On the Choice of Similarity Measures for Type-2 Fuzzy Sets

Josie McCulloch*, Christian Wagner

Lab for Uncertainty in Data and Decision Making (LUCID), School of Computer Science, University of Nottingham, United Kingdom

---

Abstract

Similarity measures are among the most common methods of comparing type-2 fuzzy sets and have been used in numerous applications. However, deciding how to measure similarity and choosing which existing measure to use can be difficult. Whilst some measures give results that highly correlate with each other, others give considerably different results. We evaluate all of the current similarity measures on type-2 fuzzy sets to discover which measures have common properties of similarity and, for those that do not, we discuss why the properties are different, demonstrate whether and what effect this has in applications, and discuss how a measure can avoid missing a property that is required. We analyse existing measures in the context of computing with words using a comprehensive collection of data-driven fuzzy sets. Specifically, we highlight and demonstrate how each method performs at clustering words of similar meaning.

Keywords: type-2 fuzzy sets; similarity measures

---

1. Introduction

Similarity measures are among the most common methods of comparing fuzzy sets to determine if they are closely related. Their utility has led to their use in numerous applications. Similarity is often used to solve classification and clustering problems [20, 37, 44], such as pattern recognition [30]. Similarity has also been used extensively in decision making to find preferences by comparing the similarity between solutions and the ideal outcome [21]. Further, similarity is often utilised in both basic and advanced Computing with Words (CW) - where the former uses fuzzy if-then rules and the latter uses natural language statements containing a mixture of numbers, intervals and words. In basic CW, similarity is commonly used to test the firing strength of a rule by calculating the similarity between the input and the rule antecedent [1, 10], and matching

*Corresponding author

Email addresses: josie.mcculloch@nottingham.ac.uk (Josie McCulloch), christian.wagner@nottingham.ac.uk (Christian Wagner)

words to concepts described using fuzzy sets [36]. In advanced CW, similarity is often used to choose the most appropriate linguistic approximation for the output of the CW system. This is achieved by finding which word (modelled as a fuzzy set) is most similar to the fuzzy output [28]. Similarity measures are also used to group similar words [40], for example, to ensure that each word has a distinct meaning.

General type-2 fuzzy sets have gained popularity due to their ability to provide higher accuracy in applications that rely on data that are often noisy [4], and due to recent advances that simplify modelling the secondary membership function [26, 35]. As a result, there has been an increase in similarity measures developed for general type-2 fuzzy sets [13, 12, 24, 30, 37, 44, 48, 23] as well as for interval-valued type-2 fuzzy sets [1, 32, 39, 46, 49].

Among the many different similarity measures developed for type-2 fuzzy sets, some give results that highly correlate with each other, while others give notably different results for the same fuzzy sets. This may be desired in respect to the intended application (different applications may focus on different aspects of the fuzzy sets), or it may be the result of underlying challenges with the specific method. For example, some existing methods are unable to identify if fuzzy sets are disjoint or identical, and some give unexpectedly high or low results of similarity in specific cases, as discussed later in the paper.

To illustrate differences in measures, consider the two interval type-2 fuzzy sets in Fig. 1 and two measures of similarity s_j^{IT2} (see eq. (19)) and s_{zl}^{IT2} (see eq. (23)) which take different approaches to comparing fuzzy sets and therefore give different results (detailed later in section 4). The results of applying these measures to the given fuzzy sets are $s_j^{IT2} = 0.006$ and $s_{zl}^{IT2} = 0.750$. The measure s_j^{IT2} gives a low value of similarity, whilst $s_{zl}^{IT2} = 0.750$ gives a much higher result. This is because s_j^{IT2} has properties commonly found in a similarity measure, whereas s_{zl}^{IT2} does not, and therefore its results may be unexpected. This demonstrates that it is important to understand the properties of similarity measures; that is, what properties they have and how they affect what results the measure gives. For example, it is useful to be aware that s_{zl}^{IT2} will give much larger values than s_j^{IT2} - an improved knowledge of a measure will enhance its use.

The variety of existing measures makes the choice of the best a difficult one. We analyse all of the known similarity measures developed for interval or general type-2 fuzzy sets to:

1. develop an understanding of how the properties of a measure affect its results
2. discover which measures have common properties of similarity and are therefore versatile in many applications
3. illustrate when a measure may be useful despite not having all common properties

Using data-driven fuzzy sets, we individually analyse the results of similarity measures and provide pairwise comparisons to highlight when the properties

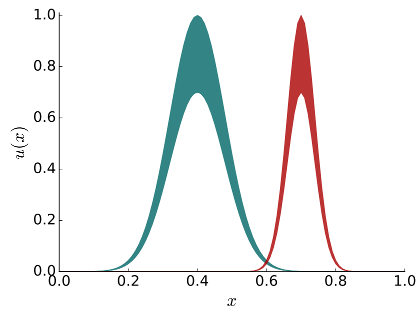


Figure 1: Two interval type-2 fuzzy sets where $s_{zl}^{IT2} = 0.750$ and $s_{j}^{IT2} = 0.006$.

of measures differ and show what affect it has on the results. Using these comparisons, we recommend choosing a measure that has all of the properties listed in section 2.

Further, we analyse the measures that do not have the common properties and demonstrate why this is the case. From this, we suggest what alternative approach should be used in a new measure when a missing property of an existing one is required.

Note that we only focus on measures for type-2 fuzzy sets as studies of similarity on crisp and type-1 fuzzy sets are available in the literature [18, 50]. These studies highlight key different approaches to compare type-1 fuzzy sets, many of which have been used as the basis for measuring type-2 fuzzy sets; we introduce two of the most common type-1 methods in the next section. The literature also highlights type-1 measures that are equivalent (i.e., provide the same rank order of results), in which case the choice of a measure is simplified to the choice of a class of measures [18]. In this paper, we highlight equivalent similarity measures on type-2 fuzzy sets.

Studies on similarity measures for interval type-2 fuzzy sets can also be found in the literature [19, 40]. These provide an overview of different approaches and compare their results using fuzzy sets constructed from real data [40]. However, although a large set of results is given, their numerical values are difficult to compare, and the reasons behind the methods giving different results are not discussed.

Wu and Mendel provide an analysis of similarity on general type-2 fuzzy sets, as well as introducing a new method [41]. A review of existing methods is given on fuzzy sets constructed from a subset of the real data used in the interval type-2 analysis [40]. However, only a small set of numerical results are given. Counter-intuitive results are highlighted, but the reasons underpinning these results are not explored.

In this paper, we compare the results of interval and general type-2 similarity measures applied to a collection of fuzzy sets constructed from real data, as provided in [40]. We visualise the results to facilitate interpretation. The resulting figures help to highlight the properties of measures and provide a clear comparison between methods. We also discuss why methods provide results that may feel counter-intuitive, and offer advice on avoiding such results when developing a new measure.

Finally, note, that the ultimate choice of a measure is data/application dependent and general recommendations are notoriously difficult. In this paper, we focus on providing generic insights to support researchers to make appropriate choices for their applications. We also provide demonstrations for specific examples - such as the grouping of fuzzy sets capturing linguistic terms in Computing with Words contexts.

In the remainder of the paper, section 2 presents a background on fuzzy sets, similarity measures, and the data used to test the measures. Then, sections 4 and 5 analyse and compare methods of measuring the similarity between interval and general type-2 fuzzy sets, respectively. Finally, section 6 presents a discussion of the results.

2. Background

In this section, we introduce different representations of type-2 fuzzy sets used in the literature, followed by a general overview of similarity measures. We then proceed to introduce similarity measures in the literature for interval and general type-2 fuzzy sets.

2.1. Type-2 Fuzzy Sets

In this paper, we analyse methods of measuring similarity between interval and general type-2 fuzzy sets. These methods rely on different equations of modelling fuzzy sets; these are the vertical slice [29], zSlice/alpha-plane [26, 35] and embedded set representations [25, 30]. In this section, we briefly introduce these equations.

Definition 1. *The vertical slice representation of a general type-2 fuzzy set* $\tilde{F}$ *is expressed as [29]*

$$\tilde{F} = \{((x,u),\ \mu_{\tilde{F}}(x,u)) \mid x \in X, u \in [\underline{u}_{\tilde{F}}(x), \overline{u}_{\tilde{F}}(x)]\}, \tag{1}$$

where x *is a variable in* X*,* u *is the primary membership of* x*,* $\mu_{\tilde{F}}(x,u)$ *is the secondary degree of membership for* x *and* u*, and*

$$\underline{u}_{\tilde{F}}(x) = \min\{u \mid u \in [0,1], \mu_{\tilde{F}}(x,u) > 0\} \tag{2}$$

$$\overline{u}_{\tilde{F}}(x) = \max\{u \mid u \in [0,1], \mu_{\tilde{F}}(x,u) > 0\}. \tag{3}$$

We refer to $\underline{u}_{\tilde{F}}$ *and* $\overline{u}_{\tilde{F}}$ *as the lower and upper membership functions of* $\tilde{F}$*, respectively. In this representation, the fuzzy set is modelled by slices along the* x*-axis where each value of* x *is mapped to a type-1 fuzzy set.*

Definition 2. *The footprint of uncertainty of a type-2 fuzzy set* $\tilde{F}$ *is*

$$FOU(\tilde{F}) = \{(x,u) \mid x \in X,\ u \in [\underline{u}_{\tilde{F}}(x), \overline{u}_{\tilde{F}}(x)]\} \tag{4}$$

A closed interval type-2 fuzzy set is a special case of a general type-2 fuzzy set in which the secondary grade $\mu_{\tilde{A}}(x, u)$ equals 1 for $x \in X$ and $u \in [\underline{u}_{\tilde{F}}(x), \overline{u}_{\tilde{F}}(x)]$. Note that, for the remainder of this paper, when we use the term interval type-2 fuzzy set, this is understood to be a closed interval type-2 fuzzy set.

Definition 3. *An interval type-2 fuzzy set $\tilde{F}$ can be expressed as [29]*

$$\tilde{F} = \{((x, u),\ \mu_{\tilde{F}}(x, u) = 1) \mid x \in X, u \in [\underline{u}_{\tilde{F}}(x), \overline{u}_{\tilde{F}}(x)]\}. \tag{5}$$

The zSlices representation [35] (also developed as the alpha-plane representation [27]) is an alternative to the vertical slice representation for type-2 fuzzy sets. Using this, a fuzzy set is represented by a collection of quasi interval type-2 fuzzy sets (referred to as zSlices).

Definition 4. *The zSlice of a fuzzy set $\tilde{F}$ at $z \in (0, 1]$ is*

$$\tilde{F}_z = \{((x, u),\ \mu_{\tilde{F}_z}(x, u) = z) \mid x \in X, u \in [\underline{\mu}_{\tilde{F}_z}(x), \overline{\mu}_{\tilde{F}_z}(x)]\}. \tag{6}$$

where

$$\underline{\mu}_{\tilde{F}_z}(x) = \min\{u \mid u \in [0, 1], \mu_{\tilde{F}}(x, u) \geq z\} \tag{7}$$

$$\overline{\mu}_{\tilde{F}_z}(x) = \max\{u \mid u \in [0, 1], \mu_{\tilde{F}}(x, u) \geq z\} \tag{8}$$

Note that the zSlice at $z = 0$ is ignored because this is the set in which all secondary membership values are 0 and, therefore, it does not contribute to the fuzzy set.

Definition 5. *A fuzzy set $\tilde{F}$ can be represented by the collection of its zSlices as follows [35]:*

$$\tilde{F} = \{\tilde{F}_z \mid z \in (0, 1]\} \tag{9}$$

As an alternative to the vertical slices and zSlices representations, a fuzzy set can be represented as a collection of its embedded fuzzy sets. It may be represented through type-2 or type-1 embedded sets [25], where secondary membership values are not included in the latter.

Definition 6. *For a type-2 fuzzy set $\tilde{F}$, a given embedded type-1 fuzzy set F_m is defined much like a type-1 fuzzy set as*

$$F_m = \{(x, u) \mid x \in X, u \in [\underline{u}_{\tilde{F}}(x), \overline{u}_{\tilde{F}}(x)]\}, \tag{10}$$

where X is discretised into N elements, only one value of u is assigned to each x, and the secondary membership values of $\tilde{F}$ are not included.

In this paper, we discuss a similarity measure that is based on a special-case of embedded fuzzy sets [30] (see eq. (34)). This uses a discrete weighted form, in which a type-2 fuzzy set $\tilde{F}$ is represented by M type-1 embedded fuzzy sets, (i.e., $\tilde{F} = \{F_1, F_2, ..., F_M\}$), and a weight is assigned to each embedded fuzzy set. In this representation, a given type-1 embedded fuzzy set $\theta_{\tilde{F}}^m$ is defined as

$$\theta_{\tilde{F}}^m = r_m(x_l)\big(\overline{\mu}_{\tilde{F}}(x_l) - \underline{\mu}_{\tilde{F}}(x_l)\big) + \underline{\mu}_{\tilde{F}}(x_l) \mid l \in \{1, 2, ..., L\}\,, \tag{11}$$

where $r_m(x_l)$ is any arbitrary number chosen within the interval $[0, 1]$, $m \in \{1, 2, ..., M\}$, and the universe of discourse X is discretised into L points, i.e. $X = \{x_1, x_2, ..., x_L\}$. The use of $r_m(x_l)$ ensures that the primary membership value of $\theta_{\tilde{F}}^m$ at x_l is any arbitrary value constrained within the interval $[\underline{\mu}_{\tilde{F}}(x_l), \overline{\mu}_{\tilde{F}}(x_l)]$.

The fuzzy set $\theta_{\tilde{F}}^m$ only accounts for the primary membership values of $\tilde{F}$. To account for the secondary membership values, each embedded fuzzy set $\theta_{\tilde{F}}^m$ has a weight associated with it, labelled as $\lambda_{\tilde{F}}^m$. This is defined by the t-norm of the secondary membership grades of $\tilde{F}$ as

$$\lambda_{\tilde{F}}^m = t\Big(\mu_{\tilde{F}}\big(x_1, \theta_{\tilde{F}}^m(x_1)\big),\ \mu_{\tilde{F}}\big(x_2, \theta_{\tilde{F}}^m(x_2)\big), ..., \mu_{\tilde{F}}\big(x_L, \theta_{\tilde{F}}^m(x_L)\big)\Big). \tag{12}$$

where t is a t-norm, $\theta_{\tilde{F}}^m(x_l)$ is the primary membership grade of $\theta_{\tilde{F}}^m$ at x_l (given in (11)), and $\mu_{\tilde{F}}(x_l, \theta_{\tilde{F}}^m(x_l))$ is the secondary membership grade of $\tilde{F}$ at x_l and $\theta_{\tilde{F}}^m(x_l)$.

Although a fuzzy set is only fully represented by an infinite number of embedded fuzzy sets, Greenfield et al. [11] show that a small number of embedded fuzzy sets is sufficient to adequately model a fuzzy set. In this paper, we use 10 embedded fuzzy sets represented by eq. (11) and eq. (12).

This concludes the overview of type-2 fuzzy sets and their representations.

2.2. Similarity Measures: A General Overview

A similarity measure is a function $s : \tilde{A} \times \tilde{B} \to [0, 1]$; or $s : \tilde{A} \times \tilde{B} \to [[0, 1], [0, 1]]$; or $s : \tilde{A} \times \tilde{B} \to \mathscr{F}$, where $\tilde{A}$ and $\tilde{B}$ are type-2 fuzzy sets, and $\mathscr{F}$ is a type-1 fuzzy set. That is, the result may be a single-value within the interval $[0, 1]$ (as is most often the case), or the result may be an interval where the lower and upper bound are restricted within the interval $[0, 1]$ (e.g. [1]) or the result may be a type-1 fuzzy set [13, 48]. The result of the measure represents how well two fuzzy sets match by comparing how close the membership degrees are of each fuzzy set.

Distance is often viewed as a non-increasing function of similarity [34, 50]. A distance function is a metric if it has the properties

D1 $d(\tilde{A}, \tilde{B}) > 0$

D2 $d(\tilde{A}, \tilde{A}) = 0$

D3 $d(\tilde{A}, \tilde{B}) = d(\tilde{B}, \tilde{A})$

D4 $d(\tilde{A}, \tilde{B}) + d(\tilde{B}, \tilde{C}) \geq d(\tilde{B}, \tilde{C})$

The properties of a similarity measure are often based on the negation of a metric. In this paper, we define a similarity measure as a function that may have the following properties:

S1 Reflexivity: $s(\tilde{A}, \tilde{B}) = 1 \iff \tilde{A} = \tilde{B}$

S2 Symmetry: $s(\tilde{A}, \tilde{B}) = s(\tilde{B}, \tilde{A})$

S3 Transitivity: If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $s(\tilde{A}, \tilde{B}) \geq s(\tilde{A}, \tilde{C})$

S4 Overlapping: If $\tilde{A} \cap \tilde{B} \neq \emptyset$, then $s(\tilde{A}, \tilde{B}) > 0$; otherwise, $s(\tilde{A}, \tilde{B}) = 0$

S5 Minimum similarity $s(D, D^c) = 0 \ \forall D \in \mathscr{P}(U)$

where $\mathscr{P}(U)$ is the set of all crisp subsets in the universe U [34, 43]. As transitivity is defined in respect to the subsethood of the fuzzy sets, we review definitions of subsethood for type-2 sets:

Definition 7. *For interval type-2 fuzzy sets $\tilde{A}$ $\tilde{B}$, $\tilde{A} \subseteq \tilde{B}$ if $\underline{u}_{\tilde{A}}(x) \leq \underline{u}_{\tilde{B}}(x)$ and $\overline{u}_{\tilde{A}}(x) \leq \overline{u}_{\tilde{B}}(x)$, $\forall x \in X$ [47].*

Definition 8. *For general type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$, $\tilde{A} \subseteq \tilde{B}$ if $\tilde{A}_z \subseteq \tilde{B}_z, \forall z \in (0, 1]$ and the subsethood of the zSlices $\tilde{A}_z$ and $\tilde{B}_z$ is the same as given in Definition 7 [13].*

It is important to note that as the term *similarity* is loosely defined, it is not necessary for a similarity measure to have all of these properties. The property of overlapping (S4) may be considered too strict, and while many existing measures have this property, this is not true for all measures. A common alternative is S5. In this paper, we include overlapping (S4), as defined above, as an important property of similarity. Of course, the properties that are desired are dependent on the context in which the measure will be used.

Definition 9. *Where a measure has properties S1-S4, we describe it as property complete.*

Similarity measures are related to restricted equivalence functions [2], which are binary operations on the unit interval built upon the theory of equivalence functions [7]. A function $e : [0, 1]^2 \to [0, 1]$ is called a restricted equivalence function if it satisfies the following conditions [2]:

E1 $e(x, y) = e(y, x)$ for all $x, y \in [0, 1]$

E2 $e(x, y) = 1$ if and only if $x = y$

E3 $e(x, y) = 0$ if and only if $x = 1$ and $y = 0$, or $x = 0$ and $y = 1$

E4 $e(x, y) = e(c(x), c(y))$ for all $x, y \in [0, 1]$, c being a strong negation

E5 For all $x, y, z \in [0, 1]$, if $x \leq y \leq z$, then $e(x, y) \geq e(x, z)$ and $e(y, z) \geq e(x, z)$, where $c(x)$ is a strong negation, such that $c(0) = 1$, $c(1) = 0$, $c(c(x)) = c$ and c is monotonic and continuous. Note that the properties of a restricted equivalence function are strictly defined by conditions E1–E5. By contrast, the properties of a similarity measure given above are general and they do not (and in fact, cannot) all apply to a given measure of similarity.

The two main approaches for measuring similarity between type-2 fuzzy sets are 1) proximity-based measures; and 2) set-theoretic measures. In proximity-based approaches, the membership degree of each $x \in X$ is compared between the two fuzzy sets, typically by calculating the difference in membership. Using this, a comparison of the fuzzy sets as a whole can be obtained by calculating the difference for every membership degree. Set-theoretic approaches are developed from methods of measuring the binary (presence / absence) similarity coefficients [6]. Set-theoretic approaches involve the measurement of how much one set is included within another, how much they intersect or measuring partial overlap. This is measured through calculations on the cardinality of the sets [50].

Similarity measures for type-2 fuzzy sets are often built as extensions of measures for type-1 fuzzy sets. It is therefore useful to introduce two common type-1 measures (proximity based and set-theoretic) which many of the measures discussed in this paper extend. A common proximity based method for type-1 fuzzy sets is [34]

$$s_p^{T1}(A,B) = 1 - \frac{\int_{x\in X}(|\mu_A(x) - \mu_B(x)|)dx}{\int_{x\in X}(\mu_A(x) + \mu_B(x))dx}. \tag{13}$$

Note that we label equations as s_a^t where t is the type of fuzzy set ($T1$, $IT2$ or $GT2$) and a is the authors' initials.

A common measure of similarity based on set-theoretic comparisons is Jaccard's ratio, written as [15]

$$s_j^{T1}(A,B) = \frac{\int_{x\in X}\min(\mu_A(x), \mu_B(x))dx}{\int_{x\in X}\max(\mu_A(x), \mu_B(x))dx}. \tag{14}$$

We discuss type-2 measures of similarity that are extensions of eq. (13) and eq. (14).

Another method of building similarity measures is to construct them from subsethood measures [1, 5, 16], enabling an even wider variety of type-2 similarity measures. Let $c : A \times B \to [0,1]$ be a subsethood measure on type-1 fuzzy sets A and B. The most common method of measuring similarity based on subsethood where $s : \tilde{A} \times \tilde{B} \to [0,1]$ is [16]

$$s^{T1}(A,B) = t(c(A,B), c(B,A)) \tag{15}$$

where t is a t-norm. Another method, where $s : \tilde{A} \times \tilde{B} \to [[0,1],[0,1]]$, is

$$s^{T1}(A,B) = [(c(A,B), c(B,A)] \tag{16}$$

However, other methods of inducing similarity from subsethood are also in the literature [5]. An example of a proximity-based subsethood measure $c(A, B)$ (denoting how much the type-1 fuzzy set A belongs to B) can be measured as [9]

$$c_g^{T1}(A,B) = \int_{x\in X} \min\{1, 1-\mu_A(x)+\mu_B(x)\}\ dx \tag{17}$$

and a common set-theoretic subsethood measure is [45]

$$c_y^{T1}(A,B) = \frac{\int_{x\in X} \min(\mu_A(x),\mu_B(x))\ dx}{\int_{x\in X} \mu_A(x)\ dx}. \tag{18}$$

Note that the result of a similarity measure on type-2 fuzzy sets is most commonly a single value. However, there do exist methods that represent similarity as an interval [1] and as a type-1 fuzzy set [13, 48]. We focus on methods that give a single-valued or interval-valued result as these are the most common in the literature. We also discuss two methods that give a type-1 fuzzy set result when this is reduced to a single value.

The literature on similarity measures for type-2 fuzzy sets is vast. In this paper, we cover measures specifically designed for interval and general type-2 fuzzy sets. However, further measures can be extracted from the literature. For example, a general method of building similarity measures is by aggregating restricted equivalence functions [3]. In addition, similarity measures on intuitionistic fuzzy sets have been applied to interval type-2 fuzzy sets [8] and measures on the former may be transferable to the latter. However, as it is not essential to cover all possible constructions and generalisations of similarity measures for type-2 fuzzy sets, we focus only on functions designed specifically for type-2 fuzzy sets.

In the remainder of this section, we introduce similarity measures on interval and general type-2 fuzzy sets. We label the sets compared by each measure as $\tilde{A}$ and $\tilde{B}$.

2.3. Similarity measures for Interval Type-2 Fuzzy Sets

Wu & Mendel [40] and Nguyen & Kreinovich [32] extended the type-1, set-theoretic Jaccard measure (14) to interval type-2 fuzzy sets as

$$s_j^{IT2}(\tilde{A},\tilde{B}) = \frac{\int_{x\in X} \min(\overline{u}_{\tilde{A}}(x),\overline{u}_{\tilde{B}}(x))dx + \int_{x\in X} \min(\underline{u}_{\tilde{A}}(x),\underline{u}_{\tilde{B}}(x))dx}{\int_{x\in X} \max(\overline{u}_{\tilde{A}}(x),\overline{u}_{\tilde{B}}(x))dx + \int_{x\in X} \max(\underline{u}_{\tilde{A}}(x),\underline{u}_{\tilde{B}}(x))dx}. \tag{19}$$

Also, akin to s_j^{IT2}, Zheng et al. [49] proposed the following:

$$s_{zwzz}^{IT2}(\tilde{A},\tilde{B}) = \frac{1}{2}\left(\frac{\int_{x\in X} \min(\overline{u}_{\tilde{A}}(x),\overline{u}_{\tilde{B}}(x))dx}{\int_{x\in X} \max(\overline{u}_{\tilde{A}}(x),\overline{u}_{\tilde{B}}(x))dx} + \frac{\int_{x\in X} \min(\underline{u}_{\tilde{A}}(x),\underline{u}_{\tilde{B}}(x))dx}{\int_{x\in X} \max(\underline{u}_{\tilde{A}}(x),\underline{u}_{\tilde{B}}(x))dx}\right). \tag{20}$$

Both s_j^{IT2} and s_{zwzz}^{IT2} are property-complete.

Wu and Mendel proposed a measure for linguistic approximation that uses both set theoretic and proximity based approaches to calculate similarity [39]. This is given as

$$s_{wm}^{IT2}(\tilde{A}, \tilde{B}) = s_j^{IT2}(\tilde{A}, \tilde{B}) s_{wm_d}(\tilde{A}, \tilde{B}), \tag{21}$$

where s_j^{IT2} is the Jaccard measure (19), and s_{wm_d} is based on the distance between fuzzy sets as

$$s_{wm_d}(\tilde{A}, B) \equiv h(d(\tilde{A}, \tilde{B})), \tag{22}$$

where $d(\tilde{A}, B) = |c(\tilde{A}) - c(\tilde{B})|$, $c(A)$ refers to the centroid of set $\tilde{A}$, and h can be any function that satisfies 1) $\lim_{x\to\infty} h(x) = 0$; 2) $h(x) = 1 \iff x = 0$; and 3) $h(x)$ decreases monotonically as x increases. Details on the chosen function of h are in [38].

Zeng and Li [46] developed a proximity-based measure of similarity based on the concept of entropy (that is, a measure of how much a fuzzy set is fuzzy [22]). Their method calculates the difference between the upper and lower membership functions of the sets, taking the average of the two results as

$$s_{zl}^{IT2}(\tilde{A}, \tilde{B}) = 1 - \frac{1}{2(b-a)} \int_a^b (|\underline{u}_{\tilde{A}}(x) - \underline{u}_{\tilde{B}}(x)| + |\overline{u}_{\tilde{A}}(x) - \overline{u}_{\tilde{B}}(x)|) dx, \tag{23}$$

where a and b denote the boundaries of the finite universe of discourse X; i.e. $X = [a, b]$.

Bustince [1] proposed a proximity-based measure that represents similarity as an interval as follows:

$$s_b^{IT2}(\tilde{A}, \tilde{B}) = [s_{b_L}(\tilde{A}, \tilde{B}), s_{b_U}(\tilde{A}, \tilde{B})] \tag{24a}$$

$$s_{b_L}(\tilde{A}, \tilde{B}) = t(\Upsilon_L(\tilde{A}, \tilde{B}), \Upsilon_L(\tilde{B}, \tilde{A})) \tag{24b}$$

$$s_{b_U}(\tilde{A}, \tilde{B}) = t(\Upsilon_U(\tilde{A}, \tilde{B}), \Upsilon_U(\tilde{B}, \tilde{A})) \tag{24c}$$

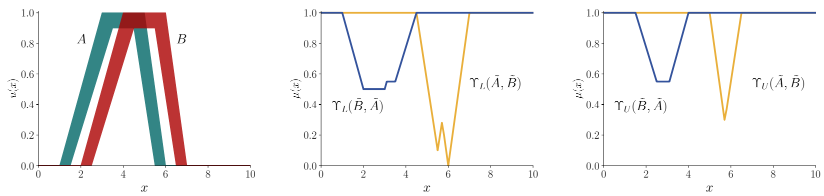
$$\Upsilon_L(\tilde{A}, \tilde{B}) = \inf_{x\in X} \{1, \min(1 - \underline{u}_{\tilde{A}}(x) + \underline{u}_{\tilde{B}}(x), 1 - \overline{u}_{\tilde{A}}(x) + \overline{u}_{\tilde{B}}(x))\} \tag{24d}$$

$$\Upsilon_U(\tilde{A}, \tilde{B}) = \inf_{x\in X} \{1, \max(1 - \underline{u}_{\tilde{A}}(x) + \underline{u}_{\tilde{B}}(x), 1 - \overline{u}_{\tilde{A}}(x) + \overline{u}_{\tilde{B}}(x))\} \tag{24e}$$

where t is any t-norm (we use the minimum t-norm throughout this paper). Note that the result of eq. (24) is an interval. Fig. 2 helps to visualise the method of s_b^{IT2}.

Fig. 2a shows two interval type-2 fuzzy sets, Fig. 2b shows the results of $\Upsilon_L(\tilde{A}, \tilde{B})$ and $\Upsilon_L(\tilde{B}, \tilde{A})$ (used in eq. (24b)), and Fig. 2c shows the results of $\Upsilon_U(\tilde{A}, \tilde{B})$ and $\Upsilon_U(\tilde{B}, \tilde{A})$ (used in eq. (24c)). Note that Υ_L and Υ_U return a single value, not a set, but we show the calculations at all values of x used to compute Υ_L and Υ_U (in Figs. 2b and 2c) to highlight the methods.

The equation $\Upsilon_L(\tilde{A}, \tilde{B})$ calculates the smallest amount by which the lower (or upper) membership value of $\tilde{B}$ is greater than the lower (or upper) membership value of $\tilde{A}$ for each value of x in X. Both $\Upsilon_L(\tilde{A}, \tilde{B})$ and $\Upsilon_L(\tilde{B}, \tilde{A})$ are calculated in (24b), meaning we also calculate how much $\tilde{A}$ has a greater membership than $\tilde{B}$. Fig. 2b shows the minimum difference for each x when



(a) (b) (c)

Figure 2: (a) Two interval type-2 fuzzy sets and (b) the calculations from $\Upsilon_L(\tilde{A},\tilde{B})$ and $\Upsilon_L(\tilde{B},\tilde{A})$ (see eq. (24b)) and (c) the calculations from $\Upsilon_U(\tilde{A},\tilde{B})$ and $\Upsilon_U(\tilde{B},\tilde{A})$ (see eq. (24c)).

calculating $\Upsilon_L(\tilde{A},\tilde{B})$ and $\Upsilon_L(\tilde{B},\tilde{A})$. Note that the figure shows the calculation at each x, but the final result of $\Upsilon_L(\tilde{A},\tilde{B})$ is the smallest of these differences across X.

The equation $\Upsilon_U(\tilde{A},\tilde{B})$ calculates the largest amount by which the lower (or upper) membership value of $\tilde{B}$ is greater than the lower (or upper) membership value of $\tilde{A}$ for each value of x in X. As before, both $\Upsilon_U(\tilde{A},\tilde{B})$ and $\Upsilon_U(\tilde{B},\tilde{A})$ are calculated in (24c). Fig. 2c shows the maximum difference for each x when calculating both $\Upsilon_U(\tilde{A},\tilde{B})$ and $\Upsilon_U(\tilde{B},\tilde{A})$. Though the figure shows the calculations at each x, the result of $\Upsilon_U(\tilde{A},\tilde{B})$ is actually the smallest of these differences across X.

Finally, the lower bound of s_b^{IT2} is the smallest value in Fig. 2b, and the upper bound is the smallest value in Fig. 2c. In this example, the lower bound of $s_b^{IT2}(\tilde{A},\tilde{B})$ is 0 because at $x = 6$, the difference in upper membership between $\tilde{A}$ and $\tilde{B}$ is 1.

This concludes similarity measures on interval type-2 fuzzy sets.

2.4. Similarity Measures for General Type-2 Fuzzy Sets

Zhao et al.[48], Hao and Mendel [13], and McCulloch and Wagner [23] take alike approaches in measuring similarity using the zSlices approach. Each method is based on the set-theoretic Jaccard measure for interval type-2 fuzzy sets (see eq. (19)). Their equations are given next.

Zhao et al. [48] proposed two new measures of similarity on zSlices-based fuzzy sets. The first measure represents similarity as a type-1 fuzzy set [48], while the second presents similarity as a single value as

$$s_{zxld}^{GT2}(A,B) = \int_0^1 s_j^{IT2}(\tilde{A}_z,\tilde{B}_z)\ dz. \tag{25}$$

where s_j^{IT2} is the Jaccard approach for interval type-2 fuzzy sets (see eq. (19)). This is also akin to the method given by Hamrawi and Coupland [12]. However, they do not specify using the Jaccard measure and instead propose using any interval type-2 similarity measure.

Hao and Mendel [13] represent similarity as a fuzzy set and compute its centroid to acquire a crisp result. The fuzzy set representing similarity is given as

$$s_{hm}^{GT2-F}(\tilde{A},\tilde{B}) = \{(z, s_j^{IT2}(\tilde{A}_z,\tilde{B}_z)) \mid z \in (0,1]\}. \tag{26}$$

In this paper, we refer to the centroid of s_{hm}^{GT2-F} as s_{hm}^{GT2}. This differs from s_{zxld}^{GT2} (eq. (25)) by weighting the similarities of the zSlices by their zLevels, whereas s_{zxld}^{GT2} takes an unweighted average.

Previously [23], we proposed a similarity measure using the zSlices-representation as follows

$$s_{mw}^{GT2}(\tilde{A},\tilde{B}) = \frac{\int_0^p z s^{IT2}(\tilde{A}_z,\tilde{B}_z)\ dz}{\int_0^p z\ dz}, \tag{27}$$

where $p = \max\{z \mid \mu_{\tilde{A}}(x,u) \geq z \wedge \mu_{\tilde{B}}(x,u) \geq z,\ z \in (0,1]\}$ is the highest non-empty zLevel. Like eq. (26), this method (eq. (27)) also weights the similarities of the zSlices by their zLevels (the value z), such that the similarity between higher zSlices contributes more to the overall similarity than those at lower zSlices.

The methods eq. (25), eq. (26), eq. (27) are all restricted to the zSlices notation of type-2 fuzzy sets. However, fuzzy sets can be represented through different notations including vertical slices [29] or embedded slices [25] (see Section 2.1). As such, Wu and Mendel [41] proposed a geometric approach which calculates the Jaccard similarity eq. (19) based on the volume of the fuzzy sets. Their measure can be applied to either the zSlices or vertical slices representation. The similarity is computed as follows:

$$s_{wm}^{GT2}(\tilde{A},\tilde{B}) = \frac{(\underline{\tilde{A}} \cap \underline{\tilde{B}})_v + (\overline{\tilde{A}} \cap \overline{\tilde{B}})_v}{(\underline{\tilde{A}} \cup \underline{\tilde{B}})_v + (\overline{\tilde{A}} \cup \overline{\tilde{B}})_v}, \tag{28}$$

In terms of the vertical slice representation $\underline{\tilde{A}}$ and $\overline{\tilde{A}}$ represent the lower and upper surfaces spanned by the portion of the vertical slices that have smaller or larger u than their corresponding apexes [41]. In terms of the zSlices representation $\underline{\tilde{A}}$ and $\overline{\tilde{A}}$ represent the lower and upper surfaces defined by the lower and upper membership functions of different zSlices. The value $(\underline{\tilde{A}} \cap \underline{\tilde{B}})_v$ is the volume of the intersection of $\underline{\tilde{A}}$ and $\underline{\tilde{B}}$, and $(\underline{\tilde{A}} \cup \underline{\tilde{B}})_v$ is the volume of the union. The volume of the fuzzy sets (and their intersection or union) can be calculated using either the vertical slice or zSlice representation by measuring the volume between each slice. Using the zSlices representation, the similarity between $\tilde{A}$ and $\tilde{B}$ is given as

$$\frac{\int_{z\in[0,1]} \left[\int_{x\in X} \min(\underline{\tilde{A}_z}(x), \underline{\tilde{B}_z}(x))dx + \int_{x\in X} \min(\overline{\tilde{A}_z}(x), \overline{\tilde{B}_z}(x))dx \right] dz}{\int_{z\in[0,1]} \left[\int_{x\in X} \max(\underline{\tilde{A}_z}(x), \underline{\tilde{B}_z}(x))dx + \int_{x\in X} \max(\overline{\tilde{A}_z}(x), \overline{\tilde{B}_z}(x))dx \right] dz}, \tag{29}$$

where $\tilde{A}_z(x) = [\underline{\tilde{A}_z}(x), \overline{\tilde{A}_z}(x)]$ is the vertical slice of the zSlice $\tilde{A}_z$ at x. Using the vertical slice representation instead, similarity is calculated as

$$\frac{\int_{x\in X}\left[\int_{z\in[0,1]}\min(\underline{\tilde{A}_z}(x),\underline{\tilde{B}_z}(x))dx+\int_{x\in X}\min(\overline{\tilde{A}_z}(x),\overline{\tilde{B}_z}(x))dz\right]dx}{\int_{x\in X}\left[\int_{z\in[0,1]}\max(\underline{\tilde{A}_z}(x),\underline{\tilde{B}_z}(x))dx+\int_{x\in X}\max(\overline{\tilde{A}_z}(x),\overline{\tilde{B}_z}(x))dz\right]dx}. \tag{30}$$

Taking either the zslices (eq. (29)) or the vertical slices approach (eq. (30)) yields the same results [41].

The methods s^{GT2}_{mw}, s^{GT2}_{hm} and s^{GT2}_{zxld} measure the Jaccard interval type-2 similarity between the zSlices of fuzzy sets and calculate the average. By contrast, Yang and Lin [20, 44] proposed measuring the Jaccard type-1 similarity between vertical slices of fuzzy sets and calculating the average. This is given as:

$$s^{GT2}_{yl}(\tilde{A},\tilde{B})=\int_{x\in X}\frac{\int_0^1\min\{u\cdot\mu_{\tilde{A}}(x,u),u\cdot\mu_{\tilde{B}}(x,u)\}\ du}{\int_0^1\max\{u\cdot\mu_{\tilde{A}}(x,u),u\cdot\mu_{\tilde{B}}(x,u)\}\ du}\ dx. \tag{31}$$

Alternatively, Hung and Yang [37] use the inverse of the Hausdorff distance to measure the proximity between secondary membership functions as

$$s^{GT2}_{hy}(\tilde{A},\tilde{B})=1-d^N(\tilde{A},\tilde{B}) \tag{32a}$$

$$d^N(\tilde{A},\tilde{B})=\int_{x\in X}H_f(\tilde{A}(x),\tilde{B}(x))\ dx \tag{32b}$$

$$H_f(A,B)=\int_0^1\alpha_i H(A_\alpha,B_\alpha)\ d\alpha \tag{32c}$$

$$H(A_\alpha,B_\alpha)=\max\{L(A_\alpha,B_\alpha),L(B_\alpha,A_\alpha)\} \tag{32d}$$

$$L(A_\alpha,B_\alpha)=\inf\left\{\lambda\in[0,\infty]|A^\lambda_\alpha\supset B_\alpha\right\} \tag{32e}$$

Within (32), $\tilde{A}$ and $\tilde{B}$ are the type-2 fuzzy sets being compared. Their similarity is measured by comparing their vertical slices $\tilde{A}(x)$ and $\tilde{B}(x)$ (in eq. (32b)), which are abbreviated as A and B (in eq. (32c)) since the vertical slices are type-1 fuzzy sets. Next, the similarity of A and B is measured by calculating the Hausdorff distance between their alpha-cuts (in eq. (32c)). The Hausdorff distance (as written in [37]) is given in eq. (32d) and (32e). However, as the α-cuts are closed intervals (assuming convexity), the Hausdorff distance between α-cuts can be rewritten in a simplified form as

$$H(A_\alpha,B_\alpha)=\max\{|\ \underline{A_\alpha}-\underline{B_\alpha}\ |,|\ \overline{A_\alpha}-\overline{B_\alpha}\ |\} \tag{33}$$

where $A_\alpha=[\underline{A_\alpha},\overline{A_\alpha}]$. Finally, in eq. (32a) the average distance (averaged over α-cuts of the vertical slices of $\tilde{A}$ and $\tilde{B}$) is used as the negation of similarity.

Mitchell [30] proposed measuring the similarity between two general type-2 fuzzy sets by comparing their embedded fuzzy sets with any type-1 similarity measure. Let $\tilde{A}$ and $\tilde{B}$ be two type-2 fuzzy sets represented by a total of M and N embedded fuzzy sets, respectively. The similarity between $\tilde{A}$ and $\tilde{B}$ is the weighted sum of the similarity between each pair of embedded fuzzy sets as [30]

$$s_m^{GT2}(\tilde{A},\tilde{B}) = \sum_{m=1}^{M}\sum_{n=1}^{N} s\Big(\theta_{\tilde{A}}^m(x), \theta_{\tilde{B}}^n(x)\Big)\Lambda\Big(\theta_{\tilde{A}}^m(x), \theta_{\tilde{B}}^n(x)\Big), \tag{34}$$

where $\theta_{\tilde{A}}^m(x)$ is the m^{th} embedded fuzzy set of $\tilde{A}$ (see eq. (11)), s can be any type-1 similarity measure, and Λ is the normalised weight

$$\Lambda(\theta_{\tilde{A}}^m(x), \theta_{\tilde{B}}^n(x)) = \frac{t(\lambda_A^m, \lambda_B^n)}{\sum_{i=1}^{M}\sum_{j=1}^{N} t(\lambda_A^i, \lambda_B^j)}, \tag{35}$$

where t is a t-norm (we use minimum) and λ_A^m is the weight associated with the embedded fuzzy set $\theta_{\tilde{A}}^m$ (see eq. (12)).

We have chosen the Jaccard ratio s_j^{T1} (see eq. (14)) for s in eq. (34) as this provides the most accurate comparison of s_m^{GT2} against other general type-2 measures that are, in most cases, based on the same ratio.

This concludes the background on similarity measures for type-2 fuzzy sets. In the next section, we introduce the methods used to evaluate these measures.

3. Methods

In this section, we introduce the data used in this paper, and two methods of using this data to evaluate the similarity measures.

We construct fuzzy sets from interval-valued data that have previously been used to demonstrate similarity measures on type-2 fuzzy sets [40, 41] and are available online [17]. The data consist of 32 words to which 174 participants assigned interval-values on the scale $[0, 10]$. Interval type-2 fuzzy sets are constructed from the interval-valued data using the HM approach [14]. Table 1 lists these words; for consistency, they are presented in the same order as given in [40] where further details on the method of data collection can be found. In Table 1, each word is assigned an ID number for ease of reference within this paper.

For each word, we use the HM approach [14] to convert interval-valued data to an interval type-2 fuzzy set. Each fuzzy set has a trapezoidal upper and lower membership function. Let the membership function μ_A be defined as $\mu_A = \text{trapmf}(x; [a, b, c, d])$ such that

$$\mu_A(x) = \begin{cases} (x-a)/(b-a) & a \le x \le b \\ 1 & b \le x \le c \\ (d-x)/(d-c) & c \le x \le d \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

The HM approach produces interval type-2 fuzzy sets. From these, we construct secondary membership functions to produce general type-2 fuzzy sets.

Table 1: Words to which survey participants assigned intervals (taken from [17]).

ID	word	ID	word
1	none to very little	17	modest amount
2	teeny-weeny	18	good amount
3	a smidgen	19	sizeable
4	tiny	20	quite a bit
5	very small	21	considerable amount
6	very little	22	substantial amount
7	a bit	23	a lot
8	little	24	high amount
9	low amount	25	very sizeable
10	small	26	large
11	somewhat small	27	very large
12	some	28	humongous amount
13	some to moderate	29	huge amount
14	moderate amount	30	very high amount
15	fair amount	31	extreme amount
16	medium	32	maximum amount

Let $\tilde{A}$ be a zSlices-based fuzzy set with a finite number of zSlices and trapezoidal upper and lower membership functions for each zSlice (denoted $\overline{\tilde{A}_z}$ and $\underline{\tilde{A}_z}$, respectively) defined as

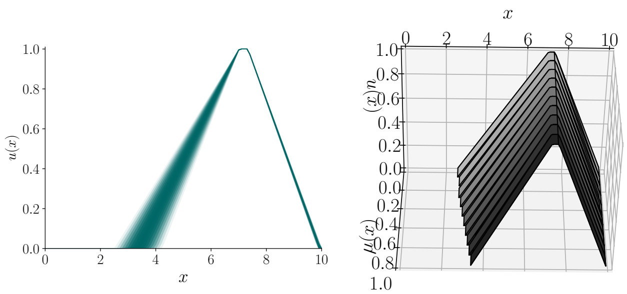
$$\begin{aligned} \overline{\tilde{A}}_z &= \text{trapmf}(x, [\overline{a}_z, \overline{b}_z, \overline{c}_z, \overline{d}_z]) \\ \underline{\tilde{A}}_z &= \text{trapmf}(x, [\underline{a}_z, \underline{b}_z, \underline{c}_z, \underline{d}_z]). \end{aligned} \tag{37}$$

We assign the maximum secondary membership in $\tilde{A}$ (i.e., $\mu_{\tilde{A}}(x,u) = 1$) where x and u are at the centre of the footprint of uncertainty. The secondary membership values decrease linearly towards the edge of the footprint of uncertainty. For example, Fig. 3 shows the general type-2 fuzzy set of word-17 in a two-dimensional figure and a three-dimensional figure, where darker shades indicate higher degrees of membership. The upper and lower membership functions of each zLevel are calculated as follows: For any given zLevel $z \in (0, 1]$, the upper membership function of $\tilde{A}_z$ is

$$\overline{\tilde{A}_z} = \text{trapmf}(x, [\overline{a_z}, \overline{b_z}, \overline{c_z}, \overline{d_z}]), \tag{38}$$

where

$$\begin{aligned} \overline{a_z} &= \overline{a} + \Big((\underline{a} - \overline{a})\big(\frac{z}{I-1}\big)\Big) \\ \overline{b_z} &= \overline{b} + \Big((\underline{b} - \overline{b})\big(\frac{z}{I-1}\big)\Big) \\ &\ldots \end{aligned} \tag{39}$$



(a) (b)

Figure 3: (a) A two-dimensional and (b) top-down three-dimensional view of word 17. Darker shades indicate a higher secondary membership.

where $\overline{a}$ and $\underline{b}$ define the lowest zSlice (in eq. (36)), z is the zLevel (i.e. the secondary membership value), and I is the total number of zSlices. Likewise, the lower membership function of $\tilde{A}_z$ is

$$\underline{\tilde{A}_z} = \text{trapmf}(x, [\underline{a_z}, \underline{b_z}, \underline{c_z}, \underline{d_z}]), \tag{40}$$

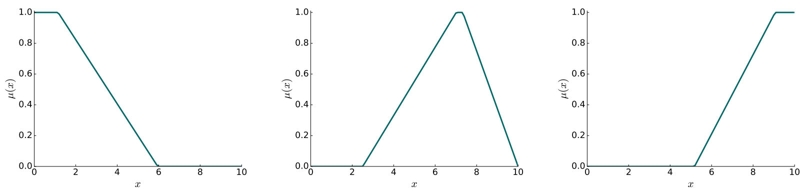
where

$$\begin{aligned} \underline{a_z} &= \underline{a} - \Big((\underline{a} - \overline{a}) \big(\frac{z}{I-1} \big) \Big) \\ \underline{b_z} &= \underline{b} - \Big((\underline{b} - \overline{b}) \big(\frac{z}{I-1} \big) \Big) \\ &\ldots \end{aligned} \tag{41}$$

Throughout the paper, we use a total of 10 zLevels.

To analyse the measures, we compare every fuzzy set with every other fuzzy set for a total of 528 comparisons. Of those, 32 comparisons are of identical fuzzy sets. In addition, 204 pairs of fuzzy sets are disjoint as a result of using the HM approach to construct type-2 fuzzy sets with the given data. We use a heatmap to visualise the results of each measure. In addition, we use scatter plots to show pairwise comparisons of the results of the measures. To illustrate these two methods, we demonstrate them with type-1 fuzzy sets here. We construct type-1 fuzzy sets based on the type-2 fuzzy set word models. For each word, the membership function of the type-1 fuzzy set is the same as the upper membership function of the interval type-2 fuzzy set.

Fig. 4 shows an example of three type-1 fuzzy sets representing words 7, 17 and 23. Fig. 5 shows a heatmap of the results of s_j^{T1} (eq. (14)) on all pairs of words, where white indicates $s = 1$ and black indicates $s = 0$. The plot is symmetrical because the measure has the property of symmetry. Clusters of similar words can be seen in light coloured sections. Word 20 stands out among neighbouring words, suggesting it should be ranked lower because it is more similar to lower ranked words that to its neighbours.



(a) (b) (c)

Figure 4: Examples of three type-1 fuzzy sets representing words 7, 17 and 23. Their membership functions are the same as the upper membership function of the HM Approach interval type-2 fuzzy sets for the same words.

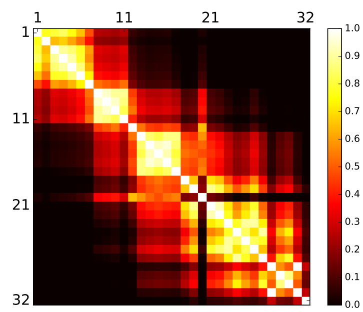


Figure 5: Heatmap showing results of s_j^{T1} comparing every word (listed in Table 1) with every other word.

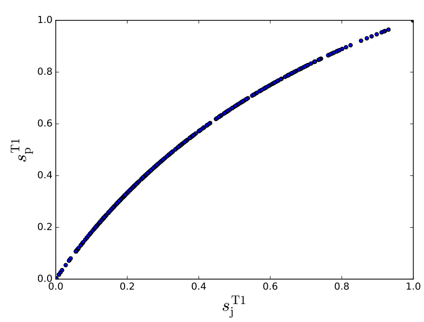


Figure 6: Results of s_j^{T1} and s_p^{T1} comparing every word with every other word. Each point on the plot shows the results for one pair of fuzzy sets according to the two measures.

Fig. 6 shows a pairwise comparison of the results of s_j^{T1} (eq. (14)) and s_p^{T1} (eq. (13)) as a scatter plot. Each point on the plot represents the similarity between one pair of fuzzy sets; therefore there are a total of 528 points. The axes represent the results from the given similarity measures. There is a perfect rank correlation between the two measures, meaning if only the rank order of similarity is important then the measures give the same results. In such cases, we say the measure are equivalent - even if the values from the measures are different. Such equivalences of type-1 fuzzy sets have been discussed in [33].

To ensure consistency in our analysis, when making pairwise comparisons, we choose one similarity measure and compare all other measures against this one. Specifically, we always compare against the same Jaccard-based property-complete measure. Note that we do not advocate in any way that Jaccard is the universally best measure, however, it being both the most popular measure in the literature, and a key example of a property-complete measure, it provides a valuable basis to observe the effects of missing properties in other measures. Finally, we note that we do not only explore measures' behaviours in respect to Jaccard, but also examine measures individually to ensure a fair analysis.

Using these data and the methods described, in the next two sections we evaluate interval and general type-2 similarity measures, respectively. We assess what approach should be taken to ensure a property-complete comparison of fuzzy sets, and which are the best similarity measures to choose from.

4. Similarity of Interval Type-2 Fuzzy Sets

In this section, we analyse each interval type-2 similarity measure on data-driven fuzzy sets that describe words. We measure the similarity of each word with each other word. Fig. 7 shows pairwise comparisons of each measure compared against the Jaccard measure (eq. 19). We choose the Jaccard measure eq. (19) as the base for these pairwise comparisons because it is property-complete and therefore helps to highlight how non-property complete measures differ. Fig. 8 shows the similarity of each word compared with each other word for each measure. Results are shown as a heatmap, where $s = 0$ is shown as black and $s = 1$ is shown as white. Each figure is symmetrical, demonstrating that each similarity measure is symmetrical.

Fig. 7a shows a comparison of s_{zwzz}^{IT2} against s_j^{IT2} across all combinations of sets. Each point in the figure represents the similarity between a pair of interval type-2 fuzzy sets according to s_j^{IT2} and s_{zwzz}^{IT2}. While the rank orders of s_{zwzz}^{IT2} and s_j^{IT2} are slightly different, they share a high correlation. In addition, their results in Fig. 8 are almost equal. Clusters of words with similar meaning can be seen, and dark regions indicate words that do not have any similarity in meaning. Both methods also show a discontinuity in similarity for word 20, suggesting it should be ranked lower in the list of words.

Fig. 7b compares s_{wm}^{IT2} with s_j^{IT2}, showing there is some correlation between the results, but the values from s_{wm}^{IT2} vary from s_j^{IT2}. The measure s_{wm}^{IT2} does not have the property of overlapping and instead will always give a low value of similarity for two disjoint fuzzy sets. However, this value is small enough that it would intuitively be considered an insignificant degree of similarity. In Fig. 8, the results of s_{wm}^{IT2} are like that of s_j^{IT2} and s_{zwzz}^{IT2}, finding the same clusters of similar words.

Fig. 7c compares s_{zl}^{IT2} with s_j^{IT2}. It is clear that s_{zl}^{IT2} produces higher values of similarity than s_j^{IT2}. This is because it does not weight similarity according to membership degrees. For example, consider two fuzzy sets $\tilde{A}$ and $\tilde{B}$, where at x_1 $\overline{\mu}_{\tilde{A}}(x_1) = 0.1$ and $\overline{\mu}_{\tilde{B}}(x_1) = 0.12$, and at x_2 $\overline{\mu}_{\tilde{A}}(x_2) = 0.9$ and $\overline{\mu}_{\tilde{B}}(x_2) = 0.92$. At x_1, the difference in upper membership values between $\tilde{A}$ and $\tilde{B}$ is 0.02 (they are similar) but the membership values are low. By contrast, at x_2 the difference in upper membership values is the same but the membership values themselves are high. Giving equal weight to the similarity at x_1 and at x_2 produces an inflated result.

This also leads to s_{zl}^{IT2} not having the property of overlapping. It instead gives a non-zero similarity for disjoint fuzzy sets. For example, consider two fuzzy sets where, for a given x, both have a membership degree of zero. The result of s_{zl}^{IT2} will increase because the fuzzy sets become more similar in the sense that they both have zero membership at x. Measuring empty vertical slices artificially increases the similarity for this approach.

These inflated results of s_{zl}^{IT2} can be seen in Fig. 8. The clusters of similar words are less well defined by s_{zl}^{IT2} than by the other methods in the same figure. Additionally, it is clear that words at the opposite ends of the scale (e.g. words 1 and 32) have resulted in a high similarity, demonstrating that the measure is unreliable. It is because s_{zl}^{IT2} gives low and high membership degrees equal weighting that the average similarity is much higher than with s_j^{IT2}. If the difference in membership degrees is weighted by the degrees themselves, the similarity result will be lower. Therefore, we recommend weighting the proximity of membership by the membership values themselves.

s_b^{IT2} provides an interval-valued measure of similarity instead of crisp values, providing a degree of uncertainty about the similarity. Fig. 7d represents this by vertical lines compared against s_j^{IT2}. A circular point is drawn where the lower and upper bounds of s_b^{IT2} are the same. The figure shows the results of s_b^{IT2} have some correlation with s_j^{IT2}. However, there are many cases in which $s_j^{IT2} > 0$ but $s_b^{IT2} = [0, 0]$. This occurs when there is a value x_1 where $\overline{\mu}_A(x_1) = 1$ and $\overline{\mu}_B(x_1) = 0$, and a value x_2 where $\underline{\mu}_A(x_2) = 1$ and $\underline{\mu}_B(x_2) = 0$, for which $\tilde{A}$ and $\tilde{B}$ are interchangeable and x_1 and x_2 may be be equal. In such cases, the similarity according to s_b^{IT2} is $[0, 0]$ even if the intersection of the fuzzy sets is not empty. Zero similarity of non-disjoint fuzzy sets was also observed in [40]. This can be seen in the results in Fig. 8. Some adjacently ranked words (e,.g. words 11 and 12) have zero similarity according to s_b^{IT2}. This can be seen as a strict form of similarity where the value with the highest membership in $\tilde{A}$ must have a non-zero membership in $\tilde{B}$ to be considered similar. Otherwise, even if the fuzzy sets overlap, their similarity is considered to be 0.

Another characteristic of s_b^{IT2} (not shown in Fig. 7d) is that the upper bound of the result is greater than 0 if the fuzzy sets are disjoint but the height

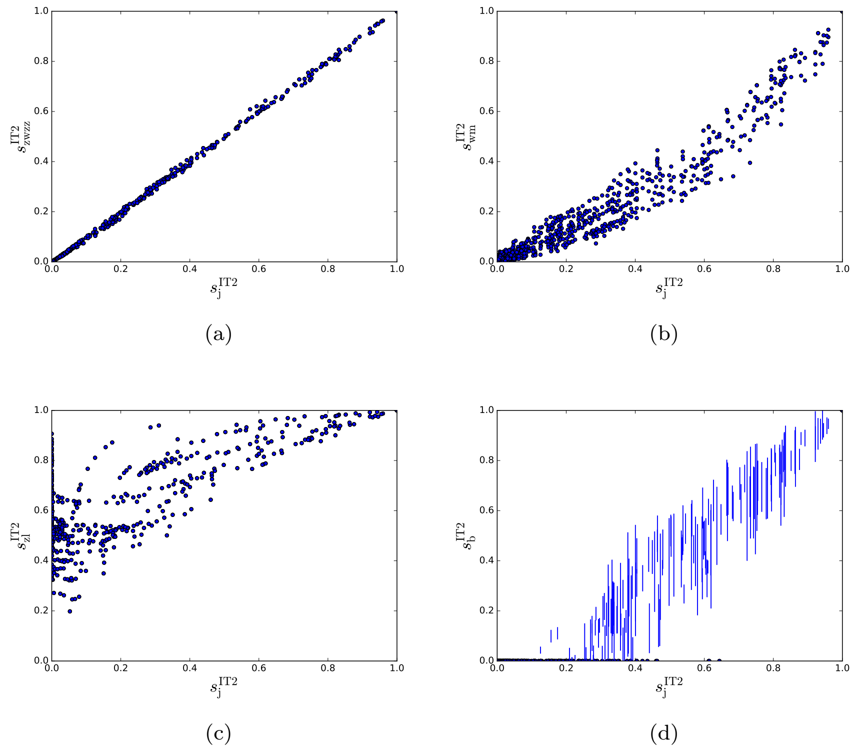


Figure 7: Results of s_j^{IT2} compared against other interval type-2 similarity measures.

of the lower or upper membership function of at least one of the sets is less than 1. This is demonstrated in [39]. The fuzzy sets used to generate Fig. 7 do not have this characteristic and so this result is not shown. Therefore, s_b^{IT2} is best restricted to data where the heights of the lower and upper membership functions are both always 1.

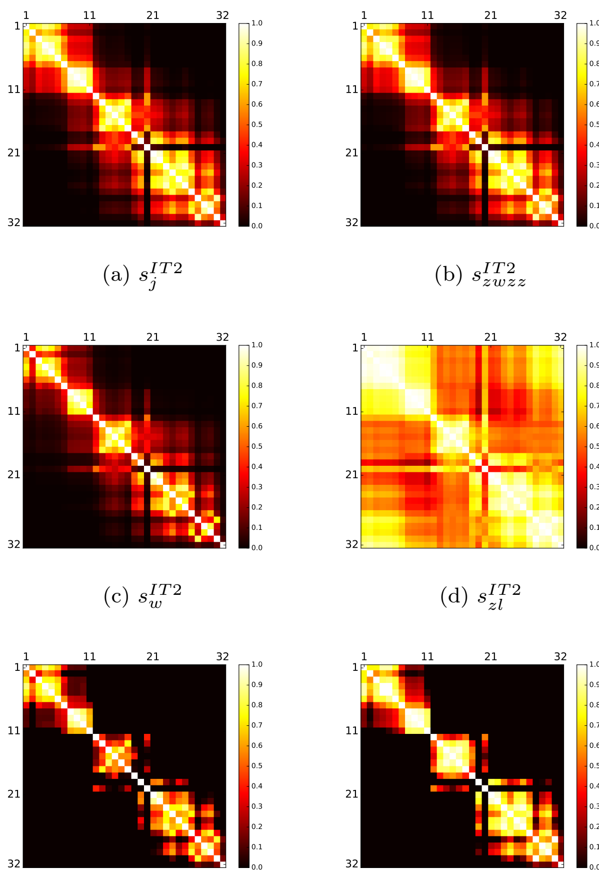
Note that s_b^{IT2} finds the same clusters of similar words in Fig. 8. It may therefore still be useful in the context of CW.

This section has presented an analysis of similarity measures for interval type-2 fuzzy sets; Table 2 provides an overview of the properties of the measures considered. s_j^{IT2} and s_{zwzz}^{IT2} are property complete and therefore the most reliable measures. Their comparison in Fig. 7a shows a very high correlation, making both measures an equally good choice. The measure s_b^{IT2} shows good interval-valued results but cannot determine if two non-normal fuzzy sets are disjoint. Likewise for s_w^{IT2}, which gives a single-valued result. Finally, s_{zl}^{IT2} is not property-complete and is likely to be unsuitable because it gives a *high* result for disjoint and barely overlapping fuzzy sets.

The next section presents an overview of measures for general type-2 fuzzy

Method	Reflex.	Symm.	Trans.	Overlap.	Min.–sim.
s_j^{IT2} [32]	✓	✓	✓	✓	✓
s_{zwzz}^{IT2} [49]	✓	✓	✓	✓	✓
s_w^{IT2} [39]	✓	✓	✓		
s_{zl}^{IT2} [46]	✓	✓			
s_b^{IT2} [1]		✓	✓		✓

Table 2: Properties of the interval type-2 similarity measures detailed in Section 2.



(a) s_j^{IT2}

(b) s_{zwzz}^{IT2}

(c) s_w^{IT2}

(d) s_{zl}^{IT2}

(e) s_b^{IT2} (lower)

(f) s_b^{IT2} (upper)

Figure 8: Results of interval type-2 similarity measures comparing each word (listed in Table 1) against each other word.

sets.

5. Similarity of General Type-2 Fuzzy Sets

In this section, we analyse each general type-2 similarity measure on data-driven fuzzy sets that describe words. We provide pairwise comparisons (see Fig. 9) as well as an individual analysis of each measure (see Fig. 12). We use the property-complete Jaccard based approach s_{mw}^{GT2} as a base for all pairwise comparisons. Note that using s_{zh}^{GT2}, s_{hm}^{GT2} or s_{wm}^{GT2} as base measure instead of s_{mw}^{GT2} would be equally useful as these three are almost equivalent as shown next. Each sub figure in Fig. 12 is symmetrical, demonstrating that each similarity measure is symmetrical.

Figures 9a, 9b and 9c show s_{hm}^{GT2}, s_{zxld}^{GT2} and s_{wm}^{GT2}, respectively, compared against s_{mw}^{GT2}. These measures are not equivalent to each other (that is, they do not give the same rank order of similarity for the sets). However, the differences between the methods are small such that they have a very high correlation ($r > 0.9$) and are almost equivalent. Such closeness between the results of these measures has been demonstrated in previous literature [23, 41]. In addition, these three measures along with s_{mw}^{GT2} and s_{m}^{GT2} show almost equal results in Fig. 12. While some differences are present (for example at words 28-31 compared with each other) the differences are small. Each method clusters the same groups of similar words and each method highlights that word 20 is too high in the rank order.

The measure s_{yl}^{GT2} gives (perhaps unexpectedly) low results. Fig. 9d shows the results of s_{yl}^{GT2} compared against s_{mw}^{GT2}. Except for identical and near-identical sets, s_{yl}^{GT2} only gives comparatively low results of similarity. This is because s_{yl}^{GT2} finds zero similarity if the footprints of uncertainty of the fuzzy sets do not overlap, even if they are close. An extreme example of such fuzzy sets (where one is a complete subset of the other) is given in Fig. 10. Although the fuzzy sets share many of the same x values, their footprints of uncertainty do not overlap. Note that this extreme example is not found in our test fuzzy sets but many pairs are affected by this characteristic of the measure. Generally, measures of similarity between type-1 fuzzy sets measure the proximity of membership functions or their intersection. It is only if fuzzy sets do not intersect that they have no similarity. Likewise, for type-2 fuzzy sets like those in Fig. 10, even though the footprints of uncertainty do not overlap, the fuzzy sets do intersect so their similarity should be greater than zero. Based on this, we propose similarity should consider intersection between fuzzy sets.

In Fig. 12, s_{yl}^{GT2} has found some of the clusters of words that the previous three measures found, however, the values of similarity are not as high and so the clusters are not as clear as with other methods. Therefore, in the context of classification and clustering s_{yl}^{GT2} is unlikely to be useful. However, the rank order of results from s_{yl}^{GT2} does not differ much from s_{mw}^{GT2} (shown in Fig. 9d). Therefore, it may still be useful where rank order of similarity is more important than the value itself (for example, in document retrieval where similar documents are listed in order of relevance).

Fig. 9e compares s_{hy}^{GT2} with s_{mw}^{GT2}. The results of s_{hy}^{GT2} are (unexpectedly) high because the distance between disjoint sets at any vertical slice is calculated from the membership degrees of the non-empty set. For example, consider the fuzzy sets in Fig. 11 and their vertical slices at $x = 0.3$; one fuzzy set is empty at this vertical slice and the other is not. The distance between these vertical slices according to eq. (32d) is 0.678 and therefore their similarity (the complement of their distance) is 0.322. However, intuitively, their similarity should be zero because there is no overlap between the vertical slices. Unless the non-empty set has a crisp membership of 1 at $u = 1$, the distance is always less than 1 and therefore the similarity is always greater than zero. This is like s_{zl}^{IT2} eq. (23), with which similarity is also higher than expected because the measure of proximity between vertical slices is absolute instead of relative to the primary membership values. In Fig. 12, s_{hy}^{GT2} gives high values than the previous four measures. The clusters of similar words are present and larger. However, it also clusters words that are disjoint. For example, words 1-7 are very similar to words 31-32 according to s_{hy}^{GT2}. The measure's inability to detect when fuzzy set are disjoint makes it an unsuitable similarity measure.

Fig. 9f compares s_{m}^{GT2} with s_{mw}^{GT2}. As pointed out in [39], Mitchell's method does not necessarily give 1 where $\tilde{A}$ and $\tilde{B}$ are identical because the embedded type-1 fuzzy sets are randomly generated and so will not necessarily be the same. Additionally, because each embedded set is randomly chosen, the result of (34) is non-deterministic and will not always be the same when comparing two fuzzy sets more than once because the selected embedded fuzzy sets will be different each time. However, although identical sets do not get the result of 1, they do always result in a high value (generally above 0.8) and non-identical sets generally do not result in such a high value. Fig. 9f shows the results of s_{m}^{GT2} have a high correlation ($r > 0.9$) with s_{mw}^{GT2}. Fig. 12 also shows that s_{m}^{GT2} finds the same clusters of similar words as the property-complete methods.

This section has presented an analysis of similarity measures for general type-2 fuzzy sets. Table 3 provides an overview of the properties of these measures. The functions s_{zxld}^{GT2}, s_{hm}^{GT2}, s_{mw}^{GT2} and s_{wm}^{GT2} are all property complete and recommended as useful measures. The functions s_{yl}^{GT2} and s_{hy}^{GT2} do not have all of the properties and may be unsuitable. Although s_{yl}^{GT2} found some clusters of similar words, the boundaries of these clusters are less clear than with other measures. The measure s_{hy}^{GT2} correctly found clusters of similar words but also incorrectly clusters words that should have no similarity, and therefore it is not a suitable measure. Finally, s_{m}^{GT2} does not have all of the properties, but it does have a high correlation with s_{mw}^{GT2} and finds the same cluster of similar words as property-complete measures. Therefore, we suggest it is a useful measure when detecting identical fuzzy sets is not crucial.

The next section provides a discussion and conclusions on which measures we suggest are best, and what methods should be considered or avoided in new measures on type-2 fuzzy sets.

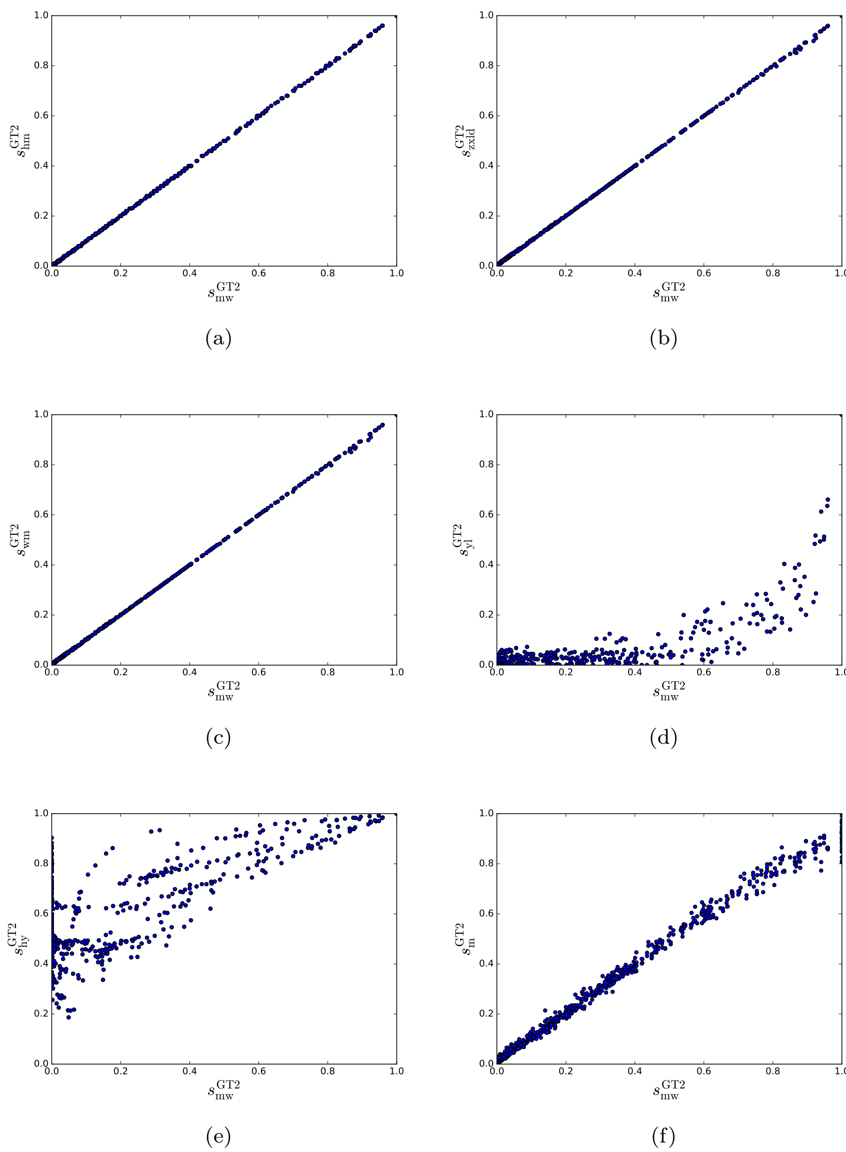
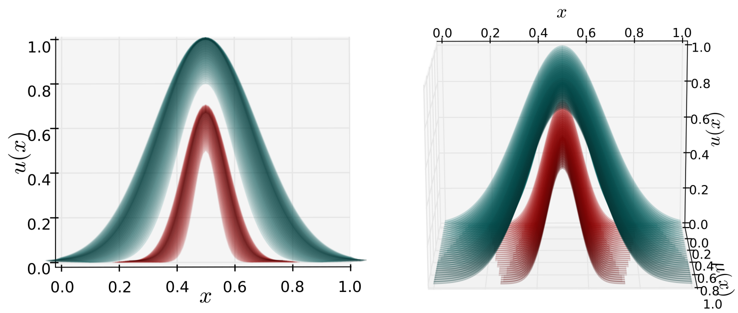
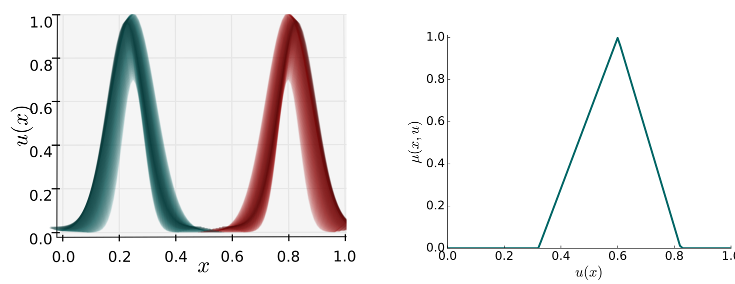


Figure 9: Results of s_{mw}^{GT2} compared against other general type-2 similarity measures.



(a) (b)

Figure 10: Non-overlapping footprints of uncertainty in general type-2 fuzzy sets show from two viewpoints.

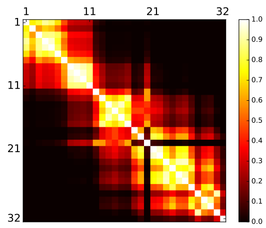


(a) (b)

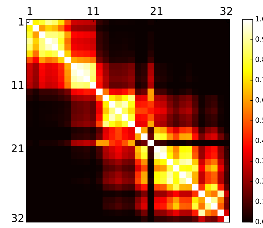
Figure 11: Disjoint general type-2 fuzzy sets (a) and their vertical slices at $x = 0.3$ (b). Note that the right-hand (red) fuzzy set is an empty set at this vertical slice.

Method	Reflex.	Symm.	Trans.	Overlap.	Min.–sim.
s_{zxld}^{GT2} [48]	✓	✓	✓	✓	✓
s_{hm}^{GT2} [13]	✓	✓	✓	✓	✓
s_{mw}^{GT2} [23]	✓	✓	✓	✓	✓
s_{yl}^{GT2} [20]	✓	✓		✓	✓
s_{hy}^{GT2} [37]	✓	✓			
s_{m}^{GT2} [30]				✓	✓

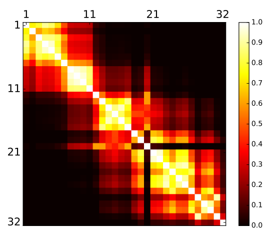
Table 3: Properties of the general type-2 similarity measures detailed in Section 2.



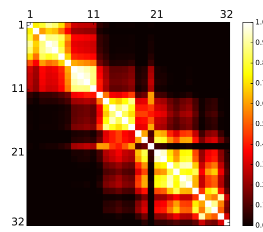
(a) s_{hm}^{GT2}



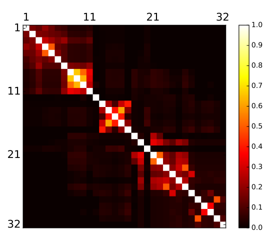
(b) s_{zxld}^{GT2}



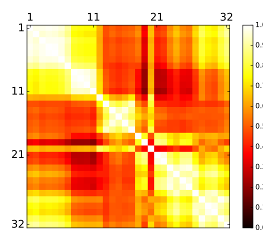
(c) s_{wm}^{GT2}



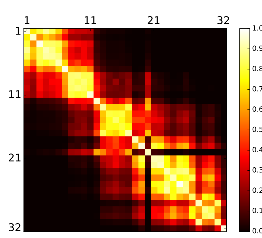
(d) s_{mw}^{GT2}



(e) s_{yl}^{GT2}



(f) s_{hy}^{GT2}



(g) s_{m}^{GT2}

Figure 12: Results of general type-2 similarity measures comparing each word (listed in Table 1) against each other word.

6. Limitations, Discussion and Conclusions

Similarity measures are vital in many applications to compare fuzzy sets. In recent years, several measures have been developed to compare type-2 fuzzy sets, but it is not always clear if and how the properties of these measures differ and, therefore, which of the existing measures is the best choice. We evaluate all of the known similarity measures developed for type-2 fuzzy sets to aid in the decision of which measure to choose. We highlight which measures are equivalent or near-equivalent and why/how the properties of key measures differ, which can lead to unexpected results.

We conducted our analysis on type-2 fuzzy sets that are constructed using the HM approach [14], with which the lower and upper bounds of the footprint of uncertainty of the fuzzy sets have a height of 1. However, differences in results may be found if using a different method, for example the Enhanced Interval Approach [42], which results in non-normal membership functions. In addition, we note that the measures discussed in this paper are restricted to continuous fuzzy sets, but there may be cases in which discrete fuzzy sets with non-convex secondary membership functions require an appropriate similarity measure [31]. Therefore, in future work, we will explore the practice of applying similarity measures to fuzzy sets where data restrict the fuzzy sets to discrete forms. We also note that this paper only includes what has been published up to now and, of course, new measures constantly arise in future work.

Among existing measures in the literature, several are near-equivalent - sharing a high correlation in practice and giving only slightly different rank orders of results – when ranking is used to summarise levels of similarity of fuzzy sets. To compare the similarity between interval type-2 fuzzy sets, s_j^{IT2} eq. (19) and s_{zwzz}^{IT2} eq. (20) are the most reliable; that is, they are the best choice when the preferred properties of a similarity measure are not known. For an interval-valued result, s_b^{IT2} eq. (24) gives results that correlate well with the aforementioned numeric measures s_j^{IT2} and s_{zwzz}^{IT2}, while also providing a measure of uncertainty in respect to the similarity – through the interval-valued output.

To compare (general) type-2 fuzzy sets, s_{zxld}^{GT2} eq. (25), s_{hm}^{GT2} eq. (26), s_{mw}^{GT2} eq. (27) and s_{wm}^{GT2} eq. (28) are all property-complete and are near-equivalent; each is a useful measure on type-2 fuzzy sets. The function s_m^{GT2} eq. (34) also provides good results, but is non-deterministic and does not identify identical fuzzy sets. Other measures that are not property-complete (s_{zl}^{IT2} eq. (23), s_{yl}^{GT2} eq. (31) and s_{hy}^{GT2} eq. (32)) will be unsuitable for most applications because they give alike results for highly overlapping *and* disjoint pairs of fuzzy sets.

In addition to evaluating the state of current measures in the literature, we highlight through empirical tests why some methods may give unexpected results and advise on how this can be avoided in new methods of similarity.

Firstly, we show that measures underestimate similarity if they compare how much the footprints of uncertainty of type-2 fuzzy sets overlap instead of measuring the proximity of their membership functions (see the analysis of eq. (31) in Section 5). For example, if we have two fuzzy sets where one is a subset of the other but their footprints of uncertainty do not overlap (see Fig. 10), methods that only measure overlap of footprints of uncertainty will falsely state the sets are entirely dissimilar even if their primary membership degrees are *close* to each other. A similarity measure should evaluate the fuzzy sets based on their intersection or relative proximity instead of whether their footprints of uncertainty overlap.

Secondly, we show that methods overestimate similarity if they do not weight the proximity of fuzzy set membership degrees according to the degree of membership (that is, if they calculate absolute difference instead of relative difference) (see the analysis of eq. (23) in Section 4). For example, if for a given value on the domain x_1, two fuzzy sets both have a low degree of membership, their similarity at x_1 is high. Likewise, if at x_2 both sets have a high degree of membership, their similarity is also high at x_2. However, giving the same degree of importance and similarity at both x_1 and x_2 leads to an overall result that overestimates the similarity of the sets (see section 4). To avoid this, the similarity at x_1 should be given a smaller weight than at x_2 when aggregating results. If the above two points are taken into consideration, a new property-complete similarity measure on type-2 fuzzy sets will be easier to develop.

We demonstrate measures in the context of CW, where we group words with similar meanings. We highlight which measures successfully group similar words and highlight which methods are unsuitable. Through these results, we show that measures do not need to be property-complete to be useful; several methods that are *missing* a property were able to find the same groups of similar words as methods that have all of the properties. However, this was not the case for all non-property-complete measures. Therefore, when the desired properties of a similarity measure are not known, we recommend using one of the property complete measures. If the desired properties are known then we recommend choosing a measure that best fits the application.

Acknowledgements

This work was partially funded by the EPSRCs Leveraging the Multi-Stakeholder Nature of Cyber Security (EP/P011918/1), by EPSRCs DTP - University of Nottingham grant, EP/M50810X/1, and RCUKs Prototyping Open Innovation Models for ICT-Enabled Manufacturing in Food and Packaging (EP/K014234/2) grants.

References

[1] H. Bustince. Indicator of inclusion grade for interval-valued fuzzy sets. Application to approximate reasoning based on interval-valued fuzzy sets. *International Journal of Approximate Reasoning*, 23(3):137–209, 2000.

[2] H. Bustince, E. Barrenechea, and M. Pagola. Restricted equivalence functions. *Fuzzy Sets and Systems*, 157(17):2333 – 2346, 2006.

[3] H. Bustince, E. Barrenechea, and M. Pagola. Image thresholding using restricted equivalence functions and maximizing the measures of similarity. *Fuzzy Sets and Systems*, 158(5):496–516, 2007.

[4] O. Castillo, L. Amador-Angulo, J. R. Castro, and M. Garcia-Valdez. A comparative study of type-1 fuzzy logic systems, interval type-2 fuzzy logic systems and generalized type-2 fuzzy logic systems in control problems. *Information Sciences*, 354:257–274, 2016.

[5] C. Chang, D. Wang, Y. Liu, and B. Qi. Fuzzy similarity measure based case retrieval method. In *2006 International Conference on Machine Learning and Cybernetics*, pages 2030–2034, Aug 2006.

[6] A. H. Cheetham and J. E. Hazel. Binary (presence-absence) similarity coefficients. *Journal of Paleontology*, pages 1130–1136, 1969.

[7] J. C. Fodor and M. Roubens. *Fuzzy preference modelling and multicriteria decision support*, volume 14. Springer Science & Business Media, 1994.

[8] H. Garg and K. Kumar. An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making. *Soft Computing*, pages 4959–4970, 2018.

[9] J. A. Goguen. The logic of inexact concepts. *Synthese*, 19(3):325–373, 1969.

[10] M. B. Gorzałczany. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems*, 21(1):1–17, 1987.

[11] S. Greenfield, R. John, and S. Coupland. A novel sampling method for type-2 defuzzification. In *Proc. UKCI 2005*, pages 120–127, 2005.

[12] H. Hamrawi and S. Coupland. Measures of uncertainty for type-2 fuzzy sets. In *Computational Intelligence (UKCI), 2010 UK Workshop on*, pages 1–6, Sept 2010.

[13] M. Hao and J. M. Mendel. Similarity measures for general type-2 fuzzy sets based on the α-plane representation. *Information Sciences*, 277(0):197–215, 2014.

[14] M. Hao and J. M. Mendel. Encoding Words Into Normal Interval Type-2 Fuzzy Sets: HM Approach. *IEEE Transactions on Fuzzy Systems*, 24(4):865–879, Aug 2016.

[15] P. Jaccard. The distribution of the flora in the alpine zone. 1. *New phytologist*, 11(2):37–50, 1912.

[16] S. Kabir, C. Wagner, T. C. Havens, and D. T. Anderson. A bidirectional subsethood based similarity measure for fuzzy sets. In *2018 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 1–7. IEEE, 2018.

[17] N. N. Karnik, Q. Liang, F. Liu, D. Wu, J. Joo, and J. M. Mendel. Type-2 Fuzzy Logic Software.

[18] M.-J. Lesot, M. Rifqi, and H. Benhadda. Similarity measures for binary and numerical data: a survey. *International Journal of Knowledge Engineering and Soft Data Paradigms*, 1(1):63–84, 2008.

[19] Y. Li, K. Qin, X. He, and D. Meng. Similarity measures of interval-valued fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 28(5):2113–2125, 2015.

[20] D.-C. Lin and M.-S. Yang. A Similarity Measure between Type-2 Fuzzy Sets with Its Application to Clustering. In *Fuzzy Systems and Knowledge Discovery, 2007. FSKD 2007. Fourth International Conference on*, volume 1, pages 726–731, 2007.

[21] O. López-Ortega and F. Castro-Espinoza. Fuzzy similarity metrics and their application to consensus reaching in group decision making. *Journal of Intelligent & Fuzzy Systems*, 36(4):3095–3104, 2019.

[22] A. D. Luca and S. Termini. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. *Information and Control*, 20(4):301–312, 1972.

[23] J. McCulloch and C. Wagner. Measuring the similarity between zSlices general type-2 fuzzy sets with non-normal secondary membership functions. In *2016 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 461–468. IEEE, 2016.

[24] J. McCulloch, C. Wagner, and U. Aickelin. Extending similarity measures of interval type-2 fuzzy sets to general type-2 fuzzy sets. In *Fuzzy Systems (FUZZ), 2013 IEEE International Conference on*, pages 1–8, 2013.

[25] J. Mendel and R. John. Type-2 fuzzy sets made simple. *Fuzzy Systems, IEEE Transactions on*, 10(2):117–127, Apr. 2002.

[26] J. Mendel and F. Liu. On new quasi-type-2 fuzzy logic systems. In *Fuzzy Systems, 2008. FUZZ-IEEE 2008. (IEEE World Congress on Computational Intelligence). IEEE International Conference on*, pages 354–360, June 2008.

[27] J. Mendel, F. Liu, and D. Zhai. α-Plane Representation for Type-2 Fuzzy Sets: Theory and Applications. *Fuzzy Systems, IEEE Transactions on*, 17(5):1189–1207, Oct. 2009.

[28] J. M. Mendel and M. R. Rajati. Advanced computing with words: status and challenges. In *Towards the Future of Fuzzy Logic*, pages 217–245. Springer, 2015.

[29] J. M. Mendel, M. R. Rajati, and P. Sussner. On clarifying some definitions and notations used for type-2 fuzzy sets as well as some recommended changes. *Information Sciences*, 340–341:337–345, 2016.

[30] H. Mitchell. Pattern recognition using type-II fuzzy sets. *Information Sciences*, 170(2–4):409–418, 2005.

[31] S.-C. Ngan. A concrete and rational approach for building type-2 fuzzy subsethood and similarity measures via a generalized foundational model. *Expert Systems with Applications*, 130:236–264, 2019.

[32] H. T. Nguyen and V. Kreinovich. Computing Degrees of Subsethood and Similarity for Interval-Valued Fuzzy Sets: Fast Algorithms. *Proceedings of the 9th International Conference on Intelligent Technologies InTec'08*, pages 47–55, 2008.

[33] J.-F. Omhover, M. Detyniecki, M. Rifqi, and B. Bouchon-Meunier. Ranking invariance between fuzzy similarity measures applied to image retrieval. In *FUZZ-IEEE*, pages 1367–1372, 2004.

[34] C. P. Pappis and N. I. Karacapilidis. A comparative assessment of measures of similarity of fuzzy values. *Fuzzy Sets and Systems*, 56(2):171–174, 1993.

[35] C. Wagner and H. Hagras. Toward General Type-2 Fuzzy Logic Systems Based on zSlices. *Fuzzy Systems, IEEE Transactions on*, 18(4):637–660, Aug. 2010.

[36] C. Wagner, S. Miller, J. Garibaldi, D. Anderson, and T. Havens. From Interval-Valued Data to General Type-2 Fuzzy Sets. *Fuzzy Systems, IEEE Transactions on*, 23(2):248–269, Apr. 2015.

[37] H. Wen-Liang and M. Yang. Similarity measures between type-2 fuzzy sets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 12(06):827–841, 2004.

[38] D. Wu and J. M. Mendel. A vector similarity measure for type-1 fuzzy sets. In *Foundations of Fuzzy Logic and Soft Computing*, pages 575–583. Springer, 2007.

[39] D. Wu and J. M. Mendel. A vector similarity measure for linguistic approximation: Interval type-2 and type-1 fuzzy sets. *Information Sciences*, 178(2):381–402, 2008.

[40] D. Wu and J. M. Mendel. A comparative study of ranking methods, similarity measures and uncertainty measures for interval type-2 fuzzy sets. *Information Sciences*, 179(8):1169–1192, 2009.

[41] D. Wu and J. M. Mendel. Similarity Measures for Closed General Type-2 Fuzzy Sets: Overview, Comparisons, and a Geometric Approach. *IEEE Transactions on Fuzzy Systems*, 2018.

[42] D. Wu, J. M. Mendel, and S. Coupland. Enhanced interval approach for encoding words into interval type-2 fuzzy sets and its convergence analysis. *Fuzzy Systems, IEEE Transactions on*, 20(3):499–513, 2012.

[43] L. Xuecheng. Entropy, distance measure and similarity measure of fuzzy sets and their relations. *Fuzzy Sets and Systems*, 52(3):305–318, 1992.

[44] M.-S. Yang and D.-C. Lin. On similarity and inclusion measures between type-2 fuzzy sets with an application to clustering. *Computers & Mathematics with Applications*, 57(6):896–907, 2009.

[45] V. R. Young. Fuzzy subsethood. *Fuzzy Sets and Systems*, 77(3):371–384, 1996.

[46] W. Zeng and H. Li. Relationship between similarity measure and entropy of interval valued fuzzy sets. *Fuzzy Sets and Systems*, 157(11):1477–1484, 2006.

[47] W. Zeng and Y. Shi. Note on Interval-Valued Fuzzy Set. In L. Wang and Y. Jin, editors, *Fuzzy Systems and Knowledge Discovery*, volume 3613 of *Lecture Notes in Computer Science*, pages 20–25. Springer Berlin Heidelberg, 2005.

[48] T. Zhao, J. Xiao, Y. Li, and X. Deng. A new approach to similarity and inclusion measures between general type-2 fuzzy sets. *Soft Computing*, 18(4):809–823, 2014.

[49] G. Zheng, J. Wang, W. Zhou, and Y. Zhang. A similarity measure between interval type-2 fuzzy sets. In *Mechatronics and Automation (ICMA), 2010 International Conference on*, pages 191–195, Aug. 2010.

[50] R. Zwick, E. Carlstein, and D. V. Budescu. Measures of similarity among fuzzy concepts: A comparative analysis. *International Journal of Approximate Reasoning*, 1(2):221–242, 1987.