Discontinuous Galerkin Methods for the Biharmonic Problem

Emmanuil H. Georgoulis* and Paul Houston†

October 12, 2007

Abstract

This work is concerned with the design and analysis of hp-version discontinuous Galerkin (DG) finite element methods for boundary-value problems involving the biharmonic operator. The first part extends the unified approach of Arnold, Brezzi, Cockburn & Marini ([1]; SIAM J. Numer. Anal. 39, 5 (2001/02), 1749-1779) developed for the Poisson problem, to the design of DG methods via an appropriate choice of numerical flux functions for fourth order problems; as an example we retrieve the interior penalty DG method developed by Süli & Mozolevski ([22]; Comput. Methods Appl. Mech. Engrg. 196, 13-16 (2007), 1851-1863). The second part of this work is concerned with a new a-priori error analysis of the hp-version interior penalty DG method, when the error is measured in terms of both the energy-norm and L^2-norm, as well certain linear functionals of the solution, for elemental polynomial degrees $p \geq 2$. Also, provided that the solution is piecewise analytic in an open neighbourhood of each element, exponential convergence is also proven for the p-version of the DG method. The sharpness of the theoretical developments is illustrated by numerical experiments.

1 Introduction

Fourth-order elliptic boundary-value problems arise, among other disciplines, in thin plate theories of elasticity. For isotropic elastic behaviour of thin plates and membranes, popular models involve the biharmonic operator together with appropriate Dirichlet and Neumann boundary conditions. Finite element methods (FEMs) have been proven extremely popular for the numerical treatment of such fourth order elliptic problems. Various finite element methods have been proposed and tested during the last 30 years. They can be broadly classified into three categories: conforming, non-conforming, and mixed FEMs.

Conforming FEMs for fourth order problems require that the finite element space is a finite dimensional subspace of the Sobolev space $H^2(\Omega)$, where Ω denotes the computational domain. To satisfy this conformity requirement, C^1-conforming elements have been, traditionally, introduced (see [6] and the references therein). The construction of such finite element spaces is highly non-trivial, especially

*Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom. E-mail: `Emmanuil.Georgoulis@mcs.le.ac.uk`

†School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom. E-mail: `Paul.Houston@nottingham.ac.uk`. The research of this author was supported by the EPSRC under grant GR/R76615.

when high order basis functions are involved, or when Ω is a three-dimensional domain, and they are rarely used in practice. To relax the strong continuity requirements across the element interfaces, non-conforming FEMs have been presented for fourth-order problems; we refer to [6, 9] for a discussion of the classical approaches, the more recent works [10, 4] and the references therein. Another approach frequently employed in the literature is to write the fourth-order problem as a system of second-order problems and use mixed finite element methods (see [5] and the references therein).

In recent years, discontinuous Galerkin (DG) finite element methods, a certain class of non-conforming methods, have been receiving considerable attention as flexible and efficient discretisations for a large class of problems ranging from computational fluid dynamics to computational mechanics and electromagnetic theory. For a historical reference (with an extensive list of references) we refer to [7] and the important paper [1]. Discontinuous Galerkin methods admit completely discontinuous finite element spaces, giving great flexibility in the mesh design, providing an ideal framework for *hp*-adaptive algorithms, i.e., algorithms where both the mesh-size h and the local polynomial degree p can vary subject to certain adaptivity criteria. For the biharmonic problem, *hp*-version interior penalty discontinuous Galerkin finite element methods have been recently presented and analysed by Mozolevski & Süli [16], Süli & Mozolevski [22] and Mozolevski, Süli & Bösing [17]. In [16, 22] the authors present the stability analysis and a-priori error bounds in the energy norm for symmetric, non-symmetric and semi-symmetric variants of the *hp*-version interior penalty DG method. Moreover, in [17] the authors extend the above a-priori error analysis to various Sobolev norms (weaker than the energy norm) and derive a-priori error bounds for the convergence of linear functionals of the solution. We remark that, all of the aforementioned error bounds have been derived under the assumption that the polynomial degree $p \geq 3$.

The purpose of this work is to study and further analyse *hp*-version DG methods for the biharmonic problem. In particular, we extend the unified approach of designing DG methods of Arnold, Brezzi, Cockburn & Marini [1] to DG methods for the biharmonic problem, and we show that the interior penalty DG method from [22] can be defined through suitable numerical fluxes. For simplicity of the presentation, we only consider the symmetric version interior penalty DG method. The fluxes for the non-symmetric and the semi-symmetric versions of the DG method (cf. [16]) are straightforward variations and are omitted to enhance the clarity of the presentation. Using this general framework it is possible to define and analyse various discontinuous Galerkin methods for fourth order problems that could be put forward in the future, in an analogous fashion to the ideas presented in [1].

Also, we present a new a-priori error analysis for this problem, based on the use of lifting operators, cf. [1, 19]. The error bounds presented below, coincide with the error bounds (in various norms) presented in [22, 17] when the solution admits finite Sobolev regularity and the elemental polynomial degree is $p \geq 3$. Additionally, this approach enables us to prove error bounds for the limiting case of $p = 2$, as well as to prove the exponential convergence of the p-version DG method when the solution is piecewise analytic on an open neighbourhood of each element. For simplicity, we only consider two-dimensional computational domains in the error analysis; the extension of the results below to three dimensions is straightforward due to the tensor-product nature of the arguments.

The paper is structured as follows. In Section 2 we define the model problem considered in this work along with the function space framework. In Section 3 we present a unified approach for the construction of discontinuous Galerkin methods for the biharmonic problem via a suitable choice of numerical flux functions. Moreover, we demonstrate that the interior penalty DG method emerges from a particular choice of these numerical fluxes. In Section 4, we give some results regarding the approximation error of the H^1 projection operator which is utilised in the subsequent error analysis presented in Section 5, where bounds for the error of the DG method in the energy- and L^2-norms are derived, as well as bounds for the error of linear functionals of the solution. The theoretical findings are illustrated by numerical experiments presented in Section 6.

2 Model problem and preliminaries

We denote by $L^p(\omega)$, $1 \leq p \leq \infty$, the standard Lebesgue spaces, $\omega \subset \mathbb{R}^d$, $d \geq 1$, with corresponding norms $\|\cdot\|_{L^p(\omega)}$; the norm of $L^2(\omega)$ will be denoted by $\|\cdot\|_\omega$ for brevity. We also denote by $H^s(\omega)$, the standard Hilbertian Sobolev space of index $s \geq 0$ of real-valued functions defined on $\omega \subset \mathbb{R}^d$, $d \geq 1$, along with the corresponding norm and seminorm $\|\cdot\|_{s,\omega}$ and $|\cdot|_{s,\omega}$, respectively.

Let Ω be a bounded open (curvilinear) polygonal domain in $\mathbb{R}^2$, and let Γ_∂ signify the union of its one-dimensional open edges, which are assumed to be sufficiently smooth (in a sense defined rigorously later). We consider the fourth-order equation

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{2.1}$$

where $f \in L^2(\Omega)$. We impose Dirichlet boundary conditions

$$\begin{aligned} u &= g_{\mathrm{D}} \text{ on } \Gamma_\partial, \\ \nabla u \cdot \mathbf{n} &= g_{\mathrm{N}} \text{ on } \Gamma_\partial, \end{aligned} \tag{2.2}$$

where $\mathbf{n}$ denotes the unit outward normal vector to Γ_∂.

Let $\mathcal{T}$ be a subdivision of Ω into disjoint open elements $\kappa \in \mathcal{T}$ such that each side of κ has at most one hanging node and define $h_\kappa := \mathrm{diam}(\bar{\kappa})$. We assume that the subdivision $\mathcal{T}$ is shape-regular (see, e.g., page 124 in [6]), constructed via mappings F_κ, where $F_\kappa : \hat{\kappa} := (-1,1)^2 \to \kappa$ is a C^∞-diffeomorphism, with non-singular Jacobian. The above mappings are assumed to be constructed so as to ensure that the union of the closures of the elements $\kappa \in \mathcal{T}$ forms a covering of the closure of Ω, i.e., $\bar{\Omega} = \cup_{\kappa \in \mathcal{T}} \bar{\kappa}$. For brevity, we denote F_κ, $\kappa \in \mathcal{T}$, collectively by $\mathbf{F} = \{F_\kappa : \kappa \in \mathcal{T}\}$.

We assign to the subdivision $\mathcal{T}$ the broken Sobolev space of composite order $\mathbf{s} := \{s_\kappa : \kappa \in \mathcal{T}\}$,

$$H^{\mathbf{s}}(\Omega, \mathcal{T}) := \left\{ u \in L^2(\Omega) : u|_\kappa \in H^{s_\kappa}(\kappa) \text{ for all } \kappa \in \mathcal{T} \right\},$$

equipped with the standard broken Sobolev norm $\|\cdot\|_{H^{\mathbf{s}}(\Omega,\mathcal{T})}$. When $s_\kappa = s$ for all $\kappa \in \mathcal{T}$, we write $H^s(\Omega, \mathcal{T})$.

Also, for a nonnegative integer p, we denote by $\mathcal{Q}_p(\hat{\kappa})$, the set of all tensor-product polynomials on $\hat{\kappa}$ of degree at most p in each coordinate direction. To each $\kappa \in \mathcal{T}$ we assign the nonnegative integer p_κ (the local polynomial degree). We collect the h_κ and p_κ into the element-wise constant functions $\mathbf{h}, \mathbf{p} : \Omega \to \mathbb{R}$, with $\mathbf{h}|_\kappa = h_\kappa$ and $\mathbf{p}|_\kappa = p_\kappa$, $\kappa \in \mathcal{T}$, respectively. We consider the finite element space

$$S_1 \equiv S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}) := \{v \in L^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{Q}_{p_\kappa}(\hat{\kappa}),\ \kappa \in \mathcal{T}\}. \tag{2.3}$$

We shall assume throughout that the meshsize function $\mathbf{h}$ and polynomial degree function $\mathbf{p}$, with $p_\kappa \geq 2$ for each $\kappa \in \mathcal{T}$, have *bounded local variation*, i.e., there exist constants $\eta,\ \rho \geq 1$, independent of $\mathbf{h}$ and $\mathbf{p}$, such that, for any pair of elements κ and κ' in $\mathcal{T}$ which share a side,

$$\eta^{-1} \leq h_\kappa / h_{\kappa'} \leq \eta \quad \text{and} \quad \rho^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho. \tag{2.4}$$

We shall refer to η and ρ collectively as *mesh parameters.*

By Γ we denote the union of all one-dimensional element edges associated with the subdivision $\mathcal{T}$ (including the boundary). Further we decompose Γ into two disjoint subsets $\Gamma = \Gamma_\partial \cup \Gamma_{\text{int}}$, where $\Gamma_{\text{int}} := \Gamma \backslash \Gamma_\partial$.

Next, we introduce some trace operators. Let κ, κ' be two (generic) elements sharing an edge $e := \partial\kappa \cap \partial\kappa' \subset \Gamma_{\text{int}}$. Define the outward normal unit vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ on e corresponding to $\partial\kappa$ and $\partial\kappa'$, respectively. For functions $q : \Omega \to \mathbb{R}$ and $\boldsymbol{\phi} : \Omega \to \mathbb{R}^2$ that may be discontinuous across Γ, we define the following quantities. For $q^+ := q|_{\partial\kappa}$, $q^- := q|_{\partial\kappa'}$ and $\boldsymbol{\phi}^+ := \boldsymbol{\phi}|_{\partial\kappa}$, $\boldsymbol{\phi}^- := \boldsymbol{\phi}|_{\partial\kappa'}$, we set

$$\{\!\!\{q\}\!\!\} := \frac{1}{2}(q^+ + q^-),\ \{\!\!\{\boldsymbol{\phi}\}\!\!\} := \frac{1}{2}(\boldsymbol{\phi}^+ + \boldsymbol{\phi}^-),\ [\![q]\!] := q^+\mathbf{n}^+ + q^-\mathbf{n}^-,\ [\![\boldsymbol{\phi}]\!] := \boldsymbol{\phi}^+ \cdot \mathbf{n}^+ + \boldsymbol{\phi}^- \cdot \mathbf{n}^-;$$

if $e \in \partial\kappa \cap \Gamma_\partial$, these definitions are modified to

$$\{\!\!\{q\}\!\!\} := q^+,\ \{\!\!\{\boldsymbol{\phi}\}\!\!\} := \boldsymbol{\phi}^+,\ [\![q]\!] := q^+\mathbf{n},\ [\![\boldsymbol{\phi}]\!] := \boldsymbol{\phi}^+ \cdot \mathbf{n}.$$

3 Unified approach for the definition of DG methods

We begin by writing the boundary-value problem (2.1), (2.2) as a first-order system:

$$\mathbf{t} = \nabla u, \quad s = \nabla\cdot\mathbf{t}, \quad \mathbf{q} = \nabla s, \quad \nabla\cdot\mathbf{q} = f, \quad \text{in } \Omega. \tag{3.1}$$

$$u = g_{\mathrm{D}}, \quad \nabla u\cdot\mathbf{n} = g_{\mathrm{N}}, \quad \text{on } \Gamma_{\partial}. \tag{3.2}$$

We consider a subdivision $\mathcal{T}$ of the computational domain Ω and we multiply the first and third equations by vector test functions $\mathbf{z}$ and $\mathbf{r}$, respectively, and the second and fourth equations by scalar test functions w and v, respectively. Then, integrating over every $\kappa\in\mathcal{T}$, and upon formal integration by parts on every element $\kappa\in\mathcal{T}$ on each equation, we obtain

$$\begin{aligned}
\int_\kappa \mathbf{t}\cdot\mathbf{z}\,\mathrm{d}x &= -\int_\kappa u\nabla\cdot\mathbf{z}\,\mathrm{d}x + \int_{\partial\kappa} u\,\mathbf{n}\cdot\mathbf{z}\,\mathrm{d}s\ ,\\
\int_\kappa sw\,\mathrm{d}x &= -\int_\kappa \mathbf{t}\cdot\nabla w\,\mathrm{d}x + \int_{\partial\kappa} w\,\mathbf{n}\cdot\mathbf{t}\,\mathrm{d}s\ ,\\
\int_\kappa \mathbf{q}\cdot\mathbf{r}\,\mathrm{d}x &= -\int_\kappa s\nabla\cdot\mathbf{r}\,\mathrm{d}x + \int_{\partial\kappa} s\,\mathbf{n}\cdot\mathbf{r}\,\mathrm{d}s\ ,\\
\int_\kappa fv\,\mathrm{d}x &= -\int_\kappa \mathbf{q}\cdot\nabla v\,\mathrm{d}x + \int_{\partial\kappa} v\,\mathbf{n}\cdot\mathbf{q}\,\mathrm{d}s\ ,
\end{aligned}$$

where $\mathbf{n}$ denotes the unit outward normal unit vector to $\partial\kappa$. Now, we restrict the choice of the trial and test functions $u,\mathbf{t},s,\mathbf{q}$ and $\mathbf{z},w,\mathbf{r},v$, respectively, to finite-dimensional subspaces. More specifically, we seek $u_h, s_h\in S_1\equiv S^{\mathbf{p}}(\Omega,\mathcal{T},\mathbf{F})$ and $\mathbf{t}_h,\mathbf{q}_h\in S_2 := [S_1]^2$ such that

$$\int_\kappa \mathbf{t}_h\cdot\mathbf{z}_h\,\mathrm{d}x = -\int_\kappa u_h\nabla\cdot\mathbf{z}_h\,\mathrm{d}x + \int_{\partial\kappa} \hat{u}\,\mathbf{n}\cdot\mathbf{z}_h\,\mathrm{d}s\ , \tag{3.3}$$

$$\int_\kappa s_h w_h\,\mathrm{d}x = -\int_\kappa \mathbf{t}_h\cdot\nabla w_h\,\mathrm{d}x + \int_{\partial\kappa} w_h\,\mathbf{n}\cdot\hat{\mathbf{t}}\,\mathrm{d}s\ , \tag{3.4}$$

$$\int_\kappa \mathbf{q}_h\cdot\mathbf{r}_h\,\mathrm{d}x = -\int_\kappa s_h\nabla\cdot\mathbf{r}_h\,\mathrm{d}x + \int_{\partial\kappa} \hat{s}\,\mathbf{n}\cdot\mathbf{r}_h\,\mathrm{d}s\ , \tag{3.5}$$

$$\int_\kappa f v_h\,\mathrm{d}x = -\int_\kappa \mathbf{q}_h\cdot\nabla v_h\,\mathrm{d}x + \int_{\partial\kappa} v_h\,\mathbf{n}\cdot\hat{\mathbf{q}}\,\mathrm{d}s\ , \tag{3.6}$$

for $w_h, v_h\in S_1$ and $\mathbf{z}_h,\mathbf{r}_h\in S_2$. The numerical fluxes $\hat{u}$, $\hat{\mathbf{t}}$, $\hat{s}$, and $\hat{\mathbf{q}}$ are approximations to u, ∇u, Δu, and $\nabla\Delta u$, respectively. It is indeed the freedom of choice of these numerical fluxes that gives rise to various discontinuous Galerkin methods.

We apply again the divergence theorem in (3.3), to obtain

$$\int_\kappa \mathbf{t}_h\cdot\mathbf{z}_h\,\mathrm{d}x = \int_\kappa \nabla u_h\cdot\mathbf{z}_h\,\mathrm{d}x + \int_{\partial\kappa} (\hat{u}-u_h)\,\mathbf{n}\cdot\mathbf{z}_h\,\mathrm{d}s. \tag{3.7}$$

Setting $\mathbf{z}_h = \nabla w_h$ in (3.7), and using the resulting equality to eliminate $\mathbf{t}_h$ from (3.4), the latter becomes

$$\int_\kappa s_h w_h \, \mathrm{d}x = -\int_\kappa \nabla u_h \cdot \nabla w_h \, \mathrm{d}x - \int_{\partial\kappa} (\hat{u} - u_h)\, \mathbf{n} \cdot \nabla w_h \, \mathrm{d}s + \int_{\partial\kappa} w_h \, \mathbf{n} \cdot \hat{\mathbf{t}} \, \mathrm{d}s. \tag{3.8}$$

Applying the divergence theorem to the first term on the right-hand side of (3.8), we obtain

$$\int_\kappa s_h w_h \, \mathrm{d}x = \int_\kappa \Delta u_h w_h \, \mathrm{d}x - \int_{\partial\kappa} (\hat{u} - u_h)\, \mathbf{n} \cdot \nabla w_h \, \mathrm{d}s + \int_{\partial\kappa} w_h \, \mathbf{n} \cdot (\hat{\mathbf{t}} - \nabla u_h) \, \mathrm{d}s. \tag{3.9}$$

We recall the identity

$$\sum_{\kappa\in\mathcal{T}} \int_{\partial\kappa} \psi \boldsymbol{\phi} \cdot \mathbf{n} \, \mathrm{d}s = \int_\Gamma [\![\psi]\!] \cdot \{\!\!\{\boldsymbol{\phi}\}\!\!\} \, \mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \{\!\!\{\psi\}\!\!\} [\![\boldsymbol{\phi}]\!] \, \mathrm{d}s\,, \tag{3.10}$$

with $\psi \in H^1(\Omega, \mathcal{T})$ and $\phi \in [H^1(\Omega, \mathcal{T})]^2$ (see, e.g., [1]). Summing up over all elements $\kappa \in \mathcal{T}$, and using (3.10) on the second and third term on the right-hand side of (3.9), we find

$$\begin{aligned} \int_\Omega s_h w_h \, \mathrm{d}x &= \int_\Omega \Delta_h u_h w_h \, \mathrm{d}x - \int_\Gamma [\![\hat{u} - u_h]\!] \cdot \{\!\!\{\nabla w_h\}\!\!\} \, \mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \{\!\!\{\hat{u} - u_h\}\!\!\} [\![\nabla w_h]\!] \, \mathrm{d}s \\ &\quad + \int_\Gamma [\![w_h]\!] \cdot \{\!\!\{\hat{\mathbf{t}} - \nabla u_h\}\!\!\} \, \mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \{\!\!\{w_h\}\!\!\} [\![\hat{\mathbf{t}} - \nabla u_h]\!] \, \mathrm{d}s. \end{aligned} \tag{3.11}$$

Choosing

$$\hat{u} = \begin{cases} \{\!\!\{u_h\}\!\!\}, & \text{if } e \subset \Gamma_{\mathrm{int}}; \\ g_{\mathrm{D}}, & \text{if } e \subset \Gamma_\partial, \end{cases} \qquad \text{and} \qquad \hat{\mathbf{t}} = \begin{cases} \{\!\!\{\nabla u_h\}\!\!\}, & \text{if } e \subset \Gamma_{\mathrm{int}}; \\ g_{\mathrm{N}}\mathbf{n}, & \text{if } e \subset \Gamma_\partial, \end{cases}$$

equation (3.11) gives

$$\begin{aligned} \int_\Omega s_h w_h \, \mathrm{d}x &= \int_\Omega \Delta_h u_h w_h \, \mathrm{d}x + \int_{\Gamma_{\mathrm{int}}} \Big([\![u_h]\!] \cdot \{\!\!\{\nabla w_h\}\!\!\} - \{\!\!\{w_h\}\!\!\} [\![\nabla u_h]\!] \Big) \, \mathrm{d}s \\ &\quad + \int_{\Gamma_\partial} \Big((u_h - g_{\mathrm{D}})(\nabla w_h \cdot \mathbf{n}) + w_h (g_{\mathrm{N}} - \nabla u_h \cdot \mathbf{n}) \Big) \, \mathrm{d}s, \end{aligned} \tag{3.12}$$

where Δ_h defines the broken Laplacian with respect to the subdivision $\mathcal{T}$.

Now, we combine (3.5) and (3.6). Setting $\mathbf{r}_h = \nabla v_h$ in (3.5) and substituting into (3.6), we deduce

$$\int_\kappa f v_h \, \mathrm{d}x = \int_\kappa s_h \Delta v_h \, \mathrm{d}x - \int_{\partial\kappa} \hat{s}\, (\nabla v_h \cdot \mathbf{n}) \, \mathrm{d}s + \int_{\partial\kappa} v_h \, \hat{\mathbf{q}} \cdot \mathbf{n} \, \mathrm{d}s\,. \tag{3.13}$$

Summing over all elements $\kappa \in \mathcal{T}$, setting $w_h = \Delta v_h$ in (3.12), and inserting this into (3.13) we get

$$\begin{aligned} \int_\Omega f v_h \, \mathrm{d}x &= \int_\Omega \Delta_h u_h \Delta_h v_h \, \mathrm{d}x + \int_{\Gamma_{\mathrm{int}}} \Big([\![u_h]\!] \cdot \{\!\!\{\nabla \Delta v_h\}\!\!\} - \{\!\!\{\Delta v_h\}\!\!\} [\![\nabla u_h]\!] \Big) \, \mathrm{d}s \\ &\quad + \int_{\Gamma_\partial} \Big((u_h - g_{\mathrm{D}})(\nabla \Delta v_h \cdot \mathbf{n}) + \Delta v_h (g_{\mathrm{N}} - \nabla u_h \cdot \mathbf{n}) \Big) \, \mathrm{d}s \\ &\quad - \sum_{\kappa\in\mathcal{T}} \int_{\partial\kappa} \hat{s}\, (\nabla v_h \cdot \mathbf{n}) \, \mathrm{d}s + \sum_{\kappa\in\mathcal{T}} \int_{\partial\kappa} v_h \, \hat{\mathbf{q}} \cdot \mathbf{n} \, \mathrm{d}s\,. \end{aligned} \tag{3.14}$$

Making use of (3.10) in the fourth and fifth terms on the right-hand side of (3.14), we deduce

$$\begin{aligned}\int_\Omega f v_h \,\mathrm{d}x &= \int_\Omega \Delta_h u_h \Delta_h v_h \,\mathrm{d}x + \int_{\Gamma_{\text{int}}} \Big([\![u_h]\!] \cdot \{\!\!\{\nabla \Delta v_h\}\!\!\} - \{\!\!\{\Delta v_h\}\!\!\}[\![\nabla u_h]\!] \Big) \,\mathrm{d}s \\ &\quad + \int_{\Gamma_\partial} \Big((u_h - g_{\mathrm{D}})(\nabla \Delta v_h \cdot \mathbf{n}) + \Delta v_h (g_{\mathrm{N}} - \nabla u_h \cdot \mathbf{n}) \Big) \,\mathrm{d}s \\ &\quad - \int_\Gamma [\![\hat{s}]\!] \cdot \{\!\!\{\nabla v_h\}\!\!\} \,\mathrm{d}s - \int_{\Gamma_{\text{int}}} \{\!\!\{\hat{s}\}\!\!\}[\![\nabla v_h]\!] \,\mathrm{d}s + \int_\Gamma [\![v_h]\!] \cdot \{\!\!\{\hat{\mathbf{q}}\}\!\!\} \,\mathrm{d}s + \int_{\Gamma_{\text{int}}} \{\!\!\{v_h\}\!\!\}[\![\hat{\mathbf{q}}]\!] \,\mathrm{d}s \,. \end{aligned} \tag{3.15}$$

Choosing

$$\hat{s} = \begin{cases} \{\!\!\{\Delta u_h\}\!\!\} - \beta [\![\nabla u_h]\!], & \text{if } e \subset \Gamma_{\text{int}}; \\ \Delta u_h - \beta(\nabla u_h \cdot \mathbf{n} - g_{\mathrm{N}}), & \text{if } e \subset \Gamma_\partial, \end{cases}$$

and

$$\hat{\mathbf{q}} = \begin{cases} \{\!\!\{\nabla \Delta u_h\}\!\!\} + \alpha [\![u_h]\!], & \text{if } e \subset \Gamma_{\text{int}}; \\ \nabla \Delta u_h + \alpha(u_h - g_{\mathrm{D}})\mathbf{n}, & \text{if } e \subset \Gamma_\partial, \end{cases}$$

with $\alpha, \beta : \Gamma \to \mathbb{R}$ positive piecewise constant functions (to be defined explicitly below), equation (3.15) gives

$$\begin{aligned} &\int_\Omega \Delta_h u_h \Delta_h v_h \,\mathrm{d}x + \int_{\Gamma_\partial} \Big(u_h (\nabla \Delta v_h \cdot \mathbf{n}) + v_h (\nabla \Delta u_h \cdot \mathbf{n}) - \Delta v_h (\nabla u_h \cdot \mathbf{n}) - \Delta u_h (\nabla v_h \cdot \mathbf{n}) \Big) \,\mathrm{d}s \\ &+ \int_{\Gamma_\partial} \Big(\alpha u_h v_h + \beta (\nabla u_h \cdot \mathbf{n})(\nabla v_h \cdot \mathbf{n}) \Big) \,\mathrm{d}s + \int_{\Gamma_{\text{int}}} \Big(\alpha [\![u_h]\!][\![v_h]\!] + \beta [\![\nabla u_h]\!][\![\nabla v_h]\!] \Big) \,\mathrm{d}s \\ &+ \int_{\Gamma_{\text{int}}} \Big([\![u_h]\!] \cdot \{\!\!\{\nabla \Delta v_h\}\!\!\} + [\![v_h]\!] \cdot \{\!\!\{\nabla \Delta u_h\}\!\!\} - \{\!\!\{\Delta v_h\}\!\!\}[\![\nabla u_h]\!] - \{\!\!\{\Delta u_h\}\!\!\}[\![\nabla v_h]\!] \Big) \,\mathrm{d}s \\ = &\int_\Omega f v_h \,\mathrm{d}x + \int_{\Gamma_\partial} \Big(g_{\mathrm{D}} (\nabla \Delta v_h \cdot \mathbf{n} + \alpha v_h) + g_{\mathrm{N}} (\beta \nabla v_h \cdot \mathbf{n} - \Delta v_h) \Big) \,\mathrm{d}s \,. \end{aligned} \tag{3.16}$$

Recalling the conventions $[\![v]\!]|_e = v\mathbf{n}$, $[\![\mathbf{r}]\!]|_e = \mathbf{r} \cdot \mathbf{n}$, $\{\!\!\{v\}\!\!\}|_e = v$, and $\{\!\!\{\mathbf{r}\}\!\!\}|_e = \mathbf{r}$, (3.16) can be written in the compressed form

$$\begin{aligned} &\int_\Omega \Delta_h u_h \Delta_h v_h \,\mathrm{d}x + \int_\Gamma \Big([\![u_h]\!] \cdot \{\!\!\{\nabla \Delta v_h\}\!\!\} + [\![v_h]\!] \cdot \{\!\!\{\nabla \Delta u_h\}\!\!\} - \{\!\!\{\Delta v_h\}\!\!\}[\![\nabla u_h]\!] - \{\!\!\{\Delta u_h\}\!\!\}[\![\nabla v_h]\!] \Big) \,\mathrm{d}s \\ &+ \int_\Gamma \Big(\alpha [\![u_h]\!][\![v_h]\!] + \beta [\![\nabla u_h]\!][\![\nabla v_h]\!] \Big) \,\mathrm{d}s \\ = &\int_\Omega f v_h \,\mathrm{d}x + \int_{\Gamma_\partial} \Big(g_{\mathrm{D}} (\nabla \Delta v_h \cdot \mathbf{n} + \alpha v_h) + g_{\mathrm{N}} (\beta \nabla v_h \cdot \mathbf{n} - \Delta v_h) \Big) \,\mathrm{d}s \,. \end{aligned} \tag{3.17}$$

Upon defining the lifting operator $\mathcal{L} : S_1(h) := S_1 + H^2(\Omega) \to S_1$ by

$$\int_\Omega \mathcal{L}(v) r \,\mathrm{d}x = \int_\Gamma \Big([\![v]\!] \cdot \{\!\!\{\nabla r\}\!\!\} - \{\!\!\{r\}\!\!\}[\![\nabla v]\!] \Big) \,\mathrm{d}s \quad \forall r \in S_1, \tag{3.18}$$

and the boundary lifting by $\mathcal{G} \in S_1$

$$\int_\Omega \mathcal{G} r \,\mathrm{d}x = \int_{\Gamma_\partial} \Big(g_{\mathrm{D}} (\nabla r \cdot \mathbf{n}) - g_{\mathrm{N}} r \Big) \,\mathrm{d}s \quad \forall r \in S_1,$$

equation (3.17) gives rise to the *symmetric interior penalty DG method* (SIP-DG):

$$\text{find } u_{\rm DG} \in S_1 \text{ such that } \quad B(u_{\rm DG}, v) = l(v) \quad \forall v \in S_1, \tag{3.19}$$

where the bilinear form $B(\cdot,\cdot)$ and the linear functional $l(\cdot)$ are given, respectively, by

$$B(u_h, v_h) = \int_\Omega \Big(\Delta_h u_h \Delta_h v_h + \mathcal{L}(u_h)\Delta_h v_h + \Delta_h u_h \mathcal{L}(v_h)\Big)\,{\rm d}x + \int_\Gamma \Big(\alpha [\![u_h]\!][\![v_h]\!] + \beta [\![\nabla u_h]\!][\![\nabla v_h]\!]\Big)\,{\rm d}s \tag{3.20}$$

and

$$l(v_h) = \int_\Omega \Big(f v_h + \mathcal{G}\Delta v_h\Big)\,{\rm d}x + \int_{\Gamma_\partial} \Big(\alpha g_{\rm D} v_h + \beta g_{\rm N}(\nabla v_h \cdot \mathbf{n})\Big)\,{\rm d}s\,. \tag{3.21}$$

Note that this formulation is inconsistent for trial and test functions belonging to the solution space $H_0^2(\Omega)$. However, when the trial and test functions belong to the finite element space, the SIP-DG (3.19) coincides with the symmetric version interior penalty method presented in Süli & Mozolevski [22].

Remark 3.1 *For simplicity of presentation, we only define numerical fluxes for the symmetric interior penalty DG method only. The non-symmetric and the semi-symmetric variants of interior penalty DG method for the biharmonic problem, introduced in [22], can be defined completely analogously. Also, the classical method of Baker [3] can be included by altering the definition of the numerical fluxes. Moreover, this general framework can include C^0-type non-conforming methods (see [4, 10] etc.) for fourth order problems by altering the definitions of the finite element spaces.*

4 Projection operators

For $\hat{I} := (-1,1)$, we define the H^1-projection operator $\hat{\lambda}_p : H^1(\hat{I}) \to \mathcal{P}_p(\hat{I})$, $p \geq 1$, $\mathcal{P}_p(\hat{I})$ being the space of polynomials of degree p or less on $\hat{I}$, by setting, for $\hat{u} \in H^1(\hat{I})$,

$$(\hat{\lambda}_p \hat{u})(x) := \int_{-1}^{x} \hat{\pi}_{p-1}(\hat{u}')(\eta)\,{\rm d}\eta + \hat{u}(-1), \qquad x \in \hat{I},$$

with $\hat{\pi}_{p-1}$ being the L^2-orthogonal projection operator onto $\mathcal{P}_{p-1}(\hat{I})$.

Let also $\hat{\kappa} \equiv (-1,1)^2$ and $\hat{u} \in \tilde{H}^1(\hat{\kappa})$. We define the H^1- projection operator $\hat{\Lambda}_p$, with polynomial degree p, by

$$\hat{\Lambda}_p = \hat{\lambda}_p^x \hat{\lambda}_p^y := (\hat{\lambda}_p \otimes I) \circ (I \otimes \hat{\lambda}_p),$$

with $\otimes$ denoting the standard tensor product, and $\hat{\lambda}_p^x, \hat{\lambda}_p^y$ denoting the one-dimensional H^1-projection operators. For a general element κ, for which there exists a mapping $F_\kappa : \hat{\kappa} \to \kappa$, we define the tensor-product H^1-projection operator Λ_p^κ by

$$\Lambda_p^\kappa u := (\hat{\Lambda}_p(u \circ F_\kappa)) \circ F_\kappa^{-1}, \qquad u \in H^1(\kappa).$$

Finally, the H^1-projection operator $\Lambda : H^1(\Omega, \mathcal{T}) \to S_1$ is defined by $(\Lambda u)|_\kappa = \Lambda_p^\kappa(u|_\kappa)$, $\kappa \in \mathcal{T}$.

Let us now introduce some notation. We denote by $\Phi(p,s)$ the quantity $\Phi(p,s) := (\Gamma(p-s+1)/\Gamma(p+s+1))^{\frac{1}{2}}$, with p, s real numbers such that $0 \le s \le p$ and $\Gamma(\cdot)$ being the Gamma function (see, e.g., [20]); we also adopt the standard convention $\Gamma(1) = 0! = 1$. We remark on the asymptotic behaviour of $\Phi(p,s)$. Making use of *Stirling's formula* (see, e.g., [20]),

$$\Gamma(n) \sim \sqrt{2\pi} n^{n-\frac{1}{2}} e^{-n}, \quad n > 0, \tag{4.1}$$

we can see that, for $p \ge 1$,

$$\Phi(p,s) \le C(s) p^{-s}, \tag{4.2}$$

with $0 \le s \le p$ and $C(s)$ denoting a constant depending only on s.

We recall some *hp*-approximation results for the error behaviour of the H^1-projection operator.

Theorem 4.1 *Let $\kappa \in \mathcal{T}$ and $h = \mathrm{diam}(\kappa)$ its diameter. Let $v \in H^{k+1}(\kappa)$, for $k \ge 1$, then the following error estimates hold:*

$$\|v - \Lambda v\|_\kappa \le C p^{-1} \Phi(p,s) h^{s+1} |v|_{s+1,\kappa}, \tag{4.3}$$

and

$$\|\nabla(v - \Lambda v)\|_\kappa \le C \Phi(p,s) h^s |v|_{s+1,\kappa}, \tag{4.4}$$

with $0 \le s \le \min\{p,k\}$, $p \ge 1$. Also, we have

$$\|v - \Lambda v\|_{\partial\kappa} \le C p^{-\frac{1}{2}} \Phi(p,t) h^{t+\frac{1}{2}} |u|_{t+1,\kappa}, \tag{4.5}$$

with $1 \le t \le \min\{p,k\}$, $p \ge 1$. Finally, for $v \in H^{k+1}(\kappa)$, with $k \ge 2$, the following error estimate holds:

$$\|\nabla(v - \Lambda v)\|_{\partial\kappa} \le C p^{\frac{1}{2}} \Phi(p,l) h^{l-\frac{1}{2}} |u|_{l+1,\kappa}, \tag{4.6}$$

with $2 \le l \le \min\{p,k\}$, $p \ge 2$.

Proof. The proof of (4.3) and (4.4) can be found in [15]. The proof of (4.5) follows from (4.3), (4.4) along with the standard trace inequality. A different proof of (4.5) and the proof of (4.6) can be found in [13].

□

We now present a bound for the Laplacian of the approximation error.

Theorem 4.2 *Let κ be a shape-regular element and $h = \mathrm{diam}(\kappa)$ its diameter. Let $v \in H^{k+2}(\kappa)$, with $k \ge 1$. Then the following bound holds:*

$$\|\Delta(v - \Lambda v)\|_\kappa \le C p^{\frac{3}{2}} \Phi(p,s) h^s |v|_{s+2,\hat{\kappa}}, \tag{4.7}$$

with $0 \le s \le \min\{p-1,k\}$.

Proof. The proof for functions defined on the reference element $\hat{\kappa}$ is given in the Appendix; (4.7) then follows from a standard scaling argument. $\square$

5 Error analysis

We define the DG-energy norm:

$$|||w||| := \left(\|\Delta_h w\|_\Omega^2 + \|\sqrt{\alpha}[\![w]\!]\|_\Gamma^2 + \|\sqrt{\beta}[\![\nabla w]\!]\|_\Gamma^2 \right)^{\frac{1}{2}},$$

for any function $w \in H^2(\Omega, \mathcal{T})$. Note that $|||\cdot|||$ is a norm in $H^2(\Omega, \mathcal{T})$ (Lemma 1 in [22]), and is equivalent to $\|\cdot\|_{H^2(\Omega,\mathcal{T})}$.

5.1 Error analysis in the energy norm

We begin the error analysis by deriving the stability of the trace and the boundary lifting.

Lemma 5.1 *Let $\mathcal{L}$ be the trace lifting defined in (3.18). Then, for $w \in S_1(h)$, the following bound holds:*

$$\|\mathcal{L}(w)\|_\Omega^2 \leq \|\sqrt{\gamma}[\![w]\!]\|_\Gamma^2 + \|\sqrt{\delta}[\![\nabla w]\!]\|_\Gamma^2, \tag{5.1}$$

for $\gamma, \delta : \Gamma \to \mathbb{R}$ piecewise constant functions, defined by $\gamma = C_\gamma \{\!\{\mathbf{p}^6/\mathbf{h}^3\}\!\}$ and $\delta = C_\delta \{\!\{\mathbf{p}^2/\mathbf{h}\}\!\}$, with C_γ and C_δ sufficiently large positive constants depending only on the mesh parameters.

Proof. Denoting by $\Pi : L^2(\Omega) \to S_1$ the (orthogonal) L^2-projection operator onto the finite element space S_1, we have

$$\begin{aligned}
\|\mathcal{L}(w)\|_\Omega &= \sup_{z\in L^2(\Omega)} \frac{\int_\Omega \mathcal{L}(w) z \,\mathrm{d}x}{\|z\|_\Omega} = \sup_{z\in L^2(\Omega)} \frac{\int_\Omega \mathcal{L}(w)\Pi z \,\mathrm{d}x}{\|z\|_\Omega} \\
&= \sup_{z\in L^2(\Omega)} \frac{\int_\Gamma \left([\![w]\!] \cdot \{\!\{\nabla(\Pi z)\}\!\} - \{\!\{\Pi z\}\!\}[\![\nabla w]\!] \right) \mathrm{d}s}{\|z\|_\Omega} \\
&\leq \sup_{z\in L^2(\Omega)} \frac{\|\sqrt{\gamma}[\![w]\!]\|_\Gamma \|\frac{1}{\sqrt{\gamma}}\{\!\{\nabla(\Pi z)\}\!\}\|_\Gamma + \|\frac{1}{\sqrt{\delta}}\{\!\{\Pi z\}\!\}\|_\Gamma \|\sqrt{\delta}[\![\nabla w]\!]\|_\Gamma}{\|z\|_\Omega} \\
&\leq \sup_{z\in L^2(\Omega)} \frac{\left(\|\frac{1}{\sqrt{\gamma}}\{\!\{\nabla(\Pi z)\}\!\}\|_\Gamma^2 + \|\frac{1}{\sqrt{\delta}}\{\!\{\Pi z\}\!\}\|_\Gamma^2 \right)^{\frac{1}{2}}}{\|z\|_\Omega} \left(\|\sqrt{\gamma}[\![w]\!]\|_\Gamma^2 + \|\sqrt{\delta}[\![\nabla w]\!]\|_\Gamma^2 \right)^{\frac{1}{2}},
\end{aligned} \tag{5.2}$$

from the definition of the L^2-norm, the orthogonality of the L^2-projection operator, the definition of the trace lifting, and exploiting the continuous and the discrete versions of the Cauchy-Schwarz inequality, respectively.

For $v \in \mathcal{Q}_{p_\kappa}(\kappa)$, $\kappa \in \mathcal{T}$, we recall the standard inverse inequalities

$$\|v\|^2_{\partial\kappa} \le C_{\rm inv}\frac{p_\kappa^2}{h_\kappa}\|v\|_\kappa \quad \text{and} \quad \|\nabla v\|^2_\kappa \le C'_{\rm inv}\frac{p_\kappa^4}{h_\kappa^2}\|v\|_\kappa,$$

(see, e.g., [21] for a proof). Then, using the shape-regularity, the mesh-regularity, the bounded local variation of the polynomial degree distribution assumptions on the finite element space S_1, we deduce

$$\|\delta^{-1/2}\{\!\!\{\Pi z\}\!\!\}\|^2_\Gamma \le \frac{1}{2}\sum_{\kappa\in\mathcal{T}}\|\delta^{-1/2}\Pi z\|^2_{\partial\kappa} \le \frac{1}{2CC_\delta}\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\|\Pi z\|^2_{\partial\kappa} \le \frac{C_{\rm inv}}{2CC_\delta}\sum_{\kappa\in\mathcal{T}}\|z\|^2_\kappa \le \frac{1}{2}\|z\|^2_\Omega,$$

where $C = C(\eta,\rho)$ is a positive constant, and $C_\delta \ge C_{\rm inv}/C$. Similarly, we have

$$\|\gamma^{-1/2}\{\!\!\{\nabla\Pi z\}\!\!\}\|^2_\Gamma \le \frac{1}{2}\sum_{\kappa\in\mathcal{T}}\|\gamma^{-1/2}\nabla(\Pi z)\|^2_{\partial\kappa} \le \frac{1}{2\tilde{C}C_\gamma}\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa^3}{p_\kappa^6}\|\nabla\Pi z\|^2_{\partial\kappa} \le \frac{C_{\rm inv}}{2\tilde{C}C_\gamma}\sum_{\kappa\in\mathcal{T}}\|z\|^2_\kappa \le \frac{1}{2}\|z\|^2_\Omega,$$

where $\tilde{C} = \tilde{C}(\eta,\rho)$ is a positive constant, and $C_\gamma \ge C'_{\rm inv}/\tilde{C}$. Inserting the last two bounds into (5.2), we deduce (5.1).

□

We are now ready to present the coercivity and continuity of the bilinear form.

Lemma 5.2 *Let $\alpha : \Gamma \to \mathbb{R}$ and $\beta : \Gamma \to \mathbb{R}$ be piecewise constant functions, such that $\alpha > 4\gamma$ and $\beta > 4\delta$. Then the bilinear form $B(\cdot\,,\cdot)$ is continuous and coercive in the sense that*

$$\begin{aligned} |B(w,v)| &\le C_1|\!|\!|w|\!|\!||\!|\!|v|\!|\!| \quad \textit{for all} \quad w,v \in S_1(h),\\ B(w,w) &\ge C_2|\!|\!|w|\!|\!|^2 \qquad \textit{for all} \quad w \in S_1,\end{aligned}$$

where C_1 and C_2 are positive constants depending only on the mesh parameters.

Proof. Let w, $v \in S_1(h)$. Then, we have

$$\begin{aligned} |B(w,v)| &\le \|\Delta_h w\|_\Omega\|\Delta_h v\|_\Omega + \|\mathcal{L}(w)\|_\Omega\|\Delta_h v\|_\Omega + \|\Delta_h w\|_\Omega\|\mathcal{L}(v)\|_\Omega\\ &\quad +\|\sqrt{\alpha}[\![w]\!]\|_\Gamma\|\sqrt{\alpha}[\![v]\!]\|_\Gamma + \|\sqrt{\beta}[\![\nabla w]\!]\|_\Gamma\|\sqrt{\beta}[\![\nabla v]\!]\|_\Gamma\\ &\le C|\!|\!|w|\!|\!||\!|\!|v|\!|\!|,\end{aligned}$$

in view of the stability of the trace lifting (5.1).

For coercivity, we have

$$\begin{aligned} B(w,w) &= |\!|\!|w|\!|\!|^2 + 2\int_\Omega \mathcal{L}(w)\Delta_h w\,\mathrm{d}x\\ &\ge |\!|\!|w|\!|\!|^2 - 2\|\mathcal{L}(w)|\!|^2_\Omega - \frac{1}{2}\|\Delta_h w\|^2_\Omega\\ &\ge |\!|\!|w|\!|\!|^2 - 2(\|\sqrt{\gamma}[\![w]\!]\|^2_\Gamma + \|\sqrt{\delta}[\![\nabla w]\!]\|^2_\Gamma) - \frac{1}{2}\|\Delta_h w\|^2_\Omega\\ &\ge \frac{1}{2}\|\Delta_h w\|^2_\Omega + \|\sqrt{\alpha-4\gamma}[\![w]\!]\|^2_\Gamma + \|\sqrt{\beta-4\delta}[\![\nabla w]\!]\|^2_\Gamma.\end{aligned}$$

Since we assumed $\alpha > 4\gamma$ and $\beta > 4\delta$, coercivity follows. $\square$

The next result (Strang's Second Lemma) gives us the abstract error bound that can be used for the error analysis of the inconsistent formulation (cf. Theorem 4.2.2 [6]).

Theorem 5.3 *Let $u \in H^2(\Omega)$ be the analytical solution of (2.1), (2.2), $u_{\rm DG} \in S_1$ the solution of the problem (3.19). Then, we have*

$$|||u - u_{\rm DG}||| \le \left(1 + \frac{C_1}{C_2}\right) \inf_{v \in S_1} |||u - v||| + \frac{1}{C_2} \sup_{w \in S_1} \frac{|R(u,w)|}{|||w|||}, \tag{5.3}$$

where $R(u,w) := B(u,w) - l(w)$ is the residual.

We continue by estimating the residual.

Lemma 5.4 *Assume that u, the analytical solution of (2.1), (2.2), has regularity $u|_\kappa \in H^{k_\kappa+2}(\kappa)$, $k_\kappa \ge 2$, $\kappa \in \mathcal{T}$. Then, for all $w \in S_1$, we have*

$$|R(u,w)| \le C\Big(\sum_{\kappa\in\mathcal{T}} p_\kappa^{-3} \Phi^2(p_\kappa, t_\kappa) h_\kappa^{2t_\kappa+2} |u|^2_{t_\kappa+3,\kappa}\Big)^{\frac{1}{2}} |||w|||, \tag{5.4}$$

with $1 \le t_\kappa \le \min\{p_\kappa, k_\kappa - 1\}$, $p_\kappa \ge 1$; the constant C is independent of u, the data f, $g_{\rm D}$, $g_{\rm N}$, and the discretisation parameters.

Proof. From the elliptic regularity of the boundary-value problem (2.1), (2.2), we have $[\![u]\!]_e = 0$ and $[\![\nabla u]\!]_e = 0$ almost everywhere for $e \in \Gamma_{\rm int}$. This gives,

$$\begin{aligned} \int_\Omega \mathcal{L}(u) r \,\mathrm{d}x &= \int_\Gamma \Big([\![u]\!]\cdot \{\!\{\nabla r\}\!\} - \{\!\{r\}\!\}[\![\nabla u]\!]\Big)\,\mathrm{d}s = \int_{\Gamma_\partial} \Big(u(\nabla r\cdot \mathbf{n}) - r(\nabla u \cdot \mathbf{n})\Big)\,\mathrm{d}s \\ &= \int_{\Gamma_\partial} \Big(g_{\rm D}(\nabla r\cdot\mathbf{n}) - r g_{\rm N}\Big)\,\mathrm{d}s = \int_\Omega \mathcal{G} r\,\mathrm{d}x \end{aligned}$$

for all $r \in S_1$; hence $\mathcal{L}(u) = \mathcal{G}$.

Next, we evaluate the residual. Let $w \in S_1$ and note that the right-hand sides of (3.17) and (3.19) coincide when $v_h = w$. Hence, the residual $R(u,w) = B(u,w) - l(w)$ is given by the difference of the left-hand sides of (3.17) and (3.19) when $v_h = w$, i.e.,

$$\begin{aligned} R(u,w) &= \int_\Omega \Big(\mathcal{L}(u)\Delta_h w + \Delta_h u \mathcal{L}(w)\Big)\,\mathrm{d}x \\ &\quad - \int_\Gamma \Big([\![u]\!]\cdot\{\!\{\nabla\Delta w\}\!\} + [\![w]\!]\cdot\{\!\{\nabla \Delta u\}\!\} - \{\!\{\Delta w\}\!\}[\![\nabla u]\!] - \{\!\{\Delta u\}\!\}[\![\nabla w]\!]\Big)\,\mathrm{d}s. \end{aligned}$$

Making use of $\mathcal{L}(u) = \mathcal{G}$, $[\![u]\!]_e = 0$ and $[\![\nabla u]\!]_e = 0$ a.e. for $e \in \Gamma_{\rm int}$, along with the orthogonality of the L^2-projection operator, we deduce

$$\begin{aligned} R(u,w) &= \int_\Omega \Big(\mathcal{G}\Delta_h w + \Pi(\Delta_h u)\mathcal{L}(w)\Big)\,\mathrm{d}x - \int_{\Gamma_\partial}\Big(g_{\rm D}(\nabla\Delta w\cdot\mathbf{n}) - \Delta w g_{\rm N}\Big)\,\mathrm{d}s \\ &\quad - \int_\Gamma \Big([\![w]\!]\cdot\{\!\{\nabla\Delta u\}\!\} - \{\!\{\Delta u\}\!\}[\![\nabla w]\!]\Big)\,\mathrm{d}s \\ &= \int_\Gamma \Big(\{\!\{\Delta u - \Pi\Delta u\}\!\}[\![\nabla w]\!] - [\![w]\!]\cdot\{\!\{\nabla(\Delta u - \Pi\Delta u)\}\!\}\Big)\,\mathrm{d}s. \end{aligned}$$

This gives

$$\begin{aligned} |R(u,w)| &\leq \|\frac{1}{\sqrt{\beta}}\{\!\{\Delta u - \Pi\Delta u\}\!\}\|_\Gamma\|\sqrt{\beta}[\![\nabla w]\!]\|_\Gamma + \|\frac{1}{\sqrt{\alpha}}\{\!\{\nabla(\Delta u - \Pi\Delta u)\}\!\}\|_\Gamma\|\sqrt{\alpha}[\![w]\!]\|_\Gamma \\ &\leq C\Big(\sum_{\kappa\in\mathcal{T}}\frac{h_\kappa}{p_\kappa^2}\|\Delta u - \Pi\Delta u\|^2_{\partial\kappa} + \frac{h_\kappa^3}{p_\kappa^6}\|\nabla(\Delta u - \Pi\Delta u)\|^2_{\partial\kappa}\Big)^{\frac{1}{2}}|\!|\!|w|\!|\!|. \qquad (5.5) \end{aligned}$$

For $v \in H^{r_\kappa+1}(\kappa)$, we recall the *hp*-approximation error bounds of the L^2-projection operator derived in [12] (see also [11] for a more detailed discussion),

$$\|v - \Pi v\|^2_{\partial\kappa} \leq Cp_\kappa^{-1}\Phi^2(p_\kappa,t_\kappa)h_\kappa^{2t_\kappa+1}|v|^2_{t_\kappa+1,\kappa},$$

and

$$\|\nabla(v - \Pi v)\|^2_{\partial\kappa} \leq Cp_\kappa^{3}\Phi^2(p_\kappa,t_\kappa)h_\kappa^{2t_\kappa-1}|v|^2_{t_\kappa+1,\kappa},$$

with $1 \leq t_\kappa \leq \min\{p_\kappa, r_\kappa\}$. Setting $v = \Delta u$ in the above bounds, it is easy to see that (5.5) can be further bounded to give (5.4).

□

We are now ready to present the main theorem of this section.

Theorem 5.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded (curvilinear) polygonal domain. Suppose that $\mathcal{T}$ is a family of shape-regular subdivisions of Ω, and $\mathbf{p}$ is a polynomial degree vector of bounded local variation. Further, assume that $u|_\kappa \in H^{k_\kappa+2}(\kappa)$, $k_\kappa \geq 2$, $\kappa \in \mathcal{T}$. Let the penalisation functions $\alpha, \beta : \Gamma \to \mathbb{R}$ be defined as in Lemma 5.2. Then, the following energy error bound holds:*

$$\begin{aligned} |\!|\!|u - u_{\rm DG}|\!|\!|^2 &\leq C\sum_{\kappa\in\mathcal{T}}\bigg(\Big(p_\kappa^3\Phi^2(p_\kappa,s_\kappa) + p_\kappa^5\Phi^2(p_\kappa,s_\kappa+1)\Big)h_\kappa^{2s_\kappa}|u|^2_{s_\kappa+2,\kappa} \\ &\qquad + p_\kappa^{-3}\Phi^2(p_\kappa,t_\kappa)h_\kappa^{2t_\kappa+2}|u|^2_{t_\kappa+3,\kappa}\bigg), \qquad (5.6) \end{aligned}$$

where $1 \leq s_\kappa \leq \min\{p_\kappa - 1, k_\kappa\}$, $1 \leq t_\kappa \leq \min\{p_\kappa, k_\kappa - 1\}$, $p_\kappa \geq 2$, $\kappa \in \mathcal{T}$, and the constant C is independent of u, p_κ and h_κ.

Proof. We begin by estimating the approximation error in the energy norm of the H^1-projection operator Λ. Using the bounds from Theorem 4.1 for $v = u$, along with (4.7), we deduce

$$\begin{aligned} |||u - \Lambda u|||^2 &\leq \sum_{\kappa\in\mathcal{T}} \|\Delta(u - \Lambda u)\|_{\kappa}^2 + C\Big(\sum_{\kappa\in\mathcal{T}} \|\sqrt{\alpha}(u - \Lambda u)\|_{\partial\kappa}^2 + \|\sqrt{\beta}\nabla(u - \Lambda u)\|_{\partial\kappa}^2\Big) \\ &\leq C\sum_{\kappa\in\mathcal{T}} \Big(p_\kappa^3 \Phi^2(p_\kappa, s_\kappa) h_\kappa^{2s_\kappa} |u|_{s_\kappa+2,\kappa} + (p_\kappa^5 + p_\kappa^3)\Phi^2(p_\kappa, q_\kappa) h_\kappa^{2q_\kappa-2} |u|_{q_\kappa+1,\kappa}\Big) \\ &\leq C\sum_{\kappa\in\mathcal{T}} \Big(p_\kappa^3 \Phi^2(p_\kappa, s_\kappa) h_\kappa^{2s_\kappa} |u|_{s_\kappa+2,\kappa} + p_\kappa^5 \Phi^2(p_\kappa, s_\kappa + 1) h_\kappa^{2s_\kappa} |u|_{s_\kappa+2,\kappa}\Big), \end{aligned}$$

with $1 \leq s_\kappa \leq \min\{p_\kappa - 1, k_\kappa\}$, where in the last step we set $q_\kappa = s_\kappa + 1$.

Finally, setting $v = \Lambda u$ in Theorem 5.3, and using Lemma 5.4, the result follows. □

Using the above result, we can remark on the convergence of the SIP-DG in this setting.

Corollary 5.6 *Consider the setting of Theorem 5.5. Further, assume that* $u|_\kappa \in H^{k_\kappa+2}(\kappa)$, $k_\kappa \geq 2$, $\kappa \in \mathcal{T}$ *and that* $p_\kappa \geq 3$. *Then, the following energy error bound holds:*

$$|||u - u_{\mathrm{DG}}|||^2 \leq C\sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa-3}} \|u\|_{k_\kappa+2,\kappa}^2 \ , \tag{5.7}$$

with $s_\kappa = \min\{p_\kappa - 1, k_\kappa\}$, *where the constant* C *is independent of* u, p_κ *and* h_κ.

Proof. Setting $t_\kappa = s_\kappa + 1$ in (5.6), we obtain

$$|||u - u_{\mathrm{DG}}|||^2 \leq C\sum_{\kappa\in\mathcal{T}} \Big(p_\kappa^3 \Phi^2(p_\kappa, s_\kappa) + p_\kappa^5 \Phi^2(p_\kappa, s_\kappa + 1)\Big) h_\kappa^{2s_\kappa} |u|_{k_\kappa+2,\kappa}^2$$

with $1 \leq s_\kappa \leq \min\{p_\kappa - 1, k_\kappa\}$. Making use of (4.2), we have $\Phi(p, s) \leq C(s)p^{-s}$, with $0 \leq s \leq p$, and $\Phi(p, t + 1) \leq C(t)p^{-t-1}$, with $-1 \leq t \leq p - 1$. Finally, observing that $p_\kappa^{-s_\kappa} |u|_{s_\kappa+2} \leq C(k_\kappa - 1)^{k_\kappa-1} p_\kappa^{-k_\kappa} \|u\|_{k_\kappa+2}$, the result follows. □

Corollary 5.6 describes the *hp*-convergence of the interior penalty DG method when $p_\kappa \geq 3$, $\kappa \in \mathcal{T}$, and coincides with the corresponding error bound first derived in [16, 22], albeit with different method of proof. We now turn our attention to the limiting case $p_\kappa = 2$, $\kappa \in \mathcal{T}$.

Corollary 5.7 *Consider the setting of Theorem 5.5. for* $u|_\kappa \in H^4(\kappa)$ *and* $p_\kappa = 2$. *Then, the following energy error bound holds:*

$$|||u - u_{\mathrm{DG}}|||^2 \leq C\sum_{\kappa\in\mathcal{T}} h_\kappa^2 \|u\|_{4,\kappa}^2 \ , \tag{5.8}$$

with the constant C *being independent of* u, p_κ *and* h_κ.

Proof. The result follows by setting $s_\kappa = t_\kappa = 1$ in (5.6). □

Finally, we show the exponential convergence of the p-version SIP-DG, provided that the solution is sufficiently smooth.

Corollary 5.8 *Consider the setting of Theorem 5.5, and assume that $u|_\kappa$ and F_κ are analytic in an open neighbourhood of κ. Then, for sufficiently large p_κ, $\kappa \in \mathcal{T}$, we have:*

$$|||u - u_{\mathrm{DG}}|||^2 \le C \sum_{\kappa \in \mathcal{T}} h_\kappa^{2s_\kappa} p_\kappa^8 \mathrm{e}^{-2b_\kappa p_\kappa} |\kappa|, \tag{5.9}$$

with $1 \le s_\kappa \le p_\kappa - 1$ for some constants $b_\kappa > 0$, $\kappa \in \mathcal{T}$, with the constant C being independent of p_κ and h_κ.

Proof. The proof is analogous to the one in [15]; here we only present the main points for completeness of presentation. Since $u|_\kappa$ and F_κ are analytic in an open neighbourhood of κ, we have

$$|u \circ F_\kappa|_{r,\hat{\kappa}} \le C d_\kappa^r \Gamma(r+1) \quad \forall r > 0,$$

for some $d_\kappa > 1$ depending on the radius of analyticity of $u|_\kappa$. Using the properties of the Gamma function, we obtain

$$\Phi^2(p_\kappa, s)|u \circ F_\kappa|_{s+2,\hat{\kappa}} \le C d_\kappa^{2s+4} \frac{\Gamma(p_\kappa - s + 1)}{\Gamma(p_\kappa + s + 1)} \big(\Gamma(s+3)\big)^2 \le C p^4 d_\kappa^{2s+4} \frac{\Gamma(p_\kappa - s)}{\Gamma(p_\kappa + s + 2)} \big(\Gamma(s+2)\big)^2. \tag{5.10}$$

Then, setting $s+1 = \alpha p$, for some $0 < \alpha < 1$ and making use of Stirling's formula (4.1), a straightforward calculation (cf. [15]) yields

$$\Phi^2(p_\kappa, s)|u \circ F_\kappa|_{s+2,\hat{\kappa}} \le C p^5 \left(\frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}} (d_\kappa \alpha)^{2\alpha} \right)^{p_\kappa}.$$

Choosing $\alpha = \alpha_{\min} := (1 + d_\kappa^2)^{-1/2}$ which minimises the function $f(\alpha, d_\kappa) := \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}} (d_\kappa \alpha)^{2\alpha}$, we deduce $f(\alpha_{\min}, d_\kappa) < 1$. Setting $b_\kappa = \frac{1}{2} |\log f(\alpha_{\min}, d_\kappa)|$, we get

$$\Phi^2(p_\kappa, s)|u \circ F_\kappa|_{s+2,\hat{\kappa}} \le C p^5 \mathrm{e}^{-2b_\kappa p_\kappa}. \tag{5.11}$$

In a completely analogous fashion (without, however, the step described in (5.10)), we also deduce

$$\Phi^2(p_\kappa, s+1)|u \circ F_\kappa|_{s+2,\hat{\kappa}} \le C p^3 \mathrm{e}^{-2b_\kappa p_\kappa}, \tag{5.12}$$

and, setting $t = \alpha p_\kappa$, we can also obtain

$$\Phi^2(p_\kappa, t)|u \circ F_\kappa|_{t+3,\hat{\kappa}} \le C p^7 \mathrm{e}^{-2b_\kappa p_\kappa}. \tag{5.13}$$

The result follows by applying a scaling argument and incorporating (5.11), (5.12), and (5.13) in the bound (5.6).

□

5.2 A priori error analysis for functionals

We consider linear functionals of the solution, of the form:

$$J(u) = \int_\Omega \psi u \, \mathrm{d}x,$$

where $\psi \in L^2(\Omega)$ is a weight function. Then the following result holds.

Theorem 5.9 *Consider the setting of Theorem 5.5, with $u|_\kappa \in H^{k_\kappa+2}(\kappa)$, $k_\kappa \geq 2$. Assume also that the domain Ω of the boundary-value problem*

$$\Delta^2 z = \psi \quad in \quad \Omega, \qquad z = 0, \qquad \nabla z \cdot \mathbf{n} = 0 \quad on \quad \partial\Omega, \tag{5.14}$$

is such that $z|_\kappa \in H^{l_\kappa+2}(\kappa)$, $l_\kappa \geq 2$. Then, the following error bound holds:

$$\begin{aligned} |J(u) - J(u_{\mathrm{DG}})| \quad \leq \quad & C \sum_{\kappa\in\mathcal{T}} \bigg(\Big(p_\kappa^3 \Phi^2(p_\kappa, s_\kappa) + p_\kappa^5 \Phi^2(p_\kappa, s_\kappa+1) \Big) h_\kappa^{2s_\kappa} |u|^2_{s_\kappa+2,\kappa} \\ & \qquad + p_\kappa^{-3} \Phi^2(p_\kappa, t_\kappa) h_\kappa^{2t_\kappa+2} |u|^2_{t_\kappa+3,\kappa} \bigg) \\ & \times \sum_{\kappa\in\mathcal{T}} \bigg(\Big(p_\kappa^3 \Phi^2(p_\kappa, q_\kappa) + p_\kappa^5 \Phi^2(p_\kappa, q_\kappa+1) \Big) h_\kappa^{2q_\kappa} |z|^2_{q_\kappa+2,\kappa} \\ & \qquad + p_\kappa^{-3} \Phi^2(p_\kappa, r_\kappa) h_\kappa^{2r_\kappa+2} |z|^2_{r_\kappa+3,\kappa} \bigg), \end{aligned} \tag{5.15}$$

where $1 \leq s_\kappa, q_\kappa \leq \min\{p_\kappa - 1, k_\kappa\}$, $1 \leq t_\kappa, r_\kappa \leq \min\{p_\kappa, k_\kappa - 1\}$, $p_\kappa \geq 2$, $\kappa \in \mathcal{T}$, and the constant C is independent of u, p_κ and h_κ.

Proof. The proof follows the same steps to the one of Theorem 3.3 in [18]. We have:

$$J(u) - J(u_{\mathrm{DG}}) = J(u - u_{\mathrm{DG}}) = l(u - u_{\mathrm{DG}}) = B(z, u - u_{\mathrm{DG}}) - R(z, u - u_{\mathrm{DG}}).$$

Now, for any $z_{hp} \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$, we have $B(z_{hp}, u - u_{\mathrm{DG}}) = B(u, z_{hp}) - l(z_{hp}) = R(u, z_{hp})$, and $R(u, z_{hp}) = -R(u, z - z_{hp})$. This implies that

$$J(u) - J(u_{\mathrm{DG}}) = B(z - z_{hp}, u - u_{\mathrm{DG}}) - R(u, z - z_{hp}) - R(z, u - u_{\mathrm{DG}}).$$

For the first term on the right-hand side, we use the continuity from Lemma 5.2; for the second term on the right-hand side, we use (5.4) with $1 \leq t_\kappa \leq \min\{p_\kappa, k_\kappa - 1\}$; and for the third term on the right-hand side, we use (5.4) with z in the place of u, r_κ in the place of t_κ, $1 \leq r_\kappa \leq \min\{p_\kappa, l_\kappa - 1\}$. Thereby, we obtain

$$\begin{aligned} |J(u) - J(u_{\mathrm{DG}})| \quad \leq \quad & C |||u - u_{\mathrm{DG}}||| \, |||z - z_{hp}||| \\ & + C \Big(\sum_{\kappa\in\mathcal{T}} p_\kappa^{-3} \Phi^2(p_\kappa, t_\kappa) h_\kappa^{2t_\kappa+2} |u|^2_{t_\kappa+3,\kappa} \Big)^{\frac{1}{2}} |||z - z_{hp}||| \\ & + C \Big(\sum_{\kappa\in\mathcal{T}} p_\kappa^{-3} \Phi^2(p_\kappa, r_\kappa) h_\kappa^{2r_\kappa+2} |u|^2_{r_\kappa+3,\kappa} \Big)^{\frac{1}{2}} |||u - u_{\mathrm{DG}}|||, \end{aligned}$$

with $1 \leq t_\kappa \leq \min\{p_\kappa, k_\kappa - 1\}$ and $1 \leq r_\kappa \leq \min\{p_\kappa, l_\kappa - 1\}$. Choosing $z_{hp} = \Lambda z$ and working as in the proof of Theorem 5.5, the result follows.

$\square$

It is also possible to prove exponential convergence for the p-version SIP-DG when the solutions of the primal and dual problems are sufficiently smooth; these results are omitted for brevity, as they are completely analogous to the arguments presented in the proof of Corollary 5.8.

5.3 Error analysis in the L^2-norm

The error bound in the L^2-norm can be easily derived from Theorem 5.9, provided the standard shift-theorem for the boundary-value problem (2.1), (2.2) holds in the Hilbertian Sobolev scale.

Theorem 5.10 *Consider the setting of Theorem 5.5, with $u|_\kappa \in H^{k_\kappa+2}(\kappa)$, $k_\kappa \geq 2$. Assume also that the domain Ω of the boundary-value problem*

$$\Delta^2 z = u - u_{\mathrm{DG}} \quad in \quad \Omega, \qquad z = 0, \qquad \nabla z \cdot \mathbf{n} = 0 \quad on \quad \partial\Omega, \tag{5.16}$$

is such that $z \in H^4(\Omega)$, and we have $\|z\|_{4,\Omega} \leq C\|u - u_{\mathrm{DG}}\|_\Omega$, with the positive constant C depending only on the domain Ω (see Remark 5.11 for a discussion on this assumption). Then, the following L^2-error bound holds:

$$\begin{aligned} \|u - u_{\mathrm{DG}}\|_\Omega^2 \quad \leq \quad & C \max_{\kappa\in\mathcal{T}} \left(\frac{h_\kappa^{2m}}{p_\kappa^{m-1}}\right) \sum_{\kappa\in\mathcal{T}} \bigg(\Big(p_\kappa^3 \Phi^2(p_\kappa, s_\kappa) + p_\kappa^5 \Phi^2(p_\kappa, s_\kappa + 1)\Big) h_\kappa^{2s_\kappa} |u|_{s_\kappa+2,\kappa}^2 \\ & + p_\kappa^{-3}\Phi^2(p_\kappa, t_\kappa) h_\kappa^{2t_\kappa+2} |u|_{t_\kappa+3,\kappa}^2 \bigg), \end{aligned} \tag{5.17}$$

where $m = \max_{\kappa\in\mathcal{T}}\{\min\{p_\kappa - 1, 2\}\}$, $1 \leq s_\kappa \leq \min\{p_\kappa - 1, k_\kappa\}$, $1 \leq t_\kappa \leq \min\{p_\kappa, k_\kappa - 1\}$, $p_\kappa \geq 2$, $\kappa \in \mathcal{T}$, and the constant C is independent of u, p_κ and h_κ.

Proof. We set $\psi = u - u_{\mathrm{DG}}$, $q_\kappa = r_\kappa = 1$, $l_\kappa = 2$, for $\kappa \in \mathcal{T}$, in Theorem 5.9. Choosing $z_{hp} = \Lambda z$, working as in the proof of Theorem 5.5 (with $k_\kappa = 2$ for $|||z - z_{hp}|||$), and using the elliptic regularity hypothesis $\|z\|_{4,\Omega} \leq C\|u - u_{\mathrm{DG}}\|_\Omega$, the result follows.

$\square$

Remark 5.11 *The assumptions of Theorem 5.10, regarding the regularity of the dual problem can be removed in the case the problem is defined on domains with smooth boundary, with sufficiently smooth data. For regularity results on domains with corners, see the expositions in [14, 8], the recent work [2] and the references therein.*

Corollary 5.12 *Consider the setting of Theorem 5.10. Further, assume that $u|_\kappa \in H^{k_\kappa+2}(\kappa)$, $k_\kappa \geq 2$, $\kappa \in \mathcal{T}$ and that $p_\kappa \geq 3$. Then, the following bound holds:*

$$\|u - u_{\mathrm{DG}}\|_\Omega^2 \leq C \max_{\kappa\in\mathcal{T}} \left(\frac{h_\kappa^{2m}}{p_\kappa^{m-1}}\right) \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa-3}} \|u\|_{k_\kappa+2,\kappa}^2 \ , \tag{5.18}$$

with $s_\kappa = \min\{p_\kappa - 1, k_\kappa\}$*, where the constant* C *is independent of* u*,* p_κ *and* h_κ*.*

Proof. The proof is completely analogous to the one of Corollary 5.5. □

The above result shows hp-convergence when $p_\kappa \geq 3$, $\kappa \in \mathcal{T}$ (cf. [22]). For the the limiting case $p_\kappa = 2$, $\kappa \in \mathcal{T}$, we have the following bound.

Corollary 5.13 *Consider the setting of Theorem 5.10. for* $u|_\kappa \in H^4(\kappa)$ *and* $p_\kappa = 2$*,* $\kappa \in \mathcal{T}$*. Then, the following energy error bound holds:*

$$\|u - u_{\mathrm{DG}}\|_\Omega^2 \leq C \max_{\kappa\in\mathcal{T}} h_\kappa^2 \sum_{\kappa\in\mathcal{T}} h_\kappa^2 \|u\|_{4,\kappa}^2 \tag{5.19}$$

with the constant C *being independent of* u*,* p_κ *and* h_κ*.*

Proof. The result follows by setting $s_\kappa = t_\kappa = 1$ in (5.17). □

Remark 5.14 *The suboptimal convergence rate for* $p_\kappa = 2$*,* $\kappa \in \mathcal{T}$ *of Corollary 5.13 has been observed numerically by Süli and Mozolevski in [22] (Example 2), cf. also the numerical experiments below.*

Again, it is also straightforward to prove exponential convergence for the L^2-error of the p-version SIP-DG, when the solutions of the primal and dual problems are sufficiently smooth; this result is again omitted for brevity.

6 Numerical Experiments

In this section we present a series of numerical experiments to illustrate the *a priori* error estimates derived in this article; see also [22, 17] for further numerical examples.

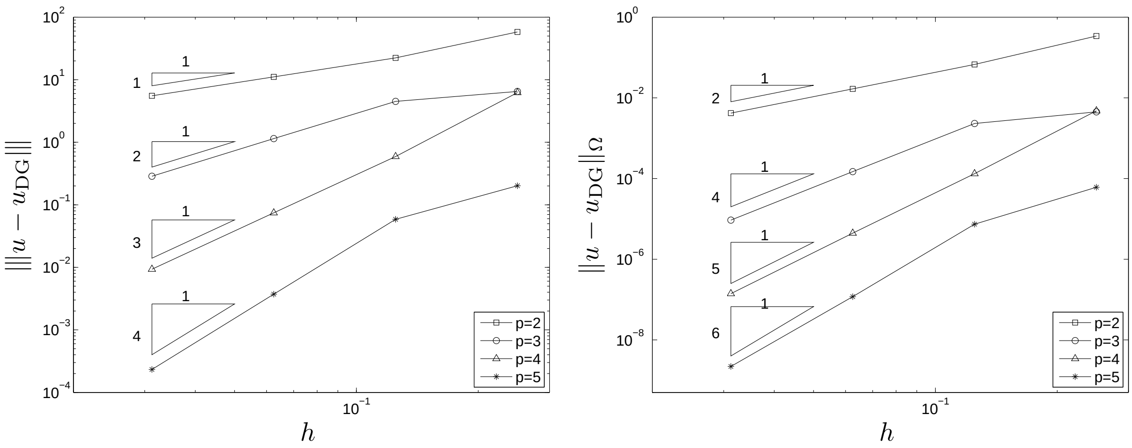
6.1 Example 1

Here, we let $\Omega = (0,1)^2$ and select f, g_{D}, and g_{N} so that the analytical solution to (2.1) and (2.2) is given by

$$u(x,y) = \sin(2\pi x)^2 \sin(2\pi y)^2;$$

this is a variant of the model problem considered in [22]. We investigate the asymptotic behaviour of the errors of the interior penalty DG method (3.19) on a sequence of successively finer square meshes for different values of the polynomial degree p.

In Figure 1(a) we first present a comparison of the DG–norm $|||\cdot|||$ of the error in the approximation to u with the mesh function h for $2 \leq p \leq 5$. Here, we observe that $|||u - u_{\mathrm{DG}}|||$ converges to zero, for



(a) (b)

Figure 1: Example 1. Convergence under h–refinement of: (a) $|||u - u_{\rm DG}|||$; (b) $\|u - u_{\rm DG}\|_{\Omega}$.

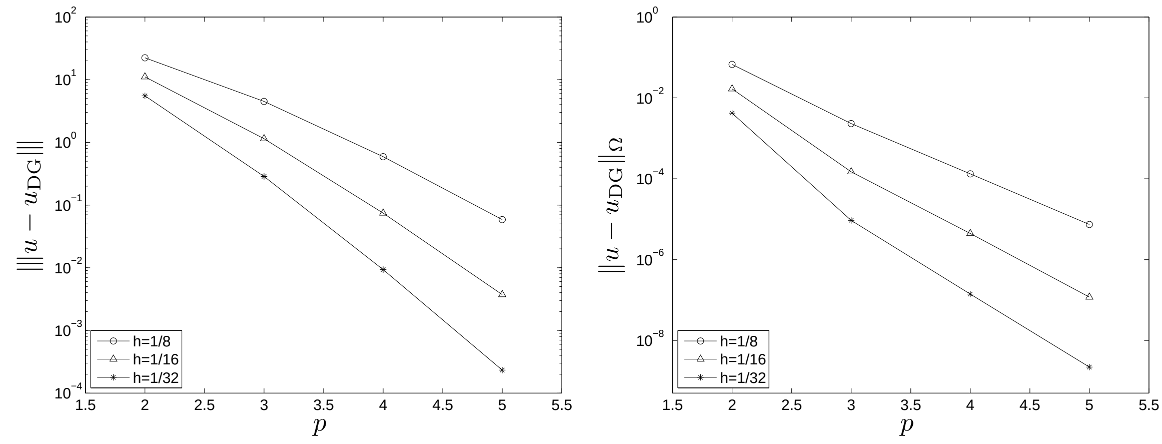
each fixed p, at the optimal rate $\mathcal{O}(h^{p-1})$ as the mesh is refined, thereby confirming Corollaries 5.6 and 5.7. Secondly, in Figure 1(b) we plot the L^2–norm of the error in the approximation to u with the mesh function h for $2 \leq p \leq 5$. For $p > 2$, here we observe optimal rates of convergence as the mesh-size is decreased, namely, $\|u - u_{\rm DG}\|_{\Omega}$ converges to zero at the rate $\mathcal{O}(h^{p+1})$, as h tends to zero, for each fixed p; cf. Corollary 5.12. However, for $p = 2$, in agreement with Corollary 5.13, we observe that the L^2 norm of the error converges to zero at the suboptimal rate $\mathcal{O}(h^2)$, as the mesh size is decreased. This lack of optimality in the L^2 norm when piecewise (discontinuous) quadratic polynomials are employed, was also numerically observed in the article [22].

Finally, we investigate the convergence of the interior penalty DG method (3.19) under p–refinement for a fixed computational mesh. To this end, in Figures 2(a) & (b), we plot the DG–norm and the L^2–norm, respectively, of the error against p on three different square meshes. In each case, we observe that on a linear–log scale, the convergence plots become straight lines as the degree of the approximating polynomial is increased, thereby indicating exponential convergence in p, cf. Corollary 5.8.

6.2 Example 2

In this second example, we investigate the performance of the interior penalty DG method (3.19) for a problem with a corner singularity in u. To this end, we let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$, and select $f = 0$. Then, writing (r, φ) to denote the system of polar coordinates, we impose an appropriate inhomogeneous boundary condition for u so that

$$u = r^{5/3} \sin(5\varphi/3).$$



(a) (b)

Figure 2: Example 1. Convergence under p–refinement of: (a) $|||u - u_{\mathrm{DG}}|||$; (b) $\|u - u_{\mathrm{DG}}\|_{\Omega}$.

Elements	Dof	$\|u - u_{\mathrm{DG}}\|_{\Omega}$	k	$|||u - u_{\mathrm{DG}}|||$	k
12	108	0.3791E-02	0.00	0.6089	0.00
48	432	0.1207E-02	1.65	0.3815	.675
192	1728	0.4218E-03	1.52	0.2413	.661
768	6912	0.1618E-03	1.38	0.1525	.662
3072	27648	0.6550E-04	1.30	0.9630E-01	.663
12288	110592	0.2727E-04	1.26	0.6075E-01	.665

Table 1: Example 2. Convergence of $\|u - u_{\mathrm{DG}}\|_{\Omega}$ and $|||u - u_{\mathrm{DG}}|||$ with $p = 2$.

The analytical solution u contains a singularity at the corner located at the origin of Ω; here, we only have $u \in H^{8/3-\varepsilon}(\Omega)$, $\varepsilon > 0$. We remark that this regularity violates the assumption required by the analysis presented in this article.

In Tables 1–4 we present a comparison of the DG–norm and L^2–norm of the error in the approximation to u, with the mesh function h on a sequence of uniform square meshes for $2 \leq p \leq 5$, respectively. In each case we show the number of elements in the computational mesh, the number of degrees of freedom in the underlying finite element space, the corresponding DG–norm and L^2–norm of the error and their respective computed rates of convergence k. Here, we observe that (asymptotically) $|||u - u_{\mathrm{DG}}|||$ tends to zero at the rate $\mathcal{O}(h^{2/3})$, as h tends to zero, which, is the optimal rate we would expect in practice. On the other hand, the L^2–norm of the error is observed to tend to zero at the (approximate) rate $\mathcal{O}(h^{1.20})$, as the mesh is enriched; a possible explanation of this convergence rate can be traced to

Elements	Dof	$\|u - u_{\text{DG}}\|_\Omega$	k	$\|\|\|u - u_{\text{DG}}\|\|\|$	k
12	192	0.7828E-03	0.00	0.3728	0.00
48	768	0.2853E-03	1.46	0.2347	.668
192	3072	0.1185E-03	1.27	0.1478	.667
768	12288	0.5052E-04	1.23	0.9313E-01	.667
3072	49152	0.2169E-04	1.22	0.5867E-01	.667
12288	196608	0.9346E-05	1.21	0.3696E-01	.667

Table 2: Example 2. Convergence of $\|u - u_{\text{DG}}\|_\Omega$ and $|||u - u_{\text{DG}}|||$ with $p = 3$.

Elements	Dof	$\|u - u_{\text{DG}}\|_\Omega$	k	$\|\|\|u - u_{\text{DG}}\|\|\|$	k
12	300	0.3608E-03	0.00	0.2763	0.00
48	1200	0.1508E-03	1.26	0.1740	.667
192	4800	0.6460E-04	1.22	0.1096	.667
768	19200	0.2782E-04	1.22	0.6904E-01	.667
3072	76800	0.1200E-04	1.21	0.4349E-01	.667
12288	307200	0.5173E-05	1.21	0.2740E-01	.667

Table 3: Example 2. Convergence of $\|u - u_{\text{DG}}\|_\Omega$ and $|||u - u_{\text{DG}}|||$ with $p = 4$.

the lack of H^4-regularity of the solution to the dual problem (cf. regularity assumptions for the dual problem in Theorem 5.10 and the discussion in Remark 5.11).

Appendix: Projection operators

We present some results of technical nature regarding the approximation properties of the H^1-projection operator defined above in high-order seminorms.

Lemma .1 *Let $\hat{u} \in H^{r+2}(\hat{I})$, with $\hat{I} := (-1, 1)$ and $r \geq 1$. Let also, $\hat{\lambda}_p : H^1(\hat{I}) \to \mathcal{P}_p$ denote the H^1-projection operator. Then the following bound holds:*

$$\|(\hat{u} - \hat{\lambda}_p \hat{u})''\|_{\hat{I}} \leq C p^{\frac{3}{2}} \Phi(p, s) \|\hat{u}^{(s+2)}\|_{\hat{I}}, \tag{.1}$$

with $0 \leq s \leq \min\{p - 1, r\}$.

Elements	Dof	$\|u-u_{\mathrm{DG}}\|_{\Omega}$	k	$\|\|\|u-u_{\mathrm{DG}}\|\|\|$	k
12	432	0.2177E-03	0.00	0.2198	0.00
48	1728	0.9323E-04	1.22	0.1385	.667
192	6912	0.4019E-04	1.21	0.8723E-01	.667
768	27648	0.1734E-04	1.21	0.5495E-01	.667
3072	110592	0.7483E-05	1.21	0.3462E-01	.667
12288	442368	0.3241E-05	1.21	0.2181E-01	.667

Table 4: Example 2. Convergence of $\|u-u_{\mathrm{DG}}\|_{\Omega}$ and $|||u-u_{\mathrm{DG}}|||$ with $p=5$.

Proof. It is easy to see that, for $v \in H^1(\hat{I})$, we have $(\hat{\lambda}_p v)' = \pi_{p-1}v'$, where π_{p-1} denotes the L^2-projection operator onto $\mathcal{P}_{p-1}$. Therefore, we have

$$\begin{aligned}\|(\hat{u}-\hat{\lambda}_p\hat{u})''\|_{\hat{I}} &\leq \|\hat{u}''-(\hat{\lambda}_p\hat{u}')'\|_{\hat{I}} + \|(\hat{\lambda}_p\hat{u}')'-(\hat{\lambda}_p\hat{u})''\|_{\hat{I}} \\ &\leq \|\hat{u}''-\pi_{p-1}\hat{u}''\|_{\hat{I}} + \|\pi_{p-1}\hat{u}''-(\pi_{p-1}\hat{u}')'\|_{\hat{I}}. \qquad (.2)\end{aligned}$$

The first term on the right-hand side of (.2) can be bounded using the standard approximation bound for the L^2-projection operator

$$\|v-\pi_q v\|_{\hat{I}} \leq \Phi(q+1,t)\|v^{(t)}\|_{\hat{I}}, \qquad (.3)$$

for $v \in H^{k+1}(\hat{I})$, $k \geq 0$,and $0 \leq t \leq \min\{q+1,k+1\}$ (see, e.g., [21]), with $v=\hat{u}''$, $q=p-1$, $t=s$, and $k=r-1$. The second term on the right-hand side of (.2) can be bounded using the "commutation error" approximation bound for the L^2-projection operator (see [11])

$$\|\pi_q v' - (\pi_q v)'\|_{\hat{I}} \leq Cq^{\frac{1}{2}}\Phi(q,t)\|v^{(t+1)}\|_{\hat{I}},$$

for $v \in H^{k+1}(\hat{I})$, $k \geq 0$, and $0 \leq t \leq \min\{q,k\}$ (see [11]), with $v=\hat{u}'$, $q=p-1$, $t=s$, and $k=r$. Then, we deduce

$$\|(\hat{u}-\hat{\lambda}_p\hat{u})''\|_{\hat{I}} \leq \Big(\Phi(p,s)+Cp^{\frac{1}{2}}\Phi(p-1,s)\Big)\|\hat{u}^{(s+2)}\|_{\hat{I}},$$

with $0 \leq s \leq \min\{p-1,r\}$. Finally, noting that $\Phi(p-1,s) \leq (2p-1)\Phi(p,s)$, the result follows. □

Now, let $\hat{\Lambda}_p := (\hat{\lambda}_p \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \hat{\lambda}_p) \equiv \hat{\lambda}_p^x \hat{\lambda}_p^y$, be the tensor-product H^1-projection operator onto $\mathcal{Q}_p(\hat{\kappa}) := (\mathcal{P}_p \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathcal{P}_p)$.

Lemma .2 *Let $\hat{u} \in H^{r+2}(\hat{\kappa})$, with $\hat{\kappa} := (-1,1)^2$ and $r \geq 1$. Then the following bound holds:*

$$\|\Delta(\hat{u}-\hat{\Lambda}_p\hat{u})\|_{\hat{\kappa}} \leq Cp^{\frac{3}{2}}\Phi(p,s)|\hat{u}|_{s+2,\hat{\kappa}}, \qquad (.4)$$

with $0 \leq s \leq \min\{p-1,r\}$.

Proof. We have

$$\|\Delta(\hat{u} - \hat{\Lambda}_p\hat{u})\|_{\hat{\kappa}} \le \|\partial_x^2(\hat{u} - \hat{\Lambda}_p\hat{u})\|_{\hat{\kappa}} + \|\partial_y^2(\hat{u} - \hat{\Lambda}_p\hat{u})\|_{\hat{\kappa}}.$$

We bound $\|\partial_x^2(\hat{u} - \hat{\Lambda}_p\hat{u})\|_{\hat{\kappa}}$; the bound for $\|\partial_y^2(\hat{u} - \hat{\Lambda}_p\hat{u})\|_{\hat{\kappa}}$ is completely analogous due to symmetry.

We set $\hat{w} = \hat{u} - \hat{\lambda}_p^y\hat{u}$, and we observe that

$$\partial_x^2(\hat{u} - \hat{\Lambda}_p\hat{u}) = \partial_x^2(\hat{u} - \hat{\lambda}_p^x\hat{u}) + \partial_x^2(\hat{u} - \hat{\lambda}_p^y\hat{u}) + \partial_x^2(\hat{w} - \hat{\lambda}_p^x\hat{w}).$$

Triangle inequality yields

$$\|\partial_x^2(\hat{u} - \hat{\Lambda}_p\hat{u})\|_{\hat{\kappa}} \le \|\partial_x^2(\hat{u} - \hat{\lambda}_p^x\hat{u})\|_{\hat{\kappa}} + \|\partial_x^2(\hat{u} - \hat{\lambda}_p^y\hat{u})\|_{\hat{\kappa}} + \|\partial_x^2(\hat{w} - \hat{\lambda}_p^x\hat{w})\|_{\hat{\kappa}}. \tag{.5}$$

The first and the third term on the right-hand side can be bound using (.2) with $s = r$ and $s = 0$, respectively, to obtain

$$\|\partial_x^2(\hat{u} - \hat{\lambda}_p^x\hat{u})\|_{\hat{\kappa}} \le Cp^{\frac{3}{2}}\Phi(p,s)\|\partial_x^{s+2}\hat{u}\|_{\hat{\kappa}},$$

and

$$\|\partial_x^2(\hat{w} - \hat{\lambda}_p^x\hat{w})\|_{\hat{\kappa}} \le Cp^{\frac{3}{2}}\|\partial_x^2\hat{w}\|_{\hat{\kappa}}. \tag{.6}$$

Observing that $\partial_x^2\hat{w} = \partial_x^2\hat{u} - \partial_x^2(\hat{\lambda}_p^y\hat{u}) = \partial_x^2\hat{u} - \hat{\lambda}_p^y(\partial_x^2\hat{u})$, the second term on the right-hand side of (.5) and the term $\|\partial_x^2\hat{w}\|_{\hat{\kappa}}$ in (.6), can be further bounded using (.3) with $v = \partial_x^2\hat{u}$, $q = p$, $t = s$, and $k = r$, as follows:

$$\|\partial_x^2(\hat{u} - \hat{\lambda}_p^y\hat{u})\|_{\hat{\kappa}} \le \Phi(p+1,s)\|\partial_x^2\partial_y^s\hat{u}\|_{\hat{\kappa}} \le \Phi(p,s)\|\partial_x^2\partial_y^s\hat{u}\|_{\hat{\kappa}},$$

with $0 \le s \le \min\{p+1, r\}$. The result now follows by inserting these bounds into (3.1). □

References

[1] Arnold, D. N., Brezzi, F., Cockburn, B., and Marini, L. D. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal. 39*, 5 (2001/02), 1749–1779 (electronic).

[2] Bacuta, C., Bramble, J. H., and Pasciak, J. E. Shift theorems for the biharmonic Dirichlet problem. In *Recent progress in computational and applied PDEs (Zhangjiajie, 2001).* Kluwer/Plenum, New York, 2002, pp. 1–26.

[3] Baker, G. A. Finite element methods for elliptic equations using nonconforming elements. *Math. Comp. 31*, 137 (1977), 45–59.

[4] Brenner, S. C., and Sung, L.-Y. C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. *J. Sci. Comput. 22/23* (2005), 83–118.

[5] Brezzi, F., and Fortin, M. *Mixed and hybrid finite element methods*, vol. 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[6] Ciarlet, P. G. *The finite element method for elliptic problems*, vol. 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original.

[7] Cockburn, B., and Karniadakis, G. E., Eds. *Discontinuous Galerkin methods*, vol. 11 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2000. Theory, computation and applications, Papers from the 1st International Symposium held in Newport, RI, May 24–26, 1999.

[8] Dauge, M. *Elliptic boundary value problems on corner domains*, vol. 1341 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.

[9] Destuynder, P., and Salaun, M. *Mathematical analysis of thin plate models*, vol. 24 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1996.

[10] Engel, G., Garikipati, K., Hughes, T. J. R., Larson, M. G., Mazzei, L., and Taylor, R. L. Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. *Comput. Methods Appl. Mech. Engrg. 191*, 34 (2002), 3669–3750.

[11] Georgoulis, E. H. Discontinuous Galerkin methods on shape-regular and anisotropic meshes. *D.Phil. Thesis, University of Oxford* (2003).

[12] Georgoulis, E. H. *hp*-version interior penalty discontinuous Galerkin finite element methods on anisotropic meshes. *Int. J. Numer. Anal. Model. 3*, 1 (2006), 52–79.

[13] Georgoulis, E. H., and Süli, E. Optimal error estimates for the hp-version interior penalty discontinuous Galerkin finite element method. *IMA J. Numer. Anal. 25*, 1 (2005), 205–220.

[14] Grisvard, P. *Elliptic problems in nonsmooth domains*, vol. 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[15] Houston, P., Schwab, C., and Süli, E. Stabilized *hp*-finite element methods for first-order hyperbolic problems. *SIAM J. Numer. Anal. 37*, 5 (2000), 1618–1643 (electronic).

[16] Mozolevski, I., and Süli, E. A priori error analysis for the *hp*-version of the discontinuous Galerkin finite element method for the biharmonic equation. *Comput. Methods Appl. Math. 3*, 4 (2003), 596–607 (electronic).

[17] Mozolevski, I., Süli, E., and Bösing, P. *hp*-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation. *J. Sci. Comput. 30*, 3 (2007), 465–491.

[18] Perugia, I., and Schötzau, D. An *hp*-analysis of the local discontinuous Galerkin method for diffusion problems. In *Proceedings of the Fifth International Conference on Spectral and High Order Methods (ICOSAHOM-01) (Uppsala)* (2002), vol. 17, pp. 561–571.

[19] Perugia, I., and Schötzau, D. The *hp*-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations. *Math. Comp. 72*, 243 (2003), 1179–1214 (electronic).

[20] Råde, L., and Westergren, B. *Mathematics handbook for science and engineering*, fourth ed. Springer-Verlag, Berlin, 1999.

[21] Schwab, C. *p- and hp- finite element methods: Theory and applications in solid and fluid mechanics.* Oxford University Press: Numerical mathematics and scientific computation, 1998.

[22] Süli, E., and Mozolevski, I. *hp*-version interior penalty DGFEMs for the biharmonic equation. *Comput. Methods Appl. Mech. Engrg. 196*, 13-16 (2007), 1851–1863.

List of Tables

List of Figures