2)} + \left(\hat{\eta}_7^{(1)} + \hat{\eta}_7^{(2)}\right) I_{18} a_i^{(1)} a_j^{(2)}\right] + \eta_3 \hat{\eta}_3^{(1)} a_i^{(1)} a_j^{(1)}, \\ \vartheta_{ij}^{(11)} &= \eta_{11}\left(1 + \hat{\eta}_{11}^{(1)}\right) a_i^{(1)} a_j^{(2)} + \left(\eta_3 \hat{\eta}_3^{(2)} a_i^{(1)} a_j^{(1)} + \eta_7 \hat{\eta}_7^{(1)} a_i^{(2)} a_j^{(2)}\right) I_{18}, \\ \vartheta_{ij}^{(13)} &= \left[\left(\eta_{13} + \eta_{11} \hat{\eta}_{11}^{(1)}\right) a_i^{(1)} a_j^{(2)} + \left(\eta_3 \hat{\eta}_3^{(3)} a_i^{(1)} a_j^{(1)} + \eta_7 \hat{\eta}_7^{(2)} a_i^{(2)} a_j^{(2)}\right) I_{18}\right]. \end{aligned} \tag{28}$$

It is fitting at this point to mention, for later use, that the linear constitutive equation (27) can be rearranged and attain an alternative matrix form:

$$\hat{\mathbf{M}} = \hat{\mathbf{H}}\hat{\mathbf{\Phi}}, \tag{29}$$

where $\hat{\mathbf{M}}$ and $\hat{\mathbf{\Phi}}$ represent the nine-component vectors:

$$\hat{\mathbf{M}} = \left(m_{11}, m_{22}, m_{33}, m_{23}, m_{32}, m_{13}, m_{31}, m_{12}, m_{21}\right)^T, \tag{30a}$$

$$\hat{\Phi} = \left(\Phi_{1,1}, \Phi_{2,2}, \Phi_{3,3}, \Phi_{(2,3)}, \Phi_{[2,3]}, \Phi_{(1,3)}, \Phi_{[1,3]}, \Phi_{(1,2)}, \Phi_{[1,2]}\right)^T. \tag{30b}$$

As will also be demonstrated with a particular example in section 4 below, the non-zero components of the implied 9×9 stiffness matrix $\hat{\mathbf{H}}$ are formed through appropriate combinations of the elastic moduli introduced in equation (23) and the components of the unit vectors $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$.

The deviatoric couple-stress constitutive equation of the conventional couple-stress theory becomes now also available, either by connecting equation (24) with equation (16) and, then, following a process analogous to that detailed in Appendix 1 or, more conveniently, by replacing everywhere in equations (21), (22), and (27) the symbol Φ with Ω and, then, deleting in the latter equation all terms involving η_0. It thus is found that:

$$\bar{m}_{ji} = \eta_1 \Omega_{(i,j)} + \eta_2 \Omega_{[j,i]} + I_4 \vartheta_{ij}^{(3)} + a_k^{(1)} \left(\eta_4 \Omega_{(j,k)} + \eta_5 \Omega_{[j,k]}\right) a_i^{(1)} + \eta_6 a_k^{(1)} \left(\Omega_{(k,i)} a_j^{(1)} - \Omega_{[k,j]} a_i^{(1)}\right) +$$
$$I_8 \vartheta_{ij}^{(7)} + a_k^{(2)} \left(\eta_8 \Omega_{(j,k)} + \eta_9 \Omega_{[j,k]}\right) a_i^{(2)} + \eta_{10} a_k^{(2)} \left(\Omega_{(k,i)} a_j^{(2)} - \Omega_{[k,j]} a_i^{(2)}\right) + I_{12} \vartheta_{ij}^{(11)} + I_{14} \vartheta_{ij}^{(13)} +$$
$$I_{18} a_k^{(2)} \left[(\eta_{12} - \eta_{16}) \Omega_{(j,k)} + (\eta_{14} + \eta_{15} + \eta_{16}) \Omega_{[j,k]}\right] a_i^{(1)} + I_{18} a_k^{(1)} \left[(\eta_{15} + \eta_{16}) \Omega_{(k,i)} - \eta_{16} \Omega_{[k,i]}\right] a_j^{(2)}, \tag{31}$$

where equation (28) still hold.

Successful solution of any relevant, well-posed boundary value problem, thus, requires from the conventional couple-stress theory to make available the ultimate forms of the displacement vector, the small strain tensor, and the actual spin-vector $\mathbf{\Omega}$, as well as the corresponding symmetric part of the stress and the deviatoric couple-stress fields.

Then, the spherical part of the couple-stress still needs to be determined through appropriate use and solution of the extra equation (26). In this regard, Appendix 1 provides additional information of the way that contraction of the free indices appearing in equation (27) leads to the following form:

$$m_{rr} = \tilde{\eta}_0 \Phi_{k,k} + \tilde{\eta}_1 a_k^{(1)} \Phi_{(k,\ell)} a_\ell^{(1)} + \tilde{\eta}_2 a_k^{(2)} \Phi_{(k,\ell)} a_\ell^{(2)} + \left(\tilde{\eta}_3 \cos\theta a_k^{(1)} \Phi_{(k,\ell)} a_\ell^{(2)} + \tilde{\eta}_4 a_k^{(1)} \Phi_{[k,\ell]} a_\ell^{(2)}\right) \cos\theta, \tag{32}$$

of the spherical part of the couple-stress, where the appearing combined elastic moduli are given in equation (86).

3.3. Determination of the spherical part of the couple-stress

In the special case of material isotropy, where $\mathbf{a}^{(1)} = \mathbf{a}^{(2)} = \mathbf{0}$, all but one combined elastic moduli appearing in equation (32) attain zero values ($\tilde{\eta}_0 = \eta_0$, $\tilde{\eta}_1 = \tilde{\eta}_2 = \tilde{\eta}_3 = \tilde{\eta}_4 = 0$) and, also, it is $\hat{\eta}_0^{(1)} = \hat{\eta}_0^{(2)} = \hat{\eta}_0^{(3)} = \hat{\eta}_0^{(4)} = 0$ in equation (26). This is the very important special case considered in Soldatos [9], where a combination of the simple forms thus attained by equations (32) and (26) suffices to eliminate in the latter equation the gradients of the auxiliary vector $\mathbf{\Phi}$ and, thus, lead to an extra PDE for the remaining unknown, m_{rr}.

Such a simple combination of equations (26) and (32) is not anymore possible in the case of any type of material anisotropy. However, the alterative form (29) of the couple-stress constitutive equation serves as a suitable starting point for the description of an alternative, although more cumbersome elimination process of the appearing gradients of $\mathbf{\Phi}$.

Since the components of the unit vectors $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$, as well as the values of all elastic moduli appearing in equation (23), depend on the anisotropic properties of the polar material of interest, that process begins with the reasonable assumption that the matrix $\hat{\mathbf{H}}$ is invertible. Inversion of the matrix equation (29) thus leads to:

$$\hat{\mathbf{\Phi}} = \hat{\mathbf{H}}^{-1} \hat{\mathbf{M}} = \hat{\mathbf{H}}^{-1} (\bar{\mathbf{M}} + \mathbf{M}^s), \tag{33}$$

where use of equation (4) implies that:

$$\bar{\mathbf{M}} = (\bar{m}_{11}, \bar{m}_{22}, \bar{m}_{33}, m_{23}, m_{32}, m_{13}, m_{31}, m_{12}, m_{21})^T,$$
$$\mathbf{M}^s = \left(\frac{1}{3} m_{rr}, \frac{1}{3} m_{rr}, \frac{1}{3} m_{rr}, 0, 0, 0, 0, 0, 0\right)^T. \tag{34}$$

The first part of equation (33) enables replacement of the unknown gradients (30b) of $\mathbf{\Phi}$, also appearing in equations (26) and (32), with the components (30a) of the non-symmetric couple-stress tensor, $\mathbf{m}$. Moreover, the split noted in the right-hand side of equation (33) enables every term appearing in the resulting equation to be expressed in terms of the unknown spherical couple-stress, m_{rr}, and/or quantities that have already become known by solving the governing equations of the conventional couple-stress theory.

As is exemplified in the subsequent section with the help of Appendix 2, the final form thus attained by equation (26) can be regarded as a first-order PDE for m_{rr}. Solution of that PDE, subject to an appropriately imposed boundary condition, then provides m_{rr} and, thus, leads to full determination of the couple-stress and, henceforth, stress fields.

It is worth noting that, with the couple-stress components thus becoming completely known, equation (33) might be re-employed to provide the gradients (30b) of the auxiliary vector $\mathbf{\Phi}$. However, potential integration of those gradients will necessitate introduction of a set of arbitrary constants and/or functions of the co-ordinate parameters. This observation thus serves as an alternative verification of the fact that $\mathbf{\Phi}$ is a virtual spin-type vector, not susceptible to unique determination.

4. Application: transverse isotropy

4.1. Locally transverse isotropic materials

This example application considers a special case in which anisotropy of the polar elastic material of interest is due to the presence of a single family of perfectly flexible fibres. In that case, the analysis detailed in the preceding section simplifies by considering that:

$$\mathbf{a}^{(1)} \equiv \mathbf{a}, \mathbf{a}^{(2)} \equiv \mathbf{0}, \tag{35}$$

and by thus accounting only for the first seven of the deformation invariants listed in equation (22).

Accordingly, the most general form of W^{Φ} is now given as follows:

$$W^{\Phi}\left(\Phi_{i,j}\right) = \frac{1}{2}\left[\eta_0 I_1(I_1 + 2\hat{\eta}_0 I_4) + \eta_1 I_2 + \eta_2 I_3 + \eta_3 I_4^2 + \eta_4 I_5 + \eta_5 I_6 + 2\eta_6 I_7\right]. \tag{36}$$

In the singular case of the conventional couple-stress theory ($\mathbf{\Phi} = \mathbf{\Omega},\ I_1 = 0$), this reduces to:

$$W^{\Omega}\left(\Omega_{i,j}\right) = \frac{1}{2}\left(\eta_1 I_2 + \eta_2 I_3 + \eta_3 I_4^2 + \eta_4 I_5 + \eta_5 I_6 + 2\eta_6 I_7\right). \tag{37}$$

Since the extra energy contribution appearing in equation (25) thus also simplifies and becomes:

$$W^m = \frac{1}{2}\eta_0 I_1(I_1 + 2\hat{\eta}_0 I_4), \tag{38}$$

the extra equation (26) that will lead to the spherical part of the couple-stress determination also attains a simplified form:

$$\Omega_\ell m_{rr,\ell} = 3\eta_0 \Phi_{r,r}\left(\Phi_{k,k} + 2\hat{\eta}_0 a_r \Phi_{(r,k)} a_k\right) - 3\left(\bar{m}_{\ell i}\Omega_i\right)_{,\ell}. \tag{39}$$

Moreover, appropriate reduction of equations (27) and (31) yields the corresponding couple-stress constitutive equation:

$$\begin{aligned} m_{ji} = {} & \eta_0 \Phi_{k,k}\left(\delta_{ij} + \hat{\eta}_0 a_i a_j\right) + \eta_1 \Phi_{(i,j)} + \eta_2 \Phi_{[j,i]} + \left(\eta_0 \hat{\eta}_0 \delta_{ij} + \eta_3 a_i a_j\right)\Phi_{(m,n)} a_m a_n + \\ & a_k\left(\eta_4 \Phi_{(j,k)} + \eta_5 \Phi_{[j,k]}\right) a_i + \eta_6 a_k\left(\Phi_{(k,i)} a_j - \Phi_{[k,j]} a_i\right), \end{aligned} \tag{40}$$

for the refined couple-stress theory, and:

$$\bar{m}_{ji} = \eta_1 \Omega_{(i,j)} + \eta_2 \Omega_{[j,i]} + \eta_3 a_i a_j \Omega_{(m,n)} a_m a_n + a_k\left(\eta_4 \Omega_{(j,k)} + \eta_5 \Omega_{[j,k]}\right) a_i + \eta_6 a_k\left(\Omega_{(k,i)} a_j - \Omega_{[k,j]} a_i\right), \tag{41}$$

for its conventional model counterpart.

Hence, through contraction of its free indices, equation (40) provides the following constitutive equation for the spherical part of the couple-stress:

$$m_{rr} = \tilde{\eta}_0 \Phi_{k,k} + \tilde{\eta}_1 a_k \Phi_{(k,\ell)} a_\ell, \tag{42}$$

where:

$$\tilde{\eta}_0 = \eta_0(3 + \hat{\eta}_0) + \eta_1, \tilde{\eta}_1 = 3\eta_0\hat{\eta}_0 + \eta_3 + \eta_4 + \eta_6. \tag{43}$$

As is previously outlined (see section 3.3) and is also demonstrated next for the further special case of straight fibres, appropriate combination of equations (41) and (39) will finally provide a more specific form attained by the extra PDE that leads to determination of the spherical part of the couple-stress.

4.2. Transverse isotropy due to single family of straight perfectly flexible fibres

In this, further special case, where anisotropy is due to the presence of a single family of perfectly flexible straight fibres, the analysis detailed in the preceding section attains further simplification by selecting:

$$\mathbf{a} = (1, 0, 0)^T, \tag{44}$$

and, hence, by placing the x_1-axis of the co-ordinate system parallel to the fibre direction.

Four of the seven invariants appearing in the strain energy functions (36) thus also attain simplified forms, namely:

$$I_4 = \Phi_{(1,1)}, I_5 = \Phi_{(1,j)}\Phi_{(j,1)}, I_6 = \Phi_{[1,j]}\Phi_{[j,1]}, I_7 = \Phi_{(1,j)}\Phi_{[j,1]}. \tag{45}$$

Hence, the extra energy contribution (38) is reduced accordingly, and the extra equation (39) becomes:

$$\Omega_\ell m_{rr,\ell} = 3\eta_0\Phi_{r,r}\left(\Phi_{k,k} + 2\hat{\eta}_0^{(1)}\Phi_{(1,1)}\right) - 3(\bar{m}_{\ell i}\Omega_i)_{,\ell}. \tag{46}$$

Moreover, the constitutive equation (40) reduces to:

$$\begin{aligned} m_{ji} = {} & \eta_0\Phi_{k,k}\left(\delta_{ij} + \vartheta_{ij}^{(0)}\right) + \eta_1\Phi_{(i,j)} + \eta_2\Phi_{[j,i]} + \Phi_{(1,1)}\left(\eta_0\hat{\eta}_0^{(1)}\delta_{ij} + \vartheta_{ij}^{(3)}\right) + \\ & \left(\eta_4\Phi_{(j,1)} + \eta_5\Phi_{[j,1]}\right)a_i + \eta_6\left(\Phi_{(1,i)}a_j - \Phi_{[1,j]}a_i\right), \end{aligned} \tag{47}$$

and the simple form of the appearing quantities:

$$\left(\vartheta_{ij}^{(0)}, \vartheta_{ij}^{(3)}\right) = \left(\hat{\eta}_0^{(1)}, \eta_3\right)a_i a_j, \tag{48}$$

enables further reduction, as follows:

$$\begin{aligned} m_{ji} = {} & \eta_0\Phi_{k,k}\left(\delta_{ij} + \hat{\eta}_0^{(1)}a_i a_j\right) + \eta_1\Phi_{(i,j)} + \eta_2\Phi_{[j,i]} + \Phi_{(1,1)}\left(\eta_0\hat{\eta}_0^{(1)}\delta_{ij} + \eta_3 a_i a_j\right) + \\ & \left(\eta_4\Phi_{(j,1)} + \eta_5\Phi_{[j,1]}\right)a_i + \eta_6\left(\Phi_{(1,i)}a_j - \Phi_{[1,j]}a_i\right). \end{aligned} \tag{49}$$

In this case, the stiffness matrix appearing in the alternative form (29) of the constitutive equation (49) attains the form:

$$\hat{\mathbf{H}} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ H_{21} & H_{22} & H_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ H_{31} & H_{32} & H_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{44} & H_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{54} & H_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_{66} & H_{67} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_{76} & H_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_{88} & H_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_{98} & H_{99} \end{bmatrix}, \tag{50}$$

where the appearing non-zero elements are as follows:

$$\begin{aligned}&H_{11}=\eta_0\big(1+2\hat{\eta}_0^{(1)}\big)+\eta_1+\eta_3+\eta_4+\eta_5+\eta_6, H_{22}=H_{33}=\eta_0+\eta_1,\\&H_{12}=H_{21}=H_{13}=H_{31}=\eta_0\big(1+\hat{\eta}_0^{(1)}\big), H_{23}=H_{32}=\eta_0,\\&H_{44}=H_{55}=\eta_1, H_{45}=-H_{54}=\eta_2,\\&H_{66}=H_{88}=\eta_1+\eta_6, H_{77}=H_{99}=-(\eta_2+\eta_5+\eta_6),\\&H_{67}=H_{89}=\eta_2, H_{76}=H_{98}=\eta_1+\eta_4.\end{aligned} \tag{51}$$

A more comprehensive view of the structure of matrix $\hat{\mathbf{H}}$ is achieved through the following, alternative representation:

$$\hat{\mathbf{H}}=\begin{bmatrix}\underset{(3\times3)}{\mathbf{H}_1} & \underset{(3\times2)}{\mathbf{0}} & \underset{(3\times2)}{\mathbf{0}} & \underset{(3\times2)}{\mathbf{0}}\\ \underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{H}_2} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}}\\ \underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{H}_3} & \underset{(2\times2)}{\mathbf{0}}\\ \underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{H}_3}\end{bmatrix}, \tag{52}$$

where the dimensions of each submatrix involved are included in associated parentheses, and:

$$\mathbf{H}_1=\begin{bmatrix}H_{11} & H_{12} & H_{12}\\ H_{12} & H_{22} & H_{23}\\ H_{12} & H_{23} & H_{22}\end{bmatrix}, \tag{53a}$$

$$\mathbf{H}_2=\begin{bmatrix}H_{44} & H_{45}\\ -H_{45} & H_{44}\end{bmatrix}, \tag{53b}$$

$$\mathbf{H}_3=\begin{bmatrix}H_{66} & H_{67}\\ H_{76} & H_{77}\end{bmatrix}, \tag{53c}$$

on account of the relationships provided in equation (51).

The form (52) of $\hat{\mathbf{H}}$ is found useful during the process of conversion of the extra equation (46) into a PDE for the unknown quantity:

$$m_{rr}=\tilde{\eta}_0\Phi_{k,k}+\tilde{\eta}_1\Phi_{(1,1)}, \tag{54}$$

which is obtained either by contracting the free indices appearing in equation (49) or through direct use of equation (42). That conversion process is detailed in Appendix 2 and enables transformation of equation (46) into the non-linear PDE:

$$\Omega_\ell m_{rr,\ell}-3\eta_0K_0m_{rr}^2-6\eta_0[K_1\bar{m}_{11}+K_2(\bar{m}_{22}+\bar{m}_{33})]m_{rr}=f(\bar{m}_{11},\bar{m}_{22},\bar{m}_{33})-3(\bar{m}_{\ell i}\Omega_i)_{,\ell}, \tag{55}$$

for the unknown m_{rr}, where the appearing constant coefficients K_0, K_1, and K_2 and the known function f are also given in Appendix 2.

5. Positive definiteness of the strain energy function

Attention is now turned onto the restrictions imposed on the elastic moduli involved in the constitutive equation (49) by the required positive definiteness of the corresponding form that the strain energy function (11) attains in the case of polar transverse isotropy. Since that form of W splits naturally into two independent parts, namely, W^e and W^Φ, the analysis will present similarities with its counterpart detailed in section 4 of Soldatos [19] for corresponding polar materials containing fibres resistant in bending. However, it will also present considerable differences, a notable one stemming from the fact that the submatrices $\mathbf{H}_2$ and $\mathbf{H}_3$ defined in equation (53) are not symmetric.

Accordingly, since non-polar linear elasticity requires from the known, transversely isotropic form of W^e to be positive definite, it is sufficient for W^{Φ} to obey conditions of a positive semi-definite quantity. This requirement is equivalent to the following:

$$\hat{\mathbf{\Phi}}^T \hat{\mathbf{H}} \hat{\mathbf{\Phi}} \geqslant 0, \tag{56}$$

where the superfix ^{T} signifies "transposed," and the equality naturally holds only in the special case of non-polar linear elasticity.

Nevertheless, equation (56) is anticipated valid not only for the case of the refined couple-stress theory but also for the singular case of its conventional counterpart, where the influence of η_0 is unaccountable. For that reason, these two cases are considered separately in what follows.

5.1. Refined couple-stress theory

The special structure (52) of $\hat{\mathbf{H}}$ enables expansion of equation (56) into the form:

$$\begin{aligned}&(\Phi_{1,1}, \Phi_{2,2}, \Phi_{3,3}) \begin{bmatrix} H_{11} & H_{12} & H_{12} \\ H_{12} & H_{22} & H_{23} \\ H_{12} & H_{23} & H_{22} \end{bmatrix} \begin{pmatrix} \Phi_{1,1} \\ \Phi_{2,2} \\ \Phi_{3,3} \end{pmatrix} + \left(\Phi_{(2,3)}, \Phi_{[2,3]}\right) \begin{bmatrix} H_{44} & H_{45} \\ -H_{45} & H_{44} \end{bmatrix} \begin{pmatrix} \Phi_{(2,3)} \\ \Phi_{[2,3]} \end{pmatrix} + \\ &\left(\Phi_{(1,3)}, \Phi_{[1,3]}\right) \begin{bmatrix} H_{66} & H_{67} \\ H_{76} & H_{77} \end{bmatrix} \begin{pmatrix} \Phi_{(1,3)} \\ \Phi_{[1,3]} \end{pmatrix} + \left(\Phi_{(1,2)}, \Phi_{[1,2]}\right) \begin{bmatrix} H_{66} & H_{67} \\ H_{76} & H_{77} \end{bmatrix} \begin{pmatrix} \Phi_{(1,2)} \\ \Phi_{[1,2]} \end{pmatrix} \geqslant 0.\end{aligned} \tag{57}$$

It follows that positive semi-definiteness of equation (56) requires from each of the submatrices $\mathbf{H}_1$,$\mathbf{H}_2$, and $\mathbf{H}_3$ defined in equation (53) to be positive semi-definite.

In this context, positive semi-definiteness of $\mathbf{H}_1$ implies that the elements of the principal diagonal as well as the 2×2 minor determinants of this matrix are non-negative, namely:

$$\begin{aligned}&H_{11} \geqslant 0, H_{22} \geqslant 0, H_{11}\left(H_{22}^2 - H_{23}^2\right) + 2H_{12}^2\left(H_{23} - H_{22}\right) \geqslant 0, \\ &H_{11}H_{22} - H_{12}^2 \geqslant 0, H_{11}H_{22} - H_{23}^2 \geqslant 0, H_{22}^2 - H_{23}^2 \geqslant 0.\end{aligned} \tag{58}$$

Connection of the first three of these inequalities with equation (51) imposes the following restrictions on the values of the relevant elastic moduli:

$$\eta_1 \geqslant 0, \eta_0 \geqslant -\eta_1, \frac{2\eta_1}{\eta_0\left(1 + 2\hat{\eta}_0^{(1)}\right) + \eta_\Omega} + \frac{(\eta_0)^2}{(\eta_0 + \eta_1)^2} \leqslant 1, \tag{59}$$

where:

$$\eta_\Omega = \eta_1 + \eta_3 + \eta_4 + \eta_5 + \eta_6. \tag{60}$$

Moreover, satisfaction of the last three of equation (58) requires imposition of the additional condition:

$$\begin{cases} \left[\eta_0\left(1 + 2\hat{\eta}_0^{(1)}\right) + \eta_\Omega\right] \geqslant \frac{{\eta_0}^2}{\eta_0 + \eta_1}, & \text{if } 0 \geqslant \hat{\eta}_0^{(1)} \geqslant -2; \\ \left[\eta_0\left(1 + 2\hat{\eta}_0^{(1)}\right) + \eta_\Omega\right] \geqslant \frac{{\eta_0}^2\left(1 + \hat{\eta}_0^{(1)}\right)^2}{\eta_0 + \eta_1}, & \text{otherwise.} \end{cases} \tag{61}$$

However, since both submatrices $\mathbf{H}_2$ and $\mathbf{H}_3$ are non-symmetric, it is adequate for this analysis (e.g., [20]) to apply positive semi-definite rules on their symmetric parts:

$$\mathbf{H}_{(2)} = \frac{1}{2}\left(\mathbf{H}_2 + \mathbf{H}_2^T\right) = \begin{bmatrix} H_{44} & 0 \\ 0 & H_{44} \end{bmatrix} = \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_1 \end{bmatrix},$$
$$\mathbf{H}_{(3)} = \frac{1}{2}\left(\mathbf{H}_3 + \mathbf{H}_3^T\right) = \begin{bmatrix} H_{66} & \frac{1}{2}(H_{67} + H_{76}) \\ \frac{1}{2}(H_{76} + H_{77}) & H_{77} \end{bmatrix} = \begin{bmatrix} \eta_1 + \eta_6 & \frac{1}{2}(\eta_1 + \eta_2 + \eta_4) \\ \frac{1}{2}(\eta_1 + \eta_2 + \eta_4) & -(\eta_2 + \eta_5 + \eta_6) \end{bmatrix}, \tag{62}$$

where use is made of equation (51).

It thus becomes immediately understood that satisfaction of the first of the conditions equation (59) suffices to guarantee positive semi-definiteness of $\mathbf{H}_2$. The final conditions sought, namely:

$$\eta_6 \geqslant -\eta_1, \eta_2 + \eta_5 + \eta_6 \leqslant 0, (\eta_1 + \eta_2 + \eta_4)^2 \leqslant -4(\eta_1 + \eta_6)(\eta_2 + \eta_5 + \eta_6), \tag{63}$$

are similarly obtained by observing that the elements of the principal diagonal as well as the determinant of $\mathbf{H}_{(3)}$ are ought to be non-negative.

5.2. The singular case of the conventional couple-stress theory

In the case of the conventional couple-stress theory, where equation (6) holds, the constitutive equation (49) reduces to:

$$\bar{m}_{ji} = \eta_1\Omega_{(i,j)} + \eta_2\Omega_{[j,i]} + \eta_3\Omega_{(1,1)}a_i a_j + \left(\eta_4\Omega_{(j,1)} + \eta_5\Omega_{[j,1]}\right)a_i + \eta_6\left(\Omega_{(1,i)}a_j - \Omega_{[1,j]}a_i\right). \tag{64}$$

Moreover, the inequality equation (56) attains the special form:

$$\hat{\mathbf{\Omega}}^T\tilde{\mathbf{H}}\hat{\mathbf{\Omega}} \geqslant 0, \tag{65}$$

where:

$$\hat{\mathbf{\Omega}} = \left(\Omega_{1,1}, \Omega_{2,2}, \Omega_{3,3}, \Omega_{(2,3)}, \Omega_{[2,3]}, \Omega_{(1,3)}, \Omega_{[1,3]}, \Omega_{(1,2)}, \Omega_{[1,2]}\right)^T, \tag{66}$$

and the stiffness matrix $\tilde{\mathbf{H}}$ is obtained by neglecting any input related to the modulus η_0 in the matrix $\hat{\mathbf{H}}$.

It is then seen that the structure of $\tilde{\mathbf{H}}$ resembles closely the structure equation (52) of $\hat{\mathbf{H}}$, with the only difference being the fact that the 3 × 3 submatrix defined in equation (53a) attains now the substantially simplified, diagonal form:

$$\tilde{\mathbf{H}}_1 = \begin{bmatrix} \eta_\Omega & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_1 \end{bmatrix}, \tag{67}$$

where the appearing, composed elastic modulus, defined in equation (60), has already exerted remarkable influence on the positive semi-definiteness conditions (61).

It is seen that, when accompanied by the first of inequalities equation (59), the additional condition:

$$\eta_\Omega \geqslant 0, \tag{68}$$

suffices to ensure positive semi-definiteness of $\tilde{\mathbf{H}}_1$.

It is accordingly concluded that, along with their well-known non-polar, transversely isotropic elasticity counterparts, the inequalities (59), (61), and (68) suffice in this case to ensure positive definiteness of the strain energy function (11), for both the conventional and the refined versions of the linear couple-stress theory.

6. On the non-elliptic structure of the governing equations—weak discontinuity surfaces

A more detailed, although still preliminary discussion of the fact that the governing equations of polar linear elasticity are not elliptic is provided in section 5 of Soldatos [19]. That discussion is associated in

Soldatos [19] with the governing equations of relevant fibrous composites containing fibres resistant in bending, but its principal mathematical arguments still hold in the present case of interest. It is accordingly found adequate for the present discussion to confine attention within the bounds of polar transverse isotropy that is due to the presence of a single family of perfectly flexible straight fibres.

In this context, the elastic moduli appearing in equations (40), (41) and, subsequently, in equation (49), are considered independent of the co-ordinate parameters and, therefore, constant. Naturally, the same is considered true for all five independent elastic moduli, λ, μ_T, μ_L, α, and β, appearing in the corresponding constitutive equation of non-polar linear elasticity:

$$\sigma_{(ij)} = \frac{\partial W^e}{\partial e_{ij}} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha\left(e_{11}\delta_{ij} + e_{kk}a_i a_j\right) + \beta e_{11} a_i a_j + 2(\mu_L - \mu_T)\left(a_i e_{j1} + a_j e_{i1}\right), \tag{69}$$

whose form implies that equation (44) is also valid.

With use of constitutive equations thus emerging from equations (69) and (41), the principal equilibrium equation (2a) may take the form:

$$\mathcal{A}_{mirj} u_{j,mr} + \mathcal{B}_{mirjnq} u_{j,mrqn} = 0, \tag{70}$$

where use of equation (5) is also implied. Here, $\mathcal{A}_{mirj}$ and $\mathcal{B}_{mirjnq}$ are the elements of corresponding tensor quantities, whose order is indicated by the number of associated indices.

In the special case of non-polar linear elasticity, where $\mathcal{B}_{mirjnq} = 0$, equation (70) acquire their well established, displacement-based (Navier-type) form. Like the displacement components themselves, the appearing second-order derivatives are considered continuous throughout the linearly elastic material of interest and, since their coefficients $\mathcal{A}_{mirj}$ are all of the same sign, the implied Navier-type equations are characterised as a set of elliptic PDEs.

However, in the present polar elasticity case, the elliptic or otherwise nature of the PDEs (70) is principally dictated by the corresponding nature of the term $\mathcal{B}_{mirjnq} u_{j,mrqn}$ that consists of fourth-order displacement derivatives. It is essentially evident that the appearing coefficients, $\mathcal{B}_{mirjnq}$, are not all of the same sign, at least because the contribution of the alternating tensor appearing in the equilibrium equation (2a) spreads around similar, if not identical terms having coefficients of opposite signs. It is accordingly seen that some of the properties of $\mathcal{A}_{mirj}$ that guarantee ellipticity of equation (70) in non-polar linear elasticity are missing in the present, polar elasticity case.

It is recognised in this context that the set (70) of displacement-based equations is not elliptic. As a result, there may exist in the polar material of interest a finite number of surfaces, known as weak discontinuity surfaces (e.g., [19,21–23]), on which fourth-order derivatives of the displacement vector might be discontinuous, regardless of the continuity status of relevant lower-order derivatives. In this context, the present section considers it sufficient to contact a short but comprehensive study regarding the relevant status of the second term appearing in the left-hand side of equation (70).

Accordingly, insertion into the second term of the left-hand side of equation (2a) of the constitutive equation (64), followed by suitable use of equation (5c), enables subsequent algebraic manipulations to yield:

$$\mathcal{B}_{mjrinq} u_{i,mrqn} = \frac{1}{8}\left(\delta_{jn}\delta_{iq} - \delta_{jq}\delta_{in}\right)\left[(\eta_1 - \eta_2)u_{q,ni\ell\ell} + \eta_6 u_{q,ni11}\right] + \frac{1}{8}(\eta_4 - \eta_5 - \eta_6)\delta_{j\alpha}\left(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}\right)u_{\delta,\beta\gamma\ell\ell}, \tag{71}$$

where Greek indices take the values 2 and 3 only.

A search for the anticipated weak discontinuity surfaces next requires determination of their unit normal $\boldsymbol{n}$. This becomes possible by denoting with:

$$\left[\left[u_{j,mrqn}\right]\right] = k_j n_m n_r n_q n_n, \tag{72}$$

the jump of $u_{j,mrqn}$ across such a surface, where $\boldsymbol{k}$ stands for the amplitude of that jump.

By taking the difference of equation (70) on the two sides of that surface, and recalling that the second-order gradients of the displacement vector are considered continuous throughout the body of the transversely isotropic solid of interest, one obtains a generalised eigenvalue problem of the form:

$$\mathcal{B}_{mirjqn} k_j n_m n_r n_q n_n = 0 \text{ or } P_{ij}(\boldsymbol{n}) k_j = 0, \tag{73}$$

where:

$$\mathbf{P} = \begin{bmatrix} 1 - n_1^2 & -n_1 n_2 & -n_1 n_3 \\ -c_1\left(n_1^2\right) n_1 n_2 & c_1\left(n_1^2\right)\left(1 - n_2^2\right) + c_2 n_3^2 & -\left[c_1\left(n_1^2\right) + c_2\right] n_2 n_3 \\ -c_1\left(n_1^2\right) n_1 n_3 & -\left[c_1\left(n_1^2\right) + c_2\right] n_2 n_3 & c_1\left(n_1^2\right)\left(1 - n_3^2\right) + c_2 n_2^2 \end{bmatrix}, \tag{74}$$

and

$$c_1\left(n_1^2\right) = \eta_1 - \eta_2 + \eta_6 n_1^2, c_2 = \eta_4 - \eta_5 - \eta_6. \tag{75}$$

The unit vector, $\boldsymbol{n}$, that determines any of the weak surfaces sought emerges as a non-trivial real solution of the eigenvalue problem (73) and, therefore, of the algebraic equation:

$$\det \boldsymbol{P}(\boldsymbol{n}) = 0. \tag{76}$$

The simplest possible such solution, namely:

$$\mathbf{n}^{(1)} \equiv \mathbf{a} = (1, 0, 0)^T, \tag{77}$$

reveals that any plane that is normal to the fibre direction is a weak surface. Such a result is interpreted as matrix failure (e.g., [21,22]) and, since it is generally not observable in cases of small elastic deformation that involve fibres resistant in bending [19], it is attributed to the fact that, in the present case, the fibres are considered perfectly flexible.

It is recalled, on the contrary, that planes parallel to the fibres are always weak discontinuity surfaces when fibres possess bending stiffness [19]. This result is interpreted as fibre de-bonding [21,22] and is still observable in the present case of perfectly flexible fibres, as can easily be verified by choosing:

$$\mathbf{n} \equiv \mathbf{n}^{(2)} = (0, 1, 0)^T \text{ or } \mathbf{n} \equiv \mathbf{n}^{(3)} = (0, 0, 1)^T. \tag{78}$$

Since the pair of these unit vectors can be chosen arbitrarily on the plane defined by equation (77), any plane containing the fibres does emerge as a weak discontinuity surface, regardless of whether fibres do [19] or do not resist bending.

Additional real solutions of equation (76) may also be possible, but their existence seems to be considerably dependent on the values of the elastic moduli involved in equation (75). Their determination and further examination may require considerable further investigation that is regarded beyond the immediate scope and purposes of this study.

7. Conclusion

The presented anisotropic extension of a recently developed, refined, linear couple-stress theory of isotropic elastic solids [9] retains the ability to determine the spherical part of the couple-stress. It is although also furnished with additional constitutive tools that enable it to model mechanical response of highly anisotropic materials that exhibit inherent linearly elastic polar material behaviour.

The type of material anisotropy considered has enough generality to embrace most of the structural materials employed in practice. This is because the derived constitutive law (section 3) enables consideration of relevant polar elastic solids possessing anisotropic properties that are as advanced as those met in the class of locally monoclinic materials. It follows that the obtained constitutive equation, presented in a suitable indicial notation form, can be simplified further and, thus, includes as special cases the important material subclasses of general and special orthotropy, as well as the subclass of transverse isotropy. In this manner, the implied refined theory is further furnished with ability to model structural analysis problems of polar fibrous composites reinforced by families of perfectly flexible fibres.

A relevant example application considered and studied in detail the subclass of polar transverse isotropy that is due to the presence of a single family of perfectly flexible fibres. That application departed with a derivation of the constitutive law that governs polar mechanical response of locally transverse isotropic materials (section 4.1) but soon after focused attention into the special but practically important case of polar transverse isotropy that is due to the presence of a single family of perfectly flexible straight fibres.

For that special case, the analysis succeeded to rearrange the relevant constitutive equation into a more comprehensive matrix, rather than indicial notation form, and thus achieved to exemplify the way that the spherical part of the couple-stress is determined. It is anticipated that similar, matrix notation forms of relevant constitutive equations will be found equally useful, and must therefore be pursued in the future, in several other advanced cases polar material anisotropy, such as those that are referred to above.

Meanwhile although, and still for the special case of polar transverse isotropy that is due to the presence of straight fibres, a detailed discussion was presented of further important issues stemming from (1) the positive definiteness of the polar form of the relevant strain energy function and (2) the lack of ellipticity of the final form attained by the governing differential equations. Accordingly, section 5 presented necessarily inequalities that, if satisfied by the elastic moduli involved in the polar part of the theory, guarantee the required positive definiteness of the strain energy function.

Moreover, the lack of ellipticity analysis detailed in section 6 arrived at the formation of an algebraic equation, whose solution suffices to determine all weak discontinuity surfaces that are present in the polar transverse isotropic material of interest. Complete solution of that equation is generally dependent on the numerical values of the aforementioned elastic moduli and must, therefore, be pursued numerically.

Nevertheless, the special structure of that algebraic problem led easily to a preliminary conclusion, according to which all material planes that are either parallel or normal to the involved family of perfectly flexible fibres emerge as weak discontinuity surfaces. It was thus noted with interest that this conclusion differs, at least partially, from its counterpart observed when fibres possess bending stiffness [19], where planes parallel to those fibres are, but planes normal to the fibres are not included within the set of the weak discontinuity surfaces sought.

The latter observation redirects attention into the fact that perfectly flexible fibres are considered in relevant mathematical models as material directions that are able to resist extension, but not compression or any other mode or kind of deformation. A connection thus seems emerging between the presented polar material analysis, including its large deformations' counterpart [10], and recent non-polar hyperelasticity developments that exclude contribution of deformation effects stemming from fibres compression (e.g., [24–26]).

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Appendix 1

Derivation of the couple-stress constitutive equations (27) and (47)

By taking into consideration the definitions equation (21), potential connection of equation (23) with equation (15b) initially requires conversion of the latter equation into the following:

$$m_{ji} = \frac{\partial W^{\Phi}}{\partial \Phi_{i,j}} = \frac{\partial W^{\Phi}}{\partial \Phi_{(m,n)}} \frac{\partial \Phi_{(m,n)}}{\partial \Phi_{i,j}} + \frac{\partial W^{\Phi}}{\partial \Phi_{[m,n]}} \frac{\partial \Phi_{[m,n]}}{\partial \Phi_{i,j}} = \frac{1}{2} \frac{\partial W^{\Phi}}{\partial \Phi_{(m,n)}} \left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) + \frac{1}{2} \frac{\partial W^{\Phi}}{\partial \Phi_{[m,n]}} \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) = \frac{\partial W^{\Phi}}{\partial \Phi_{(i,j)}} + \frac{\partial W^{\Phi}}{\partial \Phi_{[i,j]}}, \tag{79}$$

and, henceforth, into:

$$m_{ji} = \sum_{\alpha=1}^{17} \left(\frac{\partial W^{\Phi}}{\partial I_\alpha} \frac{\partial I_\alpha}{\partial \Phi_{(i,j)}} + \frac{\partial W^{\Phi}}{\partial I_\alpha} \frac{\partial I_\alpha}{\partial \Phi_{[i,j]}} \right) = \sum_{\alpha=1}^{17} \frac{\partial W^{\Phi}}{\partial I_\alpha} \left(\frac{\partial I_\alpha}{\partial \Phi_{(i,j)}} + \frac{\partial I_\alpha}{\partial \Phi_{[i,j]}} \right). \tag{80}$$

The appearing partial derivatives of W^{Φ} are next found to be:

$$
\begin{aligned}
&\frac{\partial W^{\Phi}}{\partial I_1} = \eta_0\left[I_1 + \hat{\eta}_0^{(1)} I_4 + \hat{\eta}_0^{(2)} I_8 + \left(\hat{\eta}_0^{(3)} I_{12} + \hat{\eta}_0^{(4)} I_{14}\right) I_{18}\right],\ \frac{\partial W^{\Phi}}{\partial I_2} = \frac{1}{2}\eta_1,\ \frac{\partial W^{\Phi}}{\partial I_3} = \frac{1}{2}\eta_2,\\
&\frac{\partial W^{\Phi}}{\partial I_4} = \eta_0 \hat{\eta}_0^{(1)} I_1 + \eta_3\left[I_4 + \hat{\eta}_3^{(1)} I_8 + \left(\hat{\eta}_3^{(2)} I_{12} + \hat{\eta}_3^{(3)} I_{14}\right) I_{18}\right],\ \frac{\partial W^{\Phi}}{\partial I_5} = \frac{1}{2}\eta_4,\ \frac{\partial W^{\Phi}}{\partial I_6} = \frac{1}{2}\eta_5,\\
&\frac{\partial W^{\Phi}}{\partial I_7} = \eta_6,\ \frac{\partial W^{\Phi}}{\partial I_8} = \eta_0 \hat{\eta}_0^{(2)} I_1 + \eta_3 \hat{\eta}_3^{(1)} I_4 + \eta_7\left[I_8 + \left(\hat{\eta}_7^{(1)} I_{12} + \hat{\eta}_7^{(2)} I_{14}\right) I_{18}\right],\ \frac{\partial W^{\Phi}}{\partial I_9} = \frac{1}{2}\eta_8,\\
&\frac{\partial W^{\Phi}}{\partial I_{10}} = \frac{1}{2}\eta_9,\ \frac{\partial W^{\Phi}}{\partial I_{11}} = \eta_{10},\ \frac{\partial W^{\Phi}}{\partial I_{12}} = \left(\eta_0 \hat{\eta}_0^{(3)} I_1 + \eta_3 \hat{\eta}_3^{(2)} I_4 + \eta_7 \hat{\eta}_7^{(1)} I_8\right) I_{18} + \eta_{11}\left(I_{12} + \hat{\eta}_{11}^{(1)} I_{14}\right),\\
&\frac{\partial W^{\Phi}}{\partial I_{13}} = \frac{1}{2}\eta_{12} I_{18},\ \frac{\partial W^{\Phi}}{\partial I_{14}} = \left(\eta_0 \hat{\eta}_0^{(4)} I_1 + \eta_3 \hat{\eta}_3^{(3)} I_4 + \eta_7 \hat{\eta}_7^{(2)}\right) I_{18} + \eta_{11} \hat{\eta}_{11}^{(1)} I_{12} + \eta_{13} I_{14},\\
&\frac{\partial W^{\Phi}}{\partial I_{15}} = \frac{1}{2}\eta_{14} I_{18},\ \frac{\partial W^{\Phi}}{\partial I_{16}} = \eta_{15} I_{18},\ \frac{\partial W^{\Phi}}{\partial I_{17}} = \eta_{16} I_{18},
\end{aligned}
\tag{81}
$$

and their association with the relevant derivatives of the deformation invariants, namely:

$$
\begin{aligned}
&\frac{\partial I_1}{\partial \Phi_{(i,j)}} = \delta_{ij},\ \frac{\partial I_2}{\partial \Phi_{(i,j)}} = 2\Phi_{(i,j)},\ \frac{\partial I_3}{\partial \Phi_{[i,j]}} = 2\Phi_{[j,i]},\ \frac{\partial I_4}{\partial \Phi_{(i,j)}} = a_i^{(1)} a_j^{(1)},\ \frac{\partial I_5}{\partial \Phi_{(i,j)}} = 2a_i^{(1)} \Phi_{(j,k)} a_k^{(1)},\\
&\frac{\partial I_6}{\partial \Phi_{[i,j]}} = 2a_i^{(1)} \Phi_{[j,k]} a_k^{(1)},\ \frac{\partial I_7}{\partial \Phi_{(i,j)}} = a_i^{(1)} \Phi_{[j,k]} a_k^{(1)},\ \frac{\partial I_7}{\partial \Phi_{[i,j]}} = a_k^{(1)} \Phi_{(k,i)} a_j^{(1)},\ \frac{\partial I_8}{\partial \Phi_{(i,j)}} = a_i^{(2)} a_j^{(2)},\\
&\frac{\partial I_9}{\partial \Phi_{(i,j)}} = 2a_i^{(2)} \Phi_{(j,k)} a_k^{(2)},\ \frac{\partial I_{10}}{\partial \Phi_{[i,j]}} = 2a_i^{(2)} \Phi_{[i,k]} a_k^{(2)},\ \frac{\partial I_{11}}{\partial \Phi_{(i,j)}} = a_i^{(2)} \Phi_{[j,k]} a_k^{(2)},\ \frac{\partial I_{11}}{\partial \Phi_{[i,j]}} = a_k^{(2)} \Phi_{(k,i)} a_j^{(2)},\\
&\frac{\partial I_{12}}{\partial \Phi_{(i,j)}} = \vartheta_{ij}^{1},\ \frac{\partial I_{13}}{\partial \Phi_{(i,j)}} = 2a_i^{(1)} \Phi_{(j,k)} a_k^{(2)},\ \frac{\partial I_{14}}{\partial \Phi_{[i,j]}} = a_i^{(1)} a_j^{(2)},\ \frac{\partial I_{15}}{\partial \Phi_{[i,j]}} = 2a_i^{(1)} \Phi_{[j,k]} a_k^{(2)},\\
&\frac{\partial I_{16}}{\partial \Phi_{(i,j)}} = a_i^{(1)} \Phi_{[j,k]} a_k^{(2)},\ \frac{\partial I_{16}}{\partial \Phi_{[i,j]}} = a_k^{(1)} \Phi_{(k,i)} a_j^{(2)},\\
&\frac{\partial I_{17}}{\partial \Phi_{(i,j)}} = 2\left(a_i^{(1)} \Phi_{[j,k]} a_k^{(2)} - a_k^{(1)} \Phi_{[k,i]} a_j^{(2)}\right),\ \frac{\partial I_{17}}{\partial \Phi_{[i,j]}} = 2\left(a_k^{(1)} \Phi_{(k,i)} a_j^{(2)} - a_i^{(1)} \Phi_{(j,k)} a_k^{(2)}\right),
\end{aligned}
\tag{82}
$$

leads to:

$$
\begin{aligned}
m_{ji} = {}& \eta_0\left\{\left[\delta_{ij} + \hat{\eta}_0^{(1)} a_i^{(1)} a_j^{(1)} + \hat{\eta}_0^{(2)} a_i^{(2)} a_j^{(2)} + \left(\hat{\eta}_0^{(3)} + \hat{\eta}_0^{(4)}\right) I_{18} a_i^{(1)} a_j^{(2)}\right] I_1 + \left[\hat{\eta}_0^{(1)} I_4 + \hat{\eta}_0^{(2)} I_8 + \left(\hat{\eta}_0^{(3)} I_{12} + \hat{\eta}_0^{(4)} I_{14}\right) I_{18}\right]\delta_{ij}\right\} +\\
& \eta_1 \Phi_{(i,j)} + \eta_2 \Phi_{[j,i]} + \eta_3\left[a_i^{(1)} a_j^{(1)} + \hat{\eta}_3^{(1)} a_i^{(2)} a_j^{(2)} + \left(\hat{\eta}_3^{(2)} + \hat{\eta}_3^{(3)}\right) I_{18} a_i^{(1)} a_j^{(2)}\right] I_4 + \left(\eta_4 \Phi_{(j,k)} + \eta_5 \Phi_{[j,k]}\right) a_i^{(1)} a_k^{(1)} +\\
& \eta_6\left(\Phi_{(k,i)} a_j^{(1)} - \Phi_{[k,j]} a_i^{(1)}\right) a_k^{(1)} + \left\{\eta_7\left[a_i^{(2)} a_j^{(2)} + \left(\hat{\eta}_7^{(1)} + \hat{\eta}_7^{(2)}\right) I_{18} a_i^{(1)} a_j^{(2)}\right] + \eta_3 \hat{\eta}_3^{(1)} a_i^{(1)} a_j^{(1)}\right\} I_8 +\\
& \left(\eta_8 \Phi_{(j,k)} + \eta_9 \Phi_{[j,k]}\right) a_i^{(2)} a_k^{(2)} + \eta_{10}\left(\Phi_{(k,i)} a_j^{(2)} - \Phi_{[k,j]} a_i^{(2)}\right) a_k^{(2)} +\\
& \left[\eta_{11}\left(1 + \hat{\eta}_{11}^{(1)}\right) a_i^{(1)} a_j^{(2)} + \left(\eta_3 \hat{\eta}_3^{(2)} a_i^{(1)} a_j^{(1)} + \eta_7 \hat{\eta}_7^{(1)} a_i^{(2)} a_j^{(2)}\right) I_{18}\right] I_{12} +\\
& \left[\left(\eta_{13} + \eta_{11} \hat{\eta}_{11}^{(1)}\right) a_i^{(1)} a_j^{(2)} + \left(\eta_3 \hat{\eta}_3^{(3)} a_i^{(1)} a_j^{(1)} + \eta_7 \hat{\eta}_7^{(2)} a_i^{(2)} a_j^{(2)}\right) I_{18}\right] I_{14} +\\
& \left[(\eta_{12} - \eta_{16}) \Phi_{(j,k)} + (\eta_{14} + \eta_{15} + \eta_{16}) \Phi_{[j,k]}\right] I_{18} a_i^{(1)} a_k^{(2)} + \left[(\eta_{15} + \eta_{16}) \Phi_{(k,i)} - \eta_{16} \Phi_{[k,i]}\right] I_{18} a_j^{(2)} a_k^{(1)},
\end{aligned}
\tag{83}
$$

which, after suitable rearrangement, may be more conveniently expressed in the form (27).

Contraction of the free indices appearing in equation (27) yields the spherical part of the couple-stress in the form:

$$
\begin{aligned}
m_{rr} = {} & \eta_0 \Phi_{k,k}\left(3+\vartheta_{rr}^{(0)}\right) + \eta_1 \Phi_{r,r} + I_4\left(3\eta_0\hat{\eta}_0^{(1)} + \vartheta_{rr}^{(3)}\right) + \eta_4 a_k^{(1)}\Phi_{(r,k)}a_r^{(1)} + \eta_6 a_k^{(1)}\Phi_{(k,r)}a_r^{(1)} + \\
& I_8\left(3\eta_0\hat{\eta}_0^{(2)} + \vartheta_{rr}^{(7)}\right) + \eta_8 a_k^{(2)}\Phi_{(r,k)}a_r^{(2)} + \eta_{10} a_k^{(2)}\Phi_{(k,r)}a_r^{(2)} + I_{12}\left(\vartheta_{rr}^{(11)} + 3\eta_0\hat{\eta}_0^{(3)} I_{18}\right) + \\
& \left(\vartheta_{rr}^{(13)} + 3\eta_0\hat{\eta}_0^{(4)} I_{18}\right) I_{14} + I_{18} a_k^{(2)}\left[(\eta_{12}+\eta_{15})\Phi_{(r,k)} + (\eta_{14}+\eta_{15}+2\eta_{16})\Phi_{[r,k]}\right] a_r^{(1)},
\end{aligned}
\tag{84}
$$

where:

$$
\begin{aligned}
& \vartheta_{rr}^{(0)} = \hat{\eta}_0^{(1)} + \hat{\eta}_0^{(2)} + \left(\hat{\eta}_0^{(3)} + \hat{\eta}_0^{(4)}\right) I_{18}^2,\ \vartheta_{rr}^{(3)} = \eta_3\left[1 + \hat{\eta}_3^{(1)} + \left(\hat{\eta}_3^{(2)} + \hat{\eta}_3^{(3)}\right) I_{18}^2\right], \\
& \vartheta_{rr}^{(7)} = \eta_7\left[1 + \left(\hat{\eta}_7^{(1)} + \hat{\eta}_7^{(2)}\right) I_{18}^2\right] + \eta_3\hat{\eta}_3^{(1)},\ \vartheta_{ij}^{(11)} = \left[\eta_{11}\left(1+\hat{\eta}_{11}^{(1)}\right) + \eta_3\hat{\eta}_3^{(2)} + \eta_7\hat{\eta}_7^{(1)}\right] I_{18}, \\
& \vartheta_{ij}^{(13)} = \left(\eta_{13} + \eta_{11}\hat{\eta}_{11}^{(1)} + \eta_3\hat{\eta}_3^{(3)} + \eta_7\hat{\eta}_7^{(2)}\right) I_{18}.
\end{aligned}
\tag{85}
$$

After appropriate combination with equations (22) and (85), the right-hand side of equation (84) can be rearranged to yield the simplified form equation (32) of the spherical couple-stress, where use is made of the following combined elastic moduli:

$$
\begin{aligned}
\tilde{\eta}_0 &= \eta_0\left[3 + \hat{\eta}_0^{(1)} + \hat{\eta}_0^{(2)} + \left(\hat{\eta}_0^{(3)} + \hat{\eta}_0^{(4)}\right)\cos^2\theta\right] + \eta_1, \\
\tilde{\eta}_1 &= 3\eta_0\hat{\eta}_0^{(1)} + \eta_4 + \eta_6 + \eta_3\left[1 + \hat{\eta}_3^{(1)} + \left(\hat{\eta}_3^{(2)} + \hat{\eta}_3^{(3)}\right)\cos^2\theta\right], \\
\tilde{\eta}_2 &= 3\eta_0\hat{\eta}_0^{(2)} + \eta_3\hat{\eta}_3^{(1)} + \eta_7\left[1 + \left(\hat{\eta}_7^{(1)} + \hat{\eta}_7^{(2)}\right)\cos^2\theta\right] + \eta_8 + \eta_{10}, \\
\tilde{\eta}_3 &= 3\eta_0\hat{\eta}_0^{(3)} + \eta_3\hat{\eta}_3^{(2)} + \eta_7\hat{\eta}_7^{(1)} + \eta_{11}\left(1 + \hat{\eta}_{11}^{(1)}\right) + \eta_{12} + \eta_{15}, \\
\tilde{\eta}_4 &= 3\eta_0\hat{\eta}_0^{(4)} + \eta_3\hat{\eta}_3^{(3)} + \eta_7\hat{\eta}_7^{(2)} + \eta_{11}\hat{\eta}_{11}^{(1)} + \eta_{13} + \eta_{14} + \eta_{15}.
\end{aligned}
\tag{86}
$$

Appendix 2

Formation of the PDE (55) for the unknown spherical couple-stress

The inverse of the matrix $\hat{\mathbf{H}}$ appearing in equation (52) can be represented, in a similar form, as follows:

$$
\hat{\mathbf{H}}^{-1} = \begin{bmatrix}
\underset{(3\times3)}{\mathbf{H}_1^{-1}} & \underset{(3\times2)}{\mathbf{0}} & \underset{(3\times2)}{\mathbf{0}} & \underset{(3\times2)}{\mathbf{0}} \\
\underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{H}_2^{-1}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} \\
\underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{H}_3^{-1}} & \underset{(2\times2)}{\mathbf{0}} \\
\underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{H}_3^{-1}}
\end{bmatrix}
\equiv
\begin{bmatrix}
\underset{(3\times3)}{\mathbf{\Theta}_1} & \underset{(3\times2)}{\mathbf{0}} & \underset{(3\times2)}{\mathbf{0}} & \underset{(3\times2)}{\mathbf{0}} \\
\underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{\Theta}_2} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} \\
\underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{\Theta}_3} & \underset{(2\times2)}{\mathbf{0}} \\
\underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{\Theta}_3}
\end{bmatrix}
\equiv \hat{\mathbf{\Theta}},
\tag{87}
$$

where specific values of, or expressions attained by the components of the submatrices,

$$
\mathbf{\Theta}_1 = \mathbf{H}_1^{-1} = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} & \Theta_{23} \\ \Theta_{12} & \Theta_{23} & \Theta_{22} \end{bmatrix},\ \mathbf{\Theta}_2 = \mathbf{H}_2^{-1} = \begin{bmatrix} \Theta_{44} & \Theta_{45} \\ \Theta_{54} & \Theta_{55} \end{bmatrix},\ \mathbf{\Theta}_3 = \mathbf{H}_3^{-1} = \begin{bmatrix} \Theta_{66} & \Theta_{67} \\ \Theta_{76} & \Theta_{77} \end{bmatrix},
\tag{88}
$$

are considered easily obtainable, either numerically or even analytically, and need not be pursued or quoted here.

More importantly, combination of the outlined matrix representations with equation (33) enables now the unknown gradients of the virtual spin-vector $\mathbf{\Phi}$ to be expressed as follows:

$$\begin{pmatrix} \Phi_{1,1} \\ \Phi_{2,2} \\ \Phi_{2,2} \end{pmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} & \Theta_{23} \\ \Theta_{12} & \Theta_{23} & \Theta_{22} \end{bmatrix} \left\{ \begin{pmatrix} \bar{m}_{11} \\ \bar{m}_{22} \\ \bar{m}_{33} \end{pmatrix} + \frac{1}{3} \begin{pmatrix} m_{rr} \\ m_{rr} \\ m_{rr} \end{pmatrix} \right\},$$
$$\begin{pmatrix} \Phi_{(2,3)} \\ \Phi_{[2,3]} \end{pmatrix} = \begin{bmatrix} \Theta_{44} & \Theta_{45} \\ \Theta_{54} & \Theta_{55} \end{bmatrix} \begin{pmatrix} \bar{m}_{23} \\ \bar{m}_{32} \end{pmatrix}, \begin{pmatrix} \Phi_{(1,3)} \\ \Phi_{[1,3]} \end{pmatrix} = \begin{bmatrix} \Theta_{66} & \Theta_{67} \\ \Theta_{76} & \Theta_{77} \end{bmatrix} \begin{pmatrix} \bar{m}_{13} \\ \bar{m}_{31} \end{pmatrix}, \begin{pmatrix} \Phi_{(1,2)} \\ \Phi_{[2,1]} \end{pmatrix} = \begin{bmatrix} \Theta_{66} & \Theta_{67} \\ \Theta_{76} & \Theta_{77} \end{bmatrix} \begin{pmatrix} \bar{m}_{12} \\ \bar{m}_{21} \end{pmatrix}. \tag{89}$$

It follows that:

$$\begin{aligned} &\Phi_{(1,1)} = \Theta_{11}\bar{m}_{11} + \Theta_{12}(\bar{m}_{22} + \bar{m}_{33}) + \frac{1}{3}(\Theta_{11} + 2\Theta_{12})m_{rr}, \\ &\Phi_{r,r} = \tilde{\Theta}_1\bar{m}_{11} + \tilde{\Theta}_2(\bar{m}_{22} + \bar{m}_{33}) + \tilde{\Theta}_0 m_{rr}, \end{aligned} \tag{90}$$

where:

$$\tilde{\Theta}_1 = \Theta_{11} + 2\Theta_{12}, \tilde{\Theta}_2 = \Theta_{12} + \Theta_{22} + \Theta_{23}, \tilde{\Theta}_0 = \tilde{\Theta}_1 + \tilde{\Theta}_2 + \tilde{\Theta}_3. \tag{91}$$

Hence, insertion of equation (90) into the extra equation (46) leads to the PDE (55) for the unknown spherical part of the couple-stress, where use is made of the following constant coefficients:

$$K_0 = \tilde{\Theta}_0\left(\tilde{\Theta}_0 + \frac{2}{3}\hat{\eta}_0^{(1)}\tilde{\Theta}_1\right), K_1 = \tilde{\Theta}_0\tilde{\Theta}_1 + \hat{\eta}_0^{(1)}\frac{1}{3}\tilde{\Theta}_1^2 + \hat{\eta}_0^{(1)}\tilde{\Theta}_0\Theta_{11}, K_2 = \tilde{\Theta}_0\tilde{\Theta}_2 + \hat{\eta}_0^{(1)}\frac{1}{3}\tilde{\Theta}_1\tilde{\Theta}_2 + \hat{\eta}_0^{(1)}\tilde{\Theta}_0\Theta_{12}, \tag{92}$$

as well as the function:

$$f(\bar{m}_{11}, \bar{m}_{22}, \bar{m}_{33}) = 3\eta_0\left[\tilde{\Theta}_1\bar{m}_{11} + \tilde{\Theta}_2(\bar{m}_{22} + \bar{m}_{33})\right]\left[\left(\tilde{\Theta}_1 + 2\hat{\eta}_0^{(1)}\Theta_{11}\right)\bar{m}_{11} + \left(\tilde{\Theta}_2 + 2\hat{\eta}_0^{(1)}\Theta_{12}\right)(\bar{m}_{22} + \bar{m}_{33})\right], \tag{93}$$

whose arguments become known through the solution of the conventional theory equations.