Two-Level Schwarz Preconditioners for Super Penalty Discontinuous Galerkin Methods

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Abstract. We extend the construction and analysis of the non-overlapping Schwarz preconditioners proposed in [2, 3] to the (non-consistent) super penalty discontinuos Galerkin methods introduced in [5] and [6]. We show that the resulting preconditioners are scalable, and we provide the convergence estimates. We also present numerical experiments demonstrating the theoretical results.

Key words: Schwarz preconditioners, super penalty discontinuous Galerkin methods.

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1 Introduction

Discontinuous Galerkin (DG) finite element methods have experienced a huge development in recent years. Although they have been proved to enjoy many advantages in a number of circumstances, their practical utility is still limited by the much larger number of degrees of freedom they require compared to other classical discetization methods. To handle this possible limitation, some domain decomposition preconditioners have been proposed and analysed in the past five years for *strongly consistent* and *stable* DG approximations of second order elliptic problems (cf. [2, 3, 10]).

In [3], it was numerically observed that the proposed non-overlapping Schwarz methods can also be successfully used as two-level preconditioners for *non-consistent* super penalty DG approximations; namely, the Babuška–Zlámal [5] and the Brezzi *et al.* [6] methods. Because of a non-consistency in the formulation, a super penalty procedure has to be applied in order to achieve optimal approximation properties. The over-penalization has dramatic effects on the condition number of the resulting linear system of equations. In fact, if on a given *quasi uniform* mesh $\mathcal{T}_h$ with granularity h, polynomials of degrees

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ℓ_h are used for the approximation, the condition number of the resulting stiffness matrix is of order $O(h^{-2\ell_h-2})$ (cf. [8]). In this paper we present the theoretical analysis of the Schwarz preconditioners for the Babuška–Zlámal and the Brezzi *et al.* super penalty DG methods. We show that a considerable reduction on the condition number of the preconditioned linear systems of equations, and consequently, on the number of iterates needed for convergence is achieved. Finally, we present some numerical experiments confirming our theory.

2 Super Penalty Discontinuous Galerkin Discretizations

In this section, we set up some notation, introduce the model problem we will consider, and recall the variational formulation of super penalty DG methods.

Throughout the paper, we shall use standard notation for Sobolev spaces (cf. [1]), and $x \lesssim y$ will mean that there exists a generic constant $C > 0$ (that may not be the same at different occurrences but is always mesh independent) so that $x \leq Cy$.

Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a convex bounded Lipschitz polygonal or polyhedral domain and $f \in L^2(\Omega)$. To ease the presentation, we consider the following model (toy) problem

$$-\Delta u = f \quad \text{in}\,\Omega\,, \quad u = 0 \quad \text{on}\,\partial\Omega\,. \tag{2.1}$$

Meshes. Let $\mathcal{T}_h$ be a *shape-regular* and *quasi-uniform* conforming partition of the domain Ω into disjoint open elements T, where each T is the affine image of a fixed master element $\widehat{T}$, *i.e.*, $T = F_T(\widehat{T})$, and where $\widehat{T}$ is either the open unit d-simplex or the d-hypercube in $\mathbb{R}^d$, $d = 2,3$. Letting h_T be the diameter of the element $T \in \mathcal{T}_h$, we define the mesh size h by $h = \max_{T\in\mathcal{T}_h} h_T$, and assume, for simplicity, that $h < 1$. We denote by $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ the sets of all interior and boundary faces of $\mathcal{T}_h$, respectively, and set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$.

Remark 2.1. All the theory we present in this paper can be applied, with minor changes, to the case of non-matching grids, under suitable additional assumptions on $\mathcal{T}_h$; cf. [2].

Trace operators. Let $F \in \mathcal{F}_h^I$ be an interior face shared by two elements T^+ and T^- with outward normal unit vectors $\boldsymbol{n}^{\pm}$. For piecewise smooth vector-valued and scalar functions $\boldsymbol{\tau}$ and v, respectively, we define the *jump* and *average* operators on $F \in \mathcal{F}_h^I$ by

$$\begin{aligned} [\![\boldsymbol{\tau}]\!] &= \boldsymbol{\tau}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{\tau}^- \cdot \boldsymbol{n}^-, \quad [\![v]\!] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-, \quad \text{on } F \in \mathcal{F}_h^I,\\ \{\!\{\boldsymbol{\tau}\}\!\} &= (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)/2, \quad \{\!\{v\}\!\} = (v^+ + v^-)/2, \quad \text{on } F \in \mathcal{F}_h^I, \end{aligned} \tag{2.2}$$

where $\boldsymbol{\tau}^{\pm}$ and $v^{\pm}$ denote the traces of $\boldsymbol{\tau}$ and v on $\partial K^{\pm}$ taken from within $K^{\pm}$, respectively. On a boundary face $F \in \mathcal{F}_h^B$ we set, analogously,

$$[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}\cdot\boldsymbol{n}, \quad [\![v]\!] = v\,\boldsymbol{n}, \qquad \{\!\{\boldsymbol{\tau}\}\!\} = \boldsymbol{\tau}, \quad \{\!\{v\}\!\} = v, \quad \text{on } F \in \mathcal{F}_h^B\,. \tag{2.3}$$

DG finite element space. For a given (integer) $\ell_h \geq 1$, the DG finite element space V_h is defined by

$$V_h = \{v \in L^2(\Omega) \ : \ v|_T \circ F_T \in \mathcal{M}^{\ell_h}(\widehat{T}) \ \ \forall\, T \in \mathcal{T}_h\},$$

where $\mathcal{M}^{\ell_h}(\widehat{T})$ is either the space of polynomials of degree less or equal to ℓ_h on $\widehat{T}$, if $\widehat{T}$ is the reference d-simplex, or the space of polynomials of degree at most ℓ_h in each variable on $\widehat{T}$, if $\widehat{T}$ is the reference d-hypercube.

The super penalty DG methods. For the discretization of the model problem (2.1), we consider either the Babuška–Zlàmal (BZ) [5] or the Brezzi *et al.* (BMMPR) [6] super penalty methods. More precisely, we consider the following class of DG methods:

$$\text{find } u \in V_h \text{ s.t.} \quad A_h(u,v) = (f,v) \quad \forall\, v \in V_h. \tag{2.4}$$

Here the DG bilinear form $A_h : V_h \times V_h \to \mathbb{R}$ is given by

$$A_h(u,v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla v \, \mathrm{d}x + \mathcal{S}_h(u,v) \quad \forall u,v \in V_h, \tag{2.5}$$

where the stabilization term $\mathcal{S}_h(\cdot,\cdot)$ is defined by

$$\mathcal{S}_h(u,v) = \sum_{F \in \mathcal{F}_h} \int_F \alpha\, h_F^{-2\ell_h - 1} \, [\![u]\!] \cdot [\![v]\!] \, \mathrm{d}s, \quad \mathcal{S}_h(u,v) = \sum_{F \in \mathcal{F}_h} \int_F \alpha\, h_F^{-2\ell_h} r_F([\![u]\!]) \cdot r_F([\![v]\!]) \, \mathrm{d}s,$$

for the BZ method and for the BMMPR method, respectively. In the above expressions, h_F denotes the $(d-1)$–dimensional Lebesgue measure of $F \in \mathcal{F}_h$, $\alpha > 0$ is a parameter (at our disposal) independent of the mesh size, and $r_F : [L^1(F)]^d \to [V_h]^d$ is defined by

$$\int_\Omega r_F(\boldsymbol{\varphi}) \cdot \boldsymbol{\tau} \, \mathrm{d}x = -\int_F \boldsymbol{\varphi} \cdot \{\!\{\boldsymbol{\tau}\}\!\} \, \mathrm{d}s \qquad \forall \boldsymbol{\tau} \in [V_h]^d. \tag{2.6}$$

For simplicity, we assume $\alpha \geq 1$.

3 Main Properties and Theoretical Tools

We briefly review the basic tools we shall require in the analysis of our two-level Schwarz methods.

We refer to [9] for a local inverse inequality that holds true for piecewise polynomials of a given order, and to [4] for a trace inequality that holds true for (regular enough) piecewise functions. We also recall the following equivalence (see [6] for details),

$$C_1 h_F^{-2\ell_h - 1} \| [\![v]\!] \|_{0,F}^2 \leq h_F^{-2\ell_h} \| r_F([\![v]\!]) \|_{0,\Omega}^2 \leq C_2 h_F^{-2\ell_h - 1} \| [\![v]\!] \|_{0,F}^2 \quad \forall\, F \in \mathcal{F}_h \ \ \forall\, v \in V_h, \tag{3.1}$$

where C_1 and C_2 are positive constants.

For the analysis of our Schwarz methods we consider the norm induced by the bilinear form $A_h(\cdot,\cdot)$, *i.e.*, $\|v\|_A^2 = A_h(v,v)$ for all $v \in V_h$ (recall that $A_h(\cdot,\cdot)$ is coercive provided that $\alpha > 0$). The continuity of $A_h(\cdot,\cdot)$, with respect to the norm $\|\cdot\|_A$ easily follow from the Cauchy–Schwarz inequality, *i.e.*, $A_h(u,v) \lesssim \|u\|_A \|v\|_A$ for all $u,v \in V_h$.

For an open connected polyhedral domain $D \subseteq \Omega$ that can be covered by the union of some elements in $\mathcal{T}_h$, we introduce the broken Sobolev space

$$H^s(D,\mathcal{T}_h) = \{v \in L^2(\Omega) \ : \ v|_T \in H^s(T) \ \ \forall T \in \mathcal{T}_h, \ T \subset D\}, \quad s \geq 1.$$

An important tool in the analysis of Schwarz methods is represented by a Friedrichs–Poincaré type inequality valid for broken Sobolev spaces. The next result is a small modification of the well-known result proved in [4,10].

Lemma 3.1 (Friedrichs–Poincaré inequality). *Let $D \subset \Omega \subset \mathbb{R}^d$, $d=2,3$, be a convex polygonal or polyhedral domain that can be covered by the union of some elements in $\mathcal{T}_h$. Then, there exists a positive constant C_λ, such that, for all $u \in H^1(D,\mathcal{T}_h)$ with zero average over D, it holds:*

$$\|v\|_{0,D}^2 \leq C_\lambda \ (\mathrm{diam}(D))^2 \left(\sum_{\substack{T \in \mathcal{T}_h \\ T \subset D}} |v|_{1,T}^2 + \sum_{\substack{F \in \mathcal{F}_h \\ F \subset D}} h_F^{-(2\ell_h+1)} \| [[v]] \|_{0,F}^2 \right), \tag{3.2}$$

where $C_\lambda = C' C_p \max\{1, h^{2\ell_h}\}$, with C_p the Poincaré constant, and C' depending only on the shape regularity of $\mathcal{T}_h$.

The proof goes along the lines of that in [4]. For the sake of completeness we briefly sketch it.

Proof. It is sufficient to assume that D has unit diameter; the general case follows from a standard scaling argument. Let $u \in H^1(D,\mathcal{T}_h)$ with $\int_D u \ \mathrm{d}x = 0$, we consider the auxiliary Neumann problem,

$$-\Delta \phi = u \ \text{ in } D, \quad \frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \ \text{ on } \partial D.$$

The above problem has a unique solution (up to an additive constant) $\varphi \in H^2(D)$ that verifies the elliptic regularity estimate $\|\phi\|_{2,D} \lesssim \|u\|_{0,D}$. Integration by parts, the Cauchy-Schwarz inequality and the trace inequality $h_F \|\nabla \phi \cdot \boldsymbol{n}\|_{0,F}^2 \lesssim \|\phi\|_{2,T}^2$ give

$$\begin{aligned}
\|u\|_{0,D}^2 &= \left| -\int_D u \Delta\phi \ \mathrm{d}x \right| = \left| \sum_{\substack{T \in \mathcal{T}_h \\ T \subset D}} \int_T \nabla u \cdot \nabla \phi \ \mathrm{d}x - \sum_{\substack{F \in \mathcal{F}_h^I \\ F \subset D}} \int_F [[u]] \cdot \nabla \phi \ \mathrm{d}s \right| \\
&\lesssim \left(\sum_{\substack{T \in \mathcal{T}_h \\ T \subset D}} |u|_{1,K}^2 \right)^{1/2} \left(\sum_{\substack{T \in \mathcal{T}_h \\ T \subset D}} |\phi|_{1,K}^2 \right)^{1/2} + \left(\sum_{\substack{F \in \mathcal{F}_h^I \\ F \subset D}} h_F^{-(2\ell_h+1)} \| [[u]] \|_{0,F}^2 \right)^{1/2} \left(\sum_{\substack{T \in \mathcal{T}_h \\ T \subset D}} h_F^{2\ell_h} \|\phi\|_{2,T}^2 \right)^{1/2}.
\end{aligned}$$

Then, by using the elliptic regularity of the dual problem, inequality (3.2) follows. □

We shall also use the following variant of the trace inequality shown in [10]:

$$\|u\|_{0,\partial D}^2 \lesssim H_D^{-1}\|u\|_{0,D}^2 + H_D \left(\sum_{\substack{T\in\mathcal{T}_h \\ T\subset D}} |u|_{1,T}^2 + \sum_{\substack{F\in\mathcal{F}_h \\ F\subset D}} h_F^{-(2\ell_h+1)} \|[\![u]\!]\|_{0,F}^2 \right) \quad \forall\, v \in H^1(D,\mathcal{T}_h). \tag{3.3}$$

Condition number estimate. We recall that, given a basis of V_h, any function $v \in V_h$ is uniquely determined by a set of degrees of freedom. Here and in the following, we use the bold notation to denote the spaces of degrees of freedom (vectors in $\mathbb{R}^{\mathrm{n}}$) and discrete linear operators (matrices in $\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$). If $\mathbf{A}$ is the stiffness matrix associated to the bilinear form $A_h(\cdot,\cdot)$ and the given basis, the problem (2.4) can be rewritten as the linear system of equations $\mathbf{Au} = \mathbf{f}$, with $\mathbf{A}$ symmetric, positive definite and sparse. It is a simple matter to check that the matrix $\mathbf{A}$ is ill-conditioned. In fact, in [8] it is shown that the spectral condition number of the stiffness matrix $\mathbf{A}$ arising from the BZ discretization, $\kappa(\mathbf{A})$, can be bounded by

$$\kappa(\mathbf{A}) \lesssim \frac{\alpha}{h^{2\ell_h+2}}. \tag{3.4}$$

For the BMMPR method the proof can be easily adapted and we omit the details. In practical applications such a bad condition number imply an extremely slow convergence, for example, of the *conjugate gradient* iterative solver. In the next section we devise some remedies to offset this possible limitation.

4 Schwarz Preconditioners for Super Penalty DG Methods

In this section we present the non-overlapping Schwarz preconditioners for the super penalty DG approximations introduced before.

Non-overlapping partitions. We consider three level of *nested* partitions of the domain Ω, all satisfying the previous assumptions: a subdomain partition $\mathcal{T}_N$ made of N non-overlapping subdomains, a coarse partition $\mathcal{T}_H$ (with mesh size H), and a fine partition $\mathcal{T}_h$ (with mesh size h). For each subdomain $\Omega_i \in \mathcal{T}_N$, we denote by $\mathcal{F}_{h,i}$ the set of all faces of $\mathcal{F}_h$ belonging to $\overline{\Omega}_i$, and set $\mathcal{F}_{h,i}^I = \{F \in \mathcal{F}_{h,i} \,:\, F \subset \Omega_i\}$, $\mathcal{F}_{h,i}^B = \{F \in \mathcal{F}_{h,i} \,:\, F \subset \partial\Omega_i \cap \partial\Omega\}$. The set of all (internal) faces belonging to the skeleton of the subdomain partition will be denoted by Γ, *i.e.*, $\Gamma = \bigcup_{i=1}^N \Gamma_i$ with $\Gamma_i = \{F \in \mathcal{F}_{h,i} \,:\, F \subset \partial\Omega_i\}$.

Local spaces and prolongation operators. For each $i = 1,\dots,N$, we define the local DG spaces by $V_h^i = \{u \in L^2(\Omega_i) \,:\, v|_T \circ F_T \in \mathcal{M}^{\ell_h}(\widehat{T}) \;\; \forall\, T \in \mathcal{T}_h,\;\; T \subset \Omega_i\}$, and we denote by $R_i^T : V_h^i \longrightarrow V_h$ the classical inclusion operator from V_h^i to V_h, and by R_i its transpose with respect to the L^2–inner product. We observe that $V_h = R_1^T V_h^1 \oplus \ldots \oplus R_N^T V_h^N$.

Local solvers. We consider the super penalty DG approximation of the problem:

$$-\Delta u_i = f|_{\Omega_i} \text{ in } \Omega_i, \quad u_i = 0 \text{ on } \partial\Omega_i, \quad i = 1,\dots,N.$$

Hence, in view of (2.5), the local bilinear forms $A_i : V_h^i \times V_h^i \longrightarrow \mathbb{R}$ are given by

$$A_i(u_i,v_i) = \int_{\Omega_i} \nabla_h u_i \cdot \nabla_h v_i \, \mathrm{d}x + \mathcal{S}_i(u_i,v_i). \tag{4.1}$$

Here, the local stabilization forms $\mathcal{S}_i(\cdot,\cdot)$ are defined as

$$\mathcal{S}_i(u_i,v_i) = \sum_{F\in\mathcal{F}_{h,i}} \int_F \alpha\, h_F^{-2\ell_h - 1} \, [\![u_i]\!]\cdot[\![v_i]\!] \, \mathrm{d}s, \quad \mathcal{S}_i(u_i,v_i) = \sum_{F\in\mathcal{F}_{h,i}} \int_F \alpha\, h_F^{-2\ell_h} \, r_F^i([\![u_i]\!])\cdot r_F^i([\![v_i]\!]) \, \mathrm{d}s,$$

for the BZ and the BMMPR methods, respectively, with $r_F^i : [L^1(F)]^d \rightarrow [V_h^i]^d$ defined as

$$\int_{\Omega_i} r_F^i(\boldsymbol{\varphi}_i)\cdot \boldsymbol{\tau}_i \, \mathrm{d}x = -\int_F \boldsymbol{\varphi}_i \cdot \{\!\{\boldsymbol{\tau}_i\}\!\} \, \mathrm{d}s \qquad \forall \boldsymbol{\tau}_i \in [V_h^i]^d. \tag{4.2}$$

Remark 4.1. The approximation properties of the local solvers enter directly into the analysis of the Schwarz methods. From our definition of the local solvers, it can be easily verified that, for the BZ method, $A_h(R_i^T u_i, R_i^T u_i) = A_i(u_i,u_i)$; that is, the local solvers are *exact*. For the BMMPR method, the local solvers turn out to be *approximate* in the sense that $A_h(R_i^T u_i, R_i^T u_i) \neq A_i(u_i,u_i)$. Indeed, this follows by taking into account the definition of the local and global lifting operators (4.2) and (2.6), and by noting that $F \in \Gamma_i$ is a boundary face for the local bilinear form, hence $\{\!\{v_i\}\!\} = v_i$ on $F \in \Gamma_i$, but an interior face for the global bilinear form, hence $\{\!\{R_i^T v_i\}\!\} = \frac{1}{2} v_i$ on $F \in \Gamma_i$ (cf. the definition of the average operator on interior and boundary faces (2.2)–(2.3), respectively).

Coarse solver. For a given integer ℓ_H, $0 \le \ell_H \le \ell_h$, the coarse space is given by

$$V_H \equiv V_h^0 = \{v_H \in L^2(\Omega) \; : \; v_H|_T \circ F_T \in \mathcal{M}^{\ell_H}(\widehat{T}) \;\; \forall\, T \in \mathcal{T}_H\}.$$

The coarse solver $A_0 : V_h^0 \times V_h^0 \longrightarrow \mathbb{R}$ is defined by

$$A_0(u_0,v_0) = A_h(R_0^T u_0, R_0^T v_0) \quad \forall\, u_0, v_0 \in V_h^0,$$

where $R_0^T : V_h^0 \longrightarrow V_h$ is the classical injection operator from V_h^0 to V_h.

Schwarz methods: variational and algebraic formulation. We are now ready to define the Schwarz operators. For $i = 0,\dots,N$, we set

$$\widetilde{P}_i : V_h \longrightarrow V_h^i \quad A_i(\widetilde{P}_i u, v_i) = A_h(u, R_i^T v_i) \quad \forall\, v_i \in V_h^i, \tag{4.3}$$

and define $P_i = R_i^T \widetilde{P}_i : V_h \longrightarrow V_h$. The additive and multiplicative Schwarz operators are defined by

$$P_{ad} = \sum_{i=0}^{N} P_i, \qquad P_{mu} = I - (I - P_N)(I - P_{N-1})\dots(I - P_1)(I - P_0),$$

respectively, where $I: V_h \longrightarrow V_h$ is the identity operator. We also define the error propagation operator $E_N = (I-P_N)(I-P_{N-1})\dots(I-P_0)$, and observe that $P_{mu} = I - E_N$. The Schwarz methods can be written as the product of a suitable preconditioners, namely $\mathbf{B}_{ad}$, or $\mathbf{B}_{mu}$, and $\mathbf{A}$. In fact, having fixed a basis for V_h, it is straightforward to see that, the matrix representation of the operators P_i, is given by $\mathbf{P}_i = \mathbf{R}_i^T \mathbf{A}_i^{-1} \mathbf{R}_i \mathbf{A}$ $(i=0,\dots,N)$. Then,

$$\mathbf{P}_{ad} = \sum_{i=0}^{N} \mathbf{P}_i = \sum_{i=0}^{N} \mathbf{R}_i^T \mathbf{A}_i^{-1} \mathbf{R}_i \mathbf{A} = \mathbf{B}_{ad}\mathbf{A}, \qquad \mathbf{P}_{mu} = \mathbf{I} - (\mathbf{I}-\mathbf{P}_N)\cdots(\mathbf{I}-\mathbf{P}_0) = \mathbf{B}_{mu}\mathbf{A}.$$

The additive Schwarz operator P_{ad} is self adjoint with respect to the $A_h(\cdot,\cdot)$ inner product, whereas, the multiplicative operator P_{mu} is non symmetric. Therefore, for the solution of the resulting algebraic linear system of equations, we use the *conjugate gradient* (CG) method for the former case, and the *generalized minimal residual* (GMRES) linear solver for the latter.

5 Convergence Analysis

In this section we present the convergence analysis for the proposed two-level methods. We follow the abstract convergence theory of Schwarz methods (see, *e.g.*, [7, 11]).

Since the additive operator P_{ad} is self-adjoint with respect to $A_h(\cdot,\cdot)$, we can use the Rayleigh quotient characterization of the extreme eigenvalues:

$$\lambda_{\min}(P_{ad}) = \min_{\substack{u\in V_h\\ u\neq 0}} \frac{A_h(P_{ad}u,u)}{A_h(u,u)}, \quad \lambda_{\max}(P_{ad}) = \max_{\substack{u\in V_h\\ u\neq 0}} \frac{A_h(P_{ad}u,u)}{A_h(u,u)}.$$

In Theorem 5.1 we provide a bound for the condition number of P_{ad} given by $\kappa(P_{ad}) = \lambda_{\max}(P_{ad})/\lambda_{\min}(P_{ad})$. For the multiplicative operator P_{mu}, following the abstract theory [7], we prove that a simple Richardson iteration applied to the preconditioned linear system of equations converges. This result also guarantees that our preconditioner can indeed be accelerated with the GMRES iterative solver. We remark that, the convergence result stated in Theorem 5.2 only applies to the BZ method (see Remark 5.2).

A common step in the analysis of the additive and multiplicative Schwarz methods consists in verifying the following set of assumptions:

(A1) *stable decomposition:* there exists $C_0 > 0$ such that every $u \in V_h$ admits a decomposition

$$u = \sum_{i=0}^{N} R_i^T u_i \quad \text{with } u_i \in V_i, \quad i=0,\dots,N, \quad \text{s.t.} \quad \sum_{i=0}^{N} A_i(u_i,u_i) \le C_0^2\, A_h(u,u);$$

(A2) *local stability:* there exists $\omega > 0$ such that

$$A_h(R_i^T u_i, R_i^T u_i) \le \omega A_i(u_i,u_i) \quad \forall\, u_i \in V_h^i, \quad i=1,\dots,N; \tag{5.1}$$

(A3) *strengthened Cauchy–Schwarz inequalities:* there exist $0 \le \varepsilon_{ij} \le 1$, $1 \le i,j \le N$, such that

$$\left| A_h(R_i^T u_i, R_j^T u_j) \right| \le \varepsilon_{ij} A_h(R_i^T u_i, R_i^T u_i)^{1/2} A_h(R_j^T u_j, R_j^T u_j)^{1/2} \quad \forall\, v_i \in V_h^i,\ \forall\, u_j \in V_h^j.$$

We start proving that the above assumptions hold for the proposed Schwarz preconditioners arising from both the considered BZ and BMMPR super penalty discretizations.

(A1) Stable decomposition. The next result guarantees that a *stable splitting* can be found for the family of subspaces and the corresponding bilinear forms of the super penalty DG discretizations.

Proposition 5.1 (Stable decomposition)**.** Let $A_h(\cdot,\cdot)$ be the bilinear form of the BZ or the BMMPR super penalty methods. For any $u \in V_h$, let $u = \sum_{i=0}^N R_i^T u_i$, $u_i \in V_h^i$, $i = 0,\dots,N$, where $u_0 \in V_h^0 = V_H$ is defined by

$$u_0|_D = \frac{1}{\text{meas}(D)} \int_D u \,\text{d}x \qquad D \in \mathcal{T}_H,$$

and $u_1,\dots,u_N$ are (uniquely) determined as $u - R_0^T u_0 = R_1^T u_1 + \dots + R_N^T u_N$. Then,

$$\sum_{i=0}^N A_i(u_i,u_i) \le \alpha C_0^2 A_h(u,u), \quad \text{with } C_0^2 = O\left(\frac{H}{h^{2\ell_h+1}}\right).$$

Proof. For simplicity, throughout the proof we set $\widetilde{u}_0 = R_0^T u_0$. Given $u \in V_h$, we decompose $u - \widetilde{u}_0$ as $\sum_{i=1}^N R_i^T u_i$. Then,

$$\sum_{i=0}^N A_i(u_i,u_i) = A_h(u-\widetilde{u}_0, u-\widetilde{u}_0) + A_0(u_0,u_0) - \mathcal{I}_h(u-\widetilde{u}_0, u-\widetilde{u}_0), \tag{5.2}$$

where, for the BZ method, $\mathcal{I}_h(\cdot,\cdot)$ is given by

$$\mathcal{I}_h(u,v) = \sum_{F \in \Gamma} \alpha\, h_F^{-2\ell_h - 1} \int_F \left(u_i \boldsymbol{n}_i \cdot v_j \boldsymbol{n}_j + u_j \boldsymbol{n}_j \cdot v_i \boldsymbol{n}_i \right) \,\text{d}s,$$

and, for the BMMPR method, $\mathcal{I}_h(\cdot,\cdot)$ is defined as

$$\mathcal{I}_h(u,v) = \sum_{F \in \Gamma} \alpha\, h_F^{2\ell_h} \left[\int_\Omega r_F([\![u]\!]) \cdot r_F([\![v]\!]) \,\text{d}s - \int_{\Omega_i} r_F^i([\![u_i]\!]) \cdot r_F^i([\![v_i]\!]) \,\text{d}s \right.$$
$$\left. - \int_{\Omega_j} r_F^j([\![u_j]\!]) \cdot r_F^j([\![v_j]\!]) \,\text{d}s \right].$$

We start by providing a bound for the bilinear form $\mathcal{I}_h(\cdot,\cdot)$. For the BZ method, the Cauchy–Schwarz inequality and the arithmetic-geometric mean inequality yield

$$\begin{aligned}|\mathcal{I}_h(u,u)| &\le 2\,\alpha \left(\sum_{F\in\Gamma} h_F^{-2\ell_h-1}\|u_i\|_{0,F}^2\right)^{1/2}\left(\sum_{F\in\Gamma} h_F^{-2\ell_h-1}\|u_j\|_{0,F}^2\right)^{1/2}\\ &\le \alpha \sum_{F\in\Gamma} h_F^{-2\ell_h-1}\left(\|u_i\|_{0,F}^2+\|u_j\|_{0,F}^2\right).\end{aligned}$$

Since the partitions are assumed to be nested, each subdomain Ω_i is the union of some elements $D\in\mathcal{T}_H$ and so, by setting $\Gamma_{ij}=\{F\in\Gamma\,:\,F\subset\partial\Omega_i\cap\partial\Omega_j\}$, we have

$$\sum_{\Gamma_{ij}\in\Gamma}\sum_{F\in\Gamma_{ij}} h_F^{-2\ell_h-1}\|u_i\|_{0,F}^2 \lesssim \sum_{D\in\mathcal{T}_H}\sum_{E\subset\partial D} h^{-2\ell_h-1}\|u\|_{0,E}^2, \tag{5.3}$$

where with E we denote the faces of the elements $D\in\mathcal{T}_H$, and where we have also used the shape regularity and quasi-uniformity of the mesh $\mathcal{T}_h$. Therefore, we get

$$|\mathcal{I}_h(u,u)| \lesssim \sum_{D\in\mathcal{T}_H}\sum_{E\subset\partial D} \alpha\, h^{-2\ell_h-1}\|u\|_{0,E}^2.$$

Analogously, for the BMMPR method, by using (3.1), recalling that on each $F\in\Gamma$, $\|[\![u_i]\!]\|_{0,F}=\|u_i\|_{0,F}$ and $\|[\![u]\!]\|_{0,F}^2=\|[\![R_i^T u_i+R_j^T u_j]\!]\|_{0,F}^2$, we obtain

$$\begin{aligned}|\mathcal{I}_h(u,u)| &\lesssim \sum_{F\in\Gamma}\alpha\, h_F^{-2\ell_h-1}\left(\|[\![R_i^T u_i+R_j^T u_j]\!]\|_{0,F}^2+\|u_i\|_{0,F}^2+\|u_j\|_{0,F}^2\right)\\ &\lesssim \sum_{F\in\Gamma}\alpha\, h_F^{-2\ell_h-1}\left(\|u_i\|_{0,F}^2+\|u_j\|_{0,F}^2\right) \lesssim \sum_{D\in\mathcal{T}_H}\sum_{E\subset\partial D}\alpha\, h^{-2\ell_h-1}\|u\|_{0,E}^2,\end{aligned}$$

where we have also used that $\|[\![R_i^T u_i]\!]\|_{0,F}^2=\|u_i\boldsymbol{n}_i\|_{0,F}^2=\|u_i\|_{0,F}^2$ on each $F\in\Gamma$, and the inequality (5.3). Therefore, for both the DG discretizations, by using the trace inequality (3.3) and the Friedrichs–Poincaré inequality (3.2), we find

$$|\mathcal{I}_h(u-\widetilde{u}_0,u-\widetilde{u}_0)| \lesssim \alpha\, h^{-(2\ell_h+1)}\sum_{D\in\mathcal{T}_H}\|u-\widetilde{u}_0\|_{0,\partial D}^2 \lesssim \alpha\,\frac{H}{h^{2\ell_h+1}}A_h(u,u).$$

We now estimate the term $A_0(u_0,u_0)$ (see (5.2)). Notice that, since u_0 is piecewise constant on $\mathcal{T}_H$, all the terms in $A_h(\widetilde{u}_0,\widetilde{u}_0)$ vanish except for the stability term $\mathcal{S}_h(\widetilde{u}_0,\widetilde{u}_0)$. Furthermore, in view of the equivalence (3.1), it is enough to bound the term appearing from the BZ method. Proceeding as in [3, Lemma 4.3], we obtain

$$A_h(\widetilde{u}_0,\widetilde{u}_0) \lesssim \alpha\left(1+\frac{H}{h^{2\ell_h+1}}\right)A_h(u,u).$$

Finally, the first term on the rhs in (5.2), $A_h(u-\widetilde{u}_0,u-\widetilde{u}_0)$, can be bounded by using the Cauchy–Schwarz inequality and the above estimate

$$A_h(u-\widetilde{u}_0,u-\widetilde{u}_0)\leq 2\ (A_h(u,u)+A_h(\widetilde{u}_0,\widetilde{u}_0)) \lesssim \alpha\left(1+\frac{H}{h^{2\ell_h+1}}\right)A_h(u,u).$$

Summarizing, we get

$$\sum_{i=0}^{N} A_i(u_i,u_i) \lesssim \alpha\frac{H}{h^{2\ell_h+1}}\ A_h(u,u),$$

and so the proof is complete. □

(A2) Local stability. As mentioned in Remark 4.1, for the BZ method, the local solvers are *exact*, hence inequality (5.1) is actually an identity with $\omega=1$. For the BMMPR method, we have the following result which provides a one-sided measure of the approximation properties of the local bilinear forms.

Lemma 5.1 (Local stability). *Let $A_h(\cdot,\cdot)$ be the bilinear form of the BMMPR method, and let $A_i(\cdot,\cdot)$ be the corresponding local bilinear forms. Then, there exists $\omega>0$ such that*

$$A_h(R_i^T u_i,R_i^T u_i)\leq \omega A_i(u_i,u_i) \quad \forall\ u_i\in V_h^i,\quad i=1,\ldots,N. \tag{5.4}$$

Proof. The proof easily follows by writing $A_h(R_i^T u_i,R_i^T u_i) = A_i(u_i,u_i)+\mathcal{G}_i$, where $\mathcal{G}_i = A_h(R_i^T u_i,R_i^T u_i)-A_i(u_i,u_i)$. We note that $|\mathcal{G}_i|\leq|\mathcal{G}_{i,1}|+|\mathcal{G}_{i,2}|$ with

$$\mathcal{G}_{i,1}=\sum_{F\in\Gamma_i}\int_F \alpha\ h_F^{-2\ell_h}\{\!\{r_F([\![R_i^T u_i]\!])\}\!\}\cdot\boldsymbol{n}_i u_i\ \mathrm{d}s,\qquad \mathcal{G}_{i,2}=\sum_{F\in\Gamma_i}\int_F \alpha\ h_F^{-2\ell_h} r_F^i(u_i\boldsymbol{n}_i)\cdot\boldsymbol{n}_i u_i\ \mathrm{d}s.$$

Equivalence (3.1) leads to

$$|\mathcal{G}_{i,1}|=\sum_{F\in\Gamma_i}\alpha h_F^{-2\ell_h}\|r_F([\![R_i^T u_i]\!])\|_{0,\Omega}^2\leq C_2\sum_{F\in\Gamma_i}\alpha h_F^{-2\ell_h-1}\|u_i\boldsymbol{n}_i\|_{0,F}^2\leq C_2C_1^{-1}A_i(u_i,u_i).$$

For the term $\mathcal{G}_{i,2}$, reasoning in the same way and taking into account that each $F\in\Gamma_i$ is a boundary face for the local bilinear form we obtain

$$|\mathcal{G}_{i,2}|=\sum_{F\in\Gamma_i}\alpha h_F^{-2\ell_h}\|r_F^i(u_i\boldsymbol{n}_i)\|_{0,\Omega_i}^2\leq C_2\sum_{F\in\Gamma_i}\alpha h_F^{-2\ell_h-1}\|u_i\boldsymbol{n}_i\|_{0,F}^2\leq C_2C_1^{-1}A_i(u_i,u_i).$$

Then, the above bounds and standard triangle inequality give (5.4), with $\omega=1+2C_2C_1^{-1}$. □

Remark 5.1. We wish to stress that from the expression of ω derived in the proof of Lemma 5.1, it cannot be guaranteed in general that $\omega<2$.

(A3) Strengthened Cauchy–Schwarz inequalities. From our definition of the local solvers and local subspaces, it is straightforward to see that $\varepsilon_{ii}=1$ for $i=1,\ldots,N$. For $i\neq j$, note that $A_h(R_i^T u_i, R_j^T u_j)\neq 0$ only if $\partial\Omega_i\cap\partial\Omega_j\neq\emptyset$, so $\varepsilon_{ij}=1$ in those cases, and $\varepsilon_{ij}=0$ otherwise. Then, by setting $\boldsymbol{\mathcal{E}}=\{\varepsilon_{ij}\}_{1\leq i,j\leq N}$, the spectral radius of $\boldsymbol{\mathcal{E}}$, $\rho(\boldsymbol{\mathcal{E}})$, can be bounded by $\rho(\boldsymbol{\mathcal{E}})\leq \max_i\sum_j|\varepsilon_{ij}|\leq 1+N_c$, where N_c is the maximum number of adjacent subdomains that a given subdomain might have.

We have now all ingredients to show the main results of this section.

Theorem 5.1. *Let P_{ad} be the additive Schwarz operator corresponding to the BZ or the BMMPR super penalty DG methods. Then, its condition number $\kappa(P_{ad})$ satisfies*

$$\kappa(P_{ad}) \lesssim \alpha(1+\omega[1+N_c])\frac{H}{h^{2\ell_h+1}}, \tag{5.5}$$

where ω is the local stability constant in **(A2)** *and N_c denotes the maximum number of adjacent subdomains a given subdomain can have.*

Proof. Proposition 5.1 implies that $\lambda_{\min}(P_{ad})$ is bounded from below by $C_0^{-2}=\alpha^{-1}H^{-1}h^{2\ell_h+1}$. In fact, the definition (4.3) of $\widetilde{P}_i$ and Cauchy–Schwarz inequality yield

$$\begin{aligned}
A_h(u,u)=\sum_{i=0}^{N}A_h(u,R_i^T u_i)=\sum_{i=0}^{N}A_i(\widetilde{P}_i u,u_i)&\leq\left(\sum_{i=0}^{N}A_i(\widetilde{P}_i u,\widetilde{P}_i u)\right)^{1/2}\left(\sum_{i=0}^{N}A_i(u_i,u_i)\right)^{1/2}\\
&\leq C_0\left(\sum_{i=0}^{N}A_h(u,R_i^T\widetilde{P}_i)\right)^{1/2}A_h(u,u)^{1/2}\\
&=C_0\,A_h(u,P_{ad}u)^{1/2}A_h(u,u)^{1/2}.
\end{aligned}$$

The *local stability* property and the *strengthened Cauchy–Schwarz inequalities* imply that $\lambda_{\max}(P_{ad})$ is bounded from above by $\omega\rho(\boldsymbol{\mathcal{E}})+1$. In fact,

$$\begin{aligned}
&A_h(P_0u,u)\leq A_h(P_0u,P_0u)^{1/2}A_h(u,u)^{1/2}\leq A_h(u,P_0u)^{1/2}A_h(u,u)^{1/2},\\
&A_h(\sum_{i=1}^{N}P_iu,u)\leq\omega\rho(\boldsymbol{\mathcal{E}})A_h(u,u),
\end{aligned}$$

from which the desired upper bound for $\lambda_{\max}(P_{ad})$ follows by definition. The proof is complete by recalling that $\rho(\boldsymbol{\mathcal{E}})\leq 1+N_c$ where N_c is the maximum number of adjacent subdomains that a given subdomain can have. □

The multiplicative operator is non-symmetric, and in Theorem 5.2, we show that the energy norm of the error propagation operator E_N is strictly less than one. Hence, the spectral radius of E_N is strictly less than one, and a simple Richardson iteration applied to the preconditioned system converges.

Theorem 5.2. *Let $A_h(\cdot,\cdot)$ be the bilinear form of the BZ super penalty DG method, and let P_{mu} be its multiplicative Schwarz operator. Then,*

$$\|E_N\|_A^2 = \sup_{\substack{u\in V_h\\ u\neq 0}} \frac{A_h(E_N u, E_N u)}{A_h(u,u)} \leq 1 - \frac{1}{C\alpha(1+2(N_c+1)^2)} \frac{h^{(2\ell_h+1)}}{H} < 1.$$

For the sake of conciseness we omit the proof. We note however that, once the properties **(A1)**, **(A2)** and **(A3)** are shown, the proof follows by proceeding as in [2].

Remark 5.2. The classical Schwarz theory for multiplicative methods relies upon the hypothesis that the local stability constant $\omega<2$. In view of Remark 5.1 (see also Lemma 5.1), for the BMMPR method our convergence analysis can not be applied to theoretically explain the optimal performance numerically observed.

Remark 5.3. Theorem 5.1 guarantees that the additive Schwarz preconditioner can be successfully accelerated with the CG iterative solver. Analogously, thanks to Theorem 5.2 the multiplicative Schwarz method can indeed be accelerated with the GMRES linear solver (see [11] for details).

6 Numerical Results

We take $d=2$, $\Omega=(0,1)\times(0,1)$, and we choose f so that the exact solution of the Poisson problem with non-homogeneous boundary conditions is given by $u(x,y)=\exp(xy)$. We consider subdomain partitions made of $N=4,16$ squares. The initial coarse and fine refinements consist of 2^4 and 2^8 squares, respectively, with corresponding initial mesh sizes given by $H_0=1/2^2$ and $h_0=1/2^4$. For $n=1,2,3$, we consider n successive global uniform refinements of these initial grids. For the sake of brevity we only report results obtained on Cartesian grids; analogous experiments were run on structured and unstructured triangular refinements, and the same orders have been observed. The preconditioned linear systems of equations have been solved with the CG and GMRES iterative solvers for the additive and multiplicative methods, respectively. The (relative) tolerance is set to 10^{-12}.

We first address the scalability of the additive Schwarz method, *i.e.*, the independence of the convergence rate of the number of subdomains. In Table 1, for the BZ method ($\alpha=1$), we compare the condition number estimates obtained with $N=4,16$, and $\ell_h=\ell_H=1$. The dashes mean that the coarse partition is not strictly included in the fine one, and in those cases it is meaningless to build the preconditioner. The condition number estimates for the non preconditioned systems are shown in the last row. As stated in Theorem 5.1, our preconditioner seems to be insensitive on the number of subdomains, and, as expected, a convergence rate of order $O(H/h^3)$ is clearly observed.

$\kappa(\mathbf{B}_{ad}\mathbf{A})$, $N=4$				
$H\downarrow h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	7.4360e+01	6.5867e+02	5.4275e+03	4.3961e+04
$H_0/2$	-	2.9770e+02	2.6825e+03	2.2254e+04
$H_0/4$	-	-	1.1944e+03	1.0771e+04
$H_0/8$	-	-	-	4.7526e+03
$\kappa(\mathbf{A})$	1.7321e+03	2.6835e+04	4.2604e+05	6.8037e+06

$\kappa(\mathbf{P}_{ad})$, $N=16$				
$H\downarrow h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	8.1843e+01	7.4657e+02	6.1084e+03	4.8324e+04
$H_0/2$	-	2.9355e+02	2.6374e+03	2.1707e+04
$H_0/4$	-	-	1.1828e+03	1.0770e+04
$H_0/8$	-	-	-	4.7833e+03
$\kappa(\mathbf{A})$	1.7321e+03	2.6835e+04	4.2604e+05	6.8037e+06

Table 1: BZ method ($\alpha=1$), $\ell_h=\ell_H=1$.

In Table 2, with $N=16$ and $\ell_h=\ell_H=2$, we show the condition number estimates and the CG iteration counts (between parenthesis) of the additive Schwarz method for the BZ discretization ($\alpha=1$). The cross in the last row of Table 2 means that we were not able to solve the non preconditioned system due to excessive computational requirements. Observe that, in agreement with Theorem 5.1, the condition number grows as $O(H/h^5)$.

$\kappa(\mathbf{B}_{ad}\mathbf{A})$ and CG iteration counts				
$H\downarrow h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	1.2018e+04 (88)	3.8554e+05 (176)	1.1731e+07 (259)	4.7145e+07 (339)
$H_0/2$	-	1.9072e+05 (110)	5.9690e+06 (193)	7.2780e+07 (264)
$H_0/4$	-	-	2.8401e+06 (133)	5.9919e+07 (198)
$H_0/8$	-	-	-	3.4564e+07 (119)
$\kappa(\mathbf{A})$	5.6358e+05 (739)	3.5640e+07 (1922)	2.2742e+09 (4409)	x

Table 2: BZ method ($\alpha=1$), $N=16$, $\ell_h=\ell_H=2$.

Next, we show the GMRES iteration counts computed by using the multiplicative preconditioner ($N=16$, $\alpha=1$ and $\ell_H=\ell_H=1$). For the BZ method (Table 3, left) the result reported confirm the convergence results given in Theorem 5.2. For the BMMPR method (Table 3, right) our numerical results indicate that the multiplicative preconditioner can be indeed efficiently accelerated with the GMRES iterative solver. A theoretical justification of this behaviour is still an open question.

	BZ method				BMMPR method			
$H\downarrow h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	23	39	56	63	11	44	55	55
$H_0/2$	-	21	31	38	-	23	32	25
$H_0/4$	-	-	17	22	-	-	16	17
$H_0/8$	-	-	-	11	-	-	-	10
# iter(**A**)	129	363	848	1841	129	363	848	1841

Table 3: BZ and BMMPR methods ($\alpha=1$), $\mathbf{B}_{mu}\mathbf{A}$, GMRES interation counts, $N=16$, $\ell_h=\ell_H=1$.

Finally, always with $N=16$, we present some numerical computations carried out with different values of the penalty parameter α. More precisely, in Table 4 we compared the condition number estimates of the additive operator obtained for the BZ method with $\ell_h=\ell_H=1$, and by choosing $\alpha=2$ (top) and $\alpha=10$ (bottom). From the results in Table 4 (see also Table 1 (bottom)) it is clear that, as predicted in Theorem 5.1, the condition number of the preconditioned system linearly depends on the value of the penalty parameter.

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$\kappa(\mathbf{B}_{ad}\mathbf{A})$, $\alpha=2$				
$H\downarrow h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	1.6051e+02	1.4882e+03	1.2346e+04	9.6452e+04
$H_0/2$	-	5.8421e+02	5.2702e+03	4.3160e+04
$H_0/4$	-	-	2.3627e+03	2.1537e+0
$H_0/8$	-	-	-	9.5636e+03
$\kappa(\mathbf{A})$	3.4334e+03	5.3555e+04	8.5163e+05	1.3606e+07

$\kappa(\mathbf{B}_{ad}\mathbf{A})$, $\alpha=10$				
$H\downarrow h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	7.8989e+02	7.3904e+03	5.9308e+04	4.7884e+05
$H_0/2$	-	2.8889e+03	2.6060e+04	2.1566e+05
$H_0/4$	-	-	1.1730e+04	1.0735e+05
$H_0/8$	-	-	-	4.6917e+04
$\kappa(\mathbf{A})$	1.7045e+04	2.6731e+05	4.2564e+06	6.8022e+07

Table 4: BZ method, $N=16$, $\ell_h=\ell_H=1$.

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