FOUNDATIONS FOR AN ITERATION THEORY OF ENTIRE QUASIREGULAR MAPS

WALTER BERGWEILER AND DANIEL A. NICKS

Abstract. The Fatou-Julia iteration theory of rational functions has been extended to uniformly quasiregular mappings in higher dimension by various authors, and recently some results of Fatou-Julia type have also been obtained for non-uniformly quasiregular maps. The purpose of this paper is to extend the iteration theory of transcendental entire functions to the quasiregular setting. As no examples of uniformly quasiregular maps of transcendental type are known, we work without the assumption of uniform quasiregularity. Here the Julia set is defined as the set of all points such that complement of the forward orbit of any neighbourhood has capacity zero. It is shown that for maps which are not of polynomial type the Julia set is non-empty and has many properties of the classical Julia set of transcendental entire functions.

1. Introduction and main results

In 1918-20, Fatou [14] and Julia [20] wrote long memoirs on the iteration of rational functions and thereby created the field now known as complex dynamics. The analogies (and differences) that arise in the corresponding theory for transcendental entire functions were studied by Fatou [15] in 1926.

Many of the results of the Fatou-Julia theory for rational functions, considered as self-maps of the Riemann sphere S^2, have been extended to uniformly quasiregular self-maps of the n-sphere S^n where $n \geq 2$ by Hinkkanen, Iwaniec, Martin, Mayer and others; see [19, Chapter 21], [35, Chapter 4], and [4, Section 4] for surveys. Here a quasiregular map $f\colon S^n \to S^n$ is called *uniformly quasiregular* if there exists a uniform bound on the dilatation of the iterates f^k of f. (We will recall the definition of quasiregularity, in particular the notions of dilatation and inner dilatation, in section 2.) In principle, it would also be possible to extend some of Fatou's results about transcendental entire functions $f\colon \mathbb{C} \to \mathbb{C}$ to uniformly quasiregular maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ where $n \geq 2$. However, for $n \geq 3$ no examples of such maps with an essential singularity at ∞ are known yet.

On the other hand, the iteration of quasiregular analogues of the exponential function (called the Zorich map) and the trigonometric functions were studied in [5, 8, 18]. Also [9] contains a general result about the *escaping set*

$$I(f) = \{x \in \mathbb{R}^n : f^k(x) \to \infty \text{ as } k \to \infty\}$$

of a quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n$.

The first author is supported by the Deutsche Forschungsgemeinschaft, Be 1508/7-2 and the ESF Networking Programme HCAA

Mathematics Subject Classification: Primary 37F10; Secondary 30C65, 30D05.

However, no systematic theory in the spirit of Fatou and Julia has been developed yet for quasiregular self-maps of $\mathbb{R}^n$. It is the purpose of this paper to do precisely this. Here we build on [7], which is concerned with a Fatou-Julia theory for (non-uniformly) quasiregular self-maps of S^n. The results in [7] are in turn inspired by results of Sun and Yang [38, 39, 40] dealing with the case $n = 2$.

We call quasiregular self-maps of $\mathbb{R}^n$ *entire quasiregular maps*. Such a map f is said to be of *polynomial type* if $\lim_{x\to\infty} |f(x)| = \infty$ and of *transcendental type* if $\lim_{x\to\infty} |f(x)|$ does not exist. Here and in the following $|y|$ denotes the Euclidean norm of a point $y \in \mathbb{R}^n$.

The iteration of entire quasiregular maps of polynomial type was studied in [16]. Such maps extend to quasiregular self-maps of the one-point-compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$, which we identify with the n-sphere S^n via stereographic projection. Hence entire quasiregular maps of polynomial type are also covered by the results in [7]. Thus we will mainly restrict to entire quasiregular maps of transcendental type.

The capacity of a condenser, and the distinction between sets of positive capacity and sets of zero capacity, plays an important role in the theory of quasiregular maps; see section 2 for the definitions. We write $\operatorname{cap} C = 0$ if C is a set of capacity zero and $\operatorname{cap} C > 0$ otherwise. The following definition is the same as in [7, Definition 1.1.].

Definition 1.1. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be quasiregular. Then the *Julia set* $J(f)$ of f is defined to be the set of all $x \in \mathbb{R}^n$ such that

$$\operatorname{cap}\left(\mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} f^k(U)\right) = 0 \tag{1.1}$$

for every neighbourhood U of x.

It was shown in [7] that if f is of polynomial type and the degree of f is larger than the inner dilatation of f, then $J(f)$ is not empty and has many properties of the classical Julia set. Also, with this hypothesis the above definition agrees with the classical one for uniformly quasiregular maps and thus in particular for polynomials. These results also hold in the current setting:

Theorem 1.1. *Let f be an entire quasiregular map of transcendental type. Then $J(f) \neq \emptyset$. In fact,* $\operatorname{card} J(f) = \infty$.

Here $\operatorname{card} X$ denotes the cardinality of a set X.

Theorem 1.2. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. Then the classical definition of $J(f)$ using non-normality agrees with the one given in Definition 1.1.*

As in the classical case, it is easily seen that $J(f)$ is completely invariant; that is, $f(x) \in J(f)$ if and only if $x \in J(f)$. We also note that if $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal homeomorphism and $g = \phi \circ f \circ \phi^{-1}$, then $J(g) = \phi(J(f))$.

Besides the escaping set $I(f)$ we also consider the set

$$BO(f) = \{x \in \mathbb{R}^n : (f^k(x)) \text{ is bounded}\}$$

of points with bounded orbits. For polynomials this set is called the *filled Julia set* and is usually denoted by $K(f)$, but we reserve the notation $K(f)$ for the dilatation. For polynomials and transcendental entire functions we have [12]

$$J(f) = \partial I(f) = \partial BO(f).$$

This does not hold in the present context, as Examples 7.3 and 7.4 will show that we may have $(\partial I(f) \cap \partial BO(f)) \backslash J(f) \neq \emptyset$. However, we have the following result.

Theorem 1.3. *Let f be an entire quasiregular map of transcendental type. Then $J(f) \subset \partial I(f) \cap \partial BO(f)$.*

One ingredient in the proof of Theorem 1.3 is the following result of independent interest.

Theorem 1.4. *Let f be an entire quasiregular map of transcendental type. Then $\operatorname{cap} BO(f) > 0$.*

Theorem 1.3 follows from Theorem 1.4 together with the result of [9] that $I(f)$ contains continua, implying that $\operatorname{cap} I(f) > 0$; cf. Lemma 2.12 and the remark following it.

Even for entire functions in the complex plane, $BO(f)$ need not contain continua. In this setting the Hausdorff dimension of $BO(f)$ is positive, but can be arbitrarily small [6].

We say that $\xi \in \mathbb{R}^n$ is a *periodic point* of $f \colon \mathbb{R}^n \to \mathbb{R}^n$, if there exists $p \in \mathbb{N}$ such that $f^p(\xi) = \xi$. The smallest p with this property is called the *period* of ξ. A periodic point of period 1 is called a *fixed point*. If a periodic point ξ of period p has a neighbourhood U such that $f^{pk}|_U \to \xi$ uniformly as $k \to \infty$, then ξ is called *attracting* and the set of all $x \in \mathbb{R}^n$ satisfying $f^{pk}(x) \to \xi$ as $k \to \infty$ is called the *attracting basin* of ξ and denoted by $A(\xi)$. As in [7, Theorem 1.3] we have the following result.

Theorem 1.5. *Let f be an entire quasiregular map of transcendental type with an attracting fixed point ξ. Then $J(f) \cap A(\xi) = \emptyset$ and $J(f) \subset \partial A(\xi)$.*

Again we have equality for entire functions in the complex plane, but not in the current setting; see Example 7.5 below.

The *forward orbit* $O_f^+(x)$ and the *backward orbit* $O_f^-(x)$ of a point $x \in \mathbb{R}^n$ are defined by

$$O_f^+(x) = \left\{ f^k(x) : k \in \mathbb{N} \right\}$$

and

$$O_f^-(x) = \bigcup_{k=0}^{\infty} f^{-k}(x) = \bigcup_{k=0}^{\infty} \left\{ y \in \mathbb{R}^n : f^k(y) = x \right\}.$$

For $A \subset \mathbb{R}^n$ we put

$$O_f^{\pm}(A) = \bigcup_{x \in A} O_f^{\pm}(x).$$

With this terminology (1.1) takes the form

$$\operatorname{cap}\big(\mathbb{R}^n \backslash O_f^+(U)\big) = 0. \tag{1.2}$$

The *exceptional set* $E(f)$ of a quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is the set of all points with finite backward orbit. It is a simple consequence of Picard's theorem that the exceptional set of a non-linear entire function in the complex plane contains at most one point. Rickman [30] has extended Picard's theorem to quasiregular maps and shown that there exists a constant $q = q(n, K)$ such that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiregular map of transcendental type and if $a_1, a_2, \dots, a_q \in \mathbb{R}^n$ are distinct, then there exists $j \in \{1, \dots, q\}$ such that $f^{-1}(a_j)$ is infinite. This implies that $E(f)$ contains at most $q - 1$ points.

As in [7] we obtain the analogues of some further standard results of complex dynamics under the additional hypothesis of (local) Lipschitz continuity.

Theorem 1.6. *Let f be an entire quasiregular map of transcendental type. Suppose that f is locally Lipschitz continuous. Then*

(i) $J(f) \subset \overline{O_f^-(x)}$ *for all* $x \in \mathbb{R}^n \backslash E(f)$,
(ii) $J(f) = \overline{O_f^-(x)}$ *for all* $x \in J(f) \backslash E(f)$,
(iii) $\mathbb{R}^n \backslash O_f^+(U) \subset E(f)$ *for every open set* U *intersecting* $J(f)$,
(iv) $J(f)$ *is perfect,*
(v) $J(f^p) = J(f)$ *for all* $p \in \mathbb{N}$.

Note that (iii) is considerably stronger than the property (1.1) used in the definition of $J(f)$. It follows from (iii), together with the complete invariance of $J(f)$, that

$$J(f) \backslash E(f) \subset O_f^+(U \cap J(f)) \subset J(f)$$

for every open set U intersecting $J(f)$.

We also note that (v) implies that – under the hypotheses of Theorem 1.6 – the conclusion of Theorem 1.5 holds not only for attracting fixed points, but also for attracting periodic points.

We could achieve that (v) always holds by modifying Definition 1.1 as follows: instead of requiring that (1.2) holds for all neighbourhoods U of x we would require that $\operatorname{cap}\big(\mathbb{R}^n \backslash O_{f^p}^+(U)\big) = 0$ for all neighbourhoods U of x and all $p \in \mathbb{N}$. Suitable modifications of our arguments show that with this definition of $J(f)$ our results about the Julia set would also hold, and (v) would be true automatically. However, we conjecture that Theorem 1.6 holds without the hypothesis of local Lipschitz continuity. If this is true, then, in particular, (v) always holds and so this modified definition agrees with the one given in Definition 1.1. Therefore we have used the somewhat simpler definition of $J(f)$ in Definition 1.1.

We denote the Hausdorff dimension of a subset X of $\mathbb{R}^n$ by $\dim X$.

Theorem 1.7. *Let f be as in Theorem 1.6. Then* $\dim J(f) > 0$.

In our proof of Theorem 1.1 and subsequent results we have to distinguish two cases. It turns out that the hypothesis of Lipschitz continuity in Theorem 1.6 is needed only in one of these cases. In order to motivate the terminology, we note that an entire function is said to have the "pits effect" if $|f(z)|$ is "large" except in "small" domains (which are called pits). This concept was introduced by Littlewood and Offord [22]; see also [13]. We adapt this terminology, although our definition of "large" and "small" is different from the one in the papers cited.

Definition 1.2. A quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of transcendental type is said to have the *pits effect* if there exists $N \in \mathbb{N}$ such that, for all $c > 1$ and all $\varepsilon > 0$, there exists R_0 such that if $R > R_0$, then

$$\{x \in \mathbb{R}^n : R \le |x| \le cR,\ |f(x)| \le 1\}$$

can be covered by N balls of radius εR.

For example, it follows directly that if there exists a sequence (x_k) tending to ∞ such that $|f(x_k)| \le 1$ for all $k \in \mathbb{N}$ and $\limsup_{k\to\infty} |x_{k+1}|/|x_k| < \infty$, then f does not have the pits effect. In the definition of the pits effect, we could replace the condition that $|f(x)| \le 1$ by $|f(x)| \le C$ for any positive constant C and in fact by $|f(x)| \le R^\alpha$ for any $\alpha > 0$; see Theorem 8.1.

Theorem 1.8. *Let f be an entire quasiregular map of transcendental type which does not have the pits effect. Then the conclusion of Theorem 1.6 holds.*

Theorem 1.8 is a corollary of the following result.

Theorem 1.9. *Let f be an entire quasiregular map of transcendental type which does not have the pits effect. Then* $\operatorname{cap} \overline{O_f^-(x)} > 0$ *for all $x \in \mathbb{R}^n \backslash E(f)$.*

Together with the complete invariance of $J(f)$ we obtain the following result from Theorem 1.9.

Corollary 1.1. *Let f be as in Theorem 1.9. Then* $\operatorname{cap} J(f) > 0$.

Theorems 1.8 and 1.9 apply in particular to higher dimensional analogues of the exponential and the trigonometric functions considered in [5, 8, 18]; see Examples 7.1 and 7.2 below. These functions are also locally Lipschitz continuous so that we could apply Theorem 1.6 as well.

Another condition yielding that the conclusion of Theorem 1.9 and thus Theorem 1.6 holds involves the *branch set* B_f which is defined as the set of all points where f fails to be locally injective. The *local index* $i(x, f)$ of a quasiregular map f at a point x is defined by

$$i(x, f) = \inf_U \sup_{y\in\mathbb{R}^n} \operatorname{card}\left(f^{-1}(y) \cap U\right),$$

where the infimum is taken over all neighbourhoods U of x. Thus

$$B_f = \{x : i(x, f) > 1\}.$$

With this notation we have the following analogue of [7, Theorem 1.8].

Theorem 1.10. *Let f be an entire quasiregular map of transcendental type. If $J(f) \cap B_f = \emptyset$ or, more generally, if the local index is bounded on $J(f)$, then conclusions* (ii), (iv) *and* (v) *of Theorem 1.6 hold and* $\operatorname{cap} J(f) > 0$.

If the local index is bounded on $\mathbb{R}^n$, then the conclusions of Theorems 1.6 and 1.9 hold.

We note that rational functions f for which B_f is contained in attracting basins (and thus, in particular, $J(f) \cap B_f = \emptyset$) are called *hyperbolic* or *expanding*. They play an important role in complex dynamics; cf. [25, 37]. The concept of hyperbolicity has also been extended to transcendental dynamics, see, e.g., [33].

Additional hypotheses like Lipschitz continuity or not having the pits effect are not needed in dimension 2.

Theorem 1.11. *Let $f\colon \mathbb{C}\to\mathbb{C}$ be a quasiregular map of transcendental type. Then the conclusions of Theorems 1.6 and 1.9 hold and* $\operatorname{cap} J(f)>0$.

This paper is organized as follows. In section 2 we recall the definition of quasiregularity, capacity and some other concepts needed and we collect a number of results that are used in the sequel. Section 3 deals with functions not having the pits effect. We prove Theorems 1.1 and 1.4 for such functions, and we also prove Theorem 1.9. In section 4 we consider functions with pits effect and prove Theorems 1.1 and 1.4 for such functions. Theorems 1.2, 1.3, 1.5, 1.6, 1.7, 1.8 and 1.10 are proved in section 5. The 2-dimensional case is then considered in section 6, where Theorem 1.11 is proved. Various examples illustrating our results are considered in section 7. Finally, some consequences of Harnack's inequality are discussed in section 8. In particular, we show that the definition of the pits effect can be modified as indicated above.

2. Quasiregular maps, capacity and Hausdorff measure

We refer to the monographs [29, 32] for a detailed treatment of quasiregular maps. Here we only recall the definition and the main properties needed.

For $n\geq 2$, a domain $\Omega\subset\mathbb{R}^n$ and $1\leq p<\infty$, the *Sobolev space* $W^1_{p,\mathrm{loc}}(\Omega)$ consists of the functions $f=(f_1,\dots,f_n)\colon\Omega\to\mathbb{R}^n$ for which all first order weak partial derivatives $\partial_k f_j$ exist and are locally in L^p. A continuous map $f\in W^1_{n,\mathrm{loc}}(\Omega)$ is called *quasiregular* if there exists a constant $K_O\geq 1$ such that

$$|Df(x)|^n\leq K_O J_f(x)\quad\text{a.e.},\tag{2.1}$$

where $Df(x)$ denotes the derivative,

$$|Df(x)|=\sup_{|h|=1}|Df(x)(h)|$$

its norm, and $J_f(x)$ the Jacobian determinant. Put

$$\ell(Df(x))=\inf_{|h|=1}|Df(x)(h)|.$$

The condition that (2.1) holds for some $K_O\geq 1$ is equivalent to the condition that

$$J_f(x)\leq K_I\ell(Df(x))^n\quad\text{a.e.},\tag{2.2}$$

for some $K_I\geq 1$. The smallest constants K_O and K_I satisfying (2.1) and (2.2) are called the *outer and inner dilatation* of f and are denoted by $K_O(f)$ and $K_I(f)$. We call $K(f)=\max\{K_I(f),K_O(f)\}$ the (maximal) *dilatation* of f and, for $K\geq 1$, say that f is K-*quasiregular* if $K(f)\leq K$.

If f and g are quasiregular, with f defined in the range of g, then $f\circ g$ is also quasiregular and [32, Theorem II.6.8]

$$K_I(f\circ g)\leq K_I(f)K_I(g)\quad\text{and}\quad K_O(f\circ g)\leq K_O(f)K_O(g)\tag{2.3}$$

so that $K(f\circ g)\leq K(f)K(g)$.

Quasiregularity can be defined more generally for maps between Riemannian manifolds. Here we only need the case of quasiregular maps $f\colon \Omega \to \overline{\mathbb{R}^n}$ where $\Omega \subset \overline{\mathbb{R}^n}$. Such maps are called *quasimeromorphic.* It turns out that a non-constant continuous map $f\colon \Omega \to \overline{\mathbb{R}^n}$ is quasimeromorphic if $f^{-1}(\infty)$ is discrete and f is quasiregular in $\Omega\backslash(f^{-1}(\infty) \cup \{\infty\})$.

Many properties of holomorphic functions carry over to quasiregular maps. For example, non-constant quasiregular maps are open and discrete. A key result already mentioned in the introduction is Rickman's analogue [30, 31] of Picard's Theorem.

Lemma 2.1. *For $n \geq 2$ and $K \geq 1$ there exists a constant $q = q(n, K)$ with the following property: if $a_1, \dots, a_q \in \mathbb{R}^n$ are distinct, then every K-quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n\backslash\{a_1, \dots, a_q\}$ is constant and every K-quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ for which $f^{-1}(\{a_1, \dots, a_q\})$ is finite is of polynomial type.*

The number $q(n, K)$ is called the *Rickman constant.* Note that Picard's theorem says that $q(2, 1) = 2$.

The following normal family analogue of Rickman's Theorem was obtained by Miniowitz [26, Theorem 5]. Here $\chi(x, y)$ denotes the chordal distance of two points $x, y \in \overline{\mathbb{R}^n}$. Thus $\chi(x, y) = |\pi^{-1}(x) - \pi^{-1}(y)|$ with the stereographic projection $\pi\colon S^n \to \overline{\mathbb{R}^n}$.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\mathcal{F}$ be a family of K-quasimeromorphic mappings on Ω. Let $q = q(n, K)$ be the Rickman constant and suppose that there exists $\delta > 0$ with the following property: for each $f \in \mathcal{F}$ there exist $a_1^f, \dots, a_{q+1}^f \in \overline{\mathbb{R}^n}$ satisfying $\chi(a_i^f, a_j^f) \geq \delta$ for $i \neq j$ such that $f(x) \neq a_j^f$ for all $x \in \Omega$ and $1 \leq j \leq q + 1$. Then $\mathcal{F}$ is normal.*

A family $\mathcal{F}$ of functions quasiregular (or quasimeromorphic) in a domain Ω is called *quasinormal* if every sequence in $\mathcal{F}$ has a subsequence which converges locally uniformly in $\Omega\backslash E$ for some finite subset E of Ω. Here the subset E may depend on the sequence. The following simple consequence of the maximum principle is useful in dealing with sequences converging outside a finite set; see, e.g., [3, Lemma 3.2].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let (f_k) be a non-normal sequence of functions which are K-quasiregular in Ω. If (f_k) converges locally uniformly in $\Omega\backslash E$ for some finite set E, then $f_k \to \infty$ in $\Omega\backslash E$.*

The next result is the analogue of Liouville's Theorem for quasiregular maps; see, e.g., [3, Lemma 3.4]. Here $M(r, f)$ denotes the maximum modulus of a quasiregular map f; that is, $M(r, f) = \max_{|x|=r} |f(x)|$.

Lemma 2.4. *Let f be an entire quasiregular map of transcendental type. Then*

$$\lim_{r\to\infty} \frac{\log M(r, f)}{\log r} = \infty.$$

An important tool in the theory of quasiregular mappings is the capacity of a condenser. We recall this concept briefly. For an open set $A \subset \mathbb{R}^n$ and a non-empty compact subset C of A, the pair (A, C) is called a *condenser* and its *capacity* $\operatorname{cap}(A, C)$ is defined by

$$\operatorname{cap}(A, C) = \inf_u \int_A |\nabla u|^n \, dm,$$

where the infimum is taken over all non-negative functions $u \in C_0^\infty(A)$ satisfying $u(x) \geq 1$ for all $x \in C$. Equivalently, one may take the infimum over all non-negative $u \in W^1_{n,\mathrm{loc}}(A)$ with compact support and $u(x) \geq 1$ for all $x \in C$. It follows directly from the definition that

$$\operatorname{cap}(A', C') \leq \operatorname{cap}(A, C) \quad \text{if } A' \supset A \text{ and } C' \subset C. \tag{2.4}$$

If $\operatorname{cap}(A, C) = 0$ for some bounded open set A containing C, then $\operatorname{cap}(A', C) = 0$ for every bounded open set A' containing C; see [32, Lemma III.2.2]. In this case we say that C is of *capacity zero*. Otherwise we say that C has *positive capacity*. We denote this by $\operatorname{cap} C = 0$ or $\operatorname{cap} C > 0$, respectively.

For an unbounded closed subset C of $\mathbb{R}^n$ we say that C has capacity zero if every compact subset of C has capacity zero. Equivalently, we can define condensers in $\overline{\mathbb{R}^n}$ and consider $\operatorname{cap}(A, C \cup \{\infty\})$ for open subsets A of $\overline{\mathbb{R}^n}$ with $\overline{A} \neq \overline{\mathbb{R}^n}$ which contain $C \cup \{\infty\}$.

For $a \in \mathbb{R}^n$ and $r > 0$, let $B^n(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$ be the open ball, $\overline{B^n}(a, r)$ the closed ball and $S^{n-1}(a, r) = \partial B^n(a, r)$ the sphere of radius r centred at a. We write $B^n(r)$, $\overline{B^n}(r)$ and $S^n(r)$ instead of $B^n(0, r)$, $\overline{B^n}(0, r)$ and $S^n(0, r)$. If there is no risk of confusion, we omit the superscript n for the dimension.

For a quasiregular map $f \colon \Omega \to \overline{\mathbb{R}^n}$, a point $y \in \overline{\mathbb{R}^n}$ and a Borel set E such that $\overline{E}$ is a compact subset of Ω, we denote by $n(E, y, f)$ the number of y-points of f in E, counted according to multiplicity. Thus

$$n(E, y, f) = \sum_{x \in f^{-1}(y) \cap E} i(x, f).$$

The average of $n(E, y, f)$ over all $y \in \overline{\mathbb{R}^n}$ is denoted by $A(E, f)$. Thus [32, p. 80]

$$A(E, f) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{n(E, y, f)}{(1 + |y|^2)^n} dy = \frac{1}{\omega_n} \int_E \frac{J_f(x)}{(1 + |f(x)|^2)^n} dx,$$

where ω_n denotes the volume of the n-dimensional unit ball.

Note that Rickman [32] identifies $\overline{\mathbb{R}^n}$ with $S^n(e_{n+1}/2, 1/2)$ while we have used $S^n = S^n(0, 1)$, implying that the formulas differ by a factor 2^n. Here and in the following e_k denotes the k-th unit vector.

We will write $n(r, y, f)$ and $A(r, f)$ instead of $n\big(\overline{B}(r), y, f\big)$ and $A\big(\overline{B}(r), f\big)$.

The following result [32, Theorem II.10.11] gives a connection between capacity and quasiregularity.

Lemma 2.5. *Let $f \colon \Omega \to \mathbb{R}^n$ be quasiregular, let (A, C) be a condenser in Ω and put $m = \inf_{y \in f(C)} n(C, y, f)$. Then*

$$\operatorname{cap}(f(A), f(C)) \leq \frac{K_I(f)}{m} \operatorname{cap}(A, C).$$

The following result was proved in [7, Theorem 3.2], based on ideas from [23].

Lemma 2.6. *Let $F \subset \overline{\mathbb{R}^n}$ be a set of positive capacity and let $\theta > 1$. Then there exists a constant C depending only on n, F and θ such that if $f\colon B^n(\theta r) \to \overline{\mathbb{R}^n}\backslash F$ is quasiregular, then $A(r, f) \leq C\, K_I(f)$.*

The condenser

$$E_G(t) = (B^n(1), [0, te_1])$$

is called the *Grötzsch condenser*. It has the following important extremal property [32, Lemma III.1.9].

Lemma 2.7. *Let (A, C) be a condenser with $A \subset B^n(r)$. Suppose that C is connected and that $0, x \in C$ where $x \in B^n(r)\backslash\{0\}$. Then*

$$\operatorname{cap}(A, C) \geq \operatorname{cap} E_G(|x|/r).$$

We shall also need the following bound for the capacity of the Grötzsch condenser [32, Lemma III.1.2].

Lemma 2.8. *There exists a constant λ_n depending only on n such that*

$$\operatorname{cap} E_G(t) \geq \omega_{n-1} \left(\log \frac{\lambda_n}{t} \right)^{1-n}$$

for $0 < t < 1$.

We denote the (Euclidean) diameter of a subset A of $\mathbb{R}^n$ by $\operatorname{diam} A$. For $\eta > 0$, an increasing, continuous function $h\colon (0, \eta) \to (0, \infty)$ satisfying $\lim_{t\to 0} h(t) = 0$ is called a *gauge function*. For $A \subset \mathbb{R}^n$ and $\delta > 0$, we call a sequence (A_j) of subsets of $\mathbb{R}^n$ a *δ-cover* of A if $\operatorname{diam} A_j < \delta$ for all $j \in \mathbb{N}$ and

$$A \subset \bigcup_{j=1}^{\infty} A_j.$$

We put

$$H_h^\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} h(\operatorname{diam} A_j) : (A_j) \text{ is } \delta\text{-cover of } A \right\}$$

and call

$$H_h(A) = \lim_{\delta \to 0} H_h^\delta(A)$$

the *Hausdorff measure* of A with respect to the gauge function h. Note that since $H_h^\delta(A)$ is a non-increasing function of δ, the limit defining $H_h(A)$ exists. The possibility that $H_h^\delta(A) = \infty$ is allowed, meaning that $\sum_{j=1}^{\infty} h(\operatorname{diam} A_j)$ diverges for all δ-covers (A_j).

In the special case that $h(t) = t^s$ for some $s > 0$, we call $H_h(A)$ the *s-dimensional Hausdorff measure*. There exists $d \in [0, n]$ such that $H_{t^s}(A) = \infty$ for $0 < s < d$ and $H_{t^s}(A) = 0$ for $s > d$. This value d is called the *Hausdorff dimension* of A and is denoted by $\dim A$.

Recall that a function $f\colon X \to \mathbb{R}^n$ where $X \subset \mathbb{R}^n$ is said to be *Hölder continuous* with exponent α if there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|^\alpha \quad \text{for all } x, y \in X.$$

In the special case that $\alpha = 1$ we say that f is *Lipschitz continuous* with *Lipschitz constant* L. The following result was proved in [7, Corollaries 9.1 and 9.2].

Lemma 2.9. *Let $X \subset \mathbb{R}^n$ be compact, $f\colon X \to \mathbb{R}^n$ and $\delta > 0$. Suppose that each $y \in X$ has m pre-images $x_1, \dots, x_m$ satisfying $|x_i - x_j| \geq \delta$ for $i \neq j$. Let $x \in X$.*

(i) *If f satisfies a Lipschitz condition with Lipschitz constant $L > 1$, then*

$$\dim \overline{O_f^-(x)} \geq \frac{\log m}{\log L}.$$

(ii) *If f satisfies a Hölder condition with exponent $\alpha < 1$, then*

$$H_h\left(\overline{O_f^-(x)}\right) > 0$$

for

$$h(t) = \left(\log \frac{1}{t}\right)^{(\log m)/(\log \alpha)}.$$

In [7] it is actually assumed that f maps X to X and the conclusion concerns $\dim X$ and $H_h(X)$. However, the proof yields the above result.

Part (ii) of the above result can be applied in particular to quasiregular maps by the following result [32, Theorem III.1.11].

Lemma 2.10. *Let $f\colon \Omega \to \mathbb{R}^n$ be quasiregular. Then f is locally Hölder continuous with exponent $K_I(f)^{1/(1-n)}$.*

The following result connecting capacity and Hausdorff measure can be found in [41].

Lemma 2.11. *Let $X \subset \mathbb{R}^n$ be compact and $\varepsilon > 0$. If $H_h(X) > 0$ for*

$$h(t) = \left(\log \frac{1}{t}\right)^{1-n-\varepsilon},$$

then $\operatorname{cap} X > 0$.

An immediate consequence is the following result.

Lemma 2.12. *Let $X \subset \mathbb{R}^n$ be compact. If* $\dim X > 0$*, then* $\operatorname{cap} X > 0$.

In particular, it follows that $\operatorname{cap} X > 0$ if X contains a non-degenerate continuum.

3. Functions without pits effect

In this section, we prove Theorem 1.1 and Theorem 1.4 for functions which do not have the pits effect. We also prove Theorem 1.9 which deals only with such functions.

Throughout this section, let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an entire quasiregular map of transcendental type which does not have the pits effect. We fix a large positive integer N and obtain $c > 1$, $\varepsilon > 0$ and a sequence (R_m) tending to ∞ such that

$\{x \in \mathbb{R}^n : \ R_m \leq |x| \leq cR_m, \ |f(x)| \leq 1\}$ cannot be covered by N balls of radius εR_m. We consider the functions $h_m : \mathbb{R}^n \to \mathbb{R}^n$,

$$h_m(x) = f(R_m x). \tag{3.1}$$

With $P = \{x \in \mathbb{R}^n : 1 \leq |x| \leq c\}$ we find that

$$\{x \in P : |h_m(x)| \leq 1\}$$

cannot be covered by N balls of radius ε. Thus there exist $x_1^m, \ldots, x_N^m \in P$ satisfying $|x_i^m - x_j^m| \geq \varepsilon$ for $i \neq j$ such that $|h_m(x_j^m)| \leq 1$ for $j \in \{1, \ldots, N\}$. Passing to a subsequence if necessary, we may assume that the sequences $(x_j^m)_{m \in \mathbb{N}}$ converge for $j \in \{1, \ldots, N\}$, say $x_j^m \to x_j$ as $m \to \infty$. Then $|x_i - x_j| \geq \varepsilon$ for $i \neq j$. We fix $r_1, \ldots, r_N$ satisfying $1 \leq r_1 < r_2 < \cdots < r_N \leq c$ and choose y_j^m such that $|y_j^m| = r_j$ and

$$|h_m(y_j^m)| = M(r_j, h_m) = M(R_m r_j, f).$$

Again we may assume that the sequences $(y_j^m)_{m \in \mathbb{N}}$ converge, say $y_j^m \to y_j$ as $m \to \infty$. We may choose pairwise disjoint curves γ_j which connect x_j with y_j and small neighbourhoods U_j of the curves γ_j such that their closures $\overline{U}_j$ are pairwise disjoint. Then $x_j \in U_j$ and $y_j \in U_j$ for $j \in \{1, \ldots, N\}$, and there exists $\delta > 0$ with

$$\operatorname{dist}(U_i, U_j) = \inf_{v \in U_i, w \in U_j} |v - w| \geq \delta$$

for $i \neq j$. Since $|h_m(x_j^m)| \leq 1$ and $x_j^m \to x_j$ while $|h_m(y_j^m)| \to \infty$ and $y_j^m \to y_j$, we find that the sequence (h_m) is not normal in any of the domains U_j. In fact, no subsequence of (h_m) is normal.

We shall also consider the functions $g_m : \mathbb{R}^n \to \mathbb{R}^n$,

$$g_m(x) = \frac{f(R_m x)}{R_m} = \frac{h_m(x)}{R_m}. \tag{3.2}$$

Using Lemma 2.4 we find that $|g_m(y_j^m)| \to \infty$ as $m \to \infty$. Since $|g_m(x_j^m)| \leq 1/R_m$ this implies that no subsequence of (g_m) is normal in any of the domains U_j.

Proof of Theorem 1.1 for functions without pits effect. With g_m and U_j as defined above we deduce from Lemma 2.2 that if $m \in \mathbb{N}$ is large enough, say $m \geq M$, and if $j \in \{1, \ldots, N\}$, then $g_m(U_j) \supset U_i$ for at least $N - q$ values of $i \in \{1, \ldots, N\}$. This implies that if $k \in \mathbb{N}$, $j \in \{1, \ldots, N\}$ and $m \geq M$, then, counting multiplicities, $g_m^k(U_j)$ covers at least $(N-q)^k$ of the domains U_i. This means that with $L = (N-q)^k$ there exist pairwise disjoint subsets $V_1, \ldots, V_L$ of U_j such that if $\ell \in \{1, \ldots, L\}$, then $g_m^k(V_\ell) = U_i$ for some $i \in \{1, \ldots, N\}$. Hence, for each $j \in \{1, \ldots, N\}$, there exists $i \in \{1, \ldots, N\}$ such that

$$n(U_j, y, g_m^k) \geq \frac{(N-q)^k}{N} \quad \text{for all } y \in U_i.$$

This implies that

$$A(U_j, g_m^k) \geq C_1 \frac{(N-q)^k}{N} \tag{3.3}$$

for some $C_1 > 0$ and all $j \in \{1, \ldots, N\}$.

Suppose now that $J(g_m) \cap \overline{U}_j = \emptyset$ for some $j \in \{1, \ldots, N\}$. Then for each $x \in \overline{U}_j$ there exists $r_x > 0$ such that $B(x, 2r_x) \cap J(g_m) = \emptyset$. Hence

$$A(B(x, r_x), g_m^k) \leq c_x K_I(g_m^k) \leq c_x K_I(g_m)^k$$

for some $c_x > 0$ by Lemma 2.6, (2.3) and the definition of $J(g_m)$. Since $\overline{U}_j$ can be covered by finitely many balls $B(x, r_x)$, we obtain

$$A(U_j, g_m^k) \leq C_2 K_I(g_m)^k = C_2 K_I(f)^k. \tag{3.4}$$

Choosing $N > K_I(f) + q$ we obtain a contradiction from (3.3) and (3.4). Thus $J(g_m) \cap \overline{U}_j \neq \emptyset$ for $j \in \{1, \ldots, N\}$. Since f is conjugate to g_m by the linear map $x \mapsto R_m x$, we deduce that $\operatorname{card} J(f) \geq N$. We obtain $\operatorname{card} J(f) = \infty$. □

Proof of Theorem 1.9. Let g_m and U_j be as before. Let p be the cardinality of the set of all $i \in \{1, \ldots, N\}$ such that $g_m(U_j) \supset U_i$ for at most $N/2$ values of $j \in \{1, \ldots, N\}$. Since each U_j has $N - q$ subsets mapped onto distinct domains U_i, we find that

$$p\frac{N}{2} + (N - p)N \geq N(N - q),$$

which is equivalent to $p \leq 2q$. We choose N divisible by 4 such that $N \geq 8q$. Then $N/4 \geq p$. Hence $3N/4$ of the domains U_i are contained in $N/2$ of the domains $g_m(U_j)$. With $L = 3N/4$ we may assume that $U_1, \ldots, U_L$ are contained in $N/2$ of the domains $g_m(U_j)$; that is, for each $i \in \{1, \ldots, L\}$,

$$\operatorname{card} \{j \in \{1, \ldots, N\} : g_m(U_j) \supset U_i\} \geq \frac{N}{2}.$$

Hence $\operatorname{card}\{j \in \{1, \ldots, L\} : g_m(U_j) \supset U_i\} \geq N/4$, which implies that

$$\operatorname{card} \left\{j \in \{1, \ldots, L\} : g_m\left(\overline{U}_j\right) \supset \overline{U}_i\right\} \geq \frac{N}{4},$$

for each $i \in \{1, \ldots, L\}$.

It now follows from Lemma 2.9, (ii), applied to $X = \bigcup_{j=1}^{L} \overline{U}_j$, and Lemma 2.10 that if $y \in X$, then

$$H_h\left(\overline{O_{g_m}^-(y)}\right) > 0,$$

where

$$h(t) = \left(\log \frac{1}{t}\right)^{-\beta} \quad \text{with } \beta = (n-1)\frac{\log(N/4)}{\log K_I(f)}.$$

Choosing $N > 4K_I(f) = 4K_I(g_m)$ and using Lemma 2.11 we obtain

$$\operatorname{cap} \overline{O_{g_m}^-(y)} > 0 \quad \text{for } y \in X. \tag{3.5}$$

Putting

$$V_j^m = R_m U_j = \{R_m x : x \in U_j\}$$

we obtain

$$\operatorname{cap} \overline{O_f^-(v)} > 0 \quad \text{for } v \in \bigcup_{j=1}^{L} \overline{V}_j^m, \tag{3.6}$$

provided $m \geq M$.

Let now $x \in \mathbb{R}^n \backslash E(f)$. Then $\operatorname{card} O_f^-(x) = \infty$ by the definition of $E(f)$. Recall that no subsequence of the sequence (h_m) defined by (3.1) is normal in any of the domains U_j. Lemma 2.2 implies that if $j \in \{1, \ldots, N\}$ and if m is sufficiently large, then there exists $u \in U_j$ such that $h_m(u) \in O_f^-(x)$. Putting $v = R_m u$ we obtain $v \in V_j^m$ and $f(v) \in O_f^-(x)$. This implies that $v \in O_f^-(x)$ and hence that $O_f^-(v) \subset O_f^-(x)$. Now (3.6) yields $\operatorname{cap} \overline{O_f^-(x)} > 0$. □

Proof of Theorem 1.4 for functions without pits effect. Using the terminology of the previous proof, and putting

$$U = \bigcup_{j=1}^{L} U_j \quad \text{and} \quad V = \bigcup_{j=1}^{L} V_j^m,$$

as well as $G = g_m|_U$ and $F = f|_V$, Lemma 2.9 and Lemma 2.11 actually show that (3.5) and (3.6) can be replaced by

$$\operatorname{cap} \overline{O_G^-(y)} > 0 \quad \text{for } y \in U \tag{3.7}$$

and

$$\operatorname{cap} \overline{O_F^-(v)} > 0 \quad \text{for } v \in V. \tag{3.8}$$

By a result of Siebert [36], f has infinitely many periodic points of period p for all $p \geq 2$. In particular, f has a periodic point ξ of period 2 with $\xi \notin E(f)$. As before, there exists $v \in O_f^-(\xi) \cap V$, provided m is large enough. Let l be such that $f^l(v) = \xi$. Then

$$W = O_F^-(v) \cup \left\{ f^k(v) : 0 \leq k \leq l + 1 \right\}$$

satisfies $f(W) \subset W$ and hence $f(\overline{W}) \subset \overline{W}$. Thus $\overline{W} \subset BO(f)$. Now the conclusion follows from (3.8). □

4. Functions with pits effect

In this section we prove Theorem 1.1 and Theorem 1.4 for entire quasiregular maps of transcendental type which have the pits effect. Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be such a map. Then there exists a sequence (x_m) tending to ∞ such that $|f(x_m)| \leq 1$ while $|f(x)| > 1$ for $|x_m| < |x| < 3|x_m|$. We put $R_m = |x_m|$ and define g_m by (3.2); that is, $g_m(x) = f(R_m x)/R_m$.

Using Harnack's inequality one can show that (g_m) is quasinormal; see Remark 8.1. However, the quasinormality of (g_m) is not essential and thus we briefly indicate how the argument can be completed if we assume that (g_m) is not quasinormal. Given $N \in \mathbb{N}$, we may then assume, passing to a subsequence if necessary, that there exist distinct points $z_1, \ldots z_N \in \mathbb{R}^n$ such that no subsequence of (g_m) is normal at any of these points. Choose neighbourhoods $U_1, \ldots, U_N$ of these points with pairwise disjoint closures. As in section 3 we now see that if $N > K_I(f) + q$, then $J(g_m) \cap \overline{U}_j \neq \emptyset$ for all $j \in \{1, \ldots, N\}$, provided m is large enough. Hence $J(f) \neq \emptyset$ and in fact $\operatorname{card} J(f) = \infty$, which proves Theorem 1.1 in this case.

Moreover, putting again $L = 3N/4$ and proceeding as in the proof of Theorem 1.9 we see that (3.5) holds if $N > 4K_I(f)$. The arguments used in [36] and [3] show that $\bigcup_{j=1}^{L} U_j$ contains a periodic point ξ of g_m. As in the proof of Theorem 1.4 for functions without pits effect we deduce from (3.7) that $\operatorname{cap} BO(g_m) > 0$. Hence $\operatorname{cap} BO(f) > 0$, which proves Theorem 1.4 in this case.

We will thus assume from now on that (g_m) is quasinormal. Passing to a subsequence if necessary, we may assume that (g_m) converges in $\mathbb{R}^n \backslash F$ where F is a finite set. We may assume that no subsequence of (g_m) is normal at any point of F since this can be achieved by deleting points from F and passing to a subsequence of (g_m). Using Lemma 2.4 it is easy to see that no subsequence of (g_m) is normal at 0. Thus Lemma 2.3 implies that $g_m \to \infty$ locally uniformly in $\mathbb{R}^n \backslash F$. Since $|f(x_m)| \leq 1$ we conclude that F contains a point of norm 1. Clearly we also have $0 \in F$. Thus $N = \operatorname{card} F \geq 2$. Let $F = \{y_1, \dots, y_N\}$ and choose

$$M > \max_{k=1,\dots,N} |y_k|$$

and $r > 0$ such that the closed balls of radius r around the points y_k are pairwise disjoint and contained in $B(0, M)$.

For large m we have

$$\inf_{x \in B(y_k, r)} |g_m(x)| < M \quad \text{for } 1 \leq k \leq N$$

while

$$|g_m(x)| > M \quad \text{for } x \in \overline{B}(0, M) \backslash \bigcup_{k=1}^{N} B(y_k, r).$$

Denote by $\overline{U}_{m,1}, \dots, \overline{U}_{m,\ell_m}$ the components of $g_m^{-1}(\overline{B}(0, M)) \cap B(0, M)$. Then each $B(y_k, r)$ contains at least one $\overline{U}_{m,j}$ so that $\ell_m \geq N \geq 2$. Denote by $U_{m,j}$ the interior of $\overline{U}_{m,j}$. (We do not assume here that $U_{m,j}$ is connected.) Then $g_m \colon U_{m,j} \to B(0, M)$ is a proper map. Since

$$\left| g_m\left(\frac{x_m}{R_m}\right) \right| = \frac{|f(x_m)|}{R_m} \leq \frac{1}{R_m} < M,$$

there exists j such that $x_m/R_m \in U_{m,j}$. As $g_m \colon U_{m,j} \to B(0, M)$ is proper, $U_{m,j}$ contains a zero ζ_m of g_m. Putting $z_m = R_m \zeta_m$ we thus have $f(z_m) = 0$ and $|z_m - x_m| < 2rR_m = 2r|x_m|$. Since r can be chosen arbitrarily small, we deduce that f has infinitely many zeros.

Denote by $d_{m,j}$ the degree of the proper map $g_m \colon U_{m,j} \to B(0, M)$ and put

$$d_m = \sum_{j=1}^{\ell_m} d_{m,j}.$$

Then $d_{m,j}$ equals the number of zeros of g_m in $U_{m,j}$ and thus

$$d_m = n(M, 0, g_m) = n(MR_m, 0, f) \to \infty$$

as $m \to \infty$. In particular we have $d_m > K_I(f)^2$ for large m. This condition will be needed in the proof of Theorem 1.4. For Theorem 1.1 the weaker condition $d_m > K_I(f)$ would suffice.

From now on we drop the index m and put

$$U = \bigcup_{j=1}^{\ell_m} U_{m,j},\ G = g_m,\ V_j = U_{m,j},\ L = \ell_m,\ D_j = d_{m,j} \text{ and } D = d_m.$$

Then $L \geq 2$ and $V_1, \ldots, V_L$ are open subsets of $B(0, M)$ with disjoint closures such that $G\colon V_j \to B(0, M)$ is a proper map of degree D_j for $1 \leq j \leq L$ and

$$D = \sum_{j=1}^{L} D_j > K_I(G)^2. \tag{4.1}$$

Note here that $K_I(G) = K_I(f)$.

For $i, j \in \{1, \ldots, L\}$ let $\overline{W}_1^{i,j}, \ldots, \overline{W}_t^{i,j}$ be the components of $G^{-1}(\overline{V}_j) \cap V_i$, with $t = t_{i,j} \geq 1$. Denote by $W_s^{i,j}$ the interior of $\overline{W}_s^{i,j}$. Then $G\colon W_s^{i,j} \to V_j$ is a proper map and

$$\sum_{i=1}^{L} \sum_{s=1}^{t_{i,j}} \deg\big(G\colon W_s^{i,j} \to V_j\big) = D. \tag{4.2}$$

Now $G^2\colon W_s^{i,j} \to B(0, M)$ is also a proper map and

$$\sum_{i=1}^{L} \sum_{s=1}^{t_{i,j}} \deg\big(G^2\colon W_s^{i,j} \to B(0, M)\big) = D_j D.$$

Putting

$$L_2 = \sum_{i=1}^{L} \sum_{j=1}^{L} t_{i,j}$$

and writing

$$\big\{W_s^{i,j} : 1 \leq i, j \leq L,\ 1 \leq s \leq t_{i,j}\big\} = \big\{V_\ell^2 : 1 \leq \ell \leq L_2\big\}$$

we obtain L_2 open subsets V_ℓ^2 of $B(0, M)$ with pairwise disjoint closures such that $G^2\colon V_\ell^2 \to B(0, M)$ is a proper map with

$$\sum_{\ell=1}^{L_2} \deg\big(G^2\colon V_\ell^2 \to B(0, M)\big) = D^2.$$

Note that $L_2 \geq L^2$.

Inductively we find that for $k \in \mathbb{N}$ there exist $L_k \geq L^k$ and open subsets $V_1^k, \ldots, V_{L_k}^k$ of $B(0, M)$ with pairwise disjoint closures $\overline{V}_1^k, \ldots, \overline{V}_{L_k}^k$ such that $G^k\colon V_\ell^k \to B(0, M)$ is a proper map for $1 \leq \ell \leq L_k$, with

$$\sum_{\ell=1}^{L_k} \deg\big(G^k\colon V_\ell^k \to B(0, M)\big) = D^k. \tag{4.3}$$

By construction,

$$\bigcup_{\ell=1}^{L_{k+1}} \overline{V}_\ell^{k+1} \subset \bigcup_{\ell=1}^{L_k} V_\ell^k.$$

Proof of Theorem 1.1 for functions with pits effect. Let $V_1^k, \dots, V_{L_k}^k$ be as above and put

$$V^k = \bigcup_{\ell=1}^{L_k} V_\ell^k.$$

Then $n(V^k, a, G^k) = D^k$ for all $a \in B(0, M)$ by (4.3). Thus

$$A(V^k, G^k) \geq C_1 D^k \tag{4.4}$$

for some $C_1 > 0$. Suppose now that $\overline{V}^k \cap J(G) = \emptyset$. Then for each $x \in \overline{V}^k$ there exists $r_x > 0$ such that $B(x, 2r_x) \cap J(G) = \emptyset$. As in section 3, we deduce from Lemma 2.6 and the definition of $J(G)$ that

$$A(B(x, r_x), G^k) \leq c_x K_I(G^k) \leq c_x K_I(G)^k$$

for some $c_x > 0$. Covering $\overline{V}^k$ by finitely many balls $B(x, r_x)$ we obtain

$$A(V^k, G^k) \leq C_2 K_I(G)^k$$

for some $C_2 > 0$. Since $K_I(G) = K_I(f)$ this contradicts (4.1) and (4.4) for large k. Thus $\overline{V}^k \cap J(G) \neq \emptyset$.

Since $G^k \colon V_\ell^k \to B(0, M)$ is proper and thus in particular surjective, $\overline{V}^k \subset B(0, M)$ and $J(G)$ is completely invariant, it follows that every set V_ℓ^k meets $J(G)$. Thus $\operatorname{card} J(G) \geq L_k$ for all k and hence $\operatorname{card} J(G) = \infty$. As G is conjugate to f, the conclusion follows. □

Proof of Theorem 1.4 for functions with pits effect. We again use the terminology introduced above. Suppose first that

$$L_k > K_I(G^k) \quad \text{for some } k \in \mathbb{N}. \tag{4.5}$$

By [36, Lemma 2.1.5], each V_j contains a fixed point v_j of G. Similarly as in section 3 we can now deduce from (4.5), together with Lemma 2.9 and Lemma 2.11, that $\operatorname{cap} \overline{O_{G^k}^-(v_j)} > 0$ and hence that $\operatorname{cap} BO(G) > 0$. Thus $\operatorname{cap} BO(f) > 0$.

Suppose now that (4.5) does not hold. Then

$$L_k \leq K_I(G^k) \leq K_I(G)^k \tag{4.6}$$

for all $k \in \mathbb{N}$. Put

$$X = \bigcap_{k=1}^{\infty} \bigcup_{\ell=1}^{L_k} \overline{V}_\ell^k.$$

Clearly X is compact and $X \subset BO(G)$. Thus it suffices to prove that $\operatorname{cap} X > 0$. Suppose that this is not the case. Then $\dim X = 0$ by Lemma 2.12.

This implies that for each $\eta > 0$ there exists k_0 such that

$$\operatorname{diam} V_\ell^k < \eta \quad \text{for } k \geq k_0 \text{ and } 1 \leq \ell \leq L_k. \tag{4.7}$$

In fact, suppose that there exist $\eta > 0$ and $x_j, y_j \in V_{\ell_j}^{k_j}$ with $k_j \to \infty$ such that $|x_j - y_j| \geq \eta$. We may assume that (x_j) and (y_j) converge, say $x_j \to x_0$ and $y_j \to y_0$. Then $x_0, y_0 \in X$. Since $\dim X < 1$, there exists a hyperplane H in $\mathbb{R}^n \backslash X$ that separates x_0 and y_0. For large j the points x_j and y_j are on opposite sides of the hyperplane H and thus there exists $z_j \in \overline{V}_j^{k_j} \cap H$. We may assume that (z_j) converges, say $z_j \to z_0$, since otherwise we may pass to a subsequence. Then $z_0 \in X \cap H$ since $X \cap H$ is compact. This contradicts $H \subset \mathbb{R}^n \backslash X$. Thus (4.7) holds.

It follows from (4.3) and (4.6) that for all $k \in \mathbb{N}$ there exists $\ell_k \in \{1, \ldots, L_k\}$ such that

$$\deg\big(G^k\colon V_{\ell_k}^k \to B(0,M)\big) \geq \frac{D^k}{L_k} \geq \left(\frac{D}{K_I(G)}\right)^k . \tag{4.8}$$

We fix a large k and put $W_k = V_{\ell_k}^k$ and $W_j = G^{k-j}(W_k)$ for $0 \leq j \leq k-1$ so that $W_0 = B(0,M)$. Then $G\colon W_j \to W_{j-1}$ is a proper map for $1 \leq j \leq k$. Put

$$P_j = \deg(G\colon W_j \to W_{j-1}) .$$

Then

$$\prod_{j=1}^{k} P_j = \deg\big(G^k\colon W_k \to W_0\big)$$

so that

$$\prod_{j=1}^{k} P_j \geq \left(\frac{D}{K_I(G)}\right)^k \tag{4.9}$$

by (4.8). Let $w_k \in G^{-k}(0) \cap W_k$ and put $w_j = G^{k-j}(w_k)$ for $0 \leq j \leq k-1$. Thus $w_j \in W_j$ for $0 \leq j \leq k$ and $w_0 = 0$.

Let now $\varepsilon > 0$ and choose $\delta > 0$ such that $|G(x) - G(y)| < \varepsilon$ for $x, y \in \bigcup_{j=1}^{L} V_j$ satisfying $|x - y| < \delta$. We take $\eta < \min\{\varepsilon, \delta\}$ and choose k_0 such that (4.7) is satisfied.

Assuming that $k > k_0$, we denote by Y_j, for $k_0 < j \leq k$, the component of $G^{-1}(B(w_{j-1}, \varepsilon))$ that contains w_j. Then $Y_j \supset B(w_j, \delta) \supset \overline{W}_j$.

By Lemma 2.5 we have

$$\begin{aligned}\operatorname{cap}\big(Y_j, \overline{W}_j\big) &\geq \frac{P_j}{K_I(G)} \operatorname{cap}\big(G(Y_j), G(\overline{W}_j)\big)\\ &= \frac{P_j}{K_I(G)} \operatorname{cap}\big(B(w_{j-1}, \varepsilon), \overline{W}_{j-1}\big) .\end{aligned}$$

By the extremality of the Grötzsch condenser (Lemma 2.7) and the lower bound for its capacity given by Lemma 2.8 we have

$$\begin{aligned}\operatorname{cap}\big(B(w_{j-1}, \varepsilon), \overline{W}_{j-1}\big) &\geq \operatorname{cap} E_G\left(\frac{\operatorname{diam} W_{j-1}}{\varepsilon}\right)\\ &\geq \omega_{n-1}\left(\log\left(\frac{\lambda_n \varepsilon}{\operatorname{diam} W_{j-1}}\right)\right)^{1-n} .\end{aligned}$$

On the other hand, (2.4) yields

$$\begin{aligned}\operatorname{cap}\big(Y_j,\overline{W}_j\big) &\leq \operatorname{cap}\big(B(w_j,\delta),\overline{W}_j\big)\\ &\leq \operatorname{cap}\big(B(w_j,\delta),\overline{B}(w_j,\operatorname{diam} W_j)\big)\\ &= \omega_{n-1}\left(\log\left(\frac{\delta}{\operatorname{diam} W_j}\right)\right)^{1-n}.\end{aligned}$$

Combining the last three estimates we obtain

$$\left(\log\left(\frac{\delta}{\operatorname{diam} W_j}\right)\right)^{1-n} \geq \frac{P_j}{K_I(G)}\left(\log\left(\frac{\lambda_n\varepsilon}{\operatorname{diam} W_{j-1}}\right)\right)^{1-n}$$

for $k_0 < j \leq k$. Equivalently,

$$\log\left(\frac{\lambda_n\varepsilon}{\operatorname{diam} W_{j-1}}\right) \geq \left(\frac{P_j}{K_I(G)}\right)^{1/(n-1)} \log\left(\frac{\delta}{\operatorname{diam} W_j}\right). \tag{4.10}$$

Now

$$\log\left(\frac{\lambda_n\varepsilon}{\operatorname{diam} W_{j-1}}\right) = \log\left(\frac{\delta}{\operatorname{diam} W_{j-1}}\right) + \log\frac{\lambda_n\varepsilon}{\delta}.$$

Given $\tau > 1$, we may choose η so small that $\log(\lambda_n\varepsilon/\delta) \leq (\tau-1)\log(\delta/\eta)$. Then

$$\log\left(\frac{\lambda_n\varepsilon}{\operatorname{diam} W_{j-1}}\right) \leq \tau\log\left(\frac{\delta}{\operatorname{diam} W_{j-1}}\right)$$

for $k_0 < j \leq k$. Hence (4.10) takes the form

$$\log\left(\frac{\delta}{\operatorname{diam} W_j}\right) \leq \tau\left(\frac{K_I(G)}{P_j}\right)^{1/(n-1)} \log\left(\frac{\delta}{\operatorname{diam} W_{j-1}}\right)$$

for $k_0 < j \leq k$. We conclude that

$$\log\left(\frac{\delta}{\operatorname{diam} W_k}\right) \leq \tau^{k-k_0}\left(\frac{K_I(G)^{k-k_0}}{\prod_{j=k_0+1}^{k} P_j}\right)^{1/(n-1)} \log\left(\frac{\delta}{\operatorname{diam} W_{k_0}}\right).$$

Using (4.9) we obtain

$$\log\left(\frac{\delta}{\operatorname{diam} W_k}\right) \leq C_0\left(\frac{\tau^{n-1}K_I(G)^2}{D}\right)^{k/(n-1)} \log\left(\frac{\delta}{\operatorname{diam} W_{k_0}}\right)$$

for some constant C_0. With

$$\delta_0 = \inf_{1\leq\ell\leq L_{k_0}} \operatorname{diam} V_\ell^{k_0}$$

we obtain

$$\log\left(\frac{\delta}{\eta}\right) \leq C_0\left(\frac{\tau^{n-1}K_I(G)^2}{D}\right)^{k/(n-1)} \log\left(\frac{\delta}{\delta_0}\right).$$

For τ close to 1 and large k this contradicts (4.1). $\square$

5. Proof of Theorems 1.2, 1.3, 1.5, 1.6, 1.7, 1.8 and 1.10

Proof of Theorem 1.2. Let $f\colon \mathbb{C}\to\mathbb{C}$ be an entire transcendental function and let $J_0(f)$ be the classical Julia set; that is, the set of all $z\in\mathbb{C}$ where the iterates of f do not form a normal family. We refer to [2] for the basic properties of $J_0(f)$. Let $J(f)$ be as in Definition 1.1, with $n=2$ so that $\mathbb{R}^n=\mathbb{R}^2=\mathbb{C}$.

Let $x\in J_0(f)$ and let U be a neighbourhood of x. It is a simple consequence of Montel's theorem that card $\big(\mathbb{C}\backslash O_f^+(U)\big)\leq 1$ and hence that

$$\operatorname{cap}\big(\mathbb{C}\backslash O_f^+(U)\big)=0. \tag{5.1}$$

This implies that $x\in J(f)$.

Let now $x\in J(f)$ and let U be a neighbourhood of x. Thus (5.1) holds. By a result of Baker [1], $J_0(f)$ contains continua and thus $\operatorname{cap} J_0(f)>0$ by Lemma 2.12. Therefore $J_0(f)$ is not a subset of $\mathbb{C}\backslash O_f^+(U)$, which means that $J_0(f)\cap O_f^+(U)\neq\emptyset$. By the complete invariance of $J_0(f)$ we have $J_0(f)\cap U\neq\emptyset$. As U can be taken arbitrarily small and $J_0(f)$ is closed, we now deduce that $x\in J_0(f)$. □

In order to prove Theorem 1.3, we formulate the following result of [9] already mentioned in the introduction.

Lemma 5.1. *Let f be an entire quasiregular map of transcendental type. Then $I(f)$ has at least one unbounded component.*

Proof of Theorem 1.3. Let $x\in J(f)$. Then (1.2) holds for every neighbourhood U of x. Since $\operatorname{cap} BO(f)>0$ by Theorem 1.4 and $\operatorname{cap} I(f)>0$ by Lemma 5.1 and Lemma 2.12, we have $BO(f)\cap O_f^+(U)\neq\emptyset$ and $I(f)\cap O_f^+(U)\neq\emptyset$. Since both $BO(f)$ and $I(f)$ are completely invariant, this implies that $BO(f)\cap U\neq\emptyset$ and $I(f)\cap U\neq\emptyset$. As $BO(f)\cap I(f)=\emptyset$, we deduce that $\partial BO(f)\cap U\neq\emptyset$ and $\partial I(f)\cap U\neq\emptyset$. Since this holds for every neighbourhood U of x, we conclude that $x\in\partial BO(f)$ and $x\in\partial I(f)$. □

Proof of Theorem 1.5. Let $x\in A(\xi)$. Then there exists a neighbourhood U of x such that $f^k|_U\to\xi$ uniformly. Replacing U by a smaller neighbourhood if necessary, we may achieve that $O_f^+(U)\subset B(0,R)$ for some $R>0$. Thus $\operatorname{cap}\big(\mathbb{R}^n\backslash O_f^+(U)\big)>0$ which implies that $x\notin J(f)$.

Let now $x\in J(f)$. Then $x\notin A(\xi)$ by what we have proved already. Suppose that $x\notin\partial A(\xi)$. Then there exists a neighbourhood U of x such that $U\cap A(\xi)=\emptyset$. Since $A(\xi)$ is completely invariant this implies that $O_f^+(U)\cap A(\xi)=\emptyset$ and thus $A(\xi)\subset\mathbb{R}^n\backslash O_f^+(U)$. Hence $\operatorname{cap} A(\xi)=0$ by (1.2), which is a contradiction since $A(\xi)$ is open. □

Theorems 1.6, 1.7 and 1.8 will follow from the next three results.

Theorem 5.1. *Let f be an entire quasiregular map of transcendental type. Suppose that*

$$\operatorname{cap}\overline{O_f^-(x)}>0 \quad \textit{for all } x\in\mathbb{R}^n\backslash E(f). \tag{5.2}$$

Then the conclusions (i)–(iv) *of Theorem 1.6 hold.*

Proof. Let $x \in \mathbb{R}^n \backslash E(f)$ and let U be an open set intersecting $J(f)$. It follows from (1.2) and (5.2) that $O_f^+(U) \cap \overline{O_f^-(x)} \neq \emptyset$. Since $O_f^+(U)$ is open and $\overline{O_f^-(x)}$ is closed this actually yields that $O_f^+(U) \cap O_f^-(x) \neq \emptyset$. This in turn yields both

$$x \in O_f^+(U) \tag{5.3}$$

and

$$U \cap O_f^-(x) \neq \emptyset. \tag{5.4}$$

Since (5.4) holds for every open set U intersecting $J(f)$, we have $J(f) \subset \overline{O_f^-(x)}$ and thus (i). Now (ii) follows from (i) since $J(f)$ is closed and completely invariant. Since (5.3) holds for all $x \in \mathbb{R}^n \backslash E(f)$ we obtain $\mathbb{R}^n \backslash E(f) \subset O_f^+(U)$ which immediately yields (iii).

To prove (iv) we only have to show that $J(f)$ does not contain isolated points. Now it follows immediately from (ii) that non-periodic points in $J(f)$ are not isolated in $J(f)$. Also, $J(f) \backslash E(f)$ contains non-periodic points, since if $y \in J(f) \backslash E(f)$ is periodic, then all points in $O_f^-(y) \backslash O_f^+(y)$ are non-periodic. Thus there exists a non-isolated point $x \in J(f) \backslash E(f)$. It now follows from (ii) that no point of $J(f)$ is isolated. □

Theorem 5.2. *Let f be an entire quasiregular map of transcendental type and let $p \in \mathbb{N}$. If $\operatorname{cap} J(f^p) > 0$, then $J(f^p) = J(f)$.*

Proof. It follows immediately from the definition that $J(f^p) \subset J(f)$. In order to prove the reverse inclusion, let $x \in J(f)$ and let U be a neighbourhood of x. Since $\operatorname{cap}\big(\mathbb{R}^n \backslash O_f^+(U)\big) = 0$ but $\operatorname{cap} J(f^p) > 0$, there exists $y \in O_f^+(U) \cap J(f^p)$, say $y = f^m(z)$ where $m \in \mathbb{N}$ and $z \in U$. As $V := f^m(U)$ is a neighbourhood of $y \in J(f^p)$, we have

$$\operatorname{cap}\big(\mathbb{R}^n \backslash O_{f^p}^+(V)\big) = 0. \tag{5.5}$$

We write $m = kp - l$ with $k \in \mathbb{N}$ and $l \in \{0, 1, \dots, p-1\}$. Then $V = f^{kp}(f^{-l}(U))$ and hence

$$f^l\big(O_{f^p}^+(V)\big) \subset f^l\big(O_{f^p}^+\big(f^{-l}(U)\big)\big) = O_{f^p}^+(U).$$

Noting that $f^l(\mathbb{R}^n) \supset \mathbb{R}^n \backslash E(f^l)$ we deduce that

$$\begin{aligned} \mathbb{R}^n \backslash O_{f^p}^+(U) &\subset \mathbb{R}^n \backslash f^l\big(O_{f^p}^+(V)\big) \\ &\subset \big(f^l(\mathbb{R}^n) \backslash f^l\big(O_{f^p}^+(V)\big)\big) \cup E(f^l) \\ &\subset f^l\big(\mathbb{R}^n \backslash O_{f^p}^+(V)\big) \cup E(f^l). \end{aligned} \tag{5.6}$$

Lemma 2.5 implies in particular that quasiregular mappings map sets of capacity zero to sets of capacity zero. Thus $\operatorname{cap} f^l\big(\mathbb{R}^n \backslash O_{f^p}^+(V)\big) = 0$ by (5.5). Since $E(f^l)$ is finite, we can now deduce from (5.6) that $\operatorname{cap}\big(\mathbb{R}^n \backslash O_{f^p}^+(U)\big) = 0$. Since this holds for every neighbourhood U of x, we conclude that $x \in J(f^p)$. □

Theorem 5.3. *Let f be an entire quasiregular map of transcendental type which is locally Lipschitz continuous. Then $\dim \overline{O_f^-(x)} > 0$ for all $x \in \mathbb{R}^n \backslash E(f)$.*

Proof. For functions not having the pits effect, the proof can be carried out in exactly the same way as the proof of Theorem 1.9, using part (i) of Lemma 2.9 instead of part (ii).

For functions with pits effect we proceed as in section 4 to obtain subsets $V_1, \dots, V_L$ of $B(0, M)$ with disjoint closures such that $G\colon V_j \to B(0, M)$ is proper for $1 \le j \le L$, where $G = g_m$ is conjugate to f by the map $x \mapsto R_m x$. By hypothesis, there exists $\lambda > 0$ such that

$$|G(x) - G(y)| \le \lambda |x - y| \quad \text{for } x, y \in B(0, M).$$

Lemma 2.9, (i), now yields

$$\dim \overline{O_G^-(x)} \ge \frac{\log L}{\log \lambda} > 0$$

for all $x \in B(0, M)$. It follows that $\dim \overline{O_f^-(x)} > 0$ for all $x \in B(0, MR_m)$, and as $R_m \to \infty$ this holds for all $x \in \mathbb{R}^n$. □

Proof of Theorem 1.6 and Theorem 1.7. The conclusions (i)–(iv) of Theorem 1.6 follow immediately from Theorem 5.3, Lemma 2.12 and Theorem 5.1. To prove conclusion (v) and Theorem 1.7 let $p \in \mathbb{N}$. Since f^p is also locally Lipschitz continuous we have $J(f^p) = \overline{O_{f^p}^-(x)}$ for all $x \in J(f^p) \backslash E(f^p)$ by (ii). Note here that $J(f^p) \backslash E(f^p) \ne \emptyset$ since $\operatorname{card} J(f^p) = \infty$ by Theorem 1.1 while $E(f^p)$ is finite. Thus $\dim J(f^p) > 0$ by Theorem 5.3. This proves Theorem 1.7. Moreover, together with Lemma 2.12 this yields $\operatorname{cap} J(f^p) > 0$. Conclusion (v) of Theorem 1.6 now follows from Theorem 5.2. □

Proof of Theorem 1.8. The conclusions (i)–(iv) are immediate consequences of Theorems 1.9 and 5.1. In order to prove (v), let $p \in \mathbb{N}$ and put $C = M(1, f^{p-1})$. Since f does not have the pits effect we see that if $N \in \mathbb{N}$, then there exist $c > 1$ and $\varepsilon > 0$ such that for arbitrarily large R the set

$$\{x \in \mathbb{R}^n : R \le |x| \le cR,\ |f(x)| \le 1\}$$

cannot be covered by N balls of radius εR. This implies that

$$\{x \in \mathbb{R}^n : R \le |x| \le cR,\ |f^p(x)| \le C\}$$

cannot be covered by N such balls. Thus $g(x) := f^p(Cx)/C$ does not have the pits effect. By Corollary 1.1 we have $\operatorname{cap} J(g) > 0$. Since f^p and g are conjugate this implies that $\operatorname{cap} J(f^p) > 0$. Conclusion (v) now follows from Theorem 5.2. □

Remark 5.1. In the above proof, instead of passing from f^p to g we could also have used Theorem 8.1 which implies that the inequality $|f(x)| \le 1$ in the definition of the pits effect can be replaced by $|f(x)| \le C$ for any positive constant C.

The following result will be used to prove Theorem 1.10.

Theorem 5.4. *Let f be an entire quasiregular map of transcendental type. If $x \in \mathbb{R}^n \backslash E(f)$ and the local index is bounded on $\overline{O_f^-(x)}$, then* $\operatorname{cap} \overline{O_f^-(x)} > 0$.

Proof. Let $x \in \mathbb{R}^n \backslash E(f)$ and suppose that the local index $i(y,f)$ is bounded on $\overline{O_f^-(x)}$. By Theorem 1.9 we may assume that f has the pits effect. Proceeding as in section 4, we find a subset $U \subset B(0,M)$ such that $G\colon U \to B(0,M)$ is a proper map of degree D, where $G(y) = g_m(y) = f(R_m y)/R_m$ for $y \in U$. By choosing R_m sufficiently large, we may assume that $x \in B(0, MR_m)$ and that

$$D > K_I(f) \max\{i(y,f) : y \in \overline{O_f^-(x)}\}. \tag{5.7}$$

Putting $X = \overline{O_G^-(x/R_m)}$, it will suffice to show that $\operatorname{cap} X > 0$.

As in the proof of Theorem 1.9, we can use Lemmas 2.9–2.11 to deduce that $\operatorname{cap} X > 0$ if each $y \in X$ has P pre-images $x_1, \dots, x_P$ under G satisfying $|x_i - x_j| \geq \delta$ for $i \neq j$, where $\delta > 0$ and P is the least integer greater than $K_I(f)$. We suppose that this is not the case. Then there exist $\delta_k \to 0$, $y_k \in X$ and $x_1^k, \dots, x_{P-1}^k \in G^{-1}(y_k) \subset X$ such that

$$G^{-1}(y_k) \subset \bigcup_{j=1}^{P-1} B(x_j^k, \delta_k).$$

Passing to a subsequence, we can assume that $y_k \to y_0 \in X$ and $x_j^k \to x_j \in X$ as $k \to \infty$. It can then be shown that $G^{-1}(y_0) \subset \{x_1, \dots, x_{P-1}\}$. However, $n(U, y_0, G) = D$ and so there must be some $j \in \{1, \dots, P-1\}$ for which

$$i(x_j, G) \geq D/(P-1) \geq D/K_I(f).$$

This leads to a contradiction with (5.7) since $i(x_j, G) = i(R_m x_j, f)$ and $R_m x_j \in \overline{O_f^-(x)}$. $\square$

Proof of Theorem 1.10. If the local index is bounded on $J(f)$, then applying Theorem 5.4 to some $x \in J(f) \backslash E(f)$ yields $\operatorname{cap} \overline{O_f^-(x)} > 0$. Thus $\operatorname{cap} J(f) > 0$ by complete invariance. Moreover, arguing as in the proof of Theorem 5.1 shows that conclusions (ii) and (iv) of Theorem 1.6 hold.

By the complete invariance of $J(f)$ the local index of f^p is also bounded on $J(f)$. Thus, since always $J(f^p) \subset J(f)$, the local index of f^p is bounded on $J(f^p)$. Hence $\operatorname{cap} J(f^p) > 0$ by what we have proved already. Now (v) follows from Theorem 5.2.

If the local index is bounded on $\mathbb{R}^n$, then the conclusion of Theorem 1.9 holds by Theorem 5.4. The conclusions (i)–(iv) of Theorem 1.6 then follow from Theorem 5.1. $\square$

6. The 2-dimensional case

Let $f\colon \Omega \to \mathbb{R}^n$ be quasiregular. For $n = 2$, the branch set B_f is a discrete subset of Ω. This is in contrast to the situation for $n \geq 3$, where the $(n-2)$-dimensional Hausdorff measure of $f(B_f)$ is positive unless $B_f = \emptyset$; see [32, Proposition III.5.3].

If $n = 2$, the elements of B_f are called *critical points*. For $x \in B_f$ we call $i(x,f) - 1$ the *multiplicity* of the critical point x. The following result [37, §1.3] is known as the Riemann-Hurwitz Formula.

Lemma 6.1. *Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ of connectivity c_1 and c_2, respectively. Let $f\colon \Omega_1 \to \Omega_2$ be a proper quasiregular map of degree d and denote by r the number of critical points of f, counting multiplicity; that is,*

$$r = \sum_{x \in B_f} (i(x,f) - 1).$$

Then

$$c_1 - 2 = d(c_2 - 2) + r. \tag{6.1}$$

In [37] it is assumed that f is holomorphic, but the result also holds for quasiregular maps and in fact for ramified coverings [25, p. 68]. We shall only need the case that $c_1 = c_2 = 1$. Then (6.1) simplifies to $r = d - 1$.

Proof of Theorem 1.11. By Theorem 1.8 it suffices to consider functions with pits effect. Let f be such a function and let $V_1, \dots, V_L$ and G be as in section 4. Thus $G\colon V_j \to B(0,M)$ is a proper map of degree D_j and (4.1) holds. Moreover, let also $W_s^{i,j}$ be as in section 4. Thus $W_s^{i,j} \subset V_i$ and $G\colon W_s^{i,j} \to V_j$ is a proper map for $1 \le i,j \le L$ and $1 \le s \le t_{i,j}$. We may assume that the V_j and $W_s^{i,j}$ are connected, as this is the case if $|G(c)| \ne M$ and $|G^2(c)| \ne M$ for all critical points c of G, and hence can be achieved by perturbing M slightly. By the maximum principle, the V_j and $W_s^{i,j}$ are in fact simply connected. By the Riemann-Hurwitz Formula (Lemma 6.1), $W_s^{i,j}$ contains $\deg(G\colon W_s^{i,j} \to V_j) - 1$ critical points, counting multiplicities.

Let

$$Y_j = G^{-1}(V_j) \cap B(0,M) = \bigcup_{i=1}^{L} \bigcup_{s=1}^{t_{i,j}} W_s^{i,j}$$

and denote by μ_j the number of critical points of G in Y_j. Then

$$\begin{aligned}\mu_j &= \sum_{i=1}^{L} \sum_{s=1}^{t_{i,j}} \left(\deg\left(G\colon W_s^{i,j} \to V_j\right) - 1\right)\\ &= \sum_{i=1}^{L} \sum_{s=1}^{t_{i,j}} \deg\left(G\colon W_s^{i,j} \to V_j\right) - \sum_{i=1}^{L} t_{i,j}\\ &= D - \sum_{i=1}^{L} t_{i,j}\end{aligned}$$

by (4.2). Hence

$$LD - \sum_{j=1}^{L} \sum_{i=1}^{L} t_{i,j} = \sum_{j=1}^{L} \mu_j \le D.$$

Defining L_2 and $V_1^2, \dots, V_{L_2}^2$ as in section 4 we obtain

$$L_2 = \sum_{j=1}^{L} \sum_{i=1}^{L} t_{i,j} \ge (L-1)D \ge D$$

and thus

$$L_2 > K_I(G)^2 \geq K_I(G^2) \tag{6.2}$$

by (4.1). Recall that $G^2\colon V_j^2 \to B(0,M)$ is a proper map and $\overline{V}_j^2 \subset B(0,M)$ for $1 \leq j \leq L_2$. Similarly as before we can now deduce from (6.2), Lemma 2.9 and Lemma 2.11 that $\operatorname{cap} \overline{O_{G^2}^-(x)} > 0$ for all $x \in B(0,M)$. This implies that $\operatorname{cap} \overline{O_f^-(x)} > 0$ for all $x \in \mathbb{R}^2$; that is, the conclusion of Theorem 1.9 holds. Theorem 5.1 now implies that the conclusions (i)–(iv) of Theorem 1.6 holds. Using part (ii) of this theorem we conclude that $\operatorname{cap} J(f) > 0$.

We may apply this reasoning also to the iterates of f. Thus $\operatorname{cap} J(f^p) > 0$ for all $p \in \mathbb{N}$. Conclusion (v) of Theorem 1.6 now follows from Theorem 5.2. □

7. Examples

Example 7.1. In the iteration theory of transcendental entire functions, much attention has been paid to the exponential functions $E_\lambda(z) = \lambda e^z$. Here we only mention a result of Devaney and Krych [11, p. 50] saying that if $0 < \lambda < 1/e$, then $J(E_\lambda)$ is the complement of the attracting basin of the attracting fixed point of E_λ and has the structure of a *Cantor bouquet*. By definition, this is a union of uncountably many pairwise disjoint simple curves connecting finite points in $\mathbb{C}$ (or $\mathbb{R}^n$) with infinity. For further results on exponential dynamics we refer to a detailed survey by Devaney [10], as well as papers by Rempe [28] and Schleicher [34].

Zorich [42] introduced transcendental type quasiregular mappings that are 3-dimensional analogues of the exponential function. It was shown in [5] that, for a suitable choice of parameters, Zorich maps also have attracting fixed points whose attracting basins are complements of Cantor bouquets. Moreover, results of Karpińska [21] and McMullen [24] concerning the Cantor bouquets of exponential functions were extended to Zorich maps. Here we show that – in analogy to the result of Devaney and Krych – these 3-dimensional Cantor bouquets coincide with the Julia set as defined in Definition 1.1.

To define a Zorich map, we follow [19, §6.5.4] and consider the square $Q = [-1,1]^2$ and the upper hemisphere

$$U = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| = 1,\ x_3 \geq 0\}.$$

Let $h\colon Q \to U$ be a bi-Lipschitz map, and define

$$F\colon Q \times \mathbb{R} \to \mathbb{R}^3, \qquad F(x_1, x_2, x_3) = e^{x_3} h(x_1, x_2).$$

Then F maps the infinite square beam $Q \times \mathbb{R}$ to the upper half-space. Repeated reflection along sides of square beams and the (x_1, x_2)-plane yields a map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$. This map F is quasiregular, omits the value 0 and is doubly-periodic in the x_1- and x_2-directions. We call a function F defined this way a *Zorich map* and we apply this term also to functions f_a given by

$$f_a\colon \mathbb{R}^3 \to \mathbb{R}^3, \qquad f_a(x) = F(x) - (0,0,a).$$

For a Zorich map f_a as above with parameter a chosen sufficiently large, it was proved in [5] that there exists a unique attracting fixed point ξ of f_a such that the set

$$J_0 := \mathbb{R}^3 \backslash A(\xi) = \{x \in \mathbb{R}^3 : f_a^k \not\to \xi\}$$

is a Cantor bouquet. As mentioned, we want to show that

$$J(f_a) = J_0. \tag{7.1}$$

By Theorem 1.5 we have $J(f_a) \subset J_0$ so that we only have to prove the reverse inclusion.

We will use the following notation. For $r = (r_1, r_2) \in \mathbb{Z}^2$, we put

$$P(r) = P(r_1, r_2) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 2r_1| < 1, |x_2 - 2r_2| < 1\}$$

so that $P(0,0)$ is the interior of Q. For $c \in \mathbb{R}$, we denote the half-space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > c\}$ by $H_{>c}$, and we define $H_{\geq c}$, $H_{<c}$ similarly. Observe that f_a maps $P(r_1, r_2) \times \mathbb{R}$ bijectively onto $H_{>-a}$ if $r_1 + r_2$ is even and bijectively onto $H_{<-a}$ if $r_1 + r_2$ is odd. Let $S = \{(r_1, r_2) \in \mathbb{Z}^2 : r_1 + r_2 \text{ is even}\}$.

In [5], constants $M \in \mathbb{R}$ and $\alpha \in (0,1)$ are found such that, for any $r \in S$, there exists a branch of the inverse function of f_a,

$$\Lambda^r \colon H_{\geq M} \to P(r) \times (M, \infty) =: T(r),$$

that satisfies [5, (2.3)]

$$|\Lambda^r(x) - \Lambda^r(y)| \leq \alpha |x - y| \quad \text{for } x, y \in H_{\geq M}. \tag{7.2}$$

This estimate leads to the following uniform expansion property of f_a on $\Lambda^r(H_{\geq M})$.

Lemma 7.1. *If $x \in \Lambda^r(H_{\geq M})$ for some $r \in S$ and if $R > 0$, then*

$$f_a\left(B(x, R) \cap H_{\geq M}\right) \supset B\left(f_a(x), \alpha^{-1} R\right) \cap H_{\geq M}.$$

Proof. Take $y \in B\left(f_a(x), \alpha^{-1} R\right) \cap H_{\geq M}$. Then by (7.2)

$$|x - \Lambda^r(y)| = |\Lambda^r(f_a(x)) - \Lambda^r(y)| \leq \alpha |f_a(x) - y| < R.$$

Hence $\Lambda^r(y) \in B(x, R) \cap T(r)$ and so $y = f_a(\Lambda^r(y)) \in f_a(B(x, R) \cap H_{\geq M})$. □

We shall also require the fact that [5, p. 608]

$$J_0 \subset \bigcup_{r \in S} T(r). \tag{7.3}$$

We are now ready to establish (7.1) by showing that $J_0 \subset J(f_a)$. To this end, take $x_0 \in J_0$ and let U be a neighbourhood of x_0. Write $x_k = (x_{k,1}, x_{k,2}, x_{k,3}) := f_a^k(x_0)$ and note that $x_k \in \bigcup_{r \in S} T(r) \subset H_{\geq M}$, for all $k \geq 0$, due to (7.3) and the complete invariance of J_0. It follows that we may repeatedly apply Lemma 7.1 to obtain a sequence $R_k \to \infty$ such that

$$f_a^k(U) \supset B(x_k, R_k) \cap H_{\geq M}.$$

Provided that k is large, we can find $(p_{k,1}, p_{k,2}) \in \mathbb{Z}^2$ such that the set

$$V_k = \{(y_1, y_2) \in \mathbb{R}^2 : |y_1 - 2p_{k,1}| \leq 2, |y_2 - 2p_{k,2}| \leq 2\} \times [x_{k,3}, x_{k,3} + R_k/2]$$

is contained in $B(x_k, R_k) \cap H_{\geq M}$. Note that f_a maps V_k onto the shell

$$A_k = \{y \in \mathbb{R}^3 : e^{x_{k,3}} \leq |y + (0,0,a)| \leq e^{x_{k,3} + R_k/2}\}$$

and therefore we have that $A_k \subset f_a^{k+1}(U)$. Since $x_{k,3} \geq M$ and $R_k \to \infty$, it is not difficult to see that for large k we can find $(q_{k,1}, q_{k,2}) \in \mathbb{Z}^2$ and $t_k > 0$ such that $t_k \to \infty$ and

$$\{(y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - 2q_{k,1}| \leq 2, |y_2 - 2q_{k,2}| \leq 2, |y_3| \leq t_k\} \subset A_k.$$

By considering the image of this set, we now deduce that

$$\{y \in \mathbb{R}^3 : e^{-t_k} \leq |y + (0,0,a)| \leq e^{t_k}\} \subset f_a(A_k) \subset f_a^{k+2}(U).$$

This implies that $O^+_{f_a}(U) = \mathbb{R}^3 \backslash \{(0,0,-a)\}$ and thus $x_0 \in J(f_a)$, completing the proof of (7.1).

Example 7.2. Quasiregular mappings of $\mathbb{R}^n$ that can be seen as analogues of the trigonometric functions were constructed in [8] as follows. Write $x = (x_1, \ldots, x_n)$ for points in $\mathbb{R}^n$. Let F be a bi-Lipschitz map from the half-cube $[-1,1]^{n-1} \times [0,1]$ to the upper half-ball

$$\{x \in \mathbb{R}^n : |x| \leq 1, x_n \geq 0\}$$

which maps the face $[-1,1]^{n-1} \times \{1\}$ onto the hemisphere

$$\{x \in \mathbb{R}^n : |x| = 1, x_n \geq 0\}.$$

Extend F to a mapping $F\colon [-1,1]^{n-1} \times [0,\infty) \to \{x \in \mathbb{R}^n : x_n \geq 0\}$ by

$$F(x) = e^{x_n - 1} F(x_1, \ldots, x_{n-1}, 1), \quad \text{for } x_n > 1.$$

Then F bijectively maps a half-infinite square beam onto the upper half-space. Using repeated reflections in hyperplanes, F is extended to give a quasiregular self-map of $\mathbb{R}^n$; see [8] for more details. This construction quickly leads to the fact that, for large enough $\lambda > 0$, the map $f := \lambda F$ is locally uniformly expanding. Choosing F so that it fixes the origin, and taking λ sufficiently large, this expansion property was used in [18] to prove that $O^+_f(U) = \mathbb{R}^n$ for all non-empty open subsets $U \subset \mathbb{R}^n$. Thus $J(f) = \mathbb{R}^n$. Furthermore, the periodic points of f were shown to form a dense subset of $\mathbb{R}^n$.

Example 7.3. Define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(z) = z + 1 + e^{-z}$ and, for a large positive constant M, let

$$f(z) = \begin{cases} g(z) & \text{if } \operatorname{Re} z \leq M \text{ or } \operatorname{Re} z \geq 2M, \\ g(z) + (1 + e^{-z}) \sin\left(\dfrac{\pi \operatorname{Re} z}{M}\right) & \text{if } M < \operatorname{Re} z < 2M. \end{cases}$$

It is easy to see that f is quasiregular of transcendental type if M is large. In fact, $K(f) \to 1$ as $M \to \infty$.

The function g is a classical example considered by Fatou [15, Exemple I] who showed that with the right half-plane $H = \{z : \operatorname{Re} z > 0\}$ we have $g(H) \subset H$ and $g^k|_H \to \infty$ as $k \to \infty$. We also have $f(H) \subset H$ which implies that $J(f) \cap H = \emptyset$. With $\xi = 3M/2$, we have $f(\xi) = \xi$ and $f(x) > x$ for $x > \xi$. Thus $f^k(x) \to \infty$ as $k \to \infty$ for $x > \xi$. We conclude that $(\xi, \infty) \subset I(f)$, while $\xi \in BO(f)$. Hence $\xi \in (\partial BO(f) \cap \partial I(f)) \backslash J(f)$.

Example 7.4. The quasiregular map $\tilde{f}\colon \mathbb{C} \to \mathbb{C}$ constructed in [27, §4] is of transcendental type and has the following properties.

The upper half-plane $H_+ = \{z : \operatorname{Im} z > 0\}$ is mapped into itself by $\tilde{f}$ and hence $J(\tilde{f}) \cap H_+ = \emptyset$. There is a sequence of domains $(W_k)_{k\in\mathbb{Z}}$ with closures in H_+ such that $\tilde{f}(W_k) = W_{k+1}$ and $\tilde{f}(z) = z/2$ on each W_k. Therefore, the iterates of $\tilde{f}$ converge to 0 on W_k and so $W_k \subset BO(\tilde{f})$ for $k \in \mathbb{Z}$. In contrast, all points of H_+ that are not contained in some $\overline{W_k}$ escape to infinity under iteration; that is,

$$H_+ \backslash \bigcup_{k\in\mathbb{Z}} \overline{W_k} \subset I(\tilde{f}).$$

It can then be shown that $\partial W_k \subset (\partial BO(\tilde{f}) \cap \partial I(\tilde{f}))\backslash J(\tilde{f})$ for each $k \in \mathbb{Z}$.

Example 7.5. Let $\delta > 0$, define $g\colon [0,3] \to \mathbb{R}$ by

$$g(r) = \begin{cases} (1-\delta) & \text{if } 0 \le r \le 1, \\ 1 + \delta(r-2) & \text{if } 1 < r \le 2, \\ 1 & \text{if } 2 < r \le 3, \end{cases}$$

and define $f\colon \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} zg(|z|) & \text{if } |z| \le 3, \\ z + \delta(|z|-3)e^z & \text{if } 3 < |z| \le 4, \\ z + \delta e^z & \text{if } 4 < |z|, \end{cases}$$

(cf. [4, Example 5.3]). If δ is small enough then f is quasiregular of transcendental type, and in fact $K(f) \to 1$ as $\delta \to 0$.

The function f has an attracting fixed point at 0 and it is easily seen that $B(0,2) \subset A(0)$ and $S(0,2) \subset \partial A(0)$. On the other hand, $f(B(0,3)) = B(0,3)$ and thus $S(0,2) \cap J(f) = \emptyset$. We conclude that $\partial A(0) \not\subset J(f)$.

8. Some consequences of Harnack's inequality

We show that in the condition $|f(x)| \le 1$ appearing in the definition of the pits effect (Definition 1.1) the constant 1 can be replaced by any other positive constant. In fact, we have the following result.

Theorem 8.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a quasiregular map of transcendental type having the pits effect and let $\alpha > 1$. Then there exists $N \in \mathbb{N}$ such that for all $\alpha > 1$, all $c > 1$ and all $\varepsilon > 0$ there exists R_0 such that if $R > R_0$, then*

$$\{x \in \mathbb{R}^n : R \le |x| \le cR,\ |f(x)| \le R^\alpha\}$$

can be covered by N balls of radius εR.

The following result is known as Harnack's inequality; see [32, Theorem VI.7.4, Corollary VI.2.8].

Lemma 8.1. *For $K \ge 1$ and $n \ge 2$ there exists a constant $\theta > 1$ such that if $f\colon B^n(a,2r) \to \mathbb{R}^n$ is K-quasiregular and $|f(x)| > 1$ for all $x \in B^n(a,2r)$, then*

$$\sup_{x\in B^n(a,r)} \log|f(x)| \le \theta \inf_{x\in B^n(a,r)} \log|f(x)|.$$

A simple consequence is the following result.

Lemma 8.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain, C a compact subset of Ω and $K \geq 1$. Then there exists $\beta > 1$ such that if $f\colon \Omega \to \mathbb{R}^n$ is K-quasiregular and $|f(x)| > 1$ for all $x \in \Omega$, then*
$$\max_{x\in C} \log|f(x)| \leq \beta \min_{x\in C} \log|f(x)|.$$

Proof of Theorem 8.1. Let N be as in Definition 1.2 (i.e., the definition of the pits effect) and let $c, \alpha > 1$ and $\varepsilon > 0$. Suppose that the conclusion is false. Then there exists a sequence (R_m) tending to infinity such that

$$\{x \in \mathbb{R}^n : R_m \leq |x| \leq cR_m,\ |f(x)| \leq R_m^\alpha\} \tag{8.1}$$

cannot be covered by N balls of radius εR_m. With $h_m(x) = f(R_m x)$ as in (3.1) this means that
$$\{x \in \mathbb{R}^n : 1 \leq |x| \leq c,\ |h_m(x)| \leq R_m^\alpha\}$$
cannot be covered by N balls of radius ε. On the other hand, for any $\delta > 0$ there exists R_0 such that if $R > R_0$, then $\{x \in \mathbb{R}^n : R/2 \leq |x| \leq 2cR, |f(x)| \leq 1\}$ can be covered by N balls of radius δR. In terms of h_m this means that if $R_m > R_0$, then $\{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2c, |h_m(x)| \leq 1\}$ can be covered by N balls of radius δ, say
$$\left\{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2c, |h_m(x)| \leq 1\right\} \subset \bigcup_{j=1}^N B(x_{m,j}, \delta).$$
We may assume that $(x_{m,j})_{m\in\mathbb{N}}$ converges for all $j \in \{1, \dots, N\}$, say $x_{m,j} \to x_j$ as $m \to \infty$. Then
$$\left\{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2c, |h_m(x)| \leq 1\right\} \subset \bigcup_{j=1}^N B(x_j, 2\delta)$$
for large m. We would like to apply Lemma 8.2 to
$$\Omega = \left\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2c\right\} \setminus \bigcup_{j=1}^N \overline{B}(x_j, 2\delta)$$
and the subset
$$C = \{x \in \mathbb{R}^n : 1 \leq |x| \leq c\} \setminus \bigcup_{j=1}^N B(x_j, 5N\delta).$$
However, the set Ω defined this way need not be connected. But choosing δ sufficiently small we can achieve that there exists $t \in [1, c]$ such that the sphere $S(0, t)$ is contained in C while all components of Ω that do not contain $S(0, t)$ have diameter less than $4N\delta$. Let Ω' be the component of Ω containing $S(0, t)$. Then C is a compact subset of Ω' and thus Lemma 8.2 can be applied to Ω' and C. We obtain
$$\beta \min_{x\in C} \log|h_m(x)| \geq \max_{x\in C} \log|h_m(x)| \geq \log M(t, h_m) = \log M(R_m t, f).$$

Lemma 2.4 yields

$$\min_{x\in C} \log |h_m(x)| > \alpha \log R_m$$

for large m, which means that

$$|f(x)| > R_m^\alpha \quad \text{for } R_m \le |x| \le cR_m,\ x \notin \bigcup_{j=1}^{N} B(R_m x_j, 5N\delta R_m).$$

As we may assume that $5N\delta < \varepsilon$, we see that the set given by (8.1) can be covered by N balls of radius εR_m. This is a contradiction. □

Remark 8.1. At the beginning of section 4 we remarked that the functions g_m defined by (3.2) form a quasinormal family if f has the pits effect. In order to see this, we note that by Theorem 8.1 we can cover

$$\{x \in \mathbb{R}^n : R_m \le |x| \le cR_m,\ |f(x)| \le R_m^\alpha\}$$

by N balls of radius εR_m, provided m is large. Similarly, we can cover

$$\{x \in \mathbb{R}^n : R_m/c \le |x| \le R_m,\ |f(x)| \le R_m^\alpha / c^\alpha\}$$

by N such balls. Choosing $c = 1/\varepsilon$ we can thus cover

$$B(0, 1/\varepsilon)\backslash \left\{x \in \mathbb{R}^n : |g_m(x)| \le \varepsilon^\alpha R_m^{\alpha-1}\right\}$$

by $2N+1$ balls of radius ε, provided m is large. As this holds for every $\varepsilon > 0$, and since $\alpha > 1$, we easily see that every subsequence of (g_m) has a subsequence which converges to ∞ in $\mathbb{R}^n \backslash E$ for some set E of cardinality at most $2N+1$. Thus (g_m) is quasinormal.

References

[1] I. N. Baker, The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. (Ser. A, I. Math.)* 1 (1975), 277–283.

[2] Walter Bergweiler, Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N. S.)* 29 (1993), 151–188.

[3] Walter Bergweiler, Fixed points of composite entire and quasiregular maps. *Ann. Acad. Sci. Fenn. Math.* 31 (2006), 523–540.

[4] Walter Bergweiler, Iteration of quasiregular mappings. *Comput. Methods Funct. Theory* 10 (2010), 455–481.

[5] Walter Bergweiler, Karpińska's paradox in dimension 3. *Duke Math. J.* 154 (2010), 599–630.

[6] Walter Bergweiler, On the set where the iterates of an entire function are bounded. *Proc. Amer. Math. Soc.* 140 (2012), 847–853.

[7] Walter Bergweiler, Fatou-Julia theory for non-uniformly quasiregular maps. *Ergodic Theory Dynam. Systems* 33 (2013), 1–23.

[8] Walter Bergweiler and Alexandre Eremenko, Dynamics of a higher dimensional analogue of the trigonometric functions. *Ann. Acad. Sci. Fenn. Math.* 36 (2011), 165–175.

[9] Walter Bergweiler, Alastair Fletcher, Jim Langley and Janis Meyer, The escaping set of a quasiregular mapping. *Proc. Amer. Math. Soc.* 137 (2009), 641–651.

[10] Robert L. Devaney, Complex exponential dynamics. In "*Handbook of Dynamical Systems, Vol.* 3", Elsevier, Amsterdam, 2010, 125–224.

[11] Robert L. Devaney and Michaił Krych, Dynamics of exp(z). *Ergodic Theory Dynam. Systems* 4 (1984), 35–52.

[12] A. È. Eremënko, On the iteration of entire functions. In "*Dynamical Systems and Ergodic Theory*". Banach Center Publications 23. Polish Scientific Publishers, Warsaw, 1989, 339–345.

[13] Alexandre Eremenko and Iossif Ostrovskii, On the 'pits effect' of Littlewood and Offord. *Bull. Lond. Math. Soc.* 39 (2007), 929–939.

[14] P. Fatou, Sur les équations fonctionelles. *Bull. Soc. Math. France* 47 (1919), 161–271; 48 (1920), 33–94, 208–314.

[15] P. Fatou, Sur l'itération des fonctions transcendantes entières. *Acta Math.* 47 (1926), 337–360.

[16] Alastair Fletcher and Daniel A. Nicks, Quasiregular dynamics on the n-sphere. *Ergodic Theory Dynam. Systems* 31 (2011), 23–31.

[17] Alastair Fletcher and Daniel A. Nicks, Julia sets of uniformly quasiregular mappings are uniformly perfect. *Math. Proc. Cambridge Philos. Soc.* 151 (2011), 541–550.

[18] Alastair Fletcher and Daniel A. Nicks, Chaotic dynamics of a quasiregular sine mapping. *J. Difference Equ. Appl.*, DOI: 10.1080/10236198.2012.749250

[19] Tadeusz Iwaniec and Gaven J. Martin, *Geometric Function Theory and Non-linear Analysis.* Oxford Mathematical Monographs. Oxford University Press, New York, 2001.

[20] G. Julia, Sur l'itération des fonctions rationelles. *J. Math. Pures Appl.* (7) 4 (1918), 47–245.

[21] Bogusława Karpińska, Hausdorff dimension of the hairs without endpoints for $\lambda \exp z$. *C. R. Acad. Sci. Paris Ser. I Math.* 328 (1999), 1039–1044.

[22] J. E. Littlewood and A. C. Offord, On the distribution of zeros and a-values of a random integral function. II. *Ann. of Math.* (2) 49 (1948) 885–952; errata 50 (1949), 990–991.

[23] P. Mattila and S. Rickman, Averages of the counting function of a quasiregular mapping. *Acta Math.* 143 (1979), 273–305.

[24] Curt McMullen, Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.* 300 (1987), 329–342.

[25] John Milnor, *Dynamics in One Complex Variable.* Third edition. Annals of Mathematics Studies 160. Princeton University Press, Princeton, NJ, 2006.

[26] Ruth Miniowitz, Normal families of quasimeromorphic mappings. *Proc. Amer. Math. Soc.* 84 (1982), 35–43.

[27] Daniel A. Nicks, Wandering domains in quasiregular dynamics. *Proc. Amer. Math. Soc.*, 141 (2013), 1385–1392.

[28] Lasse Rempe, Topological dynamics of exponential maps on their escaping sets. *Ergodic Theory Dynam. Systems* 26 (2006), 1939–1975.

[29] Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion.* Translations of Mathematical Monographs 73. Amer. Math. Soc., Providence, RI, 1989.

[30] Seppo Rickman, On the number of omitted values of entire quasiregular mappings. *J. Analyse Math.* 37 (1980), 100–117.

[31] Seppo Rickman, The analogue of Picard's theorem for quasiregular mappings in dimension three. *Acta Math.* 154 (1985), 195–242.

[32] Seppo Rickman, *Quasiregular Mappings.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 26. Springer-Verlag, Berlin, 1993.

[33] P. J. Rippon and G. M. Stallard, Iteration of a class of hyperbolic meromorphic functions. *Proc. Amer. Math. Soc.* 127 (1999), 3251–3258.

[34] Dierk Schleicher, Attracting dynamics of exponential maps. *Ann. Acad. Sci. Fenn. Math.* 28 (2003), 3–34.

[35] Heike Siebert, *Fixpunkte und normale Familien quasiregulärer Abbildungen.* Dissertation, University of Kiel, 2004; http://e-diss.uni-kiel.de/diss_1260.

[36] Heike Siebert, Fixed points and normal families of quasiregular mappings. *J. Anal. Math.* 98 (2006), 145–168.

[37] Norbert Steinmetz, *Rational Iteration.* De Gruyter Studies in Mathematics 16. Walter de Gruyter & Co., Berlin 1993.

[38] Daochun Sun and Lo Yang, Quasirational dynamical systems (Chinese). *Chinese Ann. Math. Ser. A* 20 (1999), 673–684.

[39] Daochun Sun and Lo Yang, Quasirational dynamic system. *Chinese Science Bull.* 45 (2000), 1277–1279.

[40] Daochun Sun and Lo Yang, Iteration of quasi-rational mapping. *Progr. Natur. Sci. (English Ed.)* 11 (2001), 16–25.

[41] Hans Wallin, Metrical characterization of conformal capacity zero. *J. Math. Anal. Appl.* 58 (1977), 298–311.

[42] V. A. Zorich, A theorem of M. A. Lavrent'ev on quasiconformal space maps. *Math. USSR Sb.* 3 (1967), 389–403; Transl. of *Mat. Sb. (N.S.)* 74 (1967), 417–433 (in Russian).

Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Str. 4, D–24098 Kiel, Germany

E-mail address: bergweiler@math.uni-kiel.de

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK

E-mail address: Dan.Nicks@nottingham.ac.uk