Free vibration analysis of curved lattice sandwich beams

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Abstract: In this paper, the effects of initial curvature and lattice core shape on the bending vibration of sandwich beams are investigated. The three-dimensional (3D) sandwich beam is simulated by combining a two-dimensional (2D) cross-sectional analysis with a one-dimensional (1D) nonlinear beam analysis. The sandwich beam is composed of two identical isotropic faces covering a lattice core. Four different lattice core structures are used to take into account the effect of core unit cell shape on the dynamic properties of the sandwich beam. The nonlinear governing equations of the sandwich beam are discretised using a time-space scheme. Numerical results show that the lattice unit cell shape affects both in-plane and out of plane stiffness values and hence changes the dynamic behaviour of the beam. Furthermore, it is observed that by changing the density ratio of the beam, modes veer away from each other at a specific value of density ratio for specific unit cell types. Moreover, the initial curvature of the beam is shown to affect the dynamics of the beam especially lower modes. Finally, it is obtained that the dynamics of the beam is different when it is initially curved or curved due to an applied end follower moment.

Keyword: Curved sandwich beam, free vibration, lattice core, exact beam formulation.

Introduction

Due to their high strength to weight ratio, sandwich structures are broadly used in many engineering applications such as aerospace, automotive, marine, etc [1]. When the length of the structure is much higher than the other two dimensions, it is possible to simulate the dynamics of the structure using beam models [2]. A sandwich beam normally is composed of three layers which are a thick core and two thin faces bounded together [1]. The faces of the sandwich beam are mainly responsible for the strength of the beam, while the core is used to support the face panels from shear failure. Both core and face panels can be made from isotropic or anisotropic materials, and the core typically can be made from foams, corrugated or honeycomb sheets [3, 4].

Many researchers investigated the dynamics of sandwich beams by considering several design parameters, and a number of governing equations were derived. Among the earliest studies, Kerwin [5] derived expressions for the flexural stiffness of sandwich beams with a damping layer between two face sheets. The vibration properties of a sandwich beam with viscoelastic core was considered by DiTaranto [6] for various boundary conditions. In this study, the differential equations of motion for the sandwich beam were derived by considering a complex shear module for the sandwich core. Mead and Markust [7] also derived a six-order differential equation for the transverse vibration of three-layered sandwich beam. The free vibration of sandwich beams using various beam models was investigated by Mead [8]. It was highlighted that the beam model should be selected based on the flexural wavelength and the core thickness. Frosting and Baruch [9] studied the effect of flexible cores on the free vibration of sandwich beams. They proposed to consider a nonlinear displacement distribution through the core to model the flexible core. Sakiyama et al. [10] investigated the free vibration of sandwich beams with elastic and viscoelastic cores with various boundary conditions. It was concluded that the core shear modules and depth affect the natural frequencies of the beam. A Galerkin element method was developed for vibration analysis of sandwich beam structures by Sainsbury and Zhang [11]. It was shown that the proposed element can accurately predict the dynamics of the beam, even higher frequencies, with less computational cost. Banerjee [12] used the dynamic stiffness method for free vibration analysis of symmetric sandwich beams. They showed that the proposed method is capable of capturing the vibration of the beam accurately. The effect of through the thickness deformation and high modulus ratio on the vibration of sandwich beams and plates were presented by Moreira and Rodrigues [13] using a layerwise model. The numerical and experimental results proofed that their proposed model was able to predict the dynamic behaviour of sandwich beams specially with soft cores. Vidal and Polit [14] studied the free vibration of sandwich and laminated beams using a family of sinus models. The continuity condition between the layers was retained, and it was shown that the developed models can accurately predict the dynamics of the beam for a range of cases. The free vibration of sandwich beams was considered by Khalili et al [15] using a dynamic stiffness method. The core to face density and thickness ratios were obtained to significantly affect the first mode of the beam irrespective of the type of the boundary condition. The literature on the free vibration of laminate composite and sandwich plates was reviewed by Sayyad and Ghugal [16]. The free vibration of sandwich beams with soft cores was analysed by Khdeir and Aldraihem [1]. A zig-zag beam theory was developed which then was shown, through comparison with available experimental results, to have high accuracy for soft core sandwich beams. Hui et al. [17] proposed a high order finite element for free vibration of sandwich structures. Their numerical results showed that the proposed model yields accurate results with less effort in compare to three dimensional finite element methods. Wang and Zhao [18] investigated the free vibration of sandwich beams with metal foam cores resting on elastic foundation. It was observed that for beams with non-uniform foam distribution, the natural frequencies are sensitive to the foam coefficient. A new high order zigzag theory for analysis of sandwich beams was developed by Garg and Chalak [19]. The developed method was then used to analyse the effect of boundary conditions on the stress distribution and frequencies of the beam. More recently, Kohsaka et al. [20] studied the vibration of sandwich beams with a lattice core. They showed that the face thickness and core strut diameter can influence the frequencies of the beam. Also, Shu et al. [21] analysed the natural and forced vibration behaviour of sandwich beams with elastic metamaterial cores. They showed that the vibration characteristics of the sandwich beam with metamaterial core is different from traditional sandwich beams.

As it was mentioned above, the core of sandwich beams can be made of different materials such as foams or periodic cellular solids (e.g. honeycombs) [22]. These cellular cores are often preferred in compare to solid cores due to their lower mass and better performance in thermal expansion, flow permeability, noise and vibration reduction and electric conductivity which can be controlled by the unit cell shape [3]. The effective material properties of the cellular cells are dependent to the geometry of the unit cell, and several researchers have proposed closed form expressions for these effective material properties [22-26]. In most of the cases, it is assumed that cell walls behave like beams, and then the material properties are obtained by solving the deformation and equilibrium of the unit cell. Brt-Smith et al. [27] studied the bending performance and failure of sandwich structures with cellular cores. It was shown that the analytical solutions to predict failure and stiffness can reliably be used for longer beams. The mechanical properties of lattice materials using different homogenization methods was considered by Arabnejad and Pasini [28]. They highlighted that the homogenization method may be selected based on the required level of accuracy and computational cost of the problem. Elsayed and Pasini [29] proposed a multiscale structural design of columns manufactured using truss lattice materials. It was determined that the buckling resistance of the column can be enhanced by properly shaping the cell element cross-section. Free vibration of sandwich beams with honeycomb-corrugated hybrid cores was investigated by Zhang et al. [30]. They concluded that the frequency of the beam is sensitive to the geometry of the hybrid core. Lou et al. [31] studied the vibration of lattice sandwich beams under various boundary conditions. It was highlighted that increasing the core thickness is the most efficient way of increasing the natural frequencies of the sandwich beam. Gu et al. [32] proposed a new lattice structure to induce bend-twist coupling for helicopter blade morphing applications. It was determined that adding a lattice core instead of a solid core increases the twist distribution of the blade. The free vibration characteristics of sandwich beams with body-centred cubic (BCC) truss core was considered by Kohsaka et al. [20]. It was proposed that using lattice sandwich panels can suppress vibration in aerospace structures.

Curved beams are extensively used in many engineering structures such as bridges, tunnels, aerospace structures, etc [33]. The dynamics and vibration of curved or twisted beams has been investigated by many researchers, and several studies reviewed the literature in this field [34-36]. Wasserman [37] obtained the exact and approximate natural frequencies and critical loads of arches with flexible supports. It was shown that both frequencies and critical loads are sensitive to the arch opening angle. The vibration and buckling behaviour of circular arches was analysed by Kang et al. [38] using differential quadrature method (DQM) for different boundary conditions. Using the DQM showed to require less computational effort than other numerical methods. Rosa and Franciosi [39] studied the dynamics of circular arches for various boundary conditions. A DQM method was used to solve the six-order differential equations of the curved beam, and the results showed that the arch opening angle affects the frequencies of the beam. The out of plane free vibration of curved beams resting on elastic foundations was studied by Lee et al. [40]. Three different boundary conditions were considered, and the effects of rotary inertia and shear deformation on the natural frequencies of the beam were obtained. Yau [41] investigated the effect of moving train on the vibration of tied-arch bridge using an analytical approach. It was highlighted that the acceleration response of the beam is dependent to the rise of the arch ribs. The in-plane free vibration of uniform circular arches was studied by Wu and Chiang [42] using finite arch element. The element stiffness matrix of the arch was determined using a simple implicit shape function. Chang and Hodges [33] studied the coupled vibration of curved beam using exact beam formulation for various boundary conditions. It was obtained that the natural frequencies of an initially curved beam are different from a straight beam and a bent beam with same geometry. The forced vibration of curved beams subjected to impulsive forces was investigated by Çalim [43]. It was concluded that the vibration amplitude and period increase when the circular beam opening angle increases. The effect of arbitrary placed lumped mass, linear and rotational springs on the in-plane vibration of curved beam was considered by Wu et al. [44]. They showed that the natural frequencies of the curved beam decrease when the subtended angle increases. Babaei et al. [45] investigated the large amplitude vibration of curved functionally graded beams. Their results showed that the beam frequencies are dependent to the beam length to curvature ratio. The effect of crack on the in-plane free vibration of curved beams was studied numerically and experimentally by Zare [46]. It was shown that the depth and crack location have significant effects on the natural frequencies of the beam. Also, it was highlighted that mode transition phenomenon is dependent to the boundary condition of the beam. More recently, the effects of initial curvature on the dynamics of rotating composite beams [47], aeroelastic stability of composite blades [48] and aeroelasticity of aircraft wing [49] were studies.

In the previous studies, the effect of lattice core unit cell shape on the bending vibration of beams with initial curvature has not been considered using exact formulation. Therefore, in this study, the bending vibration of curved sandwich beams with various lattice cores is investigated using the exact beam formulation. Furthermore, the vibration behaviour of the beam subjected to an end follower moment is studies to see if it behaves differently when it is curved due to an end moment in compare to when it has an initial curvature. To this aim, an analytical model of the sandwich beam is developed in which the core is replaced with an equivalent core with effective material properties which are dependent to the shape of the unit cell. Using these effective material properties, the stiffness and inertia values of the cross-section are obtained which then used to govern the dynamics equations of motion of the sandwich beam. Furthermore, the effect of initial curvature on the free vibration of the beam is studied. Finally, the vibration of the beam subjected to an end follower moment is investigated. The presented numerical results could give an insight into the design and analysis of curved lattice sandwich beam structures.

Problem Statement

A sandwich beam with an initial out of plane curvature with a length of L, as show in Figure 1**,** is considered. The initial curvature radius is denoted by R_r which is the inverse of the out of plane curvature k_2. The cross-section of the sandwich beam is composed of a lattice core with an overall thickness of h, two faces with a thickness of t_f, and the overall width of the cross-section is denoted by b. The core is made of cellular materials with different unit cell shapes as are described in the next section. These are the square cell (S), the hexagonal cell (H), triangular cell (T) and mixed cell (M). These unit cells can be characterised using the wall thickness (t), and wall length (l_u).

Furthermore, as shown in Figure 1-c) an end follower moment (M_2) is added to the beam next to bend the beam so that the deflected shape of the beam become similar to the beam with initial curvature. In what follows, first the material properties of the lattice core are introduced, and then the governing equations of motion are derived.

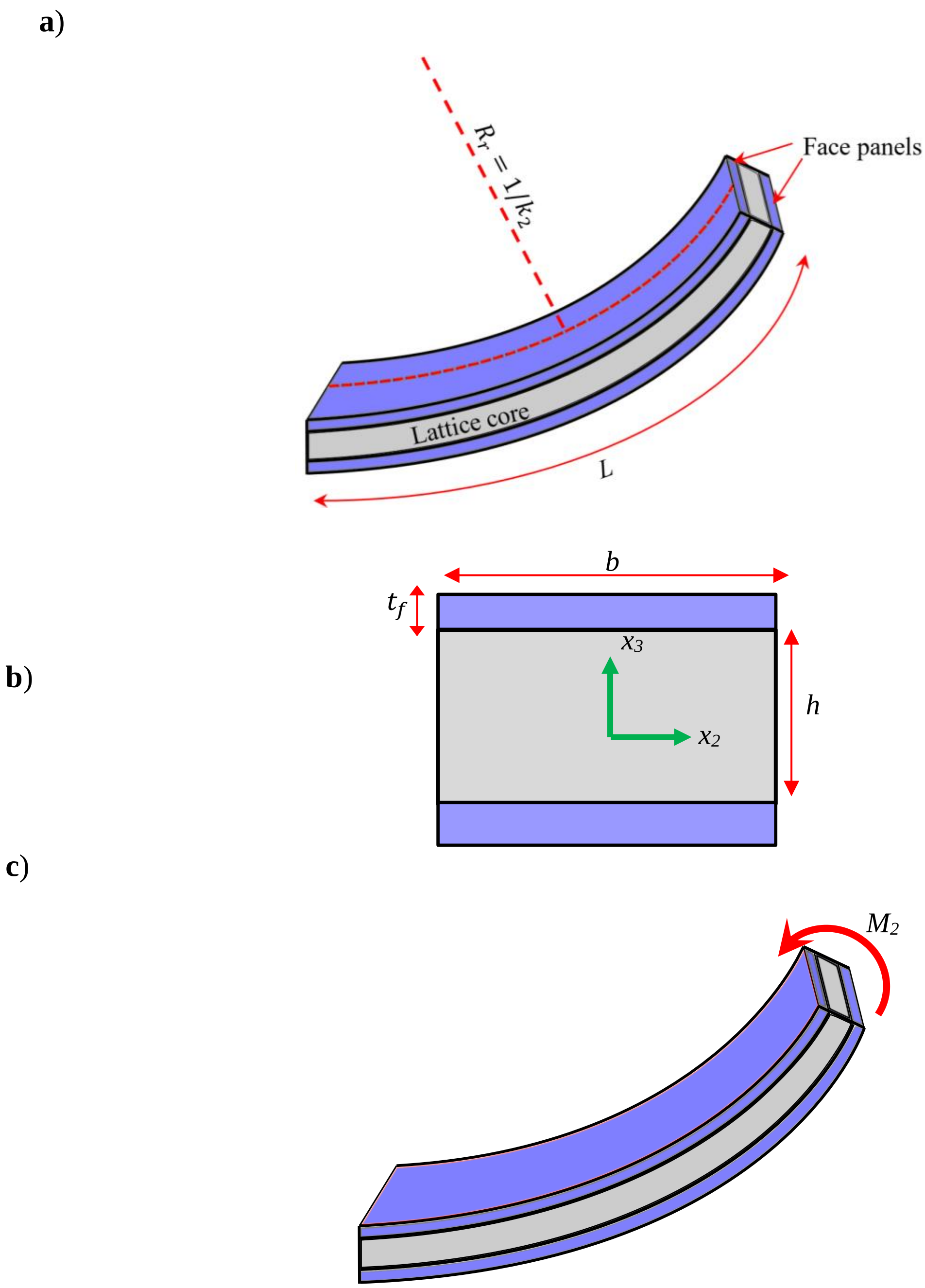

Figure 1: Schematics of a) the sandwich beam with initial curvature, b) the sandwich beam cross-section, and c) the bent beam under end follower moment

In-plane equivalent material properties of the periodic core

To investigate the effect of lattice core on the vibration behaviour of sandwich beam, it is necessary to characterise the equivalent material properties of the periodic lattice core. As it was mentioned before, in this study four different unit cells are considered. The first unit cell is a Square Cell as shown in Figure 2. It is noted that in all cases, the cell length and thickness are denoted by l_u and t, respectively. The relative density of the cell can be written as

$$r = \frac{\rho^*}{\rho_s} = \frac{2t}{l_u} \quad (1)$$

where ρ_s is the material density of the dense core elements, and ρ^* is the equivalent/effective material density of the cell. Also, the effective Young's modules of the cell can be obtained as [22]

$$E^* = \frac{r}{2} E_s \quad (2)$$

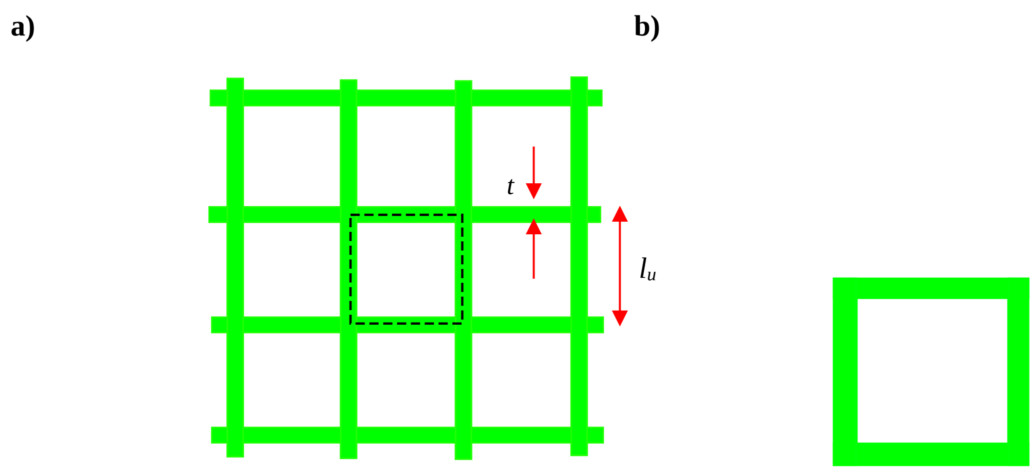

Figure 2: a) Schematic of the periodic square cell b) square unit cell

The second case is a hexagonal cell as shown in Figure 3 with the relative density and effective Young's values as follows [22]

$$r = \frac{\rho^*}{\rho_s} = \frac{2t}{\sqrt{3}l_u}$$

$$E^* = \frac{3r^3}{2}E_s \qquad (3)$$

a) **b)**

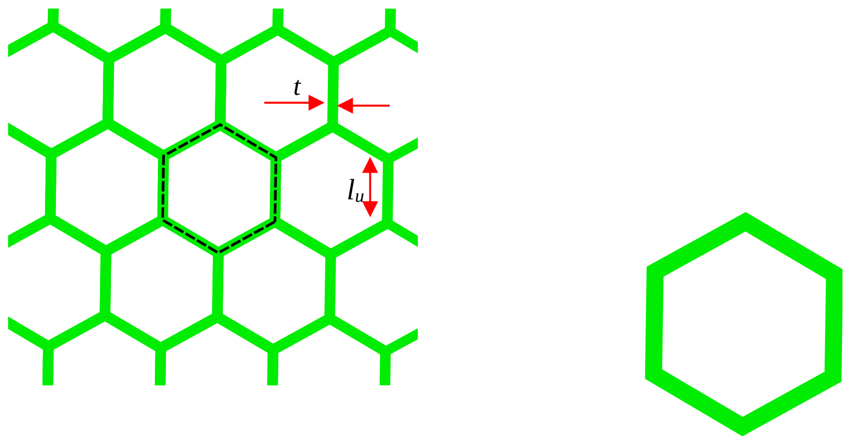

Figure 3: a) Schematic of the periodic hexagonal cell b) hexagonal unit cell

b)

a)

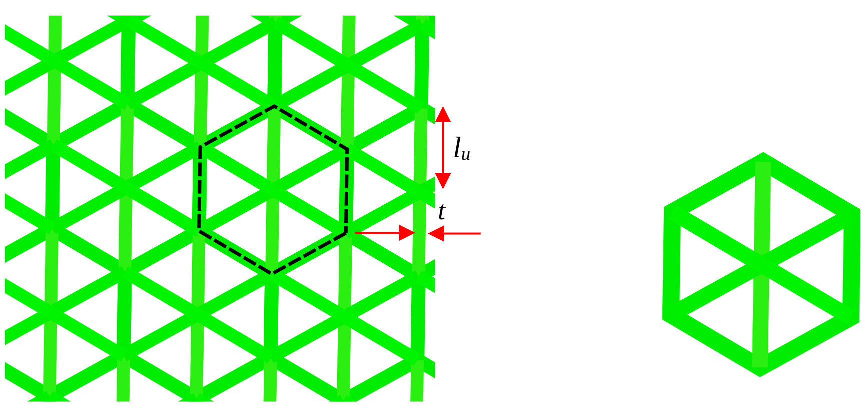

Figure 4: a) Schematic of the triangular cell b) triangular unit cell

The third case is a triangular cell as shown in Figure 4 in which its relative density and effective Young's values can be written as [22]

$$r = \frac{\rho^*}{\rho_s} = \frac{2\sqrt{3}t}{l_u}$$

$$E^* = 0.333 r E_s \tag{4}$$

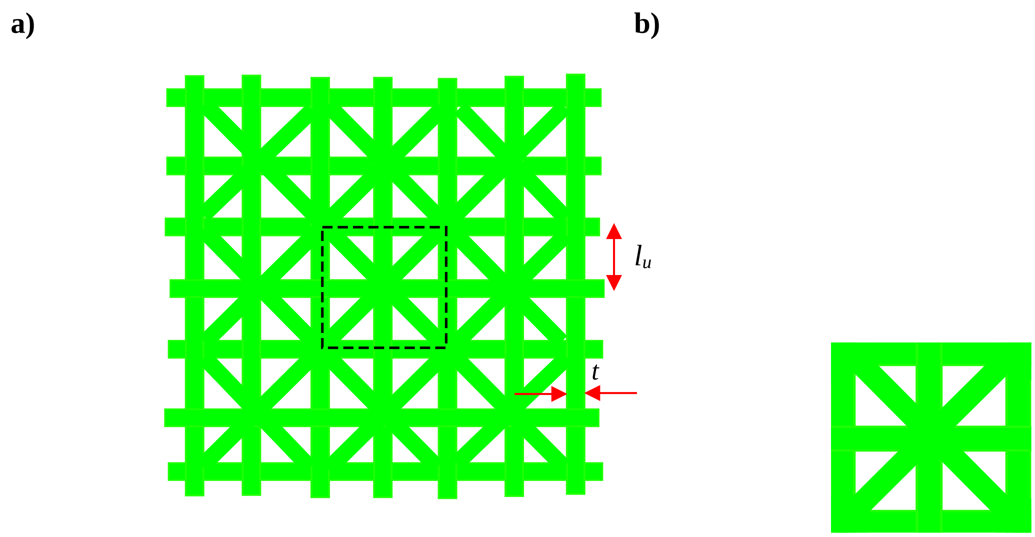

Figure 5: a) Schematic of the mixed cell b) mixed unit cell

The fourth case is a mixed cell as shown in Figure 5 with the relative density and effective Young's values as follows [22]

$$r = \frac{\rho^*}{\rho_s} = \frac{(2+\sqrt{2})t}{l_u}$$

$$E^* = 0.369 r E_s \tag{5}$$

It is noted that the effective elastic modulus of the unit cells is valid for small to moderate relative densities.

Governing Equations

The global 1D nonlinear dynamic behaviour of the sandwich beam is simulated by using the geometrically exact intrinsic beam formulation [50] as follows

$$\partial P_1/\partial t + \Omega_2 P_3 - \Omega_3 P_2 - \partial F_1/\partial x_1 - K_2 F_3 + K_3 F_2 = f_1$$

$$\partial P_2/\partial t + \Omega_3 P_1 - \Omega_1 P_3 - \partial F_2/\partial x_1 - K_3 F_1 + K_1 F_3 = f_2$$

$$\partial P_3/\partial t + \Omega_1 P_2 - \Omega_2 P_1 - \partial F_3/\partial x_1 - K_1 F_2 + K_3 F_1 = f_3$$

$$\partial H_1/\partial t + \Omega_2 H_3 - \Omega_3 H_2 + V_2 P_3 - V_3 P_2 - \partial M_1/\partial x_1 - K_2 M_3 + K_3 M_2 - 2\gamma_{12} F_3 + 2\gamma_{13} F_2 = m_1$$

$$\partial H_2/\partial t + \Omega_3 H_1 - \Omega_1 H_3 + V_3 P_1 - V_1 P_3 - \partial M_2/\partial x_1 - K_3 M_1 + K_1 M_3 - 2\gamma_{13} F_1 + (1+\gamma_{11}) F_3 = m_2$$

$$\partial H_3/\partial t + \Omega_1 H_2 - \Omega_2 H_1 + V_1 P_2 - V_2 P_1 - \partial M_3/\partial x_1 - K_1 M_2 + K_2 M_1 - (1+\gamma_{11}) F_2 + 2\gamma_{12} F_1 = m_3$$

$$\partial \gamma_{11}/\partial t - \partial V_1/\partial x_1 - K_2 V_3 + K_3 V_2 - 2\gamma_{12}\Omega_3 + 2\gamma_{13}\Omega_2 = 0 \quad (6)$$

$$2\partial \gamma_{12}/\partial t - \partial V_2/\partial x_1 - K_3 V_1 + K_1 V_3 + (1+\gamma_{11})\Omega_3 - 2\gamma_{13}\Omega_1 = 0$$

$$2\partial \gamma_{13}/\partial t - \partial V_3/\partial x_1 - K_1 V_2 + K_2 V_1 - (1+\gamma_{11})\Omega_2 + 2\gamma_{12}\Omega_1 = 0$$

$$\partial \kappa_1/\partial t - \partial \Omega_1/\partial x_1 - K_2 \Omega_3 + K_3 \Omega_2 = 0$$

$$\partial \kappa_2/\partial t - \partial \Omega_2/\partial x_1 - K_3 \Omega_1 + K_1 \Omega_3 = 0$$

$$\partial \kappa_3/\partial t - \partial \Omega_3/\partial x_1 - K_1 \Omega_2 + K_2 \Omega_1 = 0$$

where F_i, M_i, V_i and Ω_i (for *i*=1,2,3) are the internal force, internal moment, velocity and angular velocity components, respectively. Furthermore, f_i and m_i (for *i*=1,2,3) are the external force and moments applied on the beam, and x_1 is the beam arc-length. Furthermore, κ_i and γ_{1i} (for *i*=1,2,3) are the strain measures. The moment generalised strain (κ_i) can be obtained from the difference of the final curvature (**K**) and initial curvature ($\mathbf{k} = [k_1\ k_2\ k_3]$) as follows

$$\kappa_1 = K_1 - k_1$$

$$\kappa_2 = K_2 - k_2 \quad (7)$$

$$\kappa_3 = K_3 - k_3$$

where k_1, k_2 and k_3 are the initial twist and curvatures of the undeformed beam, respectively. In this study, as the effect of out of plane curvature is only considered, therefore the other two components are equal to zero (e.g. $k_1 = k_3 = 0$).

Moreover, P_i and H_i (for *i*=1,2,3) are the sectional linear and angular momenta measures which can be obtained from the linear and angular velocity using the mass/inertia matrix as follows

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \begin{bmatrix} \mu & 0 & 0 & 0 & \mu\xi_3 & -\mu\xi_2 \\ 0 & \mu & 0 & -\mu\xi_3 & 0 & 0 \\ 0 & 0 & \mu & \mu\xi_2 & 0 & 0 \\ 0 & -\mu\xi_3 & \mu\xi_2 & i_2 + i_3 & 0 & 0 \\ \mu\xi_3 & 0 & 0 & 0 & i_2 & i_{23} \\ -\mu\xi_2 & 0 & 0 & 0 & i_{23} & i_3 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{Bmatrix} \tag{8}$$

where μ is the mass per unit length of the beam, i_2, i_3, i_{23} are the cross-sectional inertia components, ξ_2, ξ_3 is the offsets between the mass centre and reference axis of the cross-section. As it was shown in Figure 1, in this study a rectangular cross-section is considered, and hence the nonzero values of the above matrix can be obtained using the following relations

$$\mu = \rho_c bh + 2\rho_f tb$$

$$i_2 = \rho_c \frac{bh^3}{12} + \rho_f \left(\frac{bt_f^3}{6} + \frac{bt_f (b + t_f)^2}{2} \right) \tag{9}$$

$$i_3 = \rho_c \frac{hb^3}{12} + \rho_f \frac{t_f b^3}{12}$$

where ρ_c and ρ_f are to the effective mass density of the core and face, respectively. It is noted that depending in the cell shape, the core mass density is obtained using Eqs. 1, 3-5.

Furthermore, the internal force and moment are related to the generalised force and moment strains through the cross-sectional stiffness matrix. It is noted that in this study both face and core are made from isotropic materials, and it is assumed that core is not soft. As the purpose of this study is to investigate the bending vibration of the beam, therefore the stiffness matrix can be simplified as follows

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} S_{55} & 0 \\ 0 & S_{66} \end{bmatrix} \begin{bmatrix} \kappa_2 \\ \kappa_3 \end{bmatrix} \quad (10)$$

The nonzero cross-sectional stiffness values of the above stiffness matrix for a sandwich beam with the dimensions shown in Figure 1 can be written as

$$S_{55} = E_c \frac{bh^3}{12} + E_f \left(\frac{bt_f^3}{6} + \frac{bt_f(b + t_f)^2}{2} \right)$$

$$S_{66} = E_c \frac{hb^3}{12} + E_f \frac{t_f b^3}{6} \quad (11)$$

where E_c and E_f refer to the effective module of elasticity of the core and face, respectively. The core modulus of elasticity is obtained using Eqs. 2-5.

By substituting Eqs. 8 and 10 into the main equation of motion (Eq.6), the simplified final equations of motion can be obtained

$$\partial P_2/\partial t - \partial F_2/\partial x_1 = 0$$

$$\partial P_3/\partial t - \partial F_3/\partial x_1 = 0$$

$$\partial H_2/\partial t - \partial M_2/\partial x_1 + F_3 = M_2$$

$$\partial H_3/\partial t - \partial M_3/\partial x_1 - F_2 = 0 \quad (12)$$

$$\partial \kappa_2/\partial t - \partial \Omega_2/\partial x_1 = 0$$

$$\partial \kappa_3/\partial t - \partial \Omega_3/\partial x_1 = 0$$

$$\partial V_2/\partial x_1 - \Omega_3 = 0$$

$$\partial V_3/\partial x_1 + \Omega_2 = 0$$

To solve these equations, a time-space finite difference discretization scheme is used [50] where all variables are discretized on the left and right had side of each node. The discretised nonlinear equations of motion can be written in a compact form as follows

$$\mathbf{a}_{ji}\dot{\mathbf{q}}_i + \mathbf{b}_{ji}\mathbf{q}_i + \mathbf{c}_{jik}\mathbf{q}_i\mathbf{q}_k = 0 \tag{13}$$

where **q** is the vector of unknowns, and **a**,**b** and **c** matrices storing the linear and nonlinear terms. To find the natural frequencies of the beam, first the nonlinear steady-state condition ($\bar{\mathbf{q}}$) of the system (Eq. 13) is obtained using the Newton-Raphson method

$$\mathbf{b}_{ji}\bar{\mathbf{q}}_i + \mathbf{c}_{jik}\bar{\mathbf{q}}_i\bar{\mathbf{q}}_k = 0 \tag{14}$$

Then, the linearised system about this steady-state condition is obtained, and the frequencies of the beam are determined by calculating the eigenvalues of this linearised system (Eq. 14)

$$\hat{\mathbf{a}}_{ji}\dot{\hat{\mathbf{q}}}_i + \hat{\mathbf{b}}_{ji}\hat{\mathbf{q}}_i = 0 \tag{15}$$

In what follows, the effects of core unit cell shape, initial curvature, and end follower moment on the free vibration of the sandwich beam are investigated.

Numerical Results

To investigate the effects of initial curvature and core unit cell shape on the dynamic behaviour of sandwich beams, first the developed model is validated in two steps. In the first step, the effect of initial curvature on the out of plane vibration of a cantilever beam with solid rectangular cross-section is obtained and presented in Table 1. It is noted that here the natural frequencies are nondimensionalised using the following relation

$$\varpi^2 = \frac{\omega^2 \mu R_r^4}{S_{55}}$$

where R_r is the radius of the arc. The results are in very good agreement showing that the developed model can predict the vibration of curved beams accurately.

Table 1: Comparison of the first four out of plane natural frequencies of a half a circle arc shape beam (opening angle of 180°)

Mode No.	Present	Rosa and Franciosi [39]
1	0.435	0.435
2	1.377	1.375
3	4.711	4.71
4	10.516	10.52

In the second step, the bending vibration of a cantilever isotropic sandwich beam is determined and compared with those reported by [17]. It is noted that in [17], a classical Timoshenko beam model is considered, while here an exact beam formulation is used. The material and geometrical properties used for this case are presented in Table 2, and the natural frequencies are nondimensionalised as follows

$$\bar{\omega}^2 = \frac{\omega^2 \rho_f L^4}{H^2 E_f}$$

where H is the overall thickness of the cross-section ($H = h + 2t_f$). The results presented in Table 3 are in satisfactory agreement (less than 1% difference) showing that the developed numerical model can obtain the natural frequencies of sandwich beams accurately.

Table 2: Material and geometrical properties of the sandwich beam

Property	Value
E_f	200 GPa
E_c	0.66 GPa
ρ_f	7800 Kg/m^3
ρ_c	60 Kg/m^3
ν_f	0.3
ν_c	0.3
L/H	100
H/b	1
t_f/b	0.015

Table 3: Comparison of the nondimensional natural frequencies of a cantilever sandwich beam

Mode No.	Present	Hui et al. [17]
1	1.0099	1.0098

2	1.4908	1.4898
3	6.3366	6.3259
4	9.3465	9.3281
5	17.7819	17.701
7	26.2165	26.081
8	34.9642	34.652
9	51.5156	51.000

By considering the above two cases, it can be concluded that the developed numerical model can be used to analyse the free vibration of sandwich beams with initial curvature accurately. In what follows, the effect of initial curvature and the core unit cell shape on the vibration of the sandwich beam is investigated.

A cantilever beam with the material properties presented in Table 4 is considered. It is assumed here that both face and core are made from a same material, and hence the material properties of face and core are equal to each other.

Table 4: The material properties of the sandwich beam with lattice core

Property	**Value**
E_s	70 GPa
ρ_s	2680 Kg/m^3
ν_s	0.3

In order to be able to compare the results when different unit cells are considered, a baseline sandwich beam configuration with material properties presented in Table 4 is considered. The baseline sandwich beam has a rectangular cross-section with a solid core. From here on, all the results are compared with respect to the values of this baseline beam. The core thickness to width ratio, the length to width ratio and the face thickness to width ratio of the beam are *h*/*b*=0.75, *L*/*b*=25 and t_f/b=0.05, respectively.

The nondimensional natural frequencies of the baseline sandwich beam is determined and presented in Table 5. As mentioned above, all values are nondimensionalised with respect to the baseline beam, and the subscript ($\bullet_b$) refers to the values of the bassline beam. It is noted that here, for simplicity, the out of plan bending (flap) and in-plane bending (lag) are denoted as (F) and (L), respectively.

Table 5: The nondimensional natural frequencies of the baseline sandwich beam

Mode	1F	1L	2F	2L	3F	3L	4F	4L

Frequency ($\widehat{\omega} = \frac{\omega^2 \mu_b L^4}{S_{55_b}}$)	3.516	4.137	22.093	25.977	62.174	73.041	122.8	144.08

The effect of relative density of the mentioned periodic unit cells on the in-plane and out of plane stiffness values of the cross-section (S_{55}, S_{66}) are obtained and shown in Figures 6-9. It is noted that all stiffness values are nondimensionalised with respect to the stiffness values of the baseline beam. It is clear that both the shape of the periodic unit cell and the density ratio of the core affect the cross-sectional stiffness values of the beam. Therefore, it is necessary to examine the effects of periodic core on the vibration behaviour of the sandwich beams.

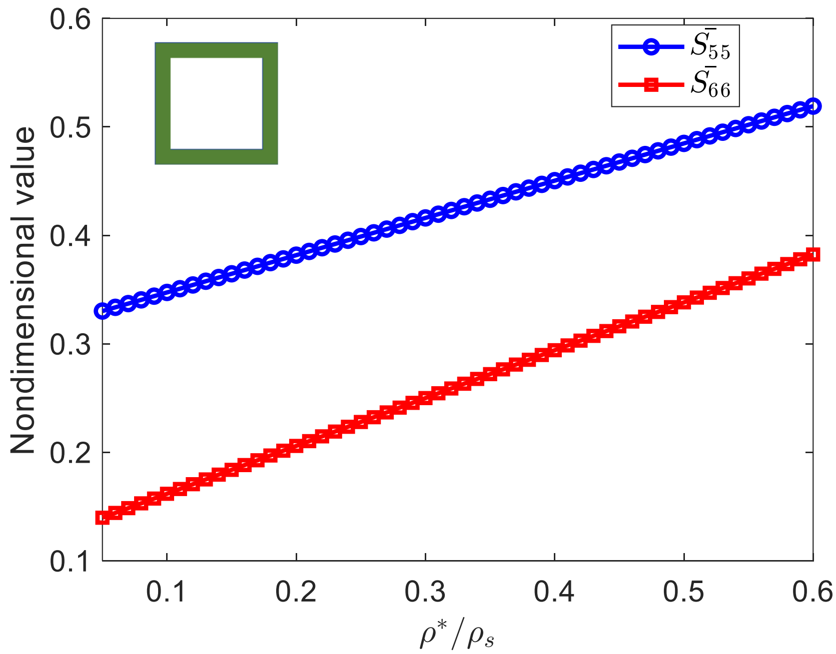

Figure 6: The stiffness values of the sandwich beam with square unit cell

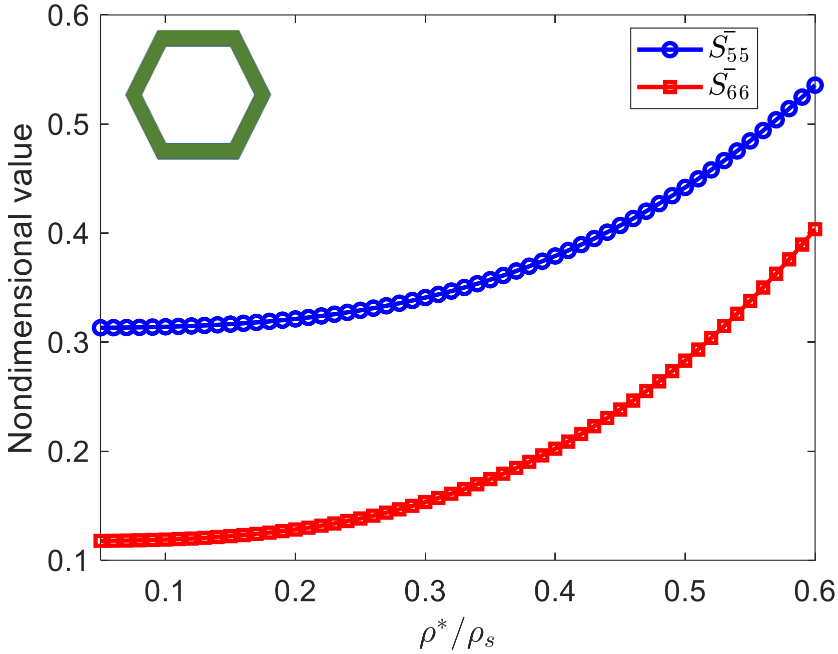

Figure 7: The stiffness values of the sandwich beam with hexagonal unit cell

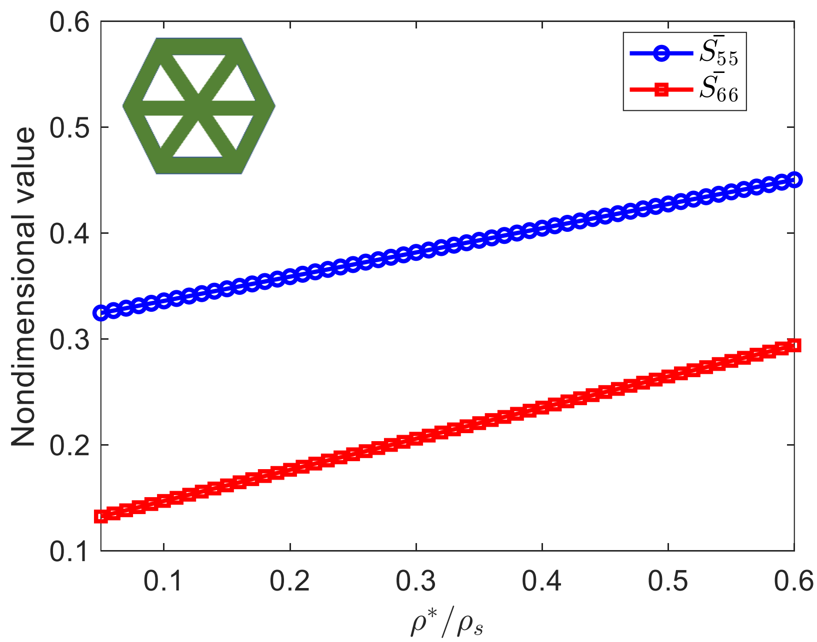

Figure 8: The stiffness values of the sandwich beam with triangular unit cell

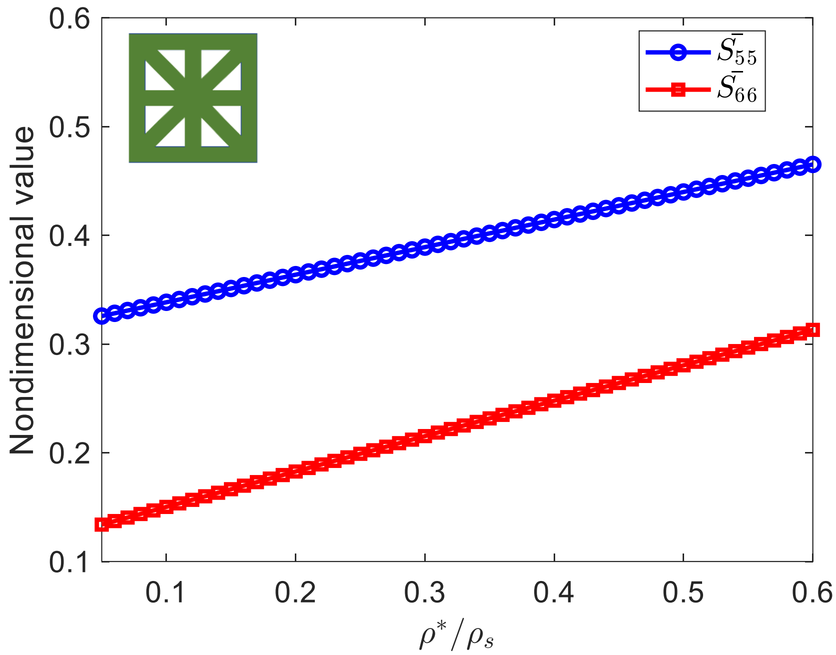

Figure 9: The stiffness values of the sandwich beam with mixed unit cell

Figures 10-13 show the effect of density ratio on the first seven nondimensional natural frequencies of the sandwich beam for various unit cells. The core density ratio affects all beam frequencies in all cases, but higher modes are more sensitive to the variation of density ratio. Also, by increasing the density ratio, all frequencies decrease for all cases except case 2 at which the frequencies first decrease, and then increase. It is noted that the density ratio at which this change of trend is seen for case 2 is not constant for all modes. Furthermore, mode veering between in-plane and our plane bending modes is seen at around $\frac{\rho^*}{\rho_s} = 0.56$ for case 1 and 2 configurations, which from this density ratio onward, the properties of bending modes are switched. Therefore, it is clear that the dynamics of the sandwich beam is dependent to the core unit cell shape and density ratio. Moreover, Table 6 presents the nondimensional frequencies of first 5 modes for various cases and density ratios.

Next, the effects of initial curvature combined with the core unit cell shape on the natural frequencies of the beam are investigated.

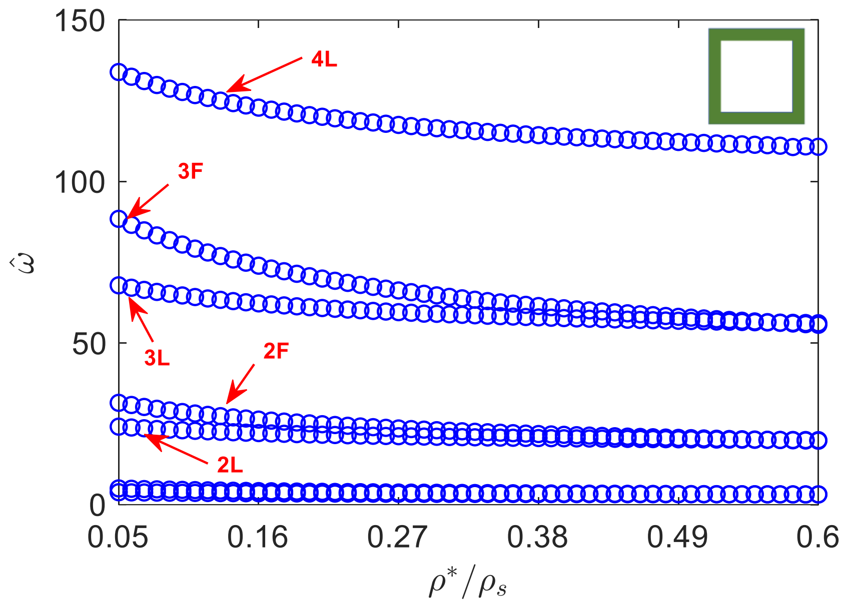

Figure 10: The effect of density ratio of the core on the natural frequencies of the sandwich beam (case 1)

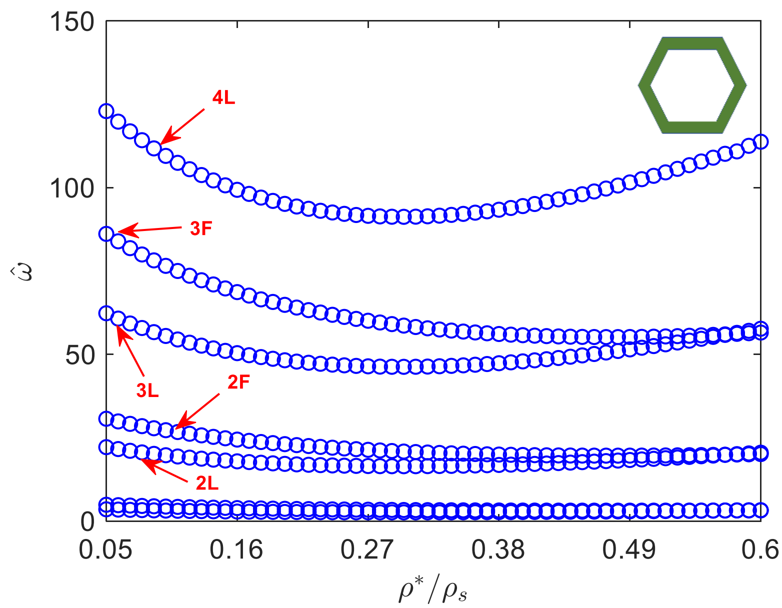

Figure 11: The effect of density ratio of the core on the natural frequencies of the sandwich beam (case 2)

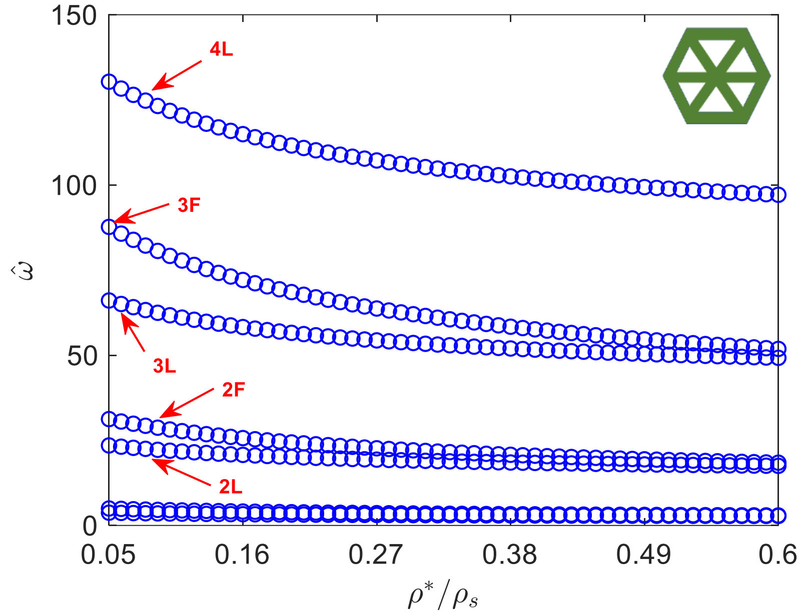

Figure 12: The effect of density ratio of the core on the natural frequencies of the sandwich beam (case 3)

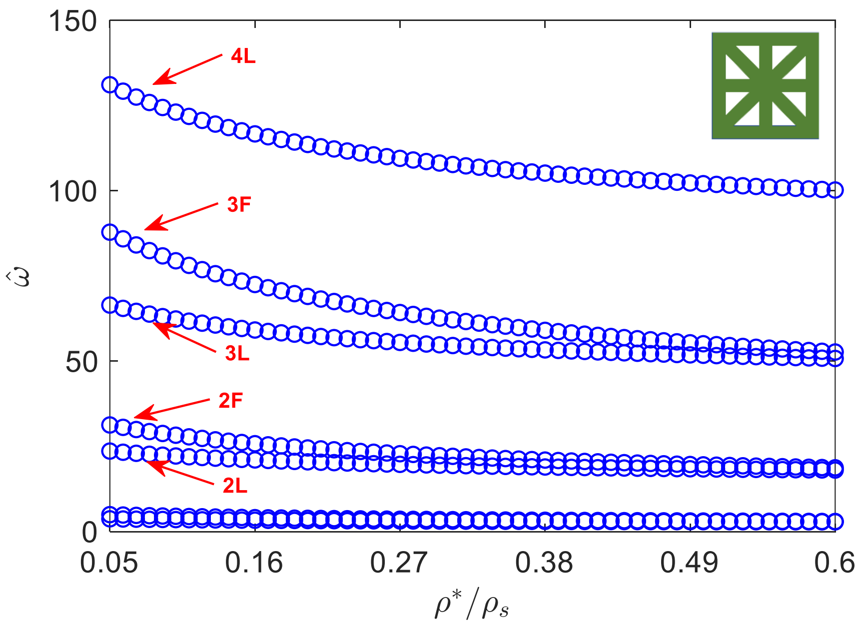

Figure 13: The effect of density ratio of the core on the natural frequencies of the sandwich beam (case 4)

Table 6: effect of density ratio and periodic core unit cell shape on the first five natural frequencies of the sandwich beam

	Density ratio	Mode 1 (1L)	Mode 2 (1F)	Mode 3 (2L)	Mode 4 (2F)	Mode5 (3L)
Case 1	**0.1**	3.6668	4.5669	23.0259	28.6607	64.7436
	0.2	3.4610	4.0057	21.7337	25.1494	61.1102
	0.3	3.3449	3.6680	21.0050	23.0347	58.6251
	0.4	3.2703	3.4402	20.5364	21.6069	54.6285
	0.5	3.2183	3.2751	20.2096	20.5723	51.9183
Case 2	**0.1**	3.1446	4.3424	19.7467	27.2517	55.5233
	0.2	2.7315	3.6749	17.1525	23.0723	48.2290
	0.3	2.6200	3.3200	16.4528	20.8489	46.2615
	0.4	2.7126	3.1556	17.0341	19.8194	47.8960
	0.5	2.9442	3.1267	18.4888	19.6402	51.9862
Case 3	**0.1**	3.4958	4.4909	21.9522	28.1835	61.7245
	0.2	3.2037	3.8835	20.1181	24.3819	56.5676
	0.3	3.0348	3.5131	19.0576	22.0615	53.5857
	0.4	2.9243	3.2602	18.3637	20.4767	51.6346
	0.5	2.8463	3.0753	17.8734	19.3169	50.2561
Case 4	**0.1**	3.5333	4.5074	22.1880	28.2870	62.3876
	0.2	3.2609	3.9101	20.4772	24.5494	57.5772
	0.3	3.1043	3.5470	19.4939	22.2749	54.8123
	0.4	3.0023	3.2998	18.8533	20.7256	53.0111
	0.5	2.9305	3.1194	18.4021	19.5943	51.7426

Effect of initial curvature on the natural frequencies

In this section, the effect of initial curvature on the natural frequencies of the sandwich beam with various core lattice unit cells is investigated. Figure 14 shows the equilibrium shape of the beam when it is subjected to various initial out of plane curvatures. It is noted that the initial curvature of the beam is uniform along the beam, and the beam deflection/shape along x_2 and x_3 coordinates are nondimensionalized with respect to the beam length (L).

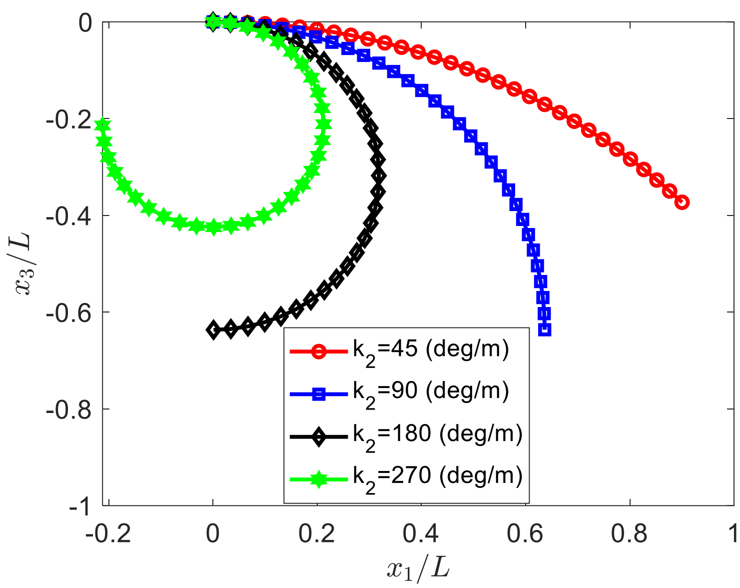

Figure 14: The deformed shape of the beam with out of plane curvature (k_2)

The effect of out of plane initial curvature on the natural frequencies of the baseline beam is shown in Figure 15. All frequencies of the beam are sensitive to the initial curvature of the beam to some extent. By increasing the initial curvature, all frequencies decrease except first and second modes. It must be noted that the first lag mode varies rapidly and tends to reach to higher modes.

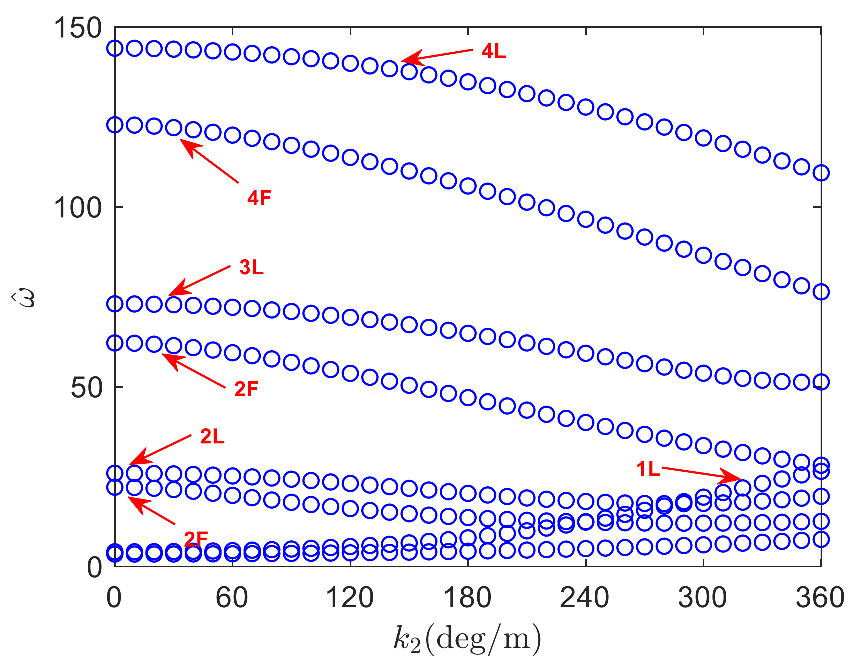

Figure 15: The effect of out of plane curvature (k_2) on the natural frequencies of the baseline beam

Figures 16-19 depict the variation of beam natural frequencies with respect to an initial curvature for two values of density ratio. As it has been observed in Figure 15, here also for all cases, the initial curvature affects all frequencies. Also, the first flap and lag modes, in all cases, increase with an increase of curvature value, while the frequency of the rest of modes decrease. Furthermore, depending on the value of density ratio, the curvature at which the modes cross each other is different for all cases. It is noted that for all curvatures, the frequencies related to density ratio $\rho^*/\rho_s = 0.1$ of is higher than the density ratio of $\rho^*/\rho_s = 0.3$. But, the rate of change is dependent to the shape of the unit cell.

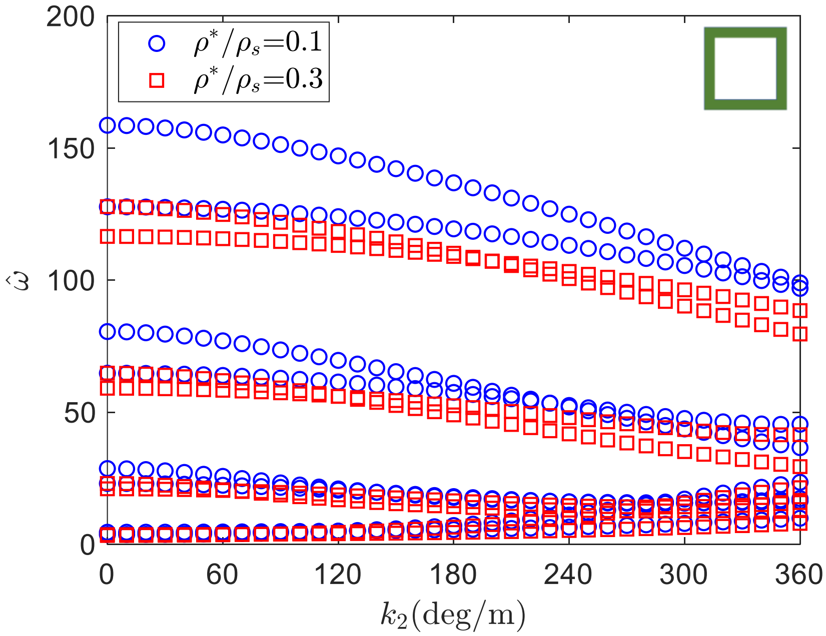

Figure 16: The effect of out of plane curvature (k_2) and density ratio on the natural frequencies of the sandwich beam (case 1)

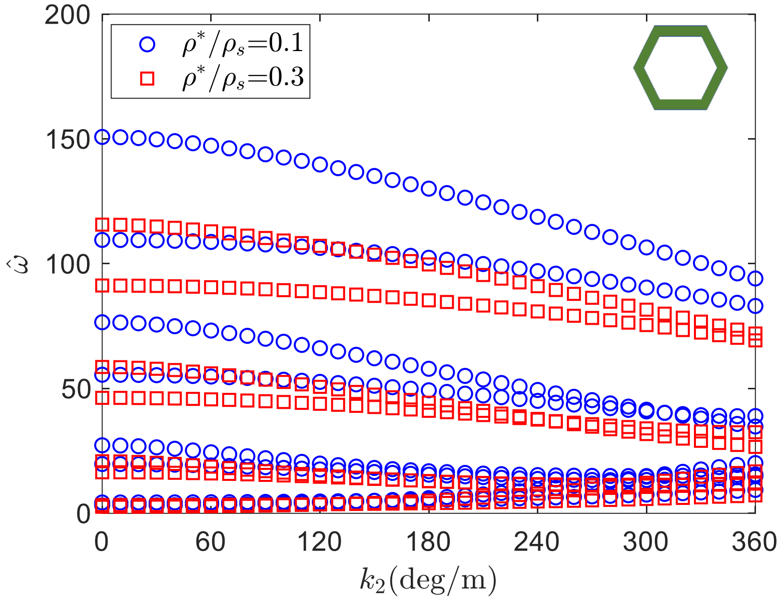

Figure 17: The effect of out of plane curvature (k_2) and density ratio on the natural frequencies of the sandwich beam (case 2)

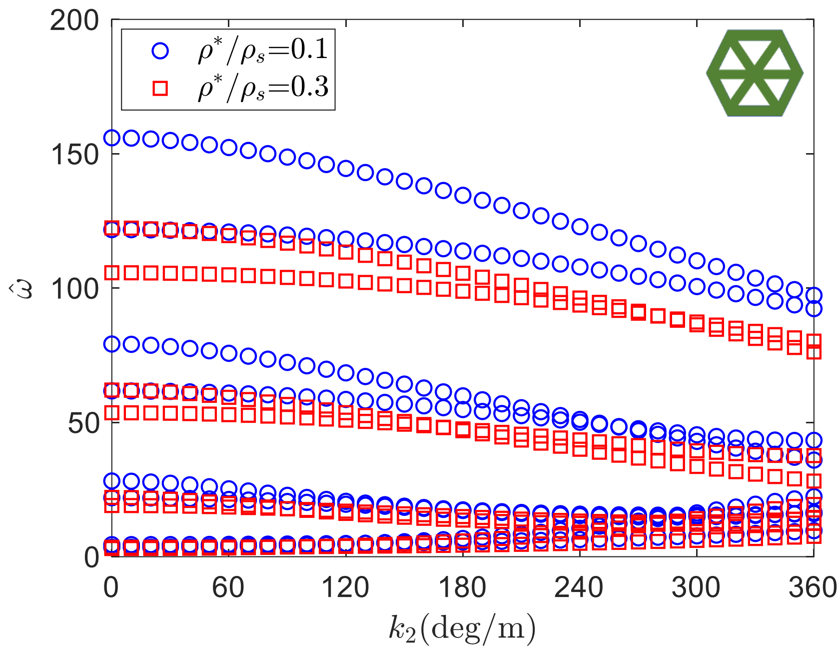

Figure 18: The effect of out of plane curvature (k_2) and density ratio on the natural frequencies of the sandwich beam (case 3)

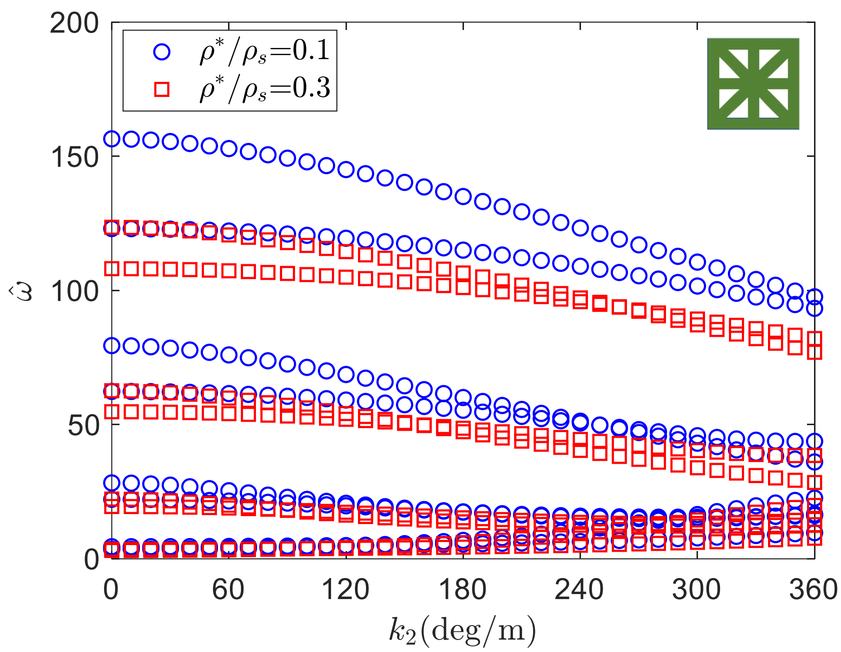

Figure 19: The effect of out of plane curvature (k_2) and density ratio on the natural frequencies of the sandwich beam (case 4)

The effect of end follower moment on the natural frequencies

In this section, it is assumed that the sandwich beam is subjected to an end follower moment which makes the beam to bend. The end moment is nondimensionalized using the following relationship

$$N = \frac{M_2 L}{2\pi\, S_{55}}$$

Figure 20 shows the nondimensional deformation of the beam for different values of moment. It is noted that the beam deformation when it is subjected to an end moment of N=0.125, N=0.25, N=0.5 and N=0.75 is exactly similar to the shape of beam with initial curvatures of k_2=45 (deg/m), k_2=90 (deg/m), k_2=180 (deg/m) and k_2=275 (deg/m), respectively. Next, the effect of the end follower moment on the dynamics of the sandwich beam is studied to investigate if the beam behaves differently when it is initially curved to curved due to the end follower force.

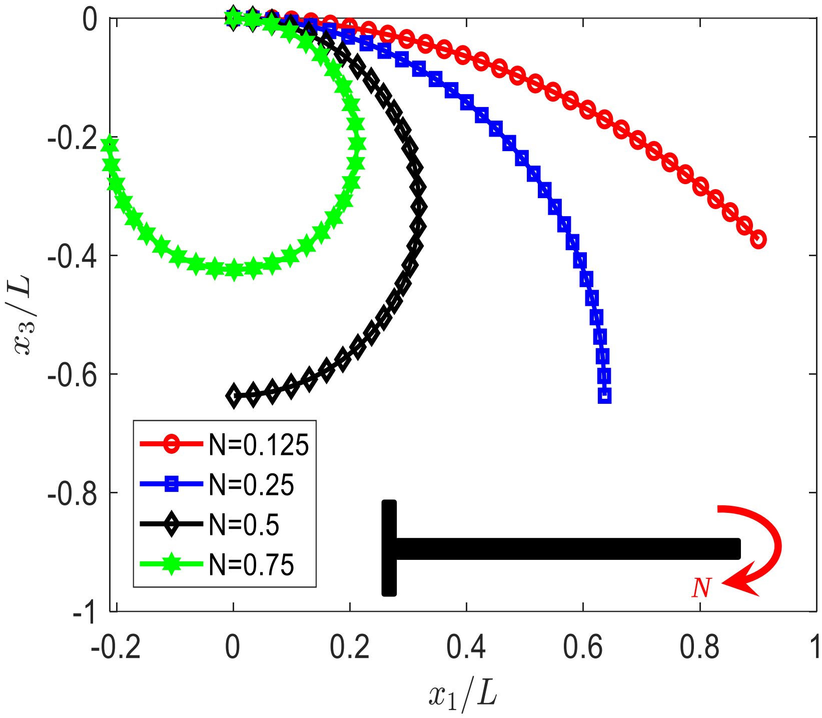

Figure 20: The deformed shape of the beam subjected to an end follower moment (*N*)

Figure 21 shows the variation of the natural frequencies of the baseline beam when it is subjected to different end follower moment values. The follower moment affects all frequencies, but not in a same way as it was observed to the initial curvature shown in Figure 15. Furthermore, at *N*=0.73, a dynamic instability (flutter) is seen in the system which is due to the coalescence of the first and second lag modes. The nondimensional frequency of instability at this case is $\hat{\omega} = 13$. By comparing Figure 21 and Figure 15, it is clear that although the equilibrium shape of the beam is the same for both cases, but the system behaves differently when it is subjected to an end follower moment or has an initial curvature. This observation was also seen by Chang and Hodges [33].

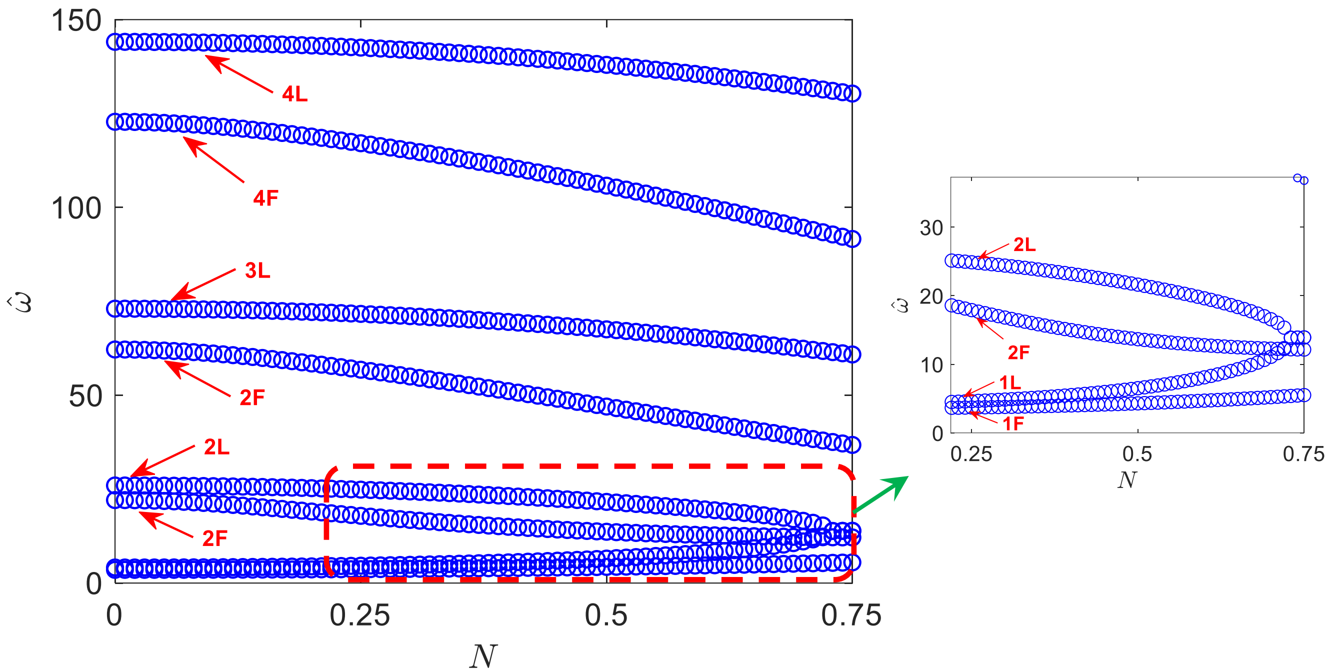

Figure 21: The effect of end follower moment on the natural frequencies of the baseline beam

Figures 22-25 show the effect of both density ratio and end follower moment on the natural frequencies of sandwich beam with various core shapes. It is noted that here the value of the end follower moment is selected so that the system doesn't suffer from dynamic instability. For all cases, when the beam is subjected to an end follower moment, all natural frequencies change. It is noted that the flap modes are more sensitive to the moment than the lag modes. Furthermore, as the density ratio increases the difference between the frequencies with and without moment tend to decrease. But, this is not correct for the fourth lag mode in case 1 and case 2 configurations in which by increasing the density ratio, the difference gets higher.

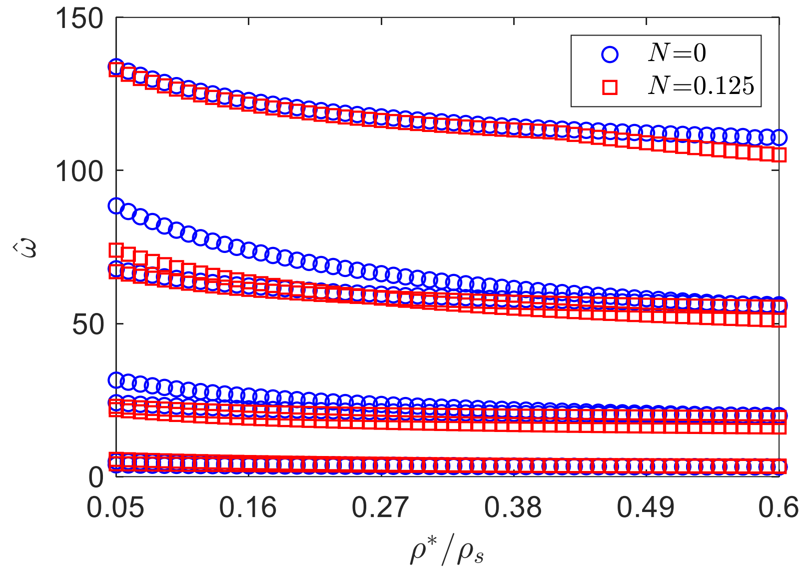

Figure 22: The effect of end follower moment and density ratio on the natural frequencies of the sandwich beam (case 1)

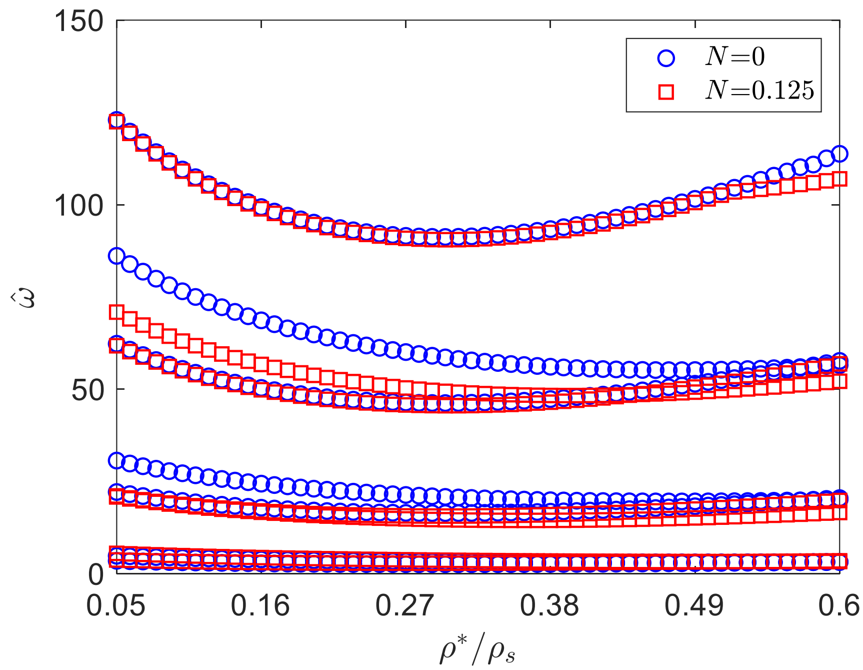

Figure 23: The effect of end follower moment and density ratio on the natural frequencies of the sandwich beam (case 2)

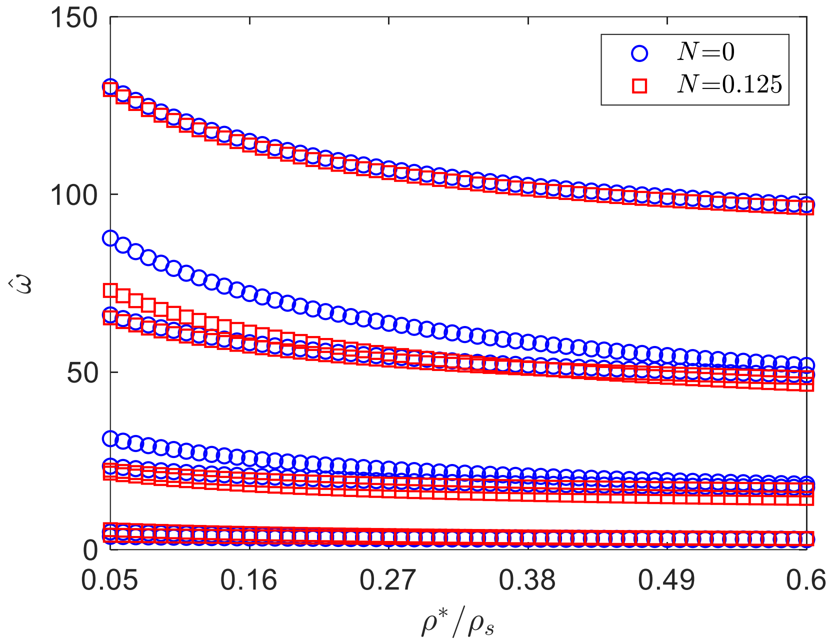

Figure 24: The effect of end follower moment and density ratio on the natural frequencies of the sandwich beam (case 3)

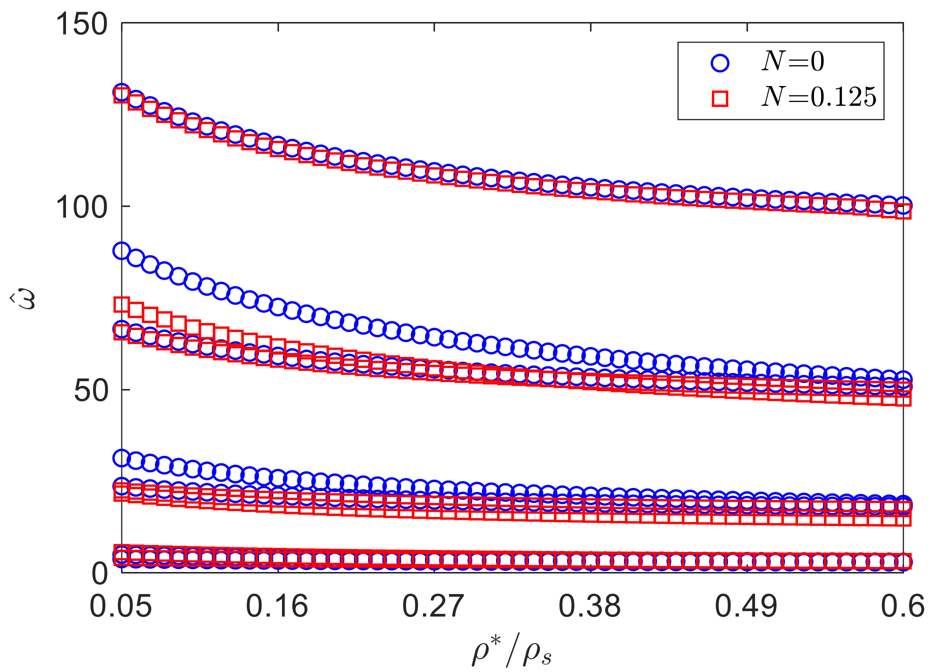

Figure 25: The effect of end follower moment and density ratio on the natural frequencies of the sandwich beam (case 4)

Conclusion

In this paper, the influence of initial curvature on the in-plane and out of plane vibration of sandwich beams with lattice core has been studied. A baseline sandwich beam with a solid core alongside four sandwich beams with lattice cores has been considered. The dynamics of the beam has been simulated using the geometrically exact beam formulation. The effect of lattice core has been taken into account by using the equivalent material properties of the core. An out of plane initial curvature has been added to the beam to see how the natural frequencies of the beam can be affected. First, the effect of core density ratio on the cross-sectional stiffness as well as the natural frequencies of the beam has been analysed. It has been observed that depending on the type of the lattice core, the natural frequencies can decrease or increase when the density ratio increases. Furthermore, the effect of initial curvature on the natural frequencies of the lattice sandwich beam has been investigated. It has been determined that the initial curvature affects. Next, the effect of an end follower moment on the natural frequencies of the sandwich beam has been studies. The results showed that the end follower moment changes the flap modes more than the lag modes. Also, a dynamic instability has been observed in the system when the beam is subjected to this follower moment. Finally, the results indicated that the natural frequencies of the beam change differently when the beam is initially bent (initial curvature) or is bent because of an end follower moment even if the final equilibrium shape of the beam is the same.

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