On Comparing and Selecting Approaches to Model Interval-Valued Data as Fuzzy Sets

Josie McCulloch (Member, IEEE), Zack Ellerby, Christian Wagner (Senior Member, IEEE)
LUCID Research Group, University of Nottingham UK
josie.mcculloch; zack.ellerby; christian.wagner@nottingham.ac.uk

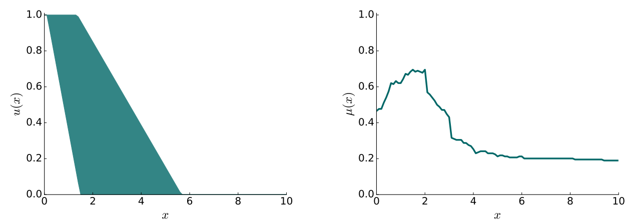
Abstract*—The capture of interval-valued data is becoming an increasingly common approach in data collection (from survey based research to the collation of sensor data) as an efficient method of obtaining information about uncertainty associated with the data in question. To best utilise this data, several methods of aggregating intervals into fuzzy sets have been proposed in the fuzzy set literature, particularly within the field of Computing with Words. Two key examples are the Interval Approach and the Interval Agreement Approach and their respective extensions. Each method takes a fundamentally different approach to constructing fuzzy sets, making different assumptions in respect to the nature and the reliability of the data. The result is noticeably different fuzzy sets that do not share the same statistical properties (such as central-tendency and standard deviation). This begs the question of how these techniques differ in respect to the relationship between the original interval-valued data and the fuzzy sets produced – and thus when and why each of the methods is the most appropriate. This paper compares the results of both methods of constructing fuzzy sets from interval-valued data. Statistical moments of the fuzzy sets are compared against the interval-valued data to evaluate how well key properties of the fuzzy sets match those of the data; for example, does the standard deviation of the fuzzy set represent the standard deviation of the raw interval-valued data? We use comparisons on real-world data to demonstrate how the methods differ and which is more appropriate given the assumptions of the data.

I. Introduction

Intervals are a useful way of capturing uncertainty in data. For example, intervals may be used to represent the error of a measurement device, where the exact correct measurement is not known but it is known to reside within the interval [1]. Another potentially valuable use of interval data is in surveys [2], [3], where the interval represents a range of values that the survey participant believes are possibly correct.

Collecting interval-valued responses in surveys has frequently been applied in the context of computing with words [3]–[6] to capture uncertainty about the meaning of words. In these examples, the interval-valued survey responses are aggregated into fuzzy sets that capture agreement or approximate the consensus of the respondents. Various methods have been proposed, the most frequently used of which are based on the Interval Approach (IA) [3] (and its extensions [4], [6]) and the Interval Agreement Approach (IAA) [7]–[9] (and its extension [10]). Using the IA, intervals are aggregated into normal, convex interval type-2 fuzzy sets that represent key areas of agreement from respondents. Whereas, using the IAA,

This work was funded by the EPSRC grant EP/P011918/1.



(a) (b)

Fig. 1. Fuzzy sets representing the word *little* based on interval-valued survey responses using the (a) IA and (b) IAA.

intervals are aggregated into type-1 fuzzy sets that may be non-normal on non-convex and represent the full range of agreement from respondents. We note that the generation of type-2 fuzzy sets using the IAA is also possible, but only arises when more than one type of uncertainty is modelled, e.g. both inter-participant (between subjects) *and* intra-participant (within subject) uncertainty.

To provide a basic illustration of the difference between the IA and IAA, Fig. 1 shows fuzzy sets constructed using both approaches using interval-valued data taken from [11] describing the word *little* on the scale $[0, 10]$. The IA uses each interval response to build an individual type-1 membership function that is interpreted as an embedded membership function of an interval type-2 fuzzy set. The IAA, however, aggregates all interval-valued data into a single type-1 membership function. This difference arises due to the distinct intended applications of the methods. The IA is designed to model the meaning of words. It therefore removes outliers that would lead to words with too broad of a meaning, and models only strong areas of agreement in responses. By contrast, the IAA is designed to model all agreement/disagreement in the data. It takes the view that outliers may provide valuable information and that, without external evidence, there is no justification for disregarding so-called ‘outliers’.

The IA and IAA-based methods take fundamentally different approaches in respect to representing interval-valued data through fuzzy sets and, therefore, it is important to consider how they compare to the data from which they are derived. One important difference between these approaches, as outlined above, is that the IA involves data pre-processing but the IAA does not. Thus, it is valuable to understand how the IA and IAA fuzzy sets differ as a result of them representing effectively different data (processed and non-processed). In this paper, we analyse fuzzy sets based on interval-valued data and contrast them with the original interval-valued data through comparing key statistical features. The results of the analysis enable us to determine how the fuzzy sets compare against the data they are constructed from, as well as how the methods and respective assumptions shape the fuzzy sets in respect to the original data. This makes the decision of which method is best for a given application an easier choice to make.

Section II provides background on methods of constructing fuzzy sets from interval-valued data taken from [11]. Then, Section III explains how we compare the fuzzy sets against the data, followed by results in Section IV. Finally, Section V presents conclusions.

II. Background

In this section, we cover two common methods of representing interval-valued data using fuzzy sets. The first type is IA-based, which includes the IA [3] and subsequent enhancements made on this [4], [6]. The second type is IAA-based, which includes the IAA [9] and the related efficient-IAA [10].

The IA [3] maps interval-valued data to an interval type-2 fuzzy set with the goal of modelling linguistic variables. First, the IA performs data pre-processing to remove unsuitable data, then the remaining data is constructed into an interval type-2 fuzzy set. Four stages of data-processing are carried out; these are: (1) bad data pre-processing, (2) outlier pre-processing, (3) tolerance limit pre-processing, and (4) reasonable-interval pre-processing. Creating a fuzzy set from the remaining data consists of nine steps, in which each remaining interval is converted into a type-1 membership function. Inadmissible membership functions are removed, and the remaining are aggregated into an interval type-2 fuzzy set. Details of the data-processing and fuzzy set production can be found in [3].

The main limitations of the IA, as discussed in [4], are that the resulting fuzzy sets are wide and the heights of the lower membership function are low. Therefore, the fuzzy sets may be too imprecise to be useful. To resolve this, the Enhanced Interval Approach [4] was developed. The EIA builds on the IA by refining the data pre-processing to have better control of the width of the remaining intervals. The construction of the lower membership function is also altered so that its height is not too low to be useful.

Hao and Mendel [6] point out that the EIA may not assign maximum membership to values that all survey participants agree belong in the set, resulting in an inaccurate representation of the data. To resolve this, they developed the HM Approach (HMA). In the HMA, the data pre-processing is the same as for the EIA, but the method of constructing the fuzzy set is improved by finding the most common overlap of intervals and constructing the fuzzy set around this point.

To visualise the above three methods, Fig. 2 shows fuzzy sets constructed using the IA, EIA and HMA using interval-valued data that represent the word *little* on the scale $[0, 10]$.

The data pre-processing employed by the IA, EIA and HMA, while common, may pose a limitation. For example, starting from 175 interval-valued data collected in a survey, the remaining data points after pre-processing is between 3% and 70% of the original data [6]; that is, in some cases up to 97% of the intervals are removed as outliers. Pre-processing is used because the method is intended to capture the meaning of words and intervals that disagree with the majority are considered not useful. In other contexts, however, it may not be safe to assume that outlying data are incorrect; they may be outliers because their proponents have more expertise or access to different information/experience, meaning their answer could be just as or even more accurate. In addition, removing such a large amount of data may not be acceptable when only a small amount of raw data is available.

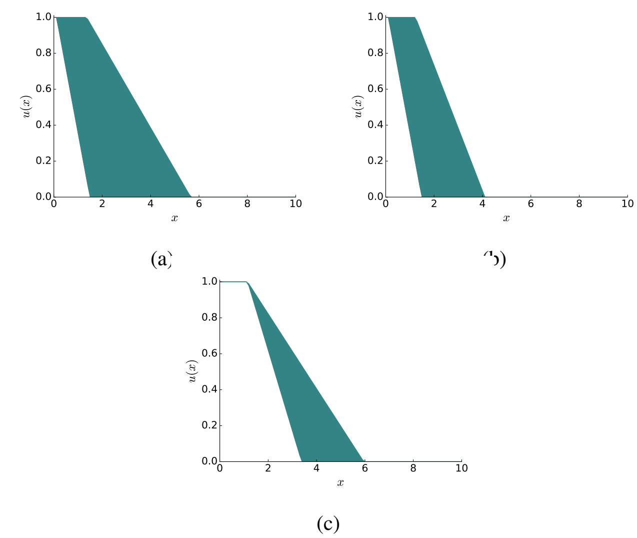
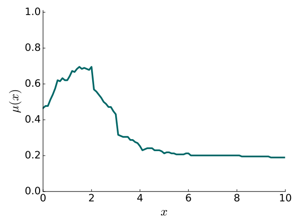


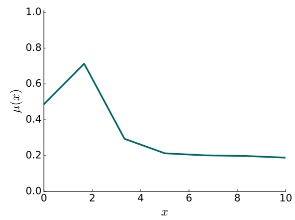
Fig. 2. Fuzzy sets representing the word *little* based on interval-valued survey responses using the (a) IA, (b) EIA and (c) HMA.

As an alternative to the IA-based methods, Wagner et al. [7]–[9] developed the Interval Agreement Approach (IAA). The IAA models *basic* interval-valued data through a type-1 fuzzy set instead of an interval type-2 fuzzy set. We refer to interval-valued data as *basic* when they represent only either intra- or inter-source uncertainty, but not both. For example, a single survey completed once by each participant is a basic interval-valued data set (capturing inter-participant uncertainty). However, if the same survey was conducted multiple times with the same participants, the data would capture both inter- and intra-participant uncertainty – which would result in type-2 fuzzy sets when modelled with the IAA. Other key differences compared with the IA-based methods are that it does not assume the data has a uni-modal distribution and it does not employ pre-processing/remove outliers.

In the IAA, the membership assigned to a value x within a fuzzy set is the percentage of intervals that contain x. Therefore, while outliers are present, their membership value is typically low. IAA fuzzy sets do not have a normal membership function unless there is a value x that is within all intervals in the data. The method also ensures, unlike IA methods, that a multi-modal fuzzy set is created if the data is multi-modal. A key benefit of the IAA is that it is clear how much agreement exists in the data by visually observing



(a)



(b)

Fig. 3. Fuzzy sets representing the word *little* based on interval-valued survey responses using the (a) IAA and (b) eIAA using seven linguistic basis functions.

the membership function. For example, Fig. 3(a) shows an IAA model of the word *little* using the same interval-valued data as used in Fig. 2. The fuzzy set shows that no more than 65% of participants agreed that the value 2 on the scale $[0, 10]$ is *little*.

A potential limitation of the IAA, as pointed out by Havens et al. [10], is that the resulting membership functions can appear unorthodox because the fuzzy sets are defined by the endpoints of intervals, resulting in spikes in membership. This is particularly evident with small data sets (see figures in [2], for example). Havens et al. argued that the IAA is a computationally inefficient representation and proposed an improved model named the efficient IAA (eIAA). This creates a simplified membership function by representing it as the weighted sum of a set of basis functions.

Fig. 3 shows fuzzy sets constructed using the IAA and eIAA using interval-valued data that represent the word *little* on the scale $[0, 10]$. The general shape of the membership function produced by the eIAA is the same as that by the IAA.

Other methods have also been proposed in the literature, which capture an additional value within the interval that represents the most certain point [12], [13]. However, collecting a third data-point is less common as it leads to a more complex survey for participants. Therefore, we do not cover these methods in this paper.

We construct fuzzy sets from interval-valued data that has frequently been used to demonstrate the IA-based approaches [3], [4], [6] and is available online [11]. In this data, a total of 174 participants gave end-points of an interval related to the meaning of 32 different words on the scale $[0, 10]$. Each resulting interval captures an interpretation of a given word by one participant. Table I lists these words; for consistency, they are presented in the same order as given in [3], [4], [6] where further details on the method of data collection can be found. In Table I, each word is assigned an ID number for ease of reference within this paper.

TABLE I
WORDS TO WHICH SURVEY PARTICIPANTS ASSIGNED INTERVALS.

ID	word	ID	word
1	none to very little	17	modest amount
2	teeny-weeny	18	good amount
3	a smidgen	19	sizeable
4	tiny	20	quite a bit
5	very small	21	considerable amount
6	very little	22	substantial amount
7	a bit	23	a lot
8	little	24	high amount
9	low amount	25	very sizeable
10	small	26	large
11	somewhat small	27	very large
12	some	28	humongous amount
13	some to moderate	29	huge amount
14	moderate amount	30	very high amount
15	fair amount	31	extreme amount
16	medium	32	maximum amount

III. METHODS

We analyse interval-valued data and fuzzy sets derived from that data by comparing their moments: central tendency (CT) and standard deviation (SD) around the CT[1]. We evaluate how well the CT and SD of the fuzzy sets match the same moments on the interval data. If the moments closely match, we say the fuzzy set represents the data well. Where moments do not match, we analyse how the fuzzy sets differ from the data. The remainder of this section explains how the moments of fuzzy sets and interval-valued data are calculated. Note that the fuzzy sets in this paper are defined within the bounds $X = [0, 10]$ and are discretised into 101 points; i.e., $X = \{0, 0.1, ...9.9, 10\}$. A total of 101 points was chosen as using a finer discretisation did not change our results. However, using fewer discrete points will lead to less accurate results.

To calculate the CT of a fuzzy set, we use its centroid as

$$m_1^{f1}(A) = \frac{\sum_{x \in X} x\mu_A(x)}{\sum_{x \in X} \mu_A(x)}, \quad (1)$$

and we calculate the SD of a type-1 fuzzy set A as [15]:

$$m_2^{f1}(A) = \sqrt{\frac{\sum_{x \in X} \left((x - m_1^{f1}(A))^2 \mu_A(x)\right)}{\sum_{x \in X} \mu_A(x)}}, \quad (2)$$

where m_1^{f1} is the centroid as given in (1). Note that the moments and centroid of a type-1 fuzzy set are crisp values.

We calculate the moments of an interval type-2 fuzzy set using the Karnik-Mendel (KM) type-reduction algorithm [16]. We use the KM algorithm to calculate the centroid of a type-2 fuzzy set by calculating the centroid of its embedded membership functions. We denote the centroid of an interval type-2 fuzzy set $\tilde{A}$ as $m_1^{f2}(\tilde{A})$.

To calculate the second moment of $\tilde{A}$ (to obtain the SD), we replace the calculation of the centroid of the type-1 embedded membership function in the KM algorithm with the calculation of the moment as given in (2), and we sort $\left[x - m_1^{f2}(\tilde{A})\right]^2$ into ascending order [17]. We denote the standard deviation of an interval type-2 fuzzy set $\tilde{A}$ as $m_2^{f2}(\tilde{A})$.

The result of the KM algorithm is an interval type-1 fuzzy set, which is defined by a left and right endpoint, which we use to define a crisp interval.

Moments of interval data are computed in terms of lower and upper bounds. Consider a set of intervals $\bar{A} = \{a_1, ...a_n\}$

[1]Note, we also analysed higher moments (skew and kurtosis) but found the excess width commonly found in calculations on intervals [14] too large for useful results.

where a given interval a is defined as $a = [a_l, a_r]$. We calculate the central-tendency of $\bar{A}$ using the mean as follows [14]:

$$m_1^r(\bar{A}) = \left[\frac{\sum_{a\in A} a_l}{n}, \frac{\sum_{a\in A} a_r}{n}\right] \quad (3)$$

To calculate the boundaries of the SD of intervals $\bar{A}$, we minimise (for the lower bound) and maximise (for the upper bound) the following constrained optimisation problem [14]:

$$\min / \max \ m_2^r(\bar{A}) = \sqrt{\frac{1}{n}\left(\sum_{a\in A} x^2\right) - \frac{1}{n^2}\left(\sum_{a\in A} x\right)^2} \quad (4)$$

such that $a_l \leq x \leq a_r$.

To summarise, the moments of multiple interval-valued data and of an interval type-2 fuzzy set are themselves intervals. By contrast, the moments of a type-1 fuzzy set are crisp values.

Let $m_k^{f1}(A)$ be the k^{th} statistical moment of a type-1 fuzzy set A, where $k = 1$ or 2 for the CT and SD, respectively. Also, let $m_k^r(\bar{A})$ be the k^{th} statistical moment of a set of interval values $\bar{A}$. To evaluate if the moment of a type-1 fuzzy set matches that of intervals, we check if $m_k^{f1}(A) \in m_k^r(\bar{A})$. Note that $m_k^{f1}(A)$ is a crisp value, whereas $m_k^r(\bar{A})$ is interval-valued.

Let $m_k^{f2}(\tilde{A})$ be the k^{th} statistical moment of an interval type-2 fuzzy set $\tilde{A}$. To evaluate if a moment of $\tilde{A}$ matches the same moment of a set of intervals $\bar{A}$, we evaluate if $m_k^{f2}(\tilde{A})_c \in m_k^r(\bar{A})$ and if $m_k^r(\bar{A})_c \in m_k^{f2}(\tilde{A})$, where $m_k^r(\bar{A})_c$ denotes the centre of the interval $m_k^r(\bar{A})$. We use this as an efficient approach comparable to the type-1 case and will explore other methods in a future publication.

If the moments on the fuzzy sets do not match the same moments on the interval-valued data, we calculate whether the result of the moment is lower or higher; i.e., we check if $m_k^{f1}(A) < m_k^r(\bar{A})$ (for type-1 fuzzy sets) and if $m_k^{f2}(\tilde{A})_c < m_k^r(\bar{A})_c$ (for type-2 fuzzy sets) to test if the moment on the fuzzy set is lower than the moment on the data.

IV. Results

In this section, we compare statistical moments of interval-valued, data-driven fuzzy sets with the respective moments computed directly for the same data. Table II show the results of the CT and SD for the data and for the IAA and eIAA fuzzy sets. The table highlights results where the moment of the fuzzy set deviates from the moment of the intervals according to our test. Using both IAA approaches, the CT of fuzzy words 1-7 is larger than the upper bound of the CT of the intervals, and the CT of words 28-32 is smaller than the lower bound of the intervals. Although the universe of discourse (UOD) is in $[0, 10]$, the CT according to the IAA is in the range $[3.79, 6.81]$, representing only a third of the scale.

The CT of the IAA is skewed further to the centre of the UOD than the data because only the endpoints of the intervals are included in the calculation of the CT on the data (see (4)), but all values within the interval are included when calculating the centroid of the IAA set (see (1)). For example, consider word 32 – Fig. 4(a) shows the IAA set. Most responses for this word are a subset of $[8, 10]$, but nearly 20% of participants gave the answer $[0, 10]$. These wide results skew the CT of the IAA towards $m_1 = 5$, but they do not skew the CT on the raw data as strongly.

By contrast, The IA methods treat the responses $[0, 10]$ as incorrect definitions of the words, resulting from a lack of understanding of the question (for example, the interval $[0, 10]$ is the *maximum amount* in terms of the possible interval size on the given scale) or due to incorrectly inputting the desired answer (see the HMA fuzzy set, also in Fig. 4, to compare the IAA with an approach that removes this data). However, the IAA models agreement and therefore retains the information provided that around 20% of the participants disagreed with the rest.

TABLE II
Results of CT and SD on the interval-valued data and the IAA and eIAA fuzzy sets for each word [11]. Results in bold show when the fuzzy set moment does not match the corresponding moment for the data according to our test.

	CT			SD		
word	raw data	IAA	eIAA	raw data	IAA	eIAA
1	[0.52, 3.09]	**4.1**	**4.15**	[1.07, 16.03]	3.06	3.19
2	[0.81, 3.59]	**4.53**	**4.56**	[2.57, 18.79]	3.04	3.17
3	[0.76, 4.05]	**4.31**	**4.33**	[1.85, 18.66]	2.98	3.1
4	[0.48, 3.0]	**3.89**	**3.89**	[1.01, 14.88]	3.06	3.18
5	[0.5, 2.85]	**3.8**	**3.8**	[1.4, 14.44]	3.13	3.24
6	[0.37, 2.91]	**3.79**	**3.78**	[0.53, 14.59]	3.1	3.21
7	[0.79, 3.62]	**3.84**	**3.83**	[1.17, 14.91]	2.93	3.03
8	[0.74, 4.02]	3.69	3.71	[0.79, 15.27]	2.85	2.95
9	[0.75, 4.02]	3.54	3.53	[0.78, 14.95]	2.85	2.94
10	[0.69, 3.91]	3.48	3.49	[0.45, 14.03]	2.78	2.88
11	[1.3, 4.31]	3.76	3.74	[1.27, 13.45]	2.71	2.8
12	[1.69, 5.76]	4.43	4.44	[0.73, 15.24]	2.53	2.6
13	[2.61, 6.66]	4.76	4.76	[0.46, 12.5]	2.27	2.33
14	[3.34, 6.98]	5.16	5.15	[0.29, 10.13]	2.22	2.29
15	[3.57, 7.3]	5.27	5.27	[0.78, 12.58]	2.34	2.42
16	[3.39, 6.85]	5.16	5.16	[0.16, 7.53]	1.99	2.09
17	[2.75, 6.21]	4.8	4.79	[0.92, 13.07]	2.44	2.51
18	[4.77, 8.49]	6.12	6.11	[0.83, 13.88]	2.48	2.58
19	[3.97, 8.44]	5.72	5.72	[1.42, 17.38]	2.61	2.69
20	[2.79, 6.37]	5.04	5.04	[3.85, 18.08]	**2.8**	**2.89**
21	[4.85, 8.67]	6.13	6.14	[0.83, 14.49]	2.52	2.58
22	[4.89, 8.97]	6.19	6.18	[0.51, 15.58]	2.57	2.66
23	[5.7, 9.21]	6.6	6.58	[0.97, 14.79]	2.69	2.77
24	[6.21, 9.47]	6.62	6.59	[0.46, 13.78]	2.72	2.81
25	[4.91, 8.94]	6.09	6.08	[1.5, 17.72]	2.73	2.82
26	[5.53, 9.48]	6.54	6.54	[0.43, 15.51]	2.68	2.75
27	[6.37, 9.66]	6.38	**6.35**	[0.4, 16.01]	2.92	3.01
38	[6.21, 9.42]	**5.84**	**5.83**	[1.64, 18.22]	2.94	3.05
29	[6.75, 9.75]	**6.43**	**6.4**	[0.65, 15.33]	2.94	3.05
30	[7.35, 9.76]	**6.81**	**6.77**	[0.39, 11.59]	2.91	3
31	[6.67, 9.67]	**5.93**	**5.92**	[1.73, 18.42]	3.08	3.2
32	[7.59, 9.84]	**5.88**	**5.87**	[0.96, 15.09]	3.09	3.21

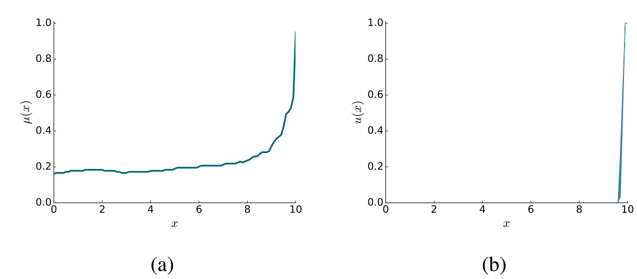


Fig. 4. Fuzzy sets representing word 32 using (a) IAA and (b) HMA.

TABLE III
RESULTS OF CT AND SD ON THE PRE-PROCESSED INTERVAL-VALUED DATA AND THE IA, EIA AND HMA FUZZY SETS FOR EACH WORD. RESULTS IN BOLD SHOW WHEN THE FUZZY SET MOMENT DOES NOT MATCH THE DATA MOMENT ACCORDING TO OUR TEST.

	CT				SD			
word	data	IA	EIA	HMA	data	IA	EIA	HMA
1	[0.0,0.56]	**[0.35, 1.68]**	[0.02,0.66]	[0.46,0.52]	[0.0,0.47]	**[0.68, 1.15]**	[0.24,0.42]	[0.35,0.4]
2	[0.03,0.49]	**[0.12, 3.31]**	[0.07,0.38]	[0.27,0.37]	[0.0,0.3]	**[1.41, 1.62]**	[0.14,0.3]	[0.2,0.28]
3	[0.07,0.87]	**[0.21, 4.38]**	[0.19,0.65]	[0.52,0.58]	[0.0,0.53]	**[1.92, 2.21]**	[0.26,0.47]	[0.36,0.41]
4	[0.03,0.68]	**[0.14, 1.48]**	[0.07,0.59]	[0.44,0.61]	[0.0,0.47]	**[0.6, 0.91]**	[0.22,0.42]	[0.35,0.46]
5	[0.03,0.72]	**[0.13, 1.49]**	[0.07,0.59]	[0.43,0.68]	[0.0,0.48]	**[0.61, 0.89]**	[0.22,0.42]	[0.35,0.52]
6	[0.02,0.88]	[0.15,1.48]	[0.09,0.88]	[0.59,0.8]	[0.0,0.61]	**[0.6, 0.91]**	[0.35,0.58]	[0.45,0.59]
7	[0.15,1.49]	[0.32,1.94]	[0.35,1.09]	[0.88,0.98]	[0.0,0.94]	**[0.79, 1.24]**	[0.45,0.75]	[0.59,0.67]
8	[0.29,2.31]	[0.46,2.26]	[0.46,1.57]	[1.2,2.03]	[0.0,1.35]	[0.92,1.48]	[0.63,1.07]	[0.9,1.46]
9	[0.19,2.4]	[0.43,2.04]	[0.48,1.84]	[1.23,2.15]	[0.0,1.45]	[0.82,1.34]	[0.73,1.24]	[0.94,1.56]
10	[0.24,2.43]	[0.41,2.14]	[0.41,1.76]	[1.24,2.33]	[0.0,1.43]	[0.87,1.39]	[0.71,1.17]	[1.0,1.71]
11	[0.72,2.87]	[0.89,4.15]	[0.98,3.17]	[1.59,1.87]	[0.0,1.5]	[0.45,1.21]	[0.2,0.94]	[1.03,1.23]
12	[1.81,4.35]	**[1.64, 7.1]**	[1.83,5.0]	[3.07,3.38]	[0.0,1.81]	[0.78,1.96]	[0.2,1.37]	[1.2,1.4]
13	[2.73,6.05]	[1.65,7.64]	[2.0,6.82]	[4.07,4.91]	[0.0,2.35]	[0.2,2.02]	[0.2,1.9]	[1.54,2.07]
14	[3.85,6.4]	[2.55,7.75]	[3.48,6.98]	[4.36,5.9]	[0.0,1.62]	[0.17,1.88]	[0.14,1.41]	[1.03,1.98]
15	[3.79,6.86]	[2.01,8.43]	[3.03,7.51]	[4.73,5.95]	[0.0,2.07]	[0.1,2.04]	[0.14,1.65]	[1.3,2.05]
16	[3.71,6.38]	[5.04,5.04]	[3.16,6.92]	[4.14,5.87]	[0.0,1.53]	[1.53,1.53]	[0.1,1.46]	[0.97,2.04]
17	[3.24,5.79]	[1.69,7.72]	[2.06,6.35]	[4.32,5.09]	[0.0,1.93]	[0.22,2.05]	[0.22,1.64]	[1.39,1.87]
18	[5.86,8.32]	[3.0,8.78]	[5.38,8.6]	[6.53,7.13]	[0.0,1.76]	[0.91,2.03]	[0.22,1.37]	[1.18,1.57]
19	[5.18,9.0]	[6.12,8.72]	[6.66,8.72]	[6.92,7.28]	[0.0,2.6]	[1.57,2.56]	[1.36,2.19]	[1.72,1.98]
20	[3.26,5.13]	[2.25,7.82]	[3.18,4.83]	[3.54,3.93]	[0.0,1.33]	[0.24,1.92]	[0.48,0.97]	[0.91,1.14]
21	[6.21,9.15]	**[3.46, 8.78]**	[5.47,8.92]	[7.2,7.96]	[0.0,2.06]	[0.71,1.87]	[0.28,1.28]	[1.35,1.9]
22	[6.41,9.44]	[6.05,9.05]	[7.41,9.09]	[7.41,7.97]	[0.0,2.06]	**[1.61, 2.56]**	[1.05,1.71]	[1.34,1.75]
23	[7.44,9.81]	[6.95,9.44]	[7.99,9.44]	[7.93,8.55]	[0.0,1.58]	**[1.25, 1.95]**	[0.81,1.36]	[1.01,1.46]
24	[7.98,9.9]	[7.64,9.62]	[8.42,9.62]	[8.37,8.76]	[0.0,1.29]	[0.97,1.51]	[0.63,1.07]	[0.88,1.16]
25	[7.18,9.64]	[6.05,9.39]	[7.93,9.34]	[8.17,8.24]	[0.0,1.71]	**[1.66, 2.42]**	[0.84,1.38]	[1.16,1.21]
26	[7.28,9.77]	[6.62,9.43]	[7.93,9.43]	[7.69,8.52]	[0.0,1.61]	**[1.4, 2.14]**	[0.83,1.39]	[1.07,1.65]
27	[8.51,10.0]	**[7.24, 9.64]**	[8.36,9.92]	[8.31,9.05]	[0.0,1.07]	**[1.16, 1.7]**	[0.67,0.92]	[0.79,1.26]
28	[9.43,10.0]	**[6.36, 9.29]**	[9.33,9.98]	[9.38,9.53]	[0.0,0.47]	**[1.5, 2.36]**	[0.24,0.42]	[0.37,0.47]
29	[9.01,10.0]	**[7.77, 9.8]**	[8.69,9.98]	[8.85,9.25]	[0.0,0.8]	**[0.94, 1.3]**	[0.51,0.69]	[0.59,0.86]
30	[8.9,10.0]	[8.28,9.85]	[8.58,9.98]	[8.65,9.19]	[0.0,0.88]	[0.71,1.02]	[0.55,0.72]	[0.66,1.0]
31	[9.51,10.0]	**[7.8, 9.76]**	[9.44,9.98]	**[9.38, 9.63]**	[0.0,0.41]	**[0.92, 1.36]**	[0.2,0.37]	**[0.32, 0.48]**
32	[9.88,10.0]	**[8.58, 9.98]**	[9.9,9.98]	[9.89,9.92]	[0.0,0.07]	**[0.55, 0.72]**	[0.0,0.1]	[0.1,0.1]

The SD of the IAA methods all fit well with the words according to our test, except for word 20, where the result is lower for the fuzzy set than the intervals. In addition, the results of the IAA and eIAA are similar, demonstrating that the eIAA provides a good approximation of the IAA.

Table III shows the results of the CT and SD of the pre-processed intervals and of the IA, EIA and HMA fuzzy sets. The table highlights results where the moment of the fuzzy set does not match the moment of the interval according to our test. Looking first at the IA fuzzy sets, the moment of the fuzzy set often does not match the moment of the intervals. This mainly occurs for words at the ends of the scale (words 1-7 and words 21-32). As with the IAA, the CT of words 1-7 is higher according to the IA set than according to the data, and for words 29-32 the CT is lower than that of the data. This is because the IA fuzzy sets are often very wide [4].

Note that the lower bound of the SD of the pre-processed data is always 0. This is because, for each word, there is a value x that is within all of the remaining intervals. Therefore, if each interval was reduced to this crisp value x, the SD of the data would be 0, hence this is the lower bound. However, in the case of a type-2 fuzzy set, the SD is only 0 if the fuzzy set is a singleton, i.e., representative of a crisp value.

The EIA and HMA perform better at modelling the pre-processed data than the IA according to our test. The moments of the EIA fuzzy sets pass our test and for the HMA all except word 31 passes our test.

Next, we compare the CT and SD of the IA-based sets with the original non-processed data. (Note, we do not include a table of the results together for space consideration. The results on the non-processed data are within Table II and the results of the IA-based fuzzy sets are within Table III). The CT of the IA sets matches that of the data. However, the CT of the EIA sets is lower than the data for words 1-7 and higher for words 30-32. Similarly, the CT of the HMA sets for words 1-3 is lower and for words 30-32 is higher than of the data. The SD of the IA-based sets is different to the data for several words. For the IA sets, the SD of the fuzzy set is smaller than the SD of the data for words 1, 3, 4-7, 11, 20, 31, and 32. For the EIA, the SD of the fuzzy set is smaller than the data for words 1-7, 11, 18, 20, 21, 25, 29-32, for the HMA it is smaller for words 1-7, 11, 20, 25, 30-32. This demonstrates that the IA methods provide a model that is fundamentally different to the full data before pre-processing. Specifically, they extract key areas of majority agreement that can provide a useful model of a word. This involves removing intervals that are perceived as non-sensical (e.g. encompassing the entire scale) or in disagreement with most other responses.

These results show that it is generally the words at the ends of the scale that are vulnerable to being misrepresented by their fuzzy membership function. In addition, there is a considerable difference between representations produced by the IA-based methods and by the IAA-based methods as a result of the data-processing performed by the former.

Overall, the results show that the EIA and HMA provide a good representation of the remaining data after pre-processing, whereas the IA often has a CT or SD that differs from the data. Results also show that the IA methods represent noticeably different data to the original unprocessed data. Therefore, while the IA fuzzy sets provide a good model of pre-processed data, it is not a suitable method if all data points need to be treated with equal importance.

The IAA-based methods do not perform any data pre-processing and therefore provide a closer representation of the original data. However, the mean of the fuzzy set may be lower (higher) than the mean of data where that data is skewed towards the high (low) end of the scale. This is because outliers skew the membership function further to the centre of the UOD than may be desired.

V. Conclusions

In recent years, the use of interval-valued data has grown in the literature, taking advantage of the fact that such data captures uncertainty around the true answer. Two key methods that use interval-valued data to build fuzzy sets have been proposed in the literature. The first are the IA based approaches, designed to capture the meaning of words by modelling the most important subset of the data with a type-2 fuzzy set. The second are the IAA based approaches, designed to capture agreement over all of the data with a type-1 fuzzy set. These unique techniques differ such that they produce visually different fuzzy sets that differ in key statistical moments.

We compare the CT and SD of interval-valued data-driven fuzzy sets with the same moments on the data to show how the fuzzy sets compare to the data from which they are constructed. As the IA and IAA methods are designed to capture different details of the data, their moments differ appreciably. We demonstrate that the EIA and HMA effectively capture the moments of the pre-processed data, whereas the IA does not do so according to our test. The fuzzy sets generated by each of these three methods however differs considerably from the original non-processed data, thus care should be taken if such strong *pruning* of the data is appropriate for the given application. The IAA effectively captures the SD of the data. The CT of the IAA sets is skewed towards the centre of the UOD when, in fact, the values of x with highest membership are near the edge of the UOD. This is due to outliers in the data and the choice of using the centroid to calculate the CT. Results indicate that a different measure of CT, such as the mean-of-maxima, would be more appropriate for IAA sets. We recommend that the EIA or HMA methods should be used only when outliers cannot provide meaningful information, otherwise, the IAA or eIAA methods are useful.

The results of this paper are limited in that we have only used a single dataset to analyse the methods. In addition, the data set is large (containing 174 intervals per word) and therefore does not show the effect of the data-processing of the IA methods on small data sets. This is an important point to investigate as these methods remove up to 90% of the data, leaving very few data points if the unprocessed dataset is small. In addition, the IA methods assume the data is unimodal whereas the IAA methods make no such assumptions. Future work will further investigate fuzzy sets constructed from interval-valued data in cases of small and multi-modal data sets.

References

[1] S. G. Rabinovich, *Measurement errors and uncertainties: theory and practice*. Springer Science & Business Media, 2006.

[2] K. J. Wallace, C. Wagner, M. J. Smith *et al.*, "Eliciting human values for conservation planning and decisions: a global issue," *Journal of environmental management*, vol. 170, pp. 160–168, 2016.

[3] F. Liu and J. Mendel, "Encoding Words Into Interval Type-2 Fuzzy Sets Using an Interval Approach," *Fuzzy Systems, IEEE Transactions on*, vol. 16, no. 6, pp. 1503–1521, 2008.

[4] D. Wu, J. M. Mendel, and S. Coupland, "Enhanced interval approach for encoding words into interval type-2 fuzzy sets and its convergence analysis," *Fuzzy Systems, IEEE Transactions on*, vol. 20, no. 3, pp. 499–513, 2012.

[5] J. M. Mendel and D. Wu, "Determining interval type-2 fuzzy set models for words using data collected from one subject: Person fous," in *Fuzzy Systems (FUZZ-IEEE), 2014 IEEE International Conference on*. IEEE, 2014, pp. 768–775.

[6] M. Hao and J. M. Mendel, "Encoding Words Into Normal Interval Type-2 Fuzzy Sets: HM Approach," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 4, pp. 865–879, Aug 2016.

[7] S. Miller, C. Wagner, J. Garibaldi, and S. Appleby, "Constructing General Type-2 fuzzy sets from interval-valued data," in *Fuzzy Systems (FUZZ-IEEE), 2012 IEEE International Conference on*, 2012, pp. 1–8.

[8] C. Wagner, S. Miller, and J. Garibaldi, "Similarity based applications for data-driven concept and word models based on type-1 and type-2 fuzzy sets," in *Fuzzy Systems (FUZZ), 2013 IEEE International Conference on*, 2013, pp. 1–9.

[9] C. Wagner, S. Miller, J. Garibaldi, D. Anderson, and T. Havens, "From Interval-Valued Data to General Type-2 Fuzzy Sets," *Fuzzy Systems, IEEE Transactions on*, vol. 23, no. 2, pp. 248–269, Apr. 2015.

[10] T. C. Havens, C. Wagner, and D. T. Anderson, "Efficient modeling and representation of agreement in interval-valued data," in *Fuzzy Systems (FUZZ-IEEE), 2017 IEEE International Conference on*. IEEE, 2017, pp. 1–6.

[11] N. N. Karnik, Q. Liang, F. Liu, D. Wu, J. Joo, and J. M. Mendel. Type-2 Fuzzy Logic Software. [Online]. Available: sipi.usc.edu/ mendel/publications/ [Accessed: 4-Jan-2019]

[12] T. Hesketh, R. Pryor, and B. Hesketh, "An application of a computerized fuzzy graphic rating scale to the psychological measurement of individual differences," *International Journal of Man-Machine Studies*, vol. 29, no. 1, pp. 21–35, 1988.

[13] P. Quirós, J. M. Alonso, and D. P. Pancho, "Descriptive and Comparative Analysis of Human Perceptions expressed through Fuzzy Rating Scale-based Questionnaires," *International Journal of Computational Intelligence Systems*, vol. 9, no. 3, pp. 450–467, 2016.

[14] K. Zaman, S. Rangavajhala, M. P. McDonald, and S. Mahadevan, "A probabilistic approach for representation of interval uncertainty," *Reliability Engineering & System Safety*, vol. 96, no. 1, pp. 117–130, 2011.

[15] L. A. Zadeh, "Probability measures of fuzzy events," *Journal of mathematical analysis and applications*, vol. 23, no. 2, pp. 421–427, 1968.

[16] N. Karnik and J. Mendel, "Centroid of a type-2 fuzzy set," *Information Sciences*, vol. 132, no. 1-4, pp. 195–220, Feb. 2001.

[17] D. Wu and J. M. Mendel, "Uncertainty measures for interval type-2 fuzzy sets," *Information sciences*, vol. 177, no. 23, pp. 5378–5393, 2007.