A Class of Domain Decomposition Preconditioners for hp–Discontinuous Galerkin Finite Element Methods

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Abstract

In this article we address the question of efficiently solving the algebraic linear system of equations arising from the discretization of a symmetric, elliptic boundary value problem using hp–version discontinuous Galerkin finite element methods. In particular, we introduce a class of domain decomposition preconditioners based on the Schwarz framework, and prove bounds on the condition number of the resulting iteration operators. Numerical results confirming the theoretical estimates are also presented.

1 Introduction

For the hp–version of the finite element method, convergence is achieved by suitably combining h–refinements (dividing elements into smaller ones) and p–refinements (increasing the polynomial approximation order). Since the discontinuous Galerkin finite element method (DGFEM) naturally handles non-conforming/hybrid meshes, as well as variable approximation orders, it is well suited for the design of hp–adaptive solution strategies. Despite the great interest over the last few years in the DGFEM and its application to a

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wide range of problems (cf. [40, 32], for example), the question of developing efficient iterative solvers for the solution of the resulting (linear) system of equations has been addressed only recently and only in the framework of the h–version of the DGFEM (h–DGFEM). For example, a wide class of domain decomposition methods for discontinuous Galerkin approximations of elliptic problems has been proposed and analyzed in [24, 34, 12, 27, 3, 5, 21]. We point out that, because of differences in the variational formulation associated with the underlying discontinuous polynomial spaces, the stiffness matrices arising from discontinuous Galerkin approximations possess different sparsity structures compared to those from conforming methods, and indeed, for a given mesh and polynomial degree distribution, the underlying matrix is typically larger in the DGFEM setting. On the other hand, the underlying non-conformity of the DGFEM spaces can be successfully exploited in the development of efficient preconditioners leading to situations which have no analogue in the conforming case. For example, in [24, 3, 5], it has been shown that in the case of h–DGFEMs for elliptic problems, optimal non-overlapping Schwarz preconditioners can be constructed and analyzed leading to spectral bounds of the preconditioned systems of order H/h, where H and h represent the granularity of the coarse and fine meshes, respectively. We remark that, for conforming discretizations, iterative solvers in the classical Schwarz framework can be constructed only by employing overlapping subdomain partitions; indeed, in this setting spectral bounds of order H/h can be obtained when the subdomain partitions possess a minimal overlap. Thereby, non-overlapping Schwarz methods for h–DGFEMs perform in a similar fashion to Schwarz methods with minimal overlap for conforming discretizations, making the Schwarz approach competitive for practical applications.

In the framework of Schwarz methods for the p–version of the conforming finite element method, it has been proved in [36, 35] that optimal overlapping Schwarz preconditioners can be developed leading to iterative solvers that are scalable (the performance of the preconditioner is independent of the number of subdomains) and optimal (the condition number of the preconditioned system is independent of the spectral degree in case of generous overlap, otherwise it depends inversely on the overlap size). Methods belonging to other families of domain decomposition methods, namely non-overlapping or iterative substructuring methods, have been extensively studied for p– and hp–conforming discretizations in [9, 37, 38, 2, 28, 29, 30], for example. In general, such methods require a suitable partitioning of the set of the shape functions into internal, face, edge and vertex modes, the design of complex coarse spaces (especially in three dimensions) and exhibit spectral bounds that are dimension-dependent, while Schwarz methods maintain their simplicity and uniform bounds in any dimension.

In this work, we focus on the p–version of the discontinuous Galerkin finite element method (p–DGFEM), addressing the question of whether the class of non-overlapping Schwarz preconditioners introduced in [24, 3, 5] for the h–DGFEM can successfully be extended to the p– and hp–versions of the DGFEM. Our starting point is that, since we want to enrich the discrete space either by refining the mesh size or increasing the local polynomial approximation order, the preconditioner has to be efficient in both regimes, namely the h– and p–versions. We extend to p–DGFEMs the results shown in [24, 3, 5] for non-overlapping Schwarz methods in the context of the h–version DGFEM. Working in a quite general setting, we prove spectral bounds for the preconditioned stiffness matrix arising in the p–DGFEM of order p^2. Combining these results with the ones known for the h–DGFEM, we obtain spectral bounds of order $p^2 H/h$ for the hp–DGFEM. As far as a comparison with the results known for conforming approximations is concerned, we show that the non-overlapping Schwarz methods for p–DGFEMs perform in an analogous fashion to Schwarz methods with minimal overlap for the p–version of conforming finite elements.

The rest of this work is organized as follows. Section 2 contains the model problem and its DG discretization, together with the spectral bounds for the stiffness matrices arising from hp–DGFEMs. In Section 3 we recall a class of non-overlapping Schwarz preconditioners, which is subsequently analyzed in Section 4. Section 5 contains some numerical results confirming the theoretical estimates.

2 Model Problem and DG Approximation

In this section, we introduce the model problem under consideration, set up some notation and fix the essential requirements for a DG method to fit into our theory. Throughout this article we use standard notation for Sobolev spaces (cf. [1]). In particular, for a bounded domain $D \subseteq \mathbb{R}^d$, $d \geq 1$, we write $|\cdot|_{s,D}$ and $\|\cdot\|_{s,D}$ to denote the standard Sobolev semi-norm and norm, respectively, defined on $H^s(D)$, $s \geq 0$.

Given a bounded polygonal domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, and a function $f \in L^2(\Omega)$, we consider, for simplicity, the weak formulation of the Poisson problem with homogeneous Dirichlet boundary conditions: find $u \in H^1_0(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall\, v \in H^1_0(\Omega), \tag{1}$$

where $(\cdot,\cdot)_\Omega$ is the standard inner product in $[L^2(\Omega)]^d$ given by $(\mathbf{u}, \mathbf{v})_\Omega := \int_\Omega \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}x$.

To avoid the proliferation of constants, we will use the notation $x \lesssim y$ to represent the inequality $x \leq C\, y$, with $C > 0$ independent of the mesh size and the polynomial approximation order. Writing $x \approx y$ will signify that there exists a constant $C > 0$ such that $C^{-1}x \leq y \leq Cx$.

2.1 Meshes, Finite Element Spaces and Trace Operators

We consider a family $\{\mathcal{T}_h, 0 < h \leq 1\}$ of *shape-regular*, not-necessarily matching partitions of Ω into disjoint open elements $\mathcal{K}$ such that $\overline{\Omega} = \cup_{\mathcal{K}\in\mathcal{T}_h}\overline{\mathcal{K}}$, where each $\mathcal{K} \in \mathcal{T}_h$ is the image of a fixed master element $\widehat{\mathcal{K}}$, *i.e.*, $\mathcal{K} = F_{\mathcal{K}}(\widehat{\mathcal{K}})$, and $\widehat{\mathcal{K}}$ is either the open unit d-simplex or the open unit hypercube in $\mathbb{R}^d$, $d = 2, 3$. For a given mesh $\mathcal{T}_h$, we denote by $h_{\mathcal{K}}$ the diameter of $\mathcal{K} \in \mathcal{T}_h$. An interior face of $\mathcal{T}_h$ is defined as the (non–empty) interior of $\partial\mathcal{K}_1 \cap \partial\mathcal{K}_2$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. Similarly, a boundary face of $\mathcal{T}_h$ is defined as the (non-empty) interior of $\partial\mathcal{K} \cap \Omega$, where $\mathcal{K}$ is a boundary element of $\mathcal{T}_h$. We collect all the interior (boundary, respectively) faces (if $d = 2$, "face" means "edge") in the set $\mathcal{F}_h^I$ ($\mathcal{F}_h^B$, respectively) and set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$. We assume that for all $\mathcal{K} \in \mathcal{T}_h$ and for all $F \in \mathcal{F}_h$, $h_{\mathcal{K}} \lesssim h_F$, where h_F is the diameter of $F \in \mathcal{F}_h$. This last assumption implies that the maximum number of hanging nodes on each face is uniformly bounded. Finally, we assume that the following *bounded local variation* property holds (cf. [25, 26]): for any pair of elements $\mathcal{K}_1$ and $\mathcal{K}_2$ sharing a $(d-1)$–dimensional face, $h_{\mathcal{K}_1} \approx h_{\mathcal{K}_2}$.

Let $\mathbf{p} := \{p_{\mathcal{K}} : \mathcal{K} \in \mathcal{T}_h\}$ be a degree vector that assigns to each element $\mathcal{K} \in \mathcal{T}_h$ a polynomial approximation order $p_{\mathcal{K}} \geq 1$. The generic *hp*–finite element space of piecewise polynomials is then given by

$$\mathcal{V}(\mathcal{T}_h, \mathbf{p}) := \{u \in L^2(\Omega) \;:\; u \circ F_{\mathcal{K}} \in \mathbb{M}^{p_{\mathcal{K}}}(\widehat{\mathcal{K}}) \quad \forall\, \mathcal{K} \in \mathcal{T}_h\}, \tag{2}$$

where $\mathcal{M}^{p_{\mathcal{K}}}(\widehat{\mathcal{K}})$ is either the space $\mathbb{P}_{p_{\mathcal{K}}}(\widehat{\mathcal{K}})$ of polynomials of degree at most $p_{\mathcal{K}}$ on $\widehat{\mathcal{K}}$, if $\widehat{\mathcal{K}}$ is the reference d-simplex, or the space $\mathbb{Q}_{p_{\mathcal{K}}}(\widehat{\mathcal{K}})$ of all tensor–product polynomials on $\widehat{\mathcal{K}}$ of degree $p_{\mathcal{K}}$ in each coordinate direction, if $\widehat{\mathcal{K}}$ is the unit reference hypercube in $\mathbb{R}^d$. We also assume that the polynomial approximation order has *local bounded variation* (cf. [25, 26]): for any pair of elements $\mathcal{K}_1$ and $\mathcal{K}_2$ sharing a $(d-1)$–dimensional face, $p_{\mathcal{K}_1} \approx p_{\mathcal{K}_2}$.

For piecewise smooth vector–valued and scalar functions $\boldsymbol{\tau}$ and z, respectively, we introduce the trace operators in the usual way [8]. Let $F \in \mathcal{F}_h^I$ be an interior face shared by two elements $\mathcal{K}_1$ and $\mathcal{K}_2$ with outward unit normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, respectively. For $i = 1, 2$, let $\boldsymbol{\tau}_i$ and

z_i denote the traces of $\boldsymbol{\tau}$ and z on $\partial\mathcal{K}_i$, taken within the interior of $\mathcal{K}_i$, respectively. We define the *jump* across F by

$$[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{\tau}_2 \cdot \boldsymbol{n}_2, \quad [\![z]\!] := z_1\boldsymbol{n}_1 + z_2\boldsymbol{n}_2,$$

and, for a given parameter $\delta \in [0,1]$, the *weighted average* across F by

$$\{\!\{\boldsymbol{\tau}\}\!\}_\delta := \delta\boldsymbol{\tau}_1 + (1-\delta)\boldsymbol{\tau}_2, \quad \{\!\{z\}\!\}_\delta := \delta z_1 + (1-\delta)z_2.$$

On a boundary face $F \in \mathcal{F}_h^B$, we set, analogously,

$$[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau} \cdot \boldsymbol{n}, \quad [\![z]\!] := z\,\boldsymbol{n}, \quad \{\!\{\boldsymbol{\tau}\}\!\}_\delta := \boldsymbol{\tau}, \quad \{\!\{z\}\!\}_\delta := z.$$

Whenever $\delta = 1/2$ (that is the *weighted average* reduces to the standard average) we neglect the subscript and simply write $\{\!\{\cdot\}\!\}$.

2.2 DG Methods in Primal and Mixed Form

In the following, we recall some of the DG methods in primal and mixed form, respectively, for which the forthcoming analysis is applicable.

For the discretization of the model problem (1), we first consider the following class of DG methods in primal form: find $u_h \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$ such that

$$\mathcal{A}(u_h, v) = (f, v)_\Omega \quad \forall\, v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p}),$$

where, denoting by ∇_h the elementwise application of the operator ∇, $\mathcal{A}:\, \mathcal{V}(\mathcal{T}_h, \mathbf{p}) \times \mathcal{V}(\mathcal{T}_h, \mathbf{p}) \to \mathbb{R}$ is defined by

$$\mathcal{A}(u,v) := (\nabla_h u, \nabla_h v)_\Omega - \sum_{F\in\mathcal{F}_h} \int_F \left(\{\!\{\nabla_h u\}\!\}_\delta \cdot [\![v]\!] + [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\}_\delta \right) \mathrm{d}s + \mathcal{S}(u,v)$$

for all $u, v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$. The symmetric interior penalty (SIP) method [7] is defined by choosing $\delta = 1/2$, and the stabilization form $\mathcal{S}(\cdot,\cdot)$ as

$$\mathcal{S}(u,v) := \sum_{F\in\mathcal{F}_h} \int_F \mathtt{a}\, [\![u]\!] \cdot [\![v]\!]\; \mathrm{d}s, \tag{3}$$

with $\mathtt{a} > 0$ a parameter (dependent on both the mesh-size and approximation order) at our disposal. With the same choice of the stabilization form $\mathcal{S}(\cdot,\cdot)$ and $\delta \neq 1/2$, we recover the method of Heinrich–Pietsch [31].

Discontinuous Galerkin methods in mixed form can be obtained by introducing the auxiliary variable $\boldsymbol{\sigma} := \nabla u$, and rewriting the second order problem (1) in mixed form, as a first order system of equations. Following [8], after discretization, the variable $\boldsymbol{\sigma}_h$ can actually be eliminated, in an element-by-element manner, obtaining again methods in the primal variable u_h only: find $u_h \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$ such that

$$\mathcal{A}(u_h, v) = (f, v)_\Omega \quad \forall\, v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p}),$$

with $\mathcal{A} : \mathcal{V}(\mathcal{T}_h, \mathbf{p}) \times \mathcal{V}(\mathcal{T}_h, \mathbf{p}) \to \mathbb{R}$ defined by

$$\mathcal{A}(u, v) := (\nabla_h u + \mathcal{R}(\llbracket u \rrbracket) + \mathcal{L}(\boldsymbol{\beta} \cdot \llbracket u \rrbracket), \nabla_h v + \mathcal{R}(\llbracket v \rrbracket) + \mathcal{L}(\boldsymbol{\beta} \cdot \llbracket v \rrbracket))_\Omega + \mathcal{S}(u, v).$$

The lifting operators $\mathcal{R}(\cdot)$ and $\mathcal{L}(\cdot)$ are defined as

$$\mathcal{R}(\boldsymbol{\tau}) := \sum_{F \in \mathcal{F}_h} r_F(\boldsymbol{\tau}), \quad \mathcal{L}(z) := \sum_{F \in \mathcal{F}_h^I} l_F(z), \tag{4}$$

where $r_F : [L^2(F)]^d \to [\mathcal{V}(\mathcal{T}_h, \mathbf{p})]^d$ and $l_F : L^2(F) \to [\mathcal{V}(\mathcal{T}_h, \mathbf{p})]^d$ are given by

$$\begin{aligned} \int_\Omega r_F(\boldsymbol{\tau}) \cdot \boldsymbol{\eta} \,\mathrm{d}x &:= -\int_F \boldsymbol{\tau} \cdot \{\!\{\boldsymbol{\eta}\}\!\} \,\mathrm{d}s \quad \forall \boldsymbol{\eta} \in [\mathcal{V}(\mathcal{T}_h, \mathbf{p})]^d \quad \forall F \in \mathcal{F}_h, \\ \int_\Omega l_F(z) \cdot \boldsymbol{\eta} \,\mathrm{d}x &:= -\int_F z \llbracket \boldsymbol{\eta} \rrbracket \,\mathrm{d}s \qquad\quad \forall \boldsymbol{\eta} \in [\mathcal{V}(\mathcal{T}_h, \mathbf{p})]^d \quad \forall F \in \mathcal{F}_h^I, \end{aligned} \tag{5}$$

respectively. By choosing, for instance, the stabilization function as in (3) and $\boldsymbol{\beta} \in \mathbb{R}^d$ (uniformly bounded independently of the mesh size and polynomial approximation order), we can recover the local discontinuous Galerkin (LDG) method of Cockburn and Shu [18].

For all the considered DG discretizations, the penalty parameter $\mathtt{a} \in L^\infty(\mathcal{F}_h)$ appearing in the stabilization function (3) will be selected as a function of the "local polynomial degree" $\mathtt{p} \in L^\infty(\mathcal{F}_h)$ defined as

$$\mathtt{p} = \mathtt{p}(\boldsymbol{x}) := \begin{cases} \max\{p_{\mathcal{K}_1}, p_{\mathcal{K}_2}\} & \text{if } \boldsymbol{x} \text{ is in the interior of } \partial\mathcal{K}_1 \cap \partial\mathcal{K}_2 \ , \\ p_{\mathcal{K}} & \text{if } \boldsymbol{x} \text{ is in the interior of } \partial\mathcal{K} \cap \partial\Omega \ , \end{cases}$$

and the "local meshsize" function $\mathtt{h} \in L^\infty(\mathcal{F}_h)$

$$\mathtt{h} = \mathtt{h}(\boldsymbol{x}) := \begin{cases} \min\{h_{\mathcal{K}_1}, h_{\mathcal{K}_2}\} & \text{if } \boldsymbol{x} \text{ is in the interior of } \partial\mathcal{K}_1 \cap \partial\mathcal{K}_2 \ , \\ h_{\mathcal{K}} & \text{if } \boldsymbol{x} \text{ is in the interior of } \partial\mathcal{K} \cap \partial\Omega \ . \end{cases}$$

More precisely, we set

$$\mathtt{a} := \alpha \frac{\mathtt{p}^2}{\mathtt{h}}, \tag{6}$$

with $\alpha \geq 1$ independent of the meshsize and the approximation order. This choice allows us to obtain continuity and coercivity bounds (in a suitable

DG norm) independent of global bounds for these quantities. Before recalling the main properties we will need in our analysis, we observe that a straightforward calculation shows that on each internal face $F \in \mathcal{F}_h^I$ it holds

$$\{\!\{u\}\!\}_\delta = \{\!\{u\}\!\} + \widetilde{\boldsymbol{\delta}} \cdot [\![u]\!]\,, \quad \{\!\{u\}\!\}_{1-\delta} = \{\!\{u\}\!\} - \widetilde{\boldsymbol{\delta}} \cdot [\![u]\!]\,,$$

with $\widetilde{\boldsymbol{\delta}} = (\delta - 1/2)\boldsymbol{n}_F$, where $\boldsymbol{n}_F$ denotes the outward unit normal vector to the face F on which the weight δ is assigned. Therefore, for all the DG methods considered, the bilinear form can be written as

$$\begin{aligned}\mathcal{A}(u,v) = &(\nabla_h u, \nabla_h v)_\Omega + \big(\nabla_h u, \mathcal{R}([\![v]\!]) + \mathcal{L}(\widetilde{\boldsymbol{\beta}} \cdot [\![v]\!])\big)_\Omega \\ &+ \big(\mathcal{R}([\![u]\!]) + \mathcal{L}(\widetilde{\boldsymbol{\beta}} \cdot [\![u]\!]), \nabla_h v\big)_\Omega \\ &+ \theta\big(\mathcal{R}([\![u]\!]) + \mathcal{L}(\widetilde{\boldsymbol{\beta}} \cdot [\![u]\!]), \mathcal{R}([\![u]\!]) + \mathcal{L}(\widetilde{\boldsymbol{\beta}} \cdot [\![v]\!])\big)_\Omega + \mathcal{S}(u,v),\end{aligned} \tag{7}$$

where $\mathcal{S}(\cdot,\cdot)$ is defined as in (3), $\theta = 0, 1$ for all the methods in primal and mixed form, respectively, and where $\widetilde{\boldsymbol{\beta}}$ is defined as follow

$$\widetilde{\boldsymbol{\beta}} := \begin{cases} \boldsymbol{\beta} & \text{for the LDG method,} \\ \widetilde{\boldsymbol{\delta}} & \text{for the Heinrich–Pietsch method,} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Remark 2.1 We point out that the formulation (7) is well defined under minimal regularity requirements, *i.e.*, for $u, v \in H^1(\mathcal{T}_h)$, where $H^1(\mathcal{T}_h)$ denotes the broken H^1 Sobolev space defined over the partition $\mathcal{T}_h$. In contrast, "classical" formulations involving integrals over the skeleton of the mesh require the assumption that $u, v \in H^2(\mathcal{T}_h)$.

Remark 2.2 We do not consider non-symmetric DG discretizations, such as the *non-symmetric* interior penalty [41] or the *incomplete* interior penalty [19] methods, for example, since our analysis does not apply to non-symmetric approximations. The abstract analysis of Schwarz methods for conforming approximations to non-symmetric elliptic problems [16] relies upon the GMRES convergence bounds [23]. According to [23], the GMRES method applied to the (preconditioned) system of equations does not stagnate provided that the skew-symmetric part of the (preconditioned) operator is "small" relative to the symmetric part (typically a low-order compact perturbation). As demonstrated in [3, 5], for non-symmetric DG approximations of the Laplace operator, the skew-symmetric part of the operator happens to be of the same order as the symmetric part, therefore convergence results cannot be attained.

2.3 Main Properties

We first recall the local inverse inequality valid for finite element or polynomial functions (cf. [17]): for any polynomial function v of degree $p_{\mathcal{K}}$ on $\mathcal{K} \in \mathcal{T}_h$, we have that

$$|v|_{1,\mathcal{K}}^2 \lesssim p_{\mathcal{K}}^4 \, h_{\mathcal{K}}^{-2} \|v\|_{0,\mathcal{K}}^2. \tag{8}$$

We will also make use of the following trace inequalities: for any $F \in \mathcal{F}_h$, which is a face of the element $\mathcal{K} \in \mathcal{T}_h$, it holds

$$\|v\|_{0,F}^2 \lesssim p_{\mathcal{K}}^2 \, h_{\mathcal{K}}^{-1} \|v\|_{0,\mathcal{K}}^2, \quad \|\nabla v \cdot \boldsymbol{n}\|_{0,F}^2 \lesssim p_{\mathcal{K}}^2 \, h_{\mathcal{K}}^{-1} \, |v|_{1,\mathcal{K}}^2 \,, \tag{9}$$

for any polynomial function v of degree $p_{\mathcal{K}}$ on $\mathcal{K} \in \mathcal{T}_h$.

We also recall the following result given [13, 14] and, for the sake of completeness, outline a sketch of the proof.

Lemma 2.3 For any $v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, it holds

$$\begin{aligned} \alpha \|r_F([\![v]\!])\|_{0,\Omega}^2 &\lesssim \|\mathtt{a}^{1/2} \, [\![v]\!] \, \|_{0,F}^2, \\ \alpha \|l_F(\widetilde{\boldsymbol{\beta}} \cdot [\![v]\!])\|_{0,\Omega}^2 &\lesssim \|\mathtt{a}^{1/2} \, [\![v]\!] \, \|_{0,F}^2, \end{aligned}$$

on each $F \in \mathcal{F}_h$.

Proof. We first consider the bound on the lifting operator $r_F(\cdot)$; to this end, we take $\boldsymbol{\eta} = r_F([\![v]\!])$ in (5), to obtain

$$\begin{aligned} \|r_F([\![v]\!])\|_{0,\Omega}^2 &\leq \|\mathtt{a}^{-1/2} \, \{\!\{r_F([\![v]\!])\}\!\} \, \|_{0,F} \|\mathtt{a}^{1/2} \, [\![v]\!] \, \|_{0,F} \\ &\lesssim \frac{1}{\sqrt{\alpha}} \|r_F([\![v]\!])\|_{0,\Omega} \|\mathtt{a}^{1/2} \, [\![v]\!] \, \|_{0,F}, \end{aligned}$$

where the last step follows from (9). The proof of the bound on $l_F(\cdot)$ follows in an analogous manner and is thereby omitted for brevity. ■

The inequalities (8)-(9), and Lemma 2.3 are the key ingredients to show that the bilinear form $\mathcal{A}(\cdot,\cdot)$ is continuous and coercive in $\mathcal{V}(\mathcal{T}_h, \mathbf{p})$ endowed with the mesh-dependent norm $\|\cdot\|_{\mathrm{DG}}$ defined by

$$\|v\|_{\mathrm{DG}}^2 := \|\nabla_h v\|_{0,\Omega}^2 + \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \; [\![v]\!] \, \|_{0,F}^2.$$

Lemma 2.4 The following continuity and coercivity bounds hold, respectively:

$$\mathcal{A}(u,v) \lesssim \|u\|_{\mathrm{DG}} \|v\|_{\mathrm{DG}} \qquad \forall \, u, v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p}), \tag{10}$$

$$\mathcal{A}(u,u) \gtrsim \|u\|_{\mathrm{DG}}^2 \qquad \forall \, u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p}). \tag{11}$$

For the LDG formulation property (11) holds, provided that the constant α appearing in the penalty parameters (cf. (6)) is bounded away from zero, whereas all the other DG schemes defined above are stable provided that $\alpha \geq \alpha_{\min} > 0$.

PROOF. We first observe that, for any $v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, by definition (4) and the application of Lemma 2.3, we have that

$$\|\mathcal{R}([\![v]\!])\|_{0,\Omega}^2 \lesssim \sum_{F\in\mathcal{F}_h} \|r_F([\![v]\!])\|_{0,\Omega}^2 \lesssim \frac{1}{\alpha} \sum_{F\in\mathcal{F}_h} \|\mathsf{a}^{1/2}\ [\![v]\!]\ \|_{0,F}^2.$$

Analogously, we can show that

$$\|\mathcal{L}(\widetilde{\boldsymbol{\beta}}\cdot[\![v]\!])\|_{0,\Omega}^2 \lesssim \sum_{F\in\mathcal{F}_h} \|l_F(\widetilde{\boldsymbol{\beta}}\cdot[\![v]\!])\|_{0,\Omega}^2 \lesssim \frac{1}{\alpha} \sum_{F\in\mathcal{F}_h} \|\mathsf{a}^{1/2}\ [\![v]\!]\ \|_{0,F}^2.$$

Next, we bound each of the terms appearing in (7). Clearly, upon application of the Cauchy-Schwarz inequality, we get

$$(\nabla_h u, \nabla_h v)_\Omega \leq \|\nabla_h u\|_{0,\Omega} \|\nabla_h v\|_{0,\Omega}.$$

Moreover, for $v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, we have

$$\begin{aligned}
(\nabla_h u, \mathcal{R}([\![v]\!]))_\Omega &\lesssim \frac{1}{\sqrt{\alpha}} \|\nabla_h u\|_{0,\Omega} \left(\sum_{F\in\mathcal{F}_h} \|\mathsf{a}^{1/2}\ [\![v]\!]\ \|_{0,F}^2 \right)^{1/2}, \\
(\nabla_h u, \mathcal{L}(\widetilde{\boldsymbol{\beta}}\cdot[\![v]\!]))_\Omega &\lesssim \frac{1}{\sqrt{\alpha}} \|\nabla_h u\|_{0,\Omega} \left(\sum_{F\in\mathcal{F}_h} \|\mathsf{a}^{1/2}\ [\![v]\!]\ \|_{0,F}^2 \right)^{1/2}.
\end{aligned} \tag{12}$$

Finally, by setting

$$\mathcal{M}(w) := \mathcal{R}([\![w]\!]) + \mathcal{L}(\widetilde{\boldsymbol{\beta}}\cdot[\![w]\!]) \quad \forall w \in \mathcal{V}(\mathcal{T}_h, \mathbf{p}),$$

we also have

$$(\mathcal{M}(u), \mathcal{M}(v))_\Omega \lesssim \frac{1}{\alpha} \left(\sum_{F\in\mathcal{F}_h} \|\mathsf{a}^{1/2}\,[\![u]\!]\ \|_{0,F}^2 \right)^{1/2} \left(\sum_{F\in\mathcal{F}_h} \|\mathsf{a}^{1/2}\ [\![v]\!]\ \|_{0,F}^2 \right)^{1/2}. \tag{13}$$

Collecting all of the previous estimates, we deduce that

$$|\mathcal{A}(u,v)| \lesssim \max\left\{1, \frac{1}{\alpha}, \frac{1}{\sqrt{\alpha}}\right\} \|u\|_{\mathrm{DG}} \|v\|_{\mathrm{DG}};$$

thereby, using the fact that $\alpha \geq 1$, gives the continuity estimate (10). As far as the coercivity bound (11) is concerned, for the LDG formulation, by applying the arithmetic-geometric mean inequality and the above estimate, we have, for every $0 < \varepsilon < 1$,

$$\begin{aligned}
\mathcal{A}(u,u) &= \|\nabla_h u + \mathcal{M}(u)\|^2_{0,\Omega} + \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \, [\![u]\!] \, \|^2_{0,F} \\
&= \|\nabla_h u\|^2_{0,\Omega} + \|\mathcal{M}(u)\|^2_{0,\Omega} + 2(\nabla_h u, \mathcal{M}(u)) + \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \, [\![u]\!] \, \|^2_{0,\Omega} \\
&\geq \, (1-\varepsilon) \, \|\nabla_h u\|^2_{0,\Omega} + \left(1 - \frac{1}{\varepsilon}\right) \|\mathcal{M}(u)\|^2_{0,\Omega} + \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \, [\![u]\!] \, \|^2_{0,F} \\
&\gtrsim \, (1-\varepsilon) \, \|\nabla_h u\|^2_{0,\Omega} + \left(1 + \frac{1}{\alpha}\left(1 - \frac{1}{\varepsilon}\right)\right) \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \, [\![u]\!] \, \|^2_{0,F}.
\end{aligned}$$

Then, (11) follows by a suitable choice of $0 < \varepsilon < 1$. For the remaining DG schemes outlined above, we have

$$\begin{aligned}
\mathcal{A}(u,u) &= \|\nabla_h u\|^2_{0,\Omega} + 2(\nabla_h u, \mathcal{M}(u)) + \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \, [\![u]\!] \, \|^2_{0,F} \\
&\geq \, (1-\varepsilon) \, \|\nabla_h u\|^2_{0,\Omega} - \frac{1}{\varepsilon}\|\mathcal{M}(u)\|^2_{0,\Omega} + \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \, [\![u]\!] \, \|^2_{0,F} \\
&\gtrsim \, (1-\varepsilon) \, \|\nabla_h u\|^2_{0,\Omega} + \left(1 - \frac{1}{\alpha\varepsilon}\right) \sum_{F \in \mathcal{F}_h} \|\mathtt{a}^{1/2} \, [\![u]\!] \, \|^2_{0,F}.
\end{aligned}$$

Arguing as before, we can conclude that there exists a suitable $\alpha_{\min} > 0$ such that (11) holds, provided that $\alpha > \alpha_{\min}$. ■

Next, we recall the following Poincaré-Friedrichs type inequalities (see [11, 7, 24]) that provide two generalizations of the standard Poincaré-Friedrichs inequality to the space of piecewise discontinuous functions.

Lemma 2.5 (Poincaré-Friedrichs inequalities) Let $D \subseteq \Omega$ be an open connected polyhedral domain with diameter H_D that can be covered by the union of some elements in $\mathcal{T}_h$. Then, for any piecewise H^1 function u defined over D it holds

$$\|u\|^2_{0,D} \lesssim H_D^2 \left(\sum_{\substack{\mathcal{K} \in \mathcal{T}_h \\ \mathcal{K} \subset D}} |u|^2_{1,\mathcal{K}} + \sum_{\substack{F \in \mathcal{F}_h \\ F \subset D}} \|\mathtt{h}^{-1/2} \, [\![u]\!] \, \|^2_{0,F} + \sum_{\substack{F \in \mathcal{F}_h \\ F \subset \partial D}} \|\mathtt{h}^{-1/2} u\|^2_{0,F} \right).$$

Moreover, if u has zero average over D, then

$$\|u\|_{0,D}^2 \lesssim H_D^2 \left(\sum_{\substack{\mathcal{K}\in\mathcal{T}_h \\ \mathcal{K}\subset D}} |u|_{1,\mathcal{K}}^2 + \sum_{\substack{F\in\mathcal{F}_h \\ F\subset D}} \|\mathtt{h}^{-1/2} [\![u]\!] \|_{0,F}^2 \right).$$

2.4 *hp*–Condition Number Estimates

Let $N_h^{\mathtt{p}}$ denote the dimension of $\mathcal{V}(\mathcal{T}_h, \mathbf{p})$. Associated with $\mathcal{V}(\mathcal{T}_h, \mathbf{p})$ there are the eigenpairs $(\lambda_j(\mathcal{A}), w_j) \in \mathbb{R} \times \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, satisfying

$$\mathcal{A}(w_j, v) = \lambda_j\, (w_j, v)_\Omega \quad \forall\, v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p}), \quad j = 1, \dots, N_h^{\mathtt{p}}.$$

Denoting by $\lambda_{\max}(\mathcal{A})$ and $\lambda_{\min}(\mathcal{A})$ the maximum and minimum eigenvalues, respectively, *i.e.*,

$$\lambda_{\max}(\mathcal{A}) := \max_{u\neq 0} \frac{\mathcal{A}(u,u)}{(u,u)_\Omega}, \quad \lambda_{\min}(\mathcal{A}) := \min_{u\neq 0} \frac{\mathcal{A}(u,u)}{(u,u)_\Omega},$$

the condition number $\kappa(\mathcal{A})$ of $\mathcal{A}$ is given by

$$\kappa(\mathcal{A}) := \frac{\lambda_{\max}(\mathcal{A})}{\lambda_{\min}(\mathcal{A})}.$$

Let φ_j, $j = 1, \dots, N_h^{\mathtt{p}}$, be the set of the basis functions that span $\mathcal{V}(\mathcal{T}_h, \mathbf{p})$, *i.e.*,

$$\mathcal{V}(\mathcal{T}_h, \mathbf{p}) = \operatorname{span}\{\varphi_j,\, j = 1, \dots, N_h^{\mathtt{p}}\},$$

then any discrete function $v \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$ may be uniquely written in the form

$$v = \sum_{j=1}^{N_h^{\mathtt{p}}} v_j\, \varphi_j,$$

where v_j, $j = 1, \dots, N_h^{\mathtt{p}}$, denotes the corresponding set of expansion coefficients. Here, and in the following, we use the bold notation to denote the spaces of degrees of freedom (vectors) and discrete linear operators (matrices). Clearly we have

$$\mathcal{A}(u, v) = \mathbf{u}^T \mathbf{A} \mathbf{v}, \quad (u, v)_\Omega = \mathbf{u}^T \mathbf{M} \mathbf{v},$$

where $\mathbf{A}$ is the stiffness matrix, and $\mathbf{M}$ is the mass matrix, *i.e.*,

$$\mathbf{A}_{ij} := \mathcal{A}(\varphi_i, \varphi_j), \quad \mathbf{M}_{ij} := (\varphi_i, \varphi_j)_\Omega, \quad i, j = 1, \dots, N_h^{\mathtt{p}}.$$

The spectral condition numbers of $\mathbf{A}$ and $\mathbf{M}$ (denoted by $\kappa(\mathbf{A})$ and $\kappa(\mathbf{M})$, respectively), are given again by the ratio of the maximum and minimum eigenvalues of $\mathbf{A}$ and $\mathbf{M}$, respectively.

The relationships among the maximum and the minimum eigenvalues of $\mathcal{A}(\cdot,\cdot)$, $\mathbf{A}$ and $\mathbf{M}$ are as follows:

$$\lambda_{\max}(\mathbf{A}) \leq \lambda_{\max}(\mathcal{A})\, \lambda_{\max}(\mathbf{M}), \qquad \lambda_{\min}(\mathcal{A})\, \lambda_{\min}(\mathbf{M}) \leq \lambda_{\min}(\mathbf{A}). \tag{14}$$

Therefore,

$$\kappa(\mathbf{A}) \leq \kappa(\mathcal{A})\, \kappa(\mathbf{M}). \tag{15}$$

The hp–bounds for the extremal eigenvalues of the bilinear form $\mathcal{A}(\cdot,\cdot)$ are provided in the next result.

Lemma 2.6 For any $u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, we have that

$$\|u\|_{0,\Omega}^2 \lesssim \mathcal{A}(u,u) \lesssim \frac{\max_{\mathcal{K}\in\mathcal{T}_h} p_{\mathcal{K}}^4}{\min_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^2} \|u\|_{0,\Omega}^2.$$

Proof. The lower bound directly follows from Lemma 2.5 with $D = \Omega$, the seminorm bound in Lemma 2.3, and the coercivity property derived in Lemma 2.4:

$$\begin{aligned}
\|u\|_{0,\Omega}^2 &\lesssim H_\Omega^2 \left(\sum_{\mathcal{K}\in\mathcal{T}_h} |u|_{1,\mathcal{K}}^2 + \sum_{F\in\mathcal{F}_h} \|\mathtt{h}^{-1/2} [\![u]\!] \|_{0,F}^2 \right) \\
&= H_\Omega^2 \left(\sum_{\mathcal{K}\in\mathcal{T}_h} |u|_{1,\mathcal{K}}^2 + \sum_{F\in\mathcal{F}_h} \frac{1}{\alpha\, \mathtt{p}^2} \|\mathtt{a}^{1/2} [\![u]\!] \|_{0,F}^2 \right) \\
&\leq H_\Omega^2 \max\left\{1, \frac{1}{\alpha \min_{\mathcal{K}\in\mathcal{T}_h} p_{\mathcal{K}}^2}\right\} \|u\|_{\mathrm{DG}}^2 \\
&\lesssim \mathcal{A}(u,u).
\end{aligned}$$

For the upper bound, we first observe that the local inverse estimate (8) implies

$$\|\nabla_h u\|_{0,\Omega}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} |u|_{1,\mathcal{K}}^2 \lesssim \sum_{\mathcal{K}\in\mathcal{T}_h} p_{\mathcal{K}}^4 h_{\mathcal{K}}^{-2} \|u\|_{0,\mathcal{K}}^2.$$

Analogously, denoting by $\mathcal{K}_1$ and $\mathcal{K}_2$ the pair of elements that share the face F (order independent), the trace estimate (9) gives

$$\sum_{F\in\mathcal{F}_h} \|\mathtt{a}^{1/2} [\![u]\!] \|_{0,F}^2 = \sum_{F\in\mathcal{F}_h} \alpha \frac{\max\{p_{\mathcal{K}_1}^2, p_{\mathcal{K}_2}^2\}}{\min\{h_{\mathcal{K}_1}, h_{\mathcal{K}_2}\}} \| [\![u]\!] \|_{0,F}^2 \lesssim \sum_{\mathcal{K}\in\mathcal{T}_h} \alpha\, p_{\mathcal{K}}^4 h_{\mathcal{K}}^{-2} \|u\|_{0,\mathcal{K}}^2,$$

where in the last step we have also used the local bounded variation property of the mesh size and polynomial approximation degrees. Therefore, we have that

$$\mathcal{A}(u,u) \lesssim \|\nabla_h u\|_{0,\Omega}^2 + \sum_{F\in\mathcal{F}_h} \|\mathtt{a}^{1/2} [\![u]\!] \|_{0,F}^2$$

$$\lesssim \max\{1,\alpha\} \sum_{\mathcal{K}\in\mathcal{T}_h} p_{\mathcal{K}}^4 h_{\mathcal{K}}^{-2} \|u\|_{0,\mathcal{K}}^2 \lesssim \alpha \frac{\max_{\mathcal{K}\in\mathcal{T}_h} p_{\mathcal{K}}^4}{\min_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^2} \|u\|_{0,\Omega}^2,$$

as required. ■

By exploiting (15) we get the following result that provides a bound on the spectral condition number of $\mathbf{A}$.

Proposition 2.7 The condition number $\kappa(\mathbf{A})$ of the stiffness matrix $\mathbf{A}$ can be bounded by

$$\kappa(\mathbf{A}) \lesssim \alpha \frac{\min_{\mathcal{K}\in\mathcal{T}_h} p_{\mathcal{K}}^4}{\min_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^2} \kappa(\mathbf{M}).$$

In order to bound $\kappa(\mathbf{A})$, it is enough to bound the eigenvalues of $\mathbf{M}$. Such bounds in general depend on the choice of the basis. Since we want to keep our analysis in a general framework, we only suppose that we have selected a set of basis functions that are orthonormal on the reference element $\widehat{\mathcal{K}}$. In such a case, $\mathbf{M}$ is a diagonal matrix with the absolute values of the Jacobian of the transformation from the physical element to the reference one as diagonal elements. The following result can be found in [39, Proposition 6.3.1], and can be proved with a standard scaling argument.

Lemma 2.8 Let $\{\varphi_i,\, i = 1,\ldots,N_h^{\mathbf{p}}\}$ be a set of basis functions that span $\mathcal{V}(\mathcal{T}_h,\mathbf{p})$ which are orthonormal on the reference element $\widehat{\mathcal{K}} \subset \mathbb{R}^d$. For any $u \in \mathcal{V}(\mathcal{T}_h,\mathbf{p})$, let $\mathbf{u}$ be the vector of expansion coefficients, then

$$\min_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^d\, \mathbf{u}^T\mathbf{u} \lesssim \mathbf{u}^T\mathbf{M}\mathbf{u} \lesssim \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^d\, \mathbf{u}^T\mathbf{u}.$$

Combining Proposition 2.7 and Lemma 2.8 we finally obtain an estimate of the spectral condition number of $\mathbf{A}$.

Corollary 2.9 For a set of basis functions which are orthonormal on the reference element $\widehat{\mathcal{K}} \subset \mathbb{R}^d$, the condition number $\kappa(\mathbf{A})$ of the stiffness matrix $\mathbf{A}$ can be bounded by

$$\kappa(\mathbf{A}) \lesssim \alpha \frac{\min_{\mathcal{K}\in\mathcal{T}_h} p_{\mathcal{K}}^4}{\min_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^2} \frac{\max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^d}{\min_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^d}.$$

Therefore, if the mesh $\mathcal{T}_h$ and the polynomial approximation orders are globally *quasi uniform* we have

$$\kappa(\mathbf{A}) \lesssim \alpha\, p^4 h^{-2}. \tag{16}$$

Next we give some details on the choice of suitable sets of basis functions. To fix the ideas, we consider the two-dimensional case; similar constructions can be considered in three dimensions.

On the reference square $\widehat{\mathcal{K}} = \{(\xi,\eta) - 1 \le \xi,\eta \le 1\}$ an orthonormal basis for $\mathbb{Q}_p(\widehat{\mathcal{K}})$ can be simply obtained by choosing the tensor product of the one-dimensional Legendre polynomials, *i.e.*,

$$\varphi_{ij}(\xi,\eta) = c_{ij} L_i(\xi) L_j(\eta) \quad i,j = 0,\dots,p,$$

with $c_{ij} := \sqrt{(2i+1)(2j+1)/4}$, and $L_i(\cdot)$ being the i-th Legendre polynomial.

On the reference triangle $\widehat{\mathcal{K}} = \{(\xi,\eta) \;:\; \xi,\eta \ge 0,\, \xi+\eta \le 1\}$ the most popular orthonormal basis is the Koornwinder–Dubiner polynomial basis [22, 42]. We consider the transformation in Figure 1 between the reference square and the reference triangle given by

$$\xi := \frac{(1+a)(1-b)}{4}, \quad \eta := \frac{(1+b)}{2}.$$

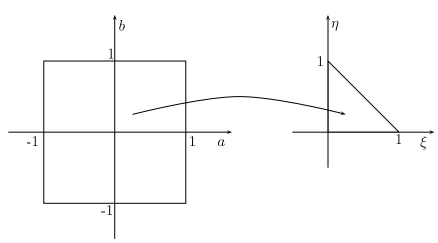

Figure 1: The mapping between the reference square to the reference triangle.

The Koornwinder–Dubiner basis is then constructed by a generalized tensor product of the Jacobi polynomials on the interval $(-1,1)$ to form a basis on the reference square, which is then transformed by the above "collapsing" mapping to a basis on the reference triangle. More precisely, the Koornwinder–Dubiner basis on the triangle $\widehat{K}$ is defined by

$$\begin{aligned}\varphi_{ij}(\xi,\eta) :=& c_{ij}(1-b)^j\, J_i^{0,0}(a)\, J_j^{2i+1,0}(b)\\ =& c_{ij}2^j(1-\eta)^j\, J_i^{0,0}\left(\frac{2\xi}{1-\eta}-1\right) J_j^{2i+1,0}\left(2\eta-1\right),\end{aligned}$$

for $i,j = 0,\dots,p$, $i+j \leq p$, where $c_{ij} := \sqrt{2(2i+1)(i+j+1)/4^i}$ and $J_i^{\alpha,\beta}(\cdot)$ is the i-th Jacobi polynomial, orthogonal under the Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$, *i.e.*,

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta J_m^{\alpha,\beta}(x)J_q^{\alpha,\beta}(x)\,\mathrm{d}x = \frac{2}{2m+1}\delta_{mq}.$$

3 Non–Overlapping Schwarz Preconditioners

In this section we introduce two level non-overlapping Schwarz preconditioners.

3.1 Subdomain Partition, Local and Coarse Solvers

We decompose the domain Ω into N non-overlapping subdomains Ω_i, *i.e.*,

$$\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i.$$

Since the choice of the subdomain partition is defined by the user, we can assume that the subdomain partition is composed of a union of convex elements, and that it is *conforming* and *quasi-uniform*. Next, we consider two levels of *nested* partitions of the domain Ω:

i) a coarse partition $\mathcal{T}_H$ (with mesh size H);

ii) a fine partition $\mathcal{T}_h$ (with mesh size h).

We will suppose that the subdomain partition does not cut any element of $\mathcal{T}_H$ (and therefore of $\mathcal{T}_h$). We remark that the hypothesis of nested grids could be weakened, under suitable additional technicalities in the definition of the inter–grid transfer operators (cf. [20]).

Next we introduce the *local* and *coarse* solvers, which represent the key ingredients of the definition of the Schwarz preconditioners.

Local solvers. For $i = 1,\dots,N$, the local DG spaces are defined accordingly to (2), but on the subdomain Ω_i, *i.e.*,

$$\mathcal{V}_i(\mathcal{T}_h,\mathbf{p}) := \{u \in L^2(\Omega_i) \ :\ u \circ F_{\mathcal{K}} \in \mathbb{M}^{p_{\mathcal{K}}}(\widehat{\mathcal{K}}) \quad \forall\, \mathcal{K} \in \mathcal{T}_h, \mathcal{K} \subset \Omega_i\}.$$

We note that a function in $\mathcal{V}_i(\mathcal{T}_h, \mathbf{p})$ is discontinuous and, as opposed to the case of conforming approximations, does not in general vanish on $\partial\Omega_i$. The classical extension (injection) operator from $\mathcal{V}_i(\mathcal{T}_h, \mathbf{p})$ to $\mathcal{V}(\mathcal{T}_h, \mathbf{p})$ is denoted by $R_i^{\mathrm{T}} : \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}) \longrightarrow \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, $i = 1, \ldots, N$. The restriction operator R_i is defined as the transpose of R_i^{T} with respect to the $L^2(\Omega_i)$ inner product. We observe that

$$\mathcal{V}(\mathcal{T}_h, \mathbf{p}) = R_1^T \mathcal{V}_1(\mathcal{T}_h, \mathbf{p}) \oplus \ldots \oplus R_N^T \mathcal{V}_N(\mathcal{T}_h, \mathbf{p}).$$

The local solvers $\mathcal{A}_i : \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}) \times \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}) \longrightarrow \mathbb{R}$ are defined as

$$\mathcal{A}_i(u_i, v_i) := \mathcal{A}(R_i^{\mathrm{T}} u_i, R_i^{\mathrm{T}} v_i) \quad \forall\, u_i, v_i \in \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}), \quad i = 1, \ldots, N.$$

Remark 3.1 *Approximate local solvers*, such as the ones proposed in [3, 5, 4, 6], could also be considered for the definition of the local components of the preconditioner. Here, for simplicity we focus on the case of *exact local solvers* (cf. [24, 34, 12]).

Coarse solver. To each $\mathcal{D} \in \mathcal{T}_H$ we assign a polynomial degree $q_{\mathcal{D}}$ such that

$$0 \leq q_{\mathcal{D}} \leq \min_{\substack{\mathcal{K} \in \mathcal{T}_h \\ \mathcal{K} \subset \mathcal{D}}} p_{\mathcal{K}}, \tag{17}$$

we store the $q_{\mathcal{D}}$ in the vector $\mathbf{q} := \{q_{\mathcal{D}} : \mathcal{D} \in \mathcal{T}_H\}$, and define the finite element space associated to the coarse partition $\mathcal{T}_H$ as

$$\mathcal{V}_0(\mathcal{T}_H, \mathbf{q}) := \{u \in L^2(\Omega) \; : \; u \circ F_{\mathcal{D}} \in \mathbb{M}^{q_{\mathcal{D}}}(\widehat{\mathcal{K}}) \quad \forall\, \mathcal{D} \in \mathcal{T}_H\}.$$

Notice that (17) guarantees that $\mathcal{V}_0(\mathcal{T}_H, \mathbf{q}) \subseteq \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, and that a coarse space made by piecewise constant functions is admitted. The *coarse solver* $\mathcal{A}_0 : \mathcal{V}_0(\mathcal{T}_H, \mathbf{q}) \times \mathcal{V}(\mathcal{T}_H, \mathbf{q}) \longrightarrow \mathbb{R}$ is defined as

$$\mathcal{A}_0(u_0, v_0) := \mathcal{A}(R_0^{\mathrm{T}} u_0, R_0^{\mathrm{T}} v_0) \quad \forall\, u_0, v_0 \in \mathcal{V}_0(\mathcal{T}_H, \mathbf{q}),$$

where $R_0^{\mathrm{T}} : \mathcal{V}_0(\mathcal{T}_H, \mathbf{q}) \longrightarrow \mathcal{V}(\mathcal{T}_h, \mathbf{p})$ is the classical injection operator from $\mathcal{V}_0(\mathcal{T}_H, \mathbf{q})$ to $\mathcal{V}(\mathcal{T}_h, \mathbf{p})$.

3.2 Variational Formulation

For $1 \leq i \leq N$, let the *local* projection operators be defined by

$$\widetilde{P}_i : \mathcal{V}(\mathcal{T}_h, \mathbf{p}) \longrightarrow \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}) : \quad \mathcal{A}_i(\widetilde{P}_i u, v_i) := \mathcal{A}(u, R_i^T v_i) \quad \forall v_i \in \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}).$$

Analogously, let

$$\widetilde{P}_0 : \mathcal{V}(\mathcal{T}_h, \mathbf{p}) \longrightarrow \mathcal{V}_0(\mathcal{T}_H, \mathbf{q}) : \quad \mathcal{A}_0(\widetilde{P}_0 u, v_0) := \mathcal{A}(u, R_0^T v_0) \quad \forall v_0 \in \mathcal{V}_0(\mathcal{T}_H, \mathbf{q}).$$

We define the projection operators

$$P_i := R_i^T \widetilde{P}_i : \mathcal{V}(\mathcal{T}_h, \mathbf{p}) \longrightarrow \mathcal{V}(\mathcal{T}_h, \mathbf{p}), \quad i = 0, 1, \dots N.$$

Then, the additive and multiplicative Schwarz operator are defined by

$$P_{\text{ad}} := \sum_{i=0}^{N} P_i, \qquad P_{\text{mu}} := I - (I - P_N)(I - P_{N-1}) \cdots (I - P_0),$$

respectively (cf. [15, 16]). We can also consider a symmetrized version of the multiplicative Schwarz operator, given by

$$P_{\text{mu}}^{\text{S}} := I - (I - P_0)^T \cdots (I - P_N)^T (I - P_N) \cdots (I - P_0),$$

which can be exploited with the conjugate gradient method.

The Schwarz operators can be written as products of suitable preconditioners, namely $\mathbf{B}_{\text{ad}}$, $\mathbf{B}_{\text{mu}}$ or $\mathbf{B}_{\text{mu}}^{S}$ of $\boldsymbol{A}$, respectively. Then, the Schwarz method consists of solving, by a suitable Krylov space-based iterative solver, the preconditioned system of equations

$$\mathbf{BAu} = \mathbf{Bf},$$

where $\mathbf{B}$ is either $\mathbf{B}_{\text{ad}}$, $\mathbf{B}_{\text{mu}}$ or $\mathbf{B}_{\text{mu}}^{\text{S}}$. We refer to [10, 44] for more details on the algebraic aspects of the Schwarz preconditioners.

4 Convergence Analysis

The abstract convergence theory of Schwarz methods for symmetric problems centers around three parameters which measure the interactions between the subspaces and bilinear forms, and their suitability in the construction of the preconditioners. We define the three parameters in the form of three assumptions.

Assumption 1 (Stable decomposition) Let $C_0 > 0$ be the minimum constant such that every $u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$ admits a decomposition

$$u = \sum_{i=0}^{N} R_i^T u_i,$$

with $u_0 \in \mathcal{V}_0(\mathcal{T}_H, \mathbf{q})$, $u_i \in \mathcal{V}_i(\mathcal{T}_h, \mathbf{p})$, $i = 1, \dots, N$, that satisfies

$$\sum_{i=0}^{N} \mathcal{A}_i(u_i, u_i) \leq C_0^2 \; \mathcal{A}(u, u);$$

Assumption 2 (Local stability) Let $1 \le \omega < 2$ be the minimum constant such that

$$\begin{aligned} &\mathcal{A}(R_i^T u_i, R_i^T u_i) \le \omega \mathcal{A}_i(u_i, u_i) \qquad \forall\, u_i \in \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}), \qquad i = 1, \dots, N; \\ &\mathcal{A}(R_0^T u_0, R_0^T u_0) \le \omega \mathcal{A}_0(u_0, u_0) \quad \forall\, u_0 \in \mathcal{V}_0(\mathcal{T}_H, \mathbf{q}). \end{aligned}$$

Assumption 3 (Strengthened Cauchy–Schwarz inequalities) Let $0 \le \varepsilon_{ij} \le 1$, $1 \le i, j \le N$, be the minimum values that satisfy

$$\left|\mathcal{A}(R_i^T u_i, R_j^T u_j)\right| \le \varepsilon_{ij} \mathcal{A}(R_i^T u_i, R_i^T u_i)^{1/2} \mathcal{A}(R_j^T u_j, R_j^T u_j)^{1/2}$$

for all $v_i \in \mathcal{V}_i(\mathcal{T}_h, \mathbf{p})$, $u_j \in \mathcal{V}_j(\mathcal{T}_h, \mathbf{p})$. Define $\rho(\boldsymbol{\mathcal{E}})$ to be the spectral radius of $\boldsymbol{\mathcal{E}} = \{\varepsilon_{ij}\}_{i,j=1,\dots,n}$.

If Assumptions 1–3 are satisfied then we can prove optimal spectral bounds for the Schwarz operators by using the classical abstract framework of Schwarz methods [44, 43].

Theorem 4.1 Let Assumptions 1–3 be satisfied. Then, the condition number of the additive Schwarz operator satisfies

$$\kappa(P_{\mathrm{ad}}) \le C_0^2 \omega \left(\rho(\boldsymbol{\mathcal{E}}) + 1\right).$$

Moreover, the error propagation operator $E_{\mathrm{mu}} := (I - P_N) \cdots (I - P_0)$ of the multiplicative Schwarz operator satisfies

$$\mathcal{A}(E_{\mathrm{mu}}, E_{\mathrm{mu}}) \le 1 - \frac{2 - \omega}{\left(2\omega^2 \rho(\boldsymbol{\mathcal{E}})^2 + 1\right) C_0^2},$$

and, the condition number of the symmetrized multiplicative Schwarz operator satisfies:

$$\kappa(P_{\mathrm{mu}}^{\mathrm{S}}) \le \frac{\left(2\omega^2 \rho(\boldsymbol{\mathcal{E}})^2 + 1\right) C_0^2}{2 - \omega}.$$

The aim of the remaining part of the section is to show that for the Schwarz operators defined in Section 3, Assumptions 1–3 are satisfied. From our definition of the local solvers and local subspaces, Assumption 2 trivially holds (and is actually an identity) with $\omega = 1$. As far as Assumption 3 is concerned, it is straightforward to see that $\varepsilon_{ii} = 1$, for $i = 1, \dots, N$. For $i \ne j$, we note that $\mathcal{A}(R_i^T u_i, R_j^T u_j) \ne 0$ only if $\partial\Omega_i \cap \partial\Omega_j \ne \emptyset$, so $\varepsilon_{ij} = 1$ in those cases, and $\varepsilon_{ij} = 0$ otherwise. Then, $\rho(\boldsymbol{\mathcal{E}})$ can be bounded by $\rho(\boldsymbol{\mathcal{E}}) \le \max_i \sum_j |\varepsilon_{ij}| \le 1 + N_c$, where N_c is the maximum number of adjacent subdomains that a given subdomain might have.

Before proving that Assumption 1 is satisfied, we derive some preliminary results. Firstly, we observe that any $u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, can be decomposed (uniquely) as

$$u = \sum_{i=1}^{N} R_i^T u_i, \quad u_i \in \mathcal{V}_i(\mathcal{T}_h, \mathbf{p}), \quad i = 1, \dots, N,$$

and the following identity holds:

$$\mathcal{A}(u,u) = \sum_{i=1}^{N} \mathcal{A}_i(u_i, u_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \mathcal{A}(R_i^T u_i, R_j^T u_j). \tag{18}$$

The next result provides an upper bound for the second term on the right hand side of (18). This result is the *hp*–version of the analogous one obtained in [3]; for the sake of completeness we include the proof below.

Lemma 4.2 For any $u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, we have

$$\left| \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \mathcal{A}(R_i^T u_i, R_j^T u_j) \right| \lesssim \|u\|_{\mathrm{DG}}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \sum_{F \in \Gamma_{ij}} \left(\|\mathtt{a}^{1/2} u_i\|_{0,F}^2 + \|\mathtt{a}^{1/2} u_j\|_{0,F}^2 \right),$$

where Γ_{ij} is the set of all faces $F \in \mathcal{F}_h$ such that $F \subset \partial\Omega_i \cap \partial\Omega_j$, $i,j = 1, \dots, N$.

Proof. Given $u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, we first observe that $\mathcal{A}(R_i^T u_i, R_j^T u_j) = 0$ if $\partial\Omega_i \cap \partial\Omega_j = \emptyset$. Also, note that

$$\left| \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \mathcal{A}(R_i^T u_i, R_j^T u_j) \right| \lesssim \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \left| \mathcal{A}(R_i^T u_i, R_j^T u_j) \right|.$$

Now let Ω_i and Ω_j be two neighboring subdomains. Setting, $\widetilde{u}_i = R_i^T u_i$ and $\widetilde{u}_j = R_j^T u_j$, we have:

$$\begin{aligned} \mathcal{A}(\widetilde{u}_i, \widetilde{u}_j) =& (\nabla_h \widetilde{u}_i, \nabla_h \widetilde{u}_j)_\Omega + (\nabla_h \widetilde{u}_i, \mathcal{M}(\widetilde{u}_j))_\Omega + (\mathcal{M}(\widetilde{u}_i), \nabla_h \widetilde{u}_j)_\Omega \\ &+ \theta(\mathcal{M}(\widetilde{u}_i), \mathcal{M}(\widetilde{u}_j))_\Omega + \mathcal{S}(\widetilde{u}_i, \widetilde{u}_j), \end{aligned}$$

where $\mathcal{M}(\phi) := \mathcal{R}(\llbracket \phi \rrbracket) + \mathcal{L}(\widetilde{\boldsymbol{\beta}} \cdot \llbracket \phi \rrbracket)$ for any $\phi \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$. Next, we bound each of the terms on the right hand side. We recall that the support of $\widetilde{u}_i$ ($\widetilde{u}_j$, respectively) is confined to Ω_i (Ω_j, respectively), which implies that

$(\nabla_h \widetilde{u}_i, \nabla_h \widetilde{u}_j)_\Omega = 0$. Recalling that $\mathcal{M}(\widetilde{u}_i) = 0$ whenever $\widetilde{u}_i = 0$, from estimate (12), it follows that

$$
\begin{aligned}
(\nabla_h \widetilde{u}_i, \mathcal{M}(\llbracket \widetilde{u}_j \rrbracket))_\Omega = (\nabla_h \widetilde{u}_i, \mathcal{M}(\llbracket \widetilde{u}_j \rrbracket))_{\Omega_i} &\leq \|\nabla_h u_i\|_{0,\Omega_i} \|\mathcal{M}(\llbracket \widetilde{u}_j \rrbracket)\|_{0,\Omega_i} \\
&\lesssim \frac{1}{\sqrt{\alpha}} \|\nabla_h \widetilde{u}_i\|_{0,\Omega_i} \left(\sum_{F \in \Gamma_{ij}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F}^2 \right)^{1/2} \\
&\lesssim \frac{1}{\sqrt{\alpha}} \left(\|\nabla_h \widetilde{u}_i\|_{0,\Omega_i}^2 + \sum_{F \in \Gamma_{ij}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F}^2 \right).
\end{aligned}
$$

Analogously, we have

$$
(\mathcal{M}(\llbracket \widetilde{u}_i \rrbracket), \nabla_h \widetilde{u}_j)_\Omega \lesssim \frac{1}{\sqrt{\alpha}} \left(\|\nabla_h \widetilde{u}_j\|_{0,\Omega_j}^2 + \sum_{F \in \Gamma_{ij}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_i \rrbracket \|_{0,F}^2 \right).
$$

From (13) we also have

$$
\begin{aligned}
(\mathcal{M}(\widetilde{u}_i), \mathcal{M}(\widetilde{u}_j))_\Omega &\lesssim \frac{1}{\alpha} \left(\sum_{\substack{F \in \mathcal{F}_h \\ F \subset \overline{\Omega}_i}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_i \rrbracket \|_{0,F}^2 \right)^{1/2} \left(\sum_{\substack{F \in \mathcal{F}_h \\ F \subset \overline{\Omega}_j}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F}^2 \right)^{1/2} \\
&\lesssim \frac{1}{\alpha} \left(\sum_{\substack{F \in \mathcal{F}_h \\ F \subset \overline{\Omega}_i}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_i \rrbracket \|_{0,F}^2 + \sum_{\substack{F \in \mathcal{F}_h \\ F \subset \overline{\Omega}_j}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F}^2 \right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathcal{S}(\widetilde{u}_i, \widetilde{u}_j) &\leq \sum_{F \in \Gamma_{ij}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_i \rrbracket \|_{0,F} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F} \\
&\lesssim \sum_{F \in \Gamma_{ij}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_i \rrbracket \|_{0,F}^2 + \sum_{F \in \Gamma_{ij}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F}^2.
\end{aligned}
$$

Collecting all the previous estimates, and using that $\max\{1/\alpha, 1/\sqrt{\alpha}\} \leq 1$ we obtain

$$
\begin{aligned}
|\mathcal{A}(\widetilde{u}_i, \widetilde{u}_j)| \lesssim &\|\nabla_h \widetilde{u}_i\|_{0,\Omega_i}^2 + \|\nabla_h \widetilde{u}_j\|_{0,\Omega_j}^2 + \sum_{F \in \Gamma_{ij}} \left(\|\mathsf{a}^{1/2} \llbracket \widetilde{u}_i \rrbracket \|_{0,F}^2 + \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F}^2 \right) \\
&+ \sum_{\substack{F \in \mathcal{F}_h \\ F \subset \overline{\Omega}_i}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_i \rrbracket \|_{0,F}^2 + \sum_{\substack{F \in \mathcal{F}_h \\ F \subset \overline{\Omega}_j}} \|\mathsf{a}^{1/2} \llbracket \widetilde{u}_j \rrbracket \|_{0,F}^2.
\end{aligned}
$$

We note that $u|_{\Omega_i} \equiv \widetilde{u}_i$, and that on each face $F \in \Gamma_{ij}$, $[\![\widetilde{u}_i]\!] = u_i \boldsymbol{n}_i$. Thereby,

$$|\mathcal{A}(\widetilde{u}_i, \widetilde{u}_j)| \lesssim \|\nabla_h u\|^2_{0,\Omega_i \cup \Omega_j} + \sum_{\substack{F \in \mathcal{F}_h \\ F \subset \overline{\Omega}_i \cup \overline{\Omega}_j \\ F \not\subset \Gamma_{ij}}} \|\mathsf{a}^{1/2} \, [\![u]\!] \, \|^2_{0,F}$$
$$+ \sum_{F \in \Gamma_{ij}} \left(\|\mathsf{a}^{1/2} \, u_i\|^2_{0,F} + \|\mathsf{a}^{1/2} \, u_j\|^2_{0,F} \right).$$

Summing over all subdomains completes the proof. ∎

The next result guarantees that a *stable splitting* can be found for the family of subspaces and the corresponding bilinear forms.

Proposition 4.3 (Stable decomposition) For any $u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$ there exists a decomposition of the form $u = \sum_{i=0}^N R_i^T u_i$, with $u_0 \in \mathcal{V}_0(\mathcal{T}_H, \mathbf{q})$ and $u_i \in \mathcal{V}_i(\mathcal{T}_h, \mathbf{p})$, $i = 1, \ldots, N$, such that

$$\sum_{i=0}^N \mathcal{A}_i(u_i, u_i) \lesssim C_0^2 \, \mathcal{A}(u, u), \qquad C_0^2 = \alpha \max_{\mathcal{D} \in \mathcal{T}_H} H_{\mathcal{D}} \frac{\max_{\substack{\mathcal{K} \in \mathcal{T}_h \\ \mathcal{K} \subset \mathcal{D}}} p_{\mathcal{K}}^2}{\min_{\substack{\mathcal{K} \in \mathcal{T}_h \\ \mathcal{K} \subset \mathcal{D}}} h_{\mathcal{K}}}.$$

Proof. Given $u \in \mathcal{V}(\mathcal{T}_h, \mathbf{p})$, let $u_0 \in \mathcal{V}_0(\mathcal{T}_H, \mathbf{q})$ be defined as

$$u_0|_{\mathcal{D}} := \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} u \, \mathrm{d}x \qquad \forall \mathcal{D} \in \mathcal{T}_H.$$

Next, we decompose uniquely $u - R_0^T u_0$ as $\sum_{i=1}^N R_i^T u_i$, and from (18) we can write

$$\mathcal{A}(u - R_0^T u_0, u - R_0^T u_0) = \sum_{i=1}^N \mathcal{A}_i(u_i, u_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^N \mathcal{A}(R_i^T u_i, R_j^T u_j).$$

Adding $\mathcal{A}_0(u_0, u_0)$ $(\equiv \mathcal{A}(R_0^T u_0, R_0^T u_0))$ to both sides, we have

$$\sum_{i=0}^N \mathcal{A}_i(u_i, u_i) = \mathcal{A}(u - R_0^T u_0, u - R_0^T u_0)$$
$$+ \mathcal{A}(R_0^T u_0, R_0^T u_0) - \sum_{\substack{i,j=1 \\ i \neq j}}^N \mathcal{A}(R_i^T u_i, R_j^T u_j),$$

and

$$\left|\sum_{i=0}^{N} \mathcal{A}_i(u_i, u_i)\right| \leq \left|\mathcal{A}(u - R_0^T u_0, u - R_0^T u_0)\right| + \left|\mathcal{A}(R_0^T u_0, R_0^T u_0)\right| + \left|\sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathcal{A}(R_i^T u_i, R_j^T u_j)\right|.$$

The first term on the right hand side can be bounded by

$$\left|\mathcal{A}(u - R_0^T u_0, u - R_0^T u_0)\right| \lesssim |\mathcal{A}(u, u)| + \left|\mathcal{A}(R_0^T u_0, R_0^T u_0)\right|.$$

Thereby,

$$\left|\sum_{i=0}^{N} \mathcal{A}_i(u_i, u_i)\right| \lesssim |\mathcal{A}(u, u)| + \left|\mathcal{A}(R_0^T u_0, R_0^T u_0)\right| + \left|\sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathcal{A}(R_i^T u_i, R_j^T u_j)\right|. \tag{19}$$

Next, we will show that the second term on the right hand side of (19) can be bounded by

$$\left|\mathcal{A}(R_0^T u_0, R_0^T u_0)\right| \lesssim \|u\|_{\mathrm{DG}}^2 + \sum_{D\in\mathcal{T}_H} \eta_{\mathcal{D}} \|u - R_0^T u_0\|_{0,\partial\mathcal{D}}^2, \tag{20}$$

where, for the sake of simplicity, we have set

$$\eta_{\mathcal{D}} =: \alpha \, \frac{\max_{\substack{\mathcal{K}\in\mathcal{T}_h\\ \mathcal{K}\subset\mathcal{D}}} p_{\mathcal{K}}^2}{\min_{\substack{\mathcal{K}\in\mathcal{T}_h\\ \mathcal{K}\subset\mathcal{D}}} h_{\mathcal{K}}} \quad \forall \mathcal{D} \in \mathcal{T}_H.$$

Indeed, by using the continuity of the bilinear form (10), recalling that $R_0^T u_0$ is piecewise constant on each element $\mathcal{D}$ of the coarse mesh, and adding and subtracting u we have

$$\begin{aligned}\left|\mathcal{A}(R_0^T u_0, R_0^T u_0)\right| &\lesssim \sum_{F\in\mathcal{F}_h} \|\mathtt{a}^{1/2} [\![R_0^T u_0]\!]\|_{0,F}^2 \\ &\lesssim \sum_{F\in\mathcal{F}_h} \|\mathtt{a}^{1/2} [\![u - R_0^T u_0]\!]\|_{0,F}^2 + \|u\|_{\mathrm{DG}}^2.\end{aligned} \tag{21}$$

For the first term in the sum on the right hand side, we can write

$$\begin{aligned}\sum_{F\in\mathcal{F}_h} \|\mathsf{a}^{1/2}\, [\![u - R_0^T u_0]\!]\|_{0,F}^2 &= \sum_{\mathcal{D}\in\mathcal{T}_H} \sum_{\substack{F\in\mathcal{F}_h \\ F\subset\mathcal{D}}} \|\mathsf{a}^{1/2}\ [\![u]\!]\ \|_{0,F}^2 \\ &\quad + \sum_{\mathcal{D}\in\mathcal{T}_H} \sum_{\substack{F\in\mathcal{F}_h \\ F\subset\partial\mathcal{D}}} \|\mathsf{a}^{1/2}\, [\![u - R_0^T u_0]\!]\|_{0,F}^2 \\ &\leq \|u\|_{\mathrm{DG}}^2 + \sum_{\mathcal{D}\in\mathcal{T}_H} \sum_{\substack{F\in\mathcal{F}_h \\ F\subset\partial\mathcal{D}}} \|\mathsf{a}^{1/2}\, [\![u - R_0^T u_0]\!]\|_{0,F}^2 \\ &\lesssim \|u\|_{\mathrm{DG}}^2 + \sum_{\mathcal{D}\in\mathcal{T}_H} \eta_{\mathcal{D}} \|u - R_0^T u_0\|_{0,\partial\mathcal{D}}^2.\end{aligned}$$

Combining the above estimate together with (21) gives (20).
For the third term on the right hand side of (19), we first observe that, due to the hypothesis of nested partitions, each subdomain Ω_i is the union of some elements $\mathcal{D} \in \mathcal{T}_H$ and therefore

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \sum_{F\in\Gamma_{ij}} \left(\|\mathsf{a}^{1/2} u_i\|_{0,F}^2 + \|\mathsf{a}^{1/2} u_j\|_{0,F}^2 \right) \lesssim \sum_{\mathcal{D}\in\mathcal{T}_H} \eta_{\mathcal{D}} \|u\|_{0,\partial\mathcal{D}}^2.$$

Employing Lemma 4.2 gives

$$\left| \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \mathcal{A}(R_i^T u_i, R_j^T u_j) \right| \lesssim \|u - R_0^T u_0\|_{\mathrm{DG}}^2 + \sum_{\mathcal{D}\in\mathcal{T}_H} \eta_{\mathcal{D}} \|u - R_0^T u_0\|_{0,\partial\mathcal{D}}^2.$$

Therefore, collecting all of the above estimates and exploiting the Cauchy-Schwarz inequality, we get

$$\left| \sum_{i=0}^{N} \mathcal{A}_i(u_i, u_i) \right| \lesssim \|u\|_{\mathrm{DG}}^2 + \sum_{\mathcal{D}\in\mathcal{T}_H} \eta_{\mathcal{D}} \|u - R_0^T u_0\|_{0,\partial\mathcal{D}}^2.$$

Next, we make use of the trace inequality shown in [24] valid for piecewise H^1 functions:

$$\|u\|_{0,\partial\mathcal{D}}^2 \leq C \left[H_{\mathcal{D}}^{-1} \|u\|_{0,\mathcal{D}}^2 + H_{\mathcal{D}} \left(\sum_{\substack{\mathcal{K}\in\mathcal{T}_h \\ \mathcal{K}\subset\mathcal{D}}} |u|_{1,\mathcal{K}}^2 + \sum_{\substack{F\in\mathcal{F}_h \\ F\subset\mathcal{D}}} h_F^{-1} \|\ [\![u]\!]\ \|_{0,F}^2 \right) \right].$$

Applying the above trace inequality and using the fact that $R_0^T u_0$ is constant on each $\mathcal{D} \in \mathcal{T}_H$, gives

$$\|u - R_0^T u_0\|_{0,\partial\mathcal{D}}^2 \lesssim H_{\mathcal{D}}^{-1} \|u - R_0^T u_0\|_{0,\mathcal{D}}^2 + H_{\mathcal{D}} \left(\sum_{\substack{\mathcal{K}\in\mathcal{T}_h \\ \mathcal{K}\subset\mathcal{D}}} |u|_{1,\mathcal{K}}^2 + \sum_{\substack{F\in\mathcal{F}_h \\ F\subset\mathcal{D}}} h_F^{-1} \| [\![u]\!] \|_{0,F}^2 \right).$$

Finally, exploiting the Poincaré inequality (2.5) yields:

$$\left| \sum_{i=0}^{N} \mathcal{A}_i(u_i, u_i) \right| \lesssim \|u\|_{\mathrm{DG}}^2 + \sum_{\mathcal{D}\in\mathcal{T}_H} \eta_{\mathcal{D}} H_{\mathcal{D}} \left(\sum_{\substack{\mathcal{K}\in\mathcal{T}_h \\ \mathcal{K}\subset\mathcal{D}}} |u|_{1,\mathcal{K}}^2 + \sum_{\substack{F\in\mathcal{F}_h \\ F\subset\mathcal{D}}} h_F^{-1} \| [\![u]\!] \|_{0,F}^2 \right).$$

The proof is then completed by employing the coercivity of the bilinear form $\mathcal{A}(\cdot, \cdot)$. ■

Remark 4.4 Whenever the coarse and fine meshes are quasi uniform and the polynomial distribution is quasi uniform, the estimate in Proposition 4.3 reduces to

$$\sum_{i=0}^{N} \mathcal{A}_i(u_i, u_i) \leq C_0^2 \, \mathcal{A}(u, u), \qquad C_0^2 = \alpha \, p^2 \frac{H}{h}.$$

The above estimate combined with Theorem 4.1 implies that

$$\kappa(P_{\mathrm{ad}}) \lesssim \alpha \, p^2 \frac{H}{h} \quad \mathcal{A}(E_{\mathrm{mu}}, E_{\mathrm{mu}}) \leq 1 - \frac{h}{p^2 H} \quad \kappa(P_{\mathrm{mu}}^{\mathrm{S}}) \lesssim \alpha \, p^2 \frac{H}{h}. \qquad (22)$$

Remark 4.5 By standard arguments, from the estimates (22) it can be proved that the additive and the symmetrized multiplicative Schwarz methods can be accelerated with the conjugate gradient (CG) iterative solver, and that the CG iteration counts are expected to behaves as $O(p)$. Analogously, the multiplicative Schwarz method can indeed be accelerated with the generalized minimal residual method (GMRES). We refer to [10, 44] for more details.

5 Numerical Results

In this section we present a series of numerical experiments to highlight the practical performance of the non-overlapping Schwarz preconditioners proposed in this article. For simplicity, we restrict ourselves to two-dimensional model problems; additionally, we note that throughout this section we select the constant appearing in the interior penalty stabilization function defined in (6) as follows: $\alpha = 10$, cf. [33], for example. We let $\Omega = (0,1) \times (0,1)$ and choose f such that the analytical solution of the model problem (1) is given by $u(x,y) = \exp(xy)(x - x^2)(y - y^2)$.

5.1 Unpreconditioned System

Firstly, we investigate the asymptotic behavior of the condition number of the unpreconditioned system, based on employing the SIP method, on a sequence of successively finer conforming structured and unstructured triangular meshes, as well as on Cartesian grids for different values of the polynomial degree p ($p_{\mathcal{K}} \equiv p$ for all $\mathcal{K} \in \mathcal{T}_h$). Based on the estimate (16) of Corollary 2.9, we expect the condition number to behave as $O(p^4 h^{-2})$. The first two levels of the structured and unstructured triangular meshes, as well as the Cartesian grids are show in Figure 2. The initial mesh sizes are denoted by h_0 and $h_0/2$; at each further step of refinement we have considered a uniform refinement of the grid at the previous level. In Table 1 we

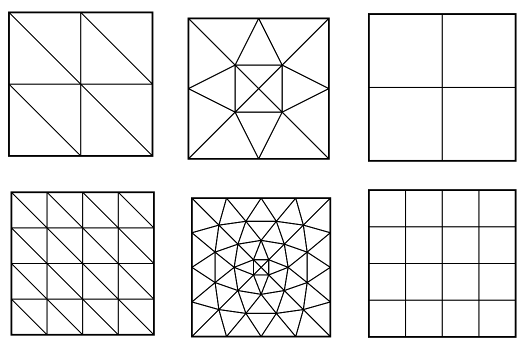

Figure 2: First two levels of the structured and unstructured triangular meshes, as well as the Cartesian grids. The corresponding mesh sizes are denoted by h_0 and $h_0/2$, respectively.

report the condition number estimates for different approximation orders and different mesh sizes on the structured triangular grids. The computed convergence rates corresponding to the last two approximation orders (last two meshes, respectively) are reported in the last column (last row, respectively). We have repeated the same set of experiments on unstructured triangular meshes as well as for the Cartesian grids: the numerical results are reported in Tables 2 & 3, respectively.

	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h–rate
$p = 1$	3.1057e+01	1.5560e+02	6.7934e+02	2.7884e+03	2.0372
$p = 2$	2.8468e+02	1.2739e+03	5.2406e+03	2.1116e+04	2.0105
$p = 3$	1.0499e+03	4.4065e+03	1.7957e+04	7.2218e+04	2.0078
$p = 4$	2.7547e+03	1.1623e+04	4.7367e+04	1.9046e+05	2.0075
$p = 5$	5.9483e+03	2.4477e+04	9.9156e+04	3.9815e+05	2.0055
$p = 6$	1.1344e+04	4.6964e+04	1.9042e+05	7.6462e+05	2.0056
$p = 7$	1.9753e+04	8.0488e+04	3.2484e+05	1.3030e+06	2.0040
$p = 8$	3.2149e+04	1.3164e+05	5.3189e+05	2.1337e+06	2.0042
$p = 9$	4.9612e+04	2.0102e+05	8.0934e+05	-	2.0094
$p = 10$	7.3378e+04	2.9848e+05	1.2031e+06	-	2.0111
p–rate	3.7148	3.7518	3.7626	3.69381	

Table 1: hp–Condition number estimates: Structured triangular grids.

	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h–rate
$p = 1$	1.1683e+02	5.1489e+02	2.1259e+03	8.5814e+03	2.0131
$p = 2$	9.4725e+02	3.9773e+03	1.6092e+04	6.4565e+04	2.0044
$p = 3$	3.3085e+03	1.3637e+04	5.4962e+04	2.2031e+05	2.0030
$p = 4$	8.6531e+03	3.5891e+04	1.4480e+05	5.8055e+05	2.0033
$p = 5$	1.8328e+04	7.5181e+04	3.0256e+05	1.2123e+06	2.0024
$p = 6$	3.4972e+04	1.4420e+05	5.8081e+05	2.3275e+06	2.0026
$p = 7$	6.0207e+04	2.4613e+05	9.8948e+05	3.9633e+06	2.0020
$p = 8$	9.8066e+04	4.0267e+05	1.6200e+06	6.4899e+06	2.0022
$p = 9$	1.5029e+05	6.1302e+05	2.4626e+06	-	2.0062
$p = 10$	2.2243e+05	9.1069e+05	3.6609e+06	-	2.0072
p–rate	3.72077	3.75661	3.76297	3.69327	

Table 2: hp–Condition number estimates: Unstructured triangular grids.

	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h–rate
$p=1$	2.6159e+01	7.1657e+01	2.6520e+02	1.0431e+03	1.9752
$p=2$	1.4744e+02	4.3077e+02	1.5766e+03	6.1675e+03	1.9678
$p=3$	4.2025e+02	1.0351e+03	3.6711e+03	1.4252e+04	1.9569
$p=4$	9.2677e+02	2.3698e+03	8.5099e+03	3.3160e+04	1.9622
$p=5$	1.6041e+03	3.8459e+03	1.3559e+04	5.2544e+04	1.9544
$p=6$	2.7011e+03	6.7179e+03	2.3960e+04	9.3171e+04	1.9593
$p=7$	3.9862e+03	9.4211e+03	3.3079e+04	1.2805e+05	1.9527
$p=8$	5.8827e+03	1.4396e+04	5.1092e+04	2.5297e+05	2.3078
$p=9$	7.9843e+03	1.8681e+04	6.5402e+04	2.5297e+05	1.9516
$p=10$	1.0891e+04	2.6327e+04	9.3083e+04	3.6104e+05	1.9556
p–rate	2.9467	3.2560	3.3497	3.3760	

Table 3: hp–Condition number estimates: Cartesian grids.

In Figure 3 we plot the computed minimum and maximum eigenvalue of $\boldsymbol{A}$ as a function of the polynomial approximation degree p for different mesh sizes: as expected the minimum eigenvalue is uniformly bounded from below by a constant, whereas $\lambda_{\max}$ grows asymptotically as p^4. Results reported in Figure 3 (top and middle) have been carried out on structured and unstructured triangular meshes, whereas the analogous ones obtained on Cartesian grids are shown in the bottom row of Figure 3.

5.2 Preconditioned System

We now investigate the performance of our preconditioners while varying h, H, and the polynomial approximation degree. For the sake of brevity we focus on the additive Schwarz version of the proposed preconditioner. We employ a uniform subdomain partition of $\Omega = (0,1)^2$ consisting of 16 squares, and consider initial coarse and fine refinements as depicted in Figure 4. We denote by H_0 and h_0 the corresponding initial coarse and fine mesh sizes, respectively, and consider $n = 1, 2, 3$, successive uniform refinements of the initial grids. In the first set of experiments, we consider a piecewise constant coarse solver, *i.e.*, $\mathcal{V}_0(\mathcal{T}_H, \mathbf{0})$: this is indeed the cheapest possible choice; numerical results obtained with a piecewise linear discontinuous coarse solver, *i.e.*, $\mathcal{V}_0(\mathcal{T}_H, \mathbf{1})$ are presented at the end of this section. The linear systems of equations have been solved by a CG iterative solver with a (relative) tolerance set equal to 10^{-9} allowing a maximum of 1000 (6000, respectively) iterations for the preconditioned (unpreconditioned, respectively) systems.

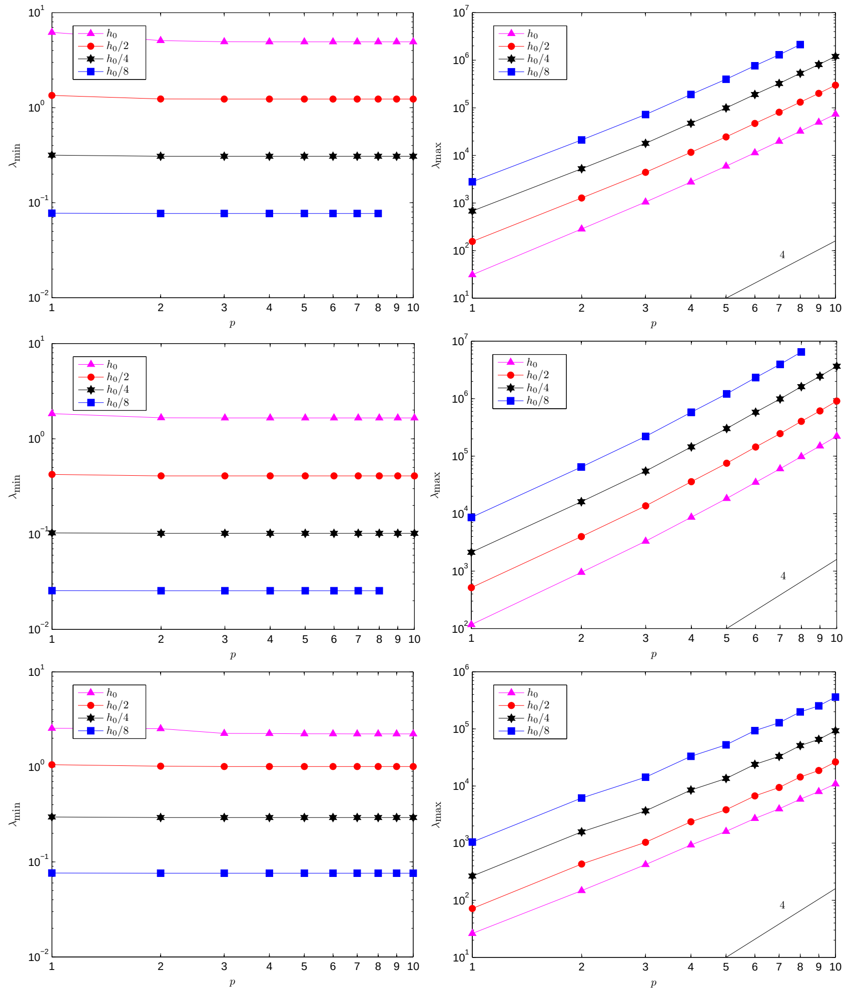

Figure 3: $\lambda_{\min}(\boldsymbol{A})$ and $\lambda_{\max}(\boldsymbol{A})$ versus the polynomial degree p for different mesh sizes: Structured triangular grids (top), unstructured triangular grids (middle) and Cartesian grids (bottom).

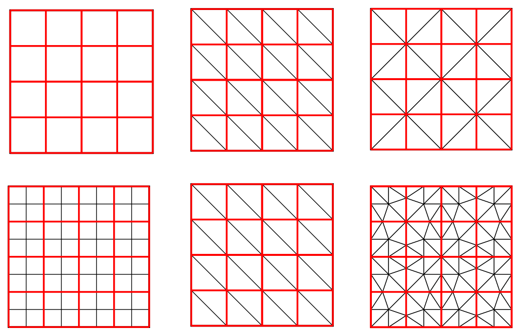

Figure 4: Initial coarse (top) and fine (bottom) refinements on Cartesian grids, structured and unstructured triangular grids, respectively, on a 16 subdomain partition.

The first set of experiments has been carried out by choosing the fine and coarse meshes depicted in Figure 4, *i.e.*, $h = h_0$, and $H = H_0$. The condition number estimates and the corresponding preconditioned conjugate gradient (PCG) iteration counts for different polynomial approximation degrees are reported in Tables 4, 5 & 6. Results in Table 4 refers to structured triangular meshes; the analogous results obtained on unstructured triangular grids and Cartesian meshes are shown in Tables 5 & 6, respectively. The results of this set of experiments are summarized in Figure 5(a). For all the cases considered it is clear that the estimates provided in (22) are sharp: the condition number of the preconditioned system behaves as p^2. Analogously, we clearly observe that the PCG iteration counts increase linearly as a function of p: this is indeed in agreement with our theoretical estimates. Moreover, comparing the condition number estimates (and the corresponding iteration counts) of the preconditioned systems with the unpreconditioned ones for a fixed p, it is also clear that the proposed preconditioner is very efficient. Indeed, we can observe a reduction of the condition number of about four orders of magnitude.

We have run the same set of experiments considering one more step of refinement for the fine mesh, *i.e.*, $h = h_0/2$, and considering the following refinement levels of the coarse mesh: $H = H_0$ and $H = H_0/2$. The condition number estimates and the PCG iteration counts of the preconditioned system for the case $H = H_0$ are shown in Figure 5(b); the analogous results obtained with $H = H_0/2$ are reported in Figure 5(c). From the results it is clear that the condition number of the preconditioned system grows as $O(p^2)$, whereas the iteration counts grow as $O(p)$. These results are indeed

	$\kappa(\mathbf{B}_{\rm ad}\mathbf{A})$	$\mathrm{it}(\mathbf{B}_{\rm ad}\mathbf{A})$	$\kappa(\mathbf{A})$	$\mathrm{it}(\mathbf{A})$
$p = 1$	5.4641e+01	59	1.3641e+05	128
$p = 2$	2.2635e+02	118	1.0083e+06	348
$p = 3$	5.1228e+02	171	3.4851e+06	647
$p = 4$	9.1275e+02	218	8.9357e+06	1036
$p = 5$	1.4277e+03	270	1.9008e+07	1511
$p = 6$	2.0573e+03	313	3.6401e+07	2091
$p = 7$	2.8013e+03	364	6.3420e+07	2760
$p = 8$	3.6599e+03	413	1.0281e+08	3514
$p = 9$	4.6330e+03	465	1.5885e+08	4368
$p = 10$	5.7205e+03	512	2.3678e+08	5333
p–rate	2.0014	0.9139	3.7891	1.8945

Table 4: hp–Condition number estimates and CG iteration counts: $h = h_0$, $H = H_0$, structured triangular grids.

	$\kappa(\mathbf{B}_{\rm ad}\mathbf{A})$	$\mathrm{it}(\mathbf{B}_{\rm ad}\mathbf{A})$	$\kappa(\mathbf{A})$	$\mathrm{it}(\mathbf{A})$
$p = 1$	5.5062e+01	58	1.1790e+05	119
$p = 2$	2.2660e+02	120	1.0025e+06	347
$p = 3$	5.1249e+02	176	3.4529e+06	644
$p = 4$	9.1297e+02	221	8.9012e+06	1034
$p = 5$	1.4280e+03	276	1.9159e+07	1517
$p = 6$	2.0575e+03	318	3.6123e+07	2083
$p = 7$	2.8016e+03	366	6.3053e+07	2752
$p = 8$	3.6601e+03	421	1.0210e+08	3502
$p = 9$	4.6332e+03	477	1.5703e+08	4343
$p = 10$	5.7208e+03	516	2.3652e+08	5330
p–rate	2.0013	0.7459	3.8873	1.9437

Table 5: hp–Condition number estimates and CG iteration counts: $h = h_0$, $H = H_0$, unstructured triangular grids.

	$\kappa(\mathbf{B}_{\mathrm{ad}}\mathbf{A})$	$\mathrm{it}(\mathbf{B}_{\mathrm{ad}}\mathbf{A})$	$\kappa(\mathbf{A})$	$\mathrm{it}(\mathbf{A})$
$p=1$	6.7594e+01	54	4.9362e+04	77
$p=2$	2.7433e+02	101	2.8803e+05	186
$p=3$	6.2000e+02	143	7.4431e+05	299
$p=4$	1.1043e+03	177	1.6561e+06	446
$p=5$	1.7272e+03	213	2.8297e+06	583
$p=6$	2.4886e+03	252	4.8341e+06	762
$p=7$	3.3886e+03	276	7.0467e+06	920
$p=8$	4.4272e+03	311	1.0556e+07	1126
$p=9$	5.6043e+03	351	1.3984e+07	1296
$p=10$	6.9199e+03	377	1.9337e+07	1524
p–rate	2.0014	0.6782	3.0762	1.5381

Table 6: *hp*–Condition number estimates and CG iteration counts: $h = h_0$, $H = H_0$, Cartesian grids.

in agreement with our theoretical estimates.

Finally, we report some numerical results carried out with a piecewise linear discontinuous coarse solver, *i.e.*, $\mathcal{V}_0(\mathcal{T}_H, \mathbf{1})$. For the sake of brevity we focus here on partitions made of Cartesian grids. In Figure 6(left) we compare the condition number estimates obtained with a piecewise constant and a piecewise linear coarse solver for the discretization steps: $h = h_0$, $H = H_0$ (top) and $h = h_0/2$, $H = H_0$ (bottom). The corresponding ratio between the condition number estimates and the PCG iteration counts for the two different choices of coarse spaces, namely a piecewise constant and a piecewise linear coarse solver are shown in Figure 6(right). A summary of the results in the case when a piecewise linear discontinuous coarse solver is employed is presented in Table 7. From the numerical computations it can be inferred that augmenting the coarse space from piecewise constants to piecewise linear elements decrease (consistently) the condition number by about a factor of 4.5, and the iteration counts by about a factor of 1.6. Such an improvement is not predicted by our theory. The dependence of the condition number of the preconditioned system on the polynomial approximation degree of the coarse solver will be the subject of further research.

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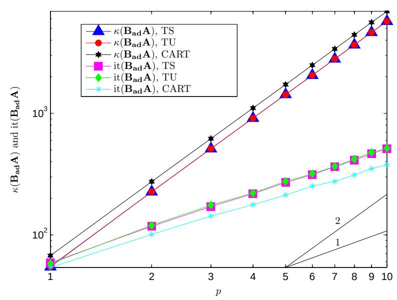
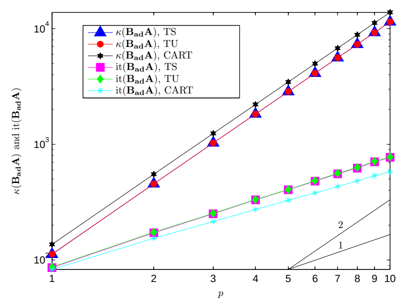
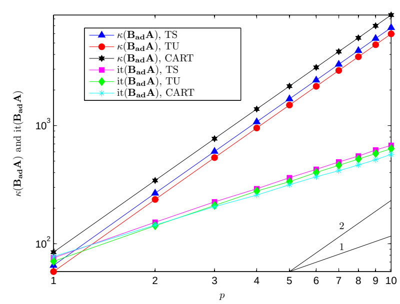


(a) $h = h_0$, $H = H_0$

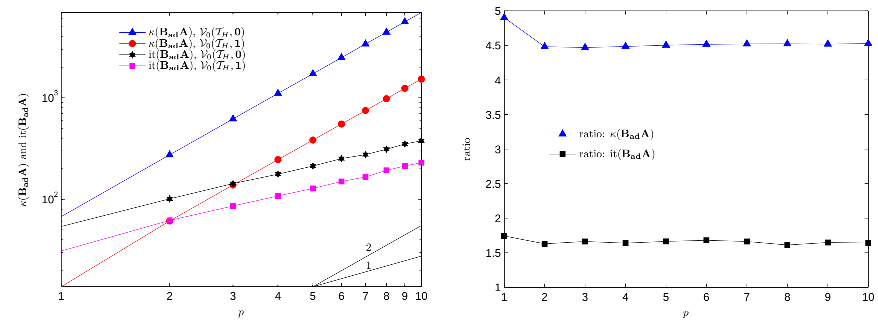


(b) $h = h_0/2$, $H = H_0$

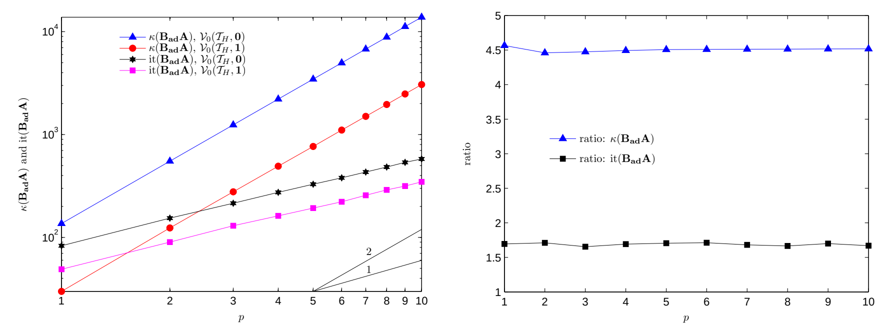


(c) $h = h_0/2$, $H = H_0/2$

Figure 5: Condition number estimates of the preconditioned system and corresponding iteration counts versus the polynomial degree p on structured triangular grids (TS), unstructured triangular grids (TU) and Cartesian grids (CART) for different discretization steps.



(a) $h = h_0$, $H = H_0$



(b) $h = h_0/2$, $H = H_0$

Figure 6: Right: Condition number estimates and iteration counts of the preconditioned system versus the polynomial degree p for two different choices of coarse spaces: $\mathcal{V}_0(\mathcal{T}_H, \mathbf{0})$ and $\mathcal{V}_0(\mathcal{T}_H, \mathbf{1})$. Left: The ratio between the computed quantities.

	$h = h_0,\ H = H_0$		$h = h_0/2,\ H = H_0$	
	$\kappa(\mathbf{B}_{\text{ad}}\mathbf{A})$	$\text{it}(\mathbf{B}_{\text{ad}}\mathbf{A})$	$\kappa(\mathbf{BA})$	$\text{it}(\mathbf{BA})$
$p = 1$	1.3790e+01	31	2.9894e+01	49
$p = 2$	6.1214e+01	62	1.2356e+02	90
$p = 3$	1.3870e+02	86	2.7773e+02	130
$p = 4$	2.4627e+02	108	4.9219e+02	162
$p = 5$	3.8356e+02	128	7.6717e+02	193
$p = 6$	5.5112e+02	150	1.1046e+03	222
$p = 7$	7.4932e+02	166	1.5031e+03	257
$p = 8$	9.7894e+02	193	1.9628e+03	290
$p = 9$	1.2400e+03	213	2.4825e+03	316
$p = 10$	1.5284e+03	230	3.0649e+03	347
p–rate	1.9844	0.7288	2.0002	0.8882

Table 7: *hp*–Condition number estimates and CG iteration counts: $\mathcal{V}_0(\mathcal{T}_H, \mathbf{1})$ coarse space, Cartesian grids.

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