Behaviour of the extended modified Volterra lattice – reductions to generalised mKdV and NLS equations

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Abstract

We consider the first member of an extended modified Volterra lattice hierarchy. This system of equations is differential with respect to one independent variable and differential-delay with respect to a second independent variable. We use asymptotic analysis to consider the long wavelength limits of the system. By considering various magnitudes for the parameters involved, we derive reduced equations related to the modified Korteweg-de Vries and nonlinear Schrödinger equations.

Highlights:
• we analyse the behaviour of solutions of the extended modified Volterra lattice
• we derive PDEs which are asymptotic approximations of the lattice
• we find similarity solutions of these limiting PDEs
• we show that in certain cases the PDEs can be transformed to mKdV or NLS

Keywords: nonlinear dynamics, modified Volterra lattice, asymptotic behaviour, integrable systems.

1. Introduction

The nonlinear Schrödinger (NLS) and modified Korteweg-de Vries (mKdV) equations arise in mathematical models of many important processes, for example, the propagation of pulses in fibre-optic cables [1], electrical transmission lines [2]. However, in such derivations, many terms are neglected, and these terms can influence the many special properties which mKdV and NLS possess. Hence it is useful to consider perturbations of these equations, and investigate which terms allow special solutions of the NLS to persist. In this paper we consider one particular class of generalised problems which reduce to these archetypal systems.

Nonisospectral extensions of completely integrable systems have been known for quite some time, e.g., [3], this then expanding the number of known integrable equations. Here we present a modified version of a nonisospectral extension of the Volterra equation of the form considered in [4] (see also [5]). In this paper we consider various asymptotic reductions of this extended modified Volterra equation for small amplitude solutions, which reveal connections between this equation and generalisations of the modified Korteweg-de Vries equation (mKdV) and the nonlinear Schrödinger equations (NLS). Similar analyses of the extended Toda lattice [6] and extended Volterra system [4] in [7, 8] demonstrated connections between these extended systems and the Korteweg-de Vries equation (KdV). The modified Volterra system is potentially much more interesting, since it supports breather solutions as well as travelling waves. In previous works we have also found connections between breathers in FPU and the mKdV system [9].

1.1. The extended modified Volterra equation

The first nontrivial member of the extended modified Volterra hierarchy is

$$\begin{aligned} q_t &= \tfrac{1}{4}\beta_1(q^2-1)(E-E^{-1})\{(1+E)^{-1}(E-1)+q(1-E)^{-1}(1+E)q\}\frac{kq_x}{(1-q^2)} \\ &\quad -\tfrac{1}{4}\beta_1(q^2-1)(E-E^{-1})\{(\beta_0-\alpha)q-\beta_0 xq\}, \end{aligned} \tag{1.1}$$

for $q(x,t)$. This equation, which we present here for the first time, is a modified version of an extended Volterra equation of the form considered in [4]. Here, E is the shift operator so that

$$Ef(x)=f(x+1), \qquad \text{and} \quad E^{-1}f(x)=f(x-1). \tag{1.2}$$

The quantity in braces in the first line is to be understood as an operator, so the second term means q multiplied by everything that comes to the right of it; thus here, $(1+E)$ acts on all of $qkq_x/(1-q^2)$.

The standard modified Volterra equation can be obtained by setting $k=0$ and $\beta_0=0$. We make the additional simplifications of setting $\alpha=1$ and $\beta_1=-4$ in (1.1) to give the standard modified Volterra equation

$$q_t = (1-q^2)(E-E^{-1})q. \tag{1.3}$$

Other values for α and β_1 can be considered simply by rescaling the time variable. We now show how this equation is related to the modified Korteweg-de Vries equation (mKdV) and nonlinear Schrödinger equation (NLS) by considering various asymptotic reductions.

1.2. Reduction of the modified Volterra equation to mKdV

To consider small amplitude solutions which vary slowly in space and over time we introduce $\epsilon \ll 1$, together with the scalings

$$y=\epsilon x, \qquad \tau=\epsilon t, \qquad T=\epsilon^3 t, \qquad q(x,t)=\epsilon Q(y,\tau,T). \tag{1.4}$$

From (1.3) we obtain

$$\epsilon^3 Q_T+\epsilon Q_\tau=(1-\epsilon^2Q^2)(2\epsilon Q_y+\tfrac{1}{3}\epsilon^3 Q_{yyy}), \tag{1.5}$$

thus at leading order we have $Q_\tau=2Q_y$ which we solve by introducing the travelling wave variable $z=y+2\tau$, and replace $Q(y,\tau,T)$ by $Q(z,T)$. The equation for $Q(z,T)$ is then given by the $O(\epsilon^3)$ terms, which gives the mKdV equation

$$Q_T=\tfrac{1}{3}Q_{yyy}-2Q^2Q_y. \tag{1.6}$$

Through a rescaling of variables, this can be mapped onto the first of (1.7).

If one considers only real values for the dependent and independent variables, then there are two distinct types of mKdV equation, namely

$$u_t+6u^2u_x-u_{xxx}=0, \qquad u_t+6u^2u_x+u_{xxx}=0. \tag{1.7}$$

The latter of these has breather solutions

$$u=-2\frac{\partial}{\partial x}\tan^{-1}\left(\frac{l\sin(k(x+t(k^2-3l^2)-t_0))}{k\cosh(l(x+t(3k^2-l^2)-x_0))}\right), \tag{1.8}$$

where k,l,x_0,t_0 are arbitrary constants. This solution can be found quoted in many texts, for example, Drazin & Johnson, [10], and its properties have been studies by Lamb [11], Ablowitz & Segur [12], Fordy [13] and Hirota [14]. If complex solutions of these equations are considered, then there are mappings between the two, which involve complex rescalings (eg $x\mapsto ix$); however, such solutions have singularities.

1.3. Reduction of the modified Volterra equation to NLS

We assume $\epsilon \ll 1$, and define

$$y = \epsilon x, \qquad \tau = \epsilon t, \qquad T = \epsilon^2 t, \qquad \theta = px - \omega t. \tag{1.9}$$

and postulate that the solution of (1.3) has the form

$$q(x,t) = \epsilon \mathrm{e}^{i\theta} F + \epsilon^2 (G_0 + \mathrm{e}^{i\theta} G_1 + \mathrm{e}^{2i\theta} G_2) + \epsilon^3 (H_0 + H_1 \mathrm{e}^{i\theta} + H_2 \mathrm{e}^{2i\theta} + H_3 \mathrm{e}^{3i\theta} \ldots) + \ldots + c.c., \tag{1.10}$$

where F, G_j, H_j are all functions of y, τ, T and 'c.c.' stands for the complex conjugate of all preceding terms. Equating equal powers of ϵ and $\mathrm{e}^{ipx-i\omega t}$, we obtain

$$\omega = -2\sin(p), \tag{1.11}$$

from $O(\epsilon \mathrm{e}^{ipx-i\omega t})$. The equations at $O(\epsilon^2 \mathrm{e}^{2ipx-2i\omega t})$ lead to $G_2 = 0$ and $O(\epsilon^2 \mathrm{e}^0)$ yields the trivial equation $0 = 0$. At $O(\epsilon^2 \mathrm{e}^{ipx-i\omega t})$ we obtain $F_\tau = 2F_y \cos p$, which implies F is a travelling wave, hence we introduce $z = y - v\tau$ with

$$v = -2\cos(p), \tag{1.12}$$

and we now write $F(y, \tau, T) = F(z, T)$. Finally, at $O(\epsilon^3 \mathrm{e}^{ipx-i\omega t})$ we find the NLS equation

$$-iF_T = (F_{zz} - 2|F|^2 F)\sin(p), \tag{1.13}$$

which is known to be integrable [11, 10, 12, 13].

However, there are two forms of NLS, firstly, the focusing NLS

$$iF_T = F_{ZZ} + 2|F|^2 F, \tag{1.14}$$

which has bright breather solutions of the form

$$F = A\mathrm{e}^{ikZ - iT(A^2-k^2)} \mathrm{sech}\,(A(Z+2kT)), \tag{1.15}$$

and also the defocusing NLS

$$iF_T = F_{ZZ} - 2|F|^2 F, \tag{1.16}$$

which has dark breathers, or hole solitons of the form

$$F = A\mathrm{e}^{ikZ + iT(2A^2+k^2)} \tanh(A(Z+2kT)). \tag{1.17}$$

Later in this paper both forms occur; however, for now, we note that (1.13) is of the defocusing case (1.16), thus dark breathers (1.17) are relevant. To be specific, (1.13) has a solution of the form

$$F(z,T) = A\exp\left\{iPz - iT(P^2+2A^2)\sin(p)\right\} \tanh\left\{A\left[z - 2PT\sin(p)\right]\right\}, \tag{1.18}$$

where A, P are arbitrary parameters. In terms of the original variables, this gives the solution

$$q(x,t) \sim 2\epsilon A \tanh\left\{\epsilon A\left[x+2t\cos(p) - 2P\epsilon t\sin(p)\right]\right\} \cos\left\{\epsilon P(x+2t\cos(p)) - t(\epsilon^2 P^2 + 2\epsilon^2 A^2)\sin(p)\right\}, \tag{1.19}$$

which is a family of moving breather solutions, the parameters being the wavenumber, p, the amplitude combination ϵA, and a higher-order wavenumber correction, ϵP. Note that the combination ϵA fully determines the width and the amplitude of the envelope, and a much smaller, second-order correction to the frequency-dependence of the linear carrier wave.

2. Reductions to generalisations of mKdV

As in the above, we choose to set $\beta_1 = -4$; other values of β_1 can be considered by rescaling time. Similarly, we set $\alpha = 1$, and consider a variety of scales for the other parameters, k and β_0, which we initially take to be small, that is, $\beta_0 = \beta \ll 1$ and $k \ll 1$.

We seek small amplitude solutions, which vary slowly in space and time, having the form

$$q(x,t) = \epsilon Q(y, T_0, T_1, T_2, T_3), \quad y = \epsilon x, \quad T_j = \epsilon^j t, \quad (j = 0, 1, 2, 3, \ldots). \tag{2.1}$$

We write (1.1) as

$$\epsilon Q_{T_0} + \epsilon^2 Q_{T_1} + \epsilon^3 Q_{T_2} + \epsilon^4 Q_{T_3} = R_1 + R_2 + R_3, \tag{2.2}$$

where

$$R_1 = (1-q^2)(E-E^{-1})(1+E)^{-1}(E-1)\frac{kq_x}{1-q^2}, \tag{2.3}$$

$$R_2 = (1-q^2)(E-E^{-1})\left(q(1-E)^{-1}(1+E)\frac{kqq_x}{1-q^2}\right), \tag{2.4}$$

$$R_3 = (1-q^2)(E-E^{-1})(q-\beta q+\beta xq). \tag{2.5}$$

We perform Taylor expansions of all the difference terms, and noting that the term R_2 includes an inverse differential operator, which is equivalent to an integral

$$(1-E)^{-1}(1+E) \sim -2\left(1+\tfrac{1}{12}\partial_y^2\right)\epsilon^{-1}\partial_y^{-1}. \tag{2.6}$$

Since this acts on a differential, it can be simplified using $\partial_y^{-1}QQ_y = \frac{1}{2}Q^2$, $\partial_y^{-1}Q^3Q_y = \frac{1}{4}Q^4$. Hence we obtain the asymptotic approximations

$$R_1 \sim k\epsilon^4 Q_{yyy} + \tfrac{1}{12}k\epsilon^6 Q_{5y} + 2k\epsilon^6(Q_y^3 + 3QQ_yQ_{yy}), \tag{2.7}$$

$$R_2 \sim -6k\epsilon^4 Q^2Q_y + k\epsilon^6 Q^4 Q_y + \tfrac{1}{3}k\epsilon^6[4Q^2Q_{yyy} + 7Q_y^3 + 22QQ_yQ_{yy}], \tag{2.8}$$

$$\begin{aligned} R_3 \sim\ & 2\epsilon^2(1-\beta)Q_y + 2\epsilon\beta(yQ)_y + \tfrac{1}{3}\beta\epsilon^3(yQ)_{yyy} + \tfrac{1}{3}\epsilon^4(1-\beta)Q_{yyy} \\ & -2\epsilon^4(1-\beta)Q^2Q_y - 2\epsilon^3\beta Q^2(yQ)_y. \end{aligned} \tag{2.9}$$

We now consider various choices for the order of magnitude for each of k and β. If $\beta = \mathcal{O}(\epsilon^4)$ or higher then there is no contribution from such terms at $\mathcal{O}(\epsilon^4)$; similarly if $k = \mathcal{O}(\epsilon)$ or higher.

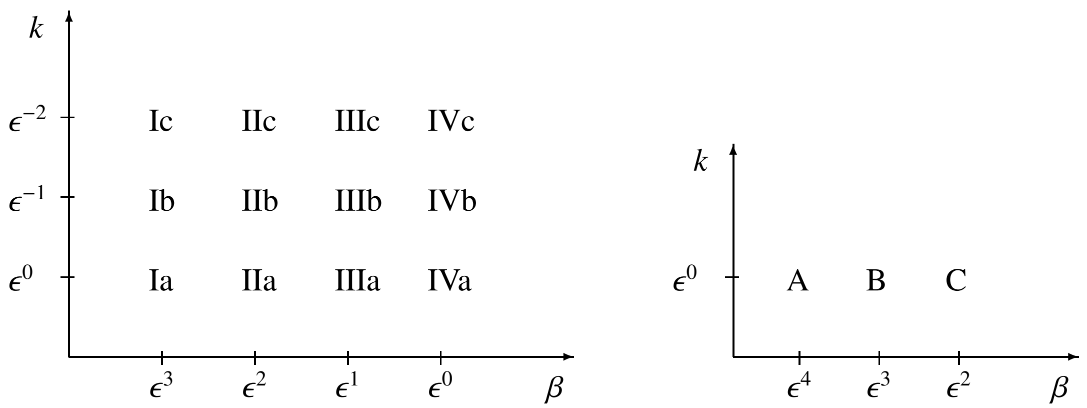

Figure 1: Illustration of the locations of the enumerated cases in parameter space. Left: the cases from which we derive generalised mKdV equations; right: the cases from which we find generalised NLS equations.

2.1. **Case Ia:** $\beta = O(\epsilon^3)$, $k = O(1)$

We start with the next simplest case after §1.2, which corresponds to the smallest possible values for β and k which make a difference to the standard modified Volterra reduction (1.2). In later subsections, we consider larger values of β and k, where their inclusion makes a more significant change to the equations.

We write $\beta = \epsilon^3 B$, with $B, k = K = O(1)$, then with the asymptotic scalings (2.1) in (1.1), there are no dynamics on the T_0 or T_2 timescales. The dominant terms are $O(\epsilon^2)$ with first corrections at $O(\epsilon^4)$, which are together given by

$$\epsilon^2 Q_{T_1} + \epsilon^4 Q_{T_3} = K\epsilon^4 Q_{yyy} - 6K\epsilon^4 Q^2 Q_y + 2\epsilon^2 Q_y + 2\epsilon^4 B(yQ)_y + \tfrac{1}{3}\epsilon^4 Q_{yyy} - 2\epsilon^4 Q^2 Q_y. \qquad (2.10)$$

Since the only $O(\epsilon)$ term is Q_{T_0}, this must be zero, and the solution can be assumed to exhibit no dynamics on that timescale. We transform to a moving coordinate frame $z = y + s(T_1)$, where the position $s(T_1)$ is to be determined, obtaining

$$\epsilon^{-2} Q_z \left(s'(T_1) - 2 + 2\epsilon^2 B s(T_1)\right) + Q_{T_3} = KQ_{zzz} - 6KQ^2 Q_z + 2B(zQ)_z + \tfrac{1}{3}Q_{zzz} - 2Q^2 Q_z. \qquad (2.11)$$

If we choose $s(T_1)$ to make the first bracket zero, via setting

$$\frac{\mathrm{d}s}{\mathrm{d}T_1} + 2B\epsilon^2 s = 2, \qquad (2.12)$$

so that

$$s(T_1) = \frac{1}{B\epsilon^2}\left(1 - \mathrm{e}^{-2B\epsilon^2 T_1}\right) + s_0 \mathrm{e}^{-2B\epsilon^2 T_1}. \qquad (2.13)$$

We are now left with a generalised mKdV equation for $Q(z, T_3)$, namely

$$Q_{T_3} = (K + \tfrac{1}{3})(Q_{zzz} - 6Q^2 Q_z) + 2B(zQ)_z. \qquad (2.14)$$

Following Calogero and Degasperis [15, 16], this can be transformed onto the mKdV equation $\widetilde{Q}_T = \widetilde{Q}_{\zeta\zeta\zeta} - 6\widetilde{Q}^2\widetilde{Q}_\zeta$ by the transformations

$$\zeta = z\mathrm{e}^{2BT_3}, \qquad Q(z, T_3) = \mathrm{e}^{2BT_3}\widetilde{Q}(\zeta, T), \qquad T = (K + \tfrac{1}{3})\frac{(\mathrm{e}^{6BT_3} - 1)}{6B}. \qquad (2.15)$$

In terms of the original variables, the solution is

$$q(x,t) = \epsilon\mathrm{e}^{2B\epsilon^3 t}\widetilde{Q}\left((\epsilon x\mathrm{e}^{2B\epsilon^3 t} + \frac{(\mathrm{e}^{2B\epsilon^3 t} - 1)}{B\epsilon^2} + s_0), \frac{(3K+1)}{18B}(\mathrm{e}^{6B\epsilon^3 t} - 1)\right). \qquad (2.16)$$

2.2. **Case Ib:** $\beta = O(\epsilon^3)$, $k = O(\epsilon^{-1})$

Writing $\beta = \epsilon^3 B$ and $k = K\epsilon^{-1}$, we obtain

$$\epsilon^2\left[Q_{T_1} - 2Q_y\right] + \epsilon^3 Q_{T_2} = \epsilon^3\left[KQ_{yyy} - 6KQ^2 Q_y\right] + O(\epsilon^4), \qquad (2.17)$$

again we have $Q_{T_0} = 0$ and so there is no dependence on T_0, there are dynamics on both the T_1 and T_2 timescales. On making the transformation to a moving coordinate frame $z = y + 2T_1$, the leading order terms cancel, leaving the terms at $O(\epsilon^3)$ as an mKdV equation for the function $Q(z, T_2)$. Thus, in this case the solutions have the form

$$q(x,t) = \epsilon Q(\epsilon(x + 2t), \epsilon^2 t), \qquad (2.18)$$

where $Q(z, T_2)$ solves the mKdV equation $Q_{T_2} = KQ_{zzz} - 6KQ^2 Q_z$.

2.3. **Case Ic:** $\beta = O(\epsilon^3)$**,** $k = O(\epsilon^{-2})$

Writing $\beta = \epsilon^3 B$ and $k = K\epsilon^{-2}$, we obtain at $O(\epsilon^2)$ the equation

$$Q_{T_1} = KQ_{yyy} - 6KQ^2 Q_y + 2Q_y, \tag{2.19}$$

which becomes an mKdV equation on writing $z = y + 2T_1$ and $Q = Q(z, T_1)$. Thus, the mKdV dynamics occur on a faster timescale than in Case Ib, as the solution has the form

$$q(x,t) = \epsilon Q(\epsilon(x + 2t), \epsilon t), \tag{2.20}$$

where $Q(z, T_1)$ again solves $Q_{T_1} = KQ_{zzz} - 6KQ^2 Q_z$.

2.4. **Case IIa:** $\beta = O(\epsilon^2)$**,** $k = O(1)$

We define $\beta = \epsilon^2 B$ with $k = K$ and,$B, K = O(1)$, then an expansion of (2.2)–(2.5) leads to

$$\begin{aligned} \epsilon^2 Q_{T_1} + \epsilon^3 Q_{T_2} + \epsilon^4 Q_{T_3} &= K\epsilon^4 Q_{yyy} - 6K\epsilon^4 Q^2 Q_y + 2\epsilon^2 Q_y - 2B\epsilon^4 Q_y + 2\epsilon^3 B(yQ)_y \\ &\quad + \tfrac{1}{3}\epsilon^4 Q_{yyy} - 2\epsilon^4 Q^2 Q_y. \end{aligned} \tag{2.21}$$

we once again transform to a moving coordinate frame via $z = y + s(T_1)$

$$\epsilon^{-2} Q_z \left[s'(T_1) - 2 + 2B\epsilon s(T_1) + 2B\epsilon^2\right] + \epsilon^{-1}\left[Q_{T_2} - 2B(zQ)_z\right] + Q_{T_3} = (K + \tfrac{1}{3})\left(Q_{zzz} - 6Q^2 Q_z\right). \tag{2.22}$$

Thus at leading order, $O(\epsilon^{-2})$ in the above, we obtain

$$s(T_1) = s_0 \mathrm{e}^{-2B\epsilon T_1} + \frac{(1 - B\epsilon^2)}{B\epsilon}(1 - \mathrm{e}^{-2B\epsilon T_1}). \tag{2.23}$$

At $O(\epsilon^{-1})$, we have $Q_{T_2} = 2B(zQ)_z$, which has solutions of the form $Q = \mathrm{e}^{2BT_2}\widetilde{Q}(z\mathrm{e}^{2BT_2})$.

If we now seek solutions of (2.22) at $O(1)$ of the form

$$Q(z, T_2, T_3) = \mathrm{e}^{2BT_2}\widetilde{Q}(\xi, T), \quad \text{where} \quad \xi = z\mathrm{e}^{2BT_2}, \quad \text{and} \quad T = T_3 \mathrm{e}^{6BT_2}, \tag{2.24}$$

we find

$$\widetilde{Q}_T = (K + \tfrac{1}{3})(\widetilde{Q}_{\xi\xi\xi} - 6\widetilde{Q}^2 \widetilde{Q}_\xi). \tag{2.25}$$

Thus this system can also exhibit mKdV dynamics, albeit only after the similarity rescaling (2.24), which results in

$$q(x,t) = \epsilon \mathrm{e}^{2B\epsilon^2 t}\widetilde{Q}\left((\epsilon x \mathrm{e}^{2B\epsilon^2 t} + s_0 + \left(\frac{1 - B\epsilon^2}{B\epsilon}\right)(\mathrm{e}^{2B\epsilon^2 t} - 1)), \epsilon^3 t \mathrm{e}^{6B\epsilon^2 t}\right). \tag{2.26}$$

2.5. **Case IIb:** $\beta = O(\epsilon^2)$**,** $k = O(\epsilon^{-1})$

This case is similar to Case I, we write $\beta = \epsilon^2 B$ and $k = \epsilon^{-1} K$ so that the first two terms in the expansion of (2.2)–(2.5) are

$$\epsilon^2 Q_{T_1} + \epsilon^3 Q_{T_2} = K\epsilon^3 Q_{yyy} - 6K\epsilon^3 Q^2 Q_y + 2\epsilon^2 Q_y + 2\epsilon^3 B(yQ)_y. \tag{2.27}$$

Writing $z = y + s(T_1)$ leads to

$$\frac{1}{\epsilon} Q_z \left(s'(T_1) - 2 + 2B\epsilon s\right) + Q_{T_2} = KQ_{zzz} - 6KQ^2 Q_z + 2B(zQ)_z. \tag{2.28}$$

If we set

$$s(T_1) = s_0 e^{-2B\epsilon T_1} + \frac{1}{B\epsilon}(1 - e^{-2B\epsilon T_1}), \tag{2.29}$$

then the bracketed term on the LHS of (2.28) is zero and we are left with

$$Q_{T_2} = KQ_{zzz} - 6KQ^2Q_z + 2B(zQ)_z, \tag{2.30}$$

which is a generalised mKdV equation of a form similar to (2.14) but on the T_2– rather than the even slower T_3–timescale. As noted by Calogero and Degasperis [15, 16], this equation can be transformed onto mKdV. We set

$$\zeta = ze^{2BT_2}, \qquad Q(z, T_2) = e^{2BT_2}Q(\zeta, T), \qquad T = \frac{e^{6BT_2} - 1}{6B}, \tag{2.31}$$

which maps the equation (2.30) onto the mKdV eq $\widetilde{Q}_T = K\widetilde{Q}_{\zeta\zeta\zeta} - 6K\widetilde{Q}^2\widetilde{Q}_\zeta$. In terms of the original variables, the solution has the form

$$q(x,t) = \epsilon e^{2B\epsilon^2 t}\widetilde{Q}\left(\epsilon x e^{2B\epsilon^2 t} + s_0 + \frac{(e^{2B\epsilon^2 t} - 1)}{B\epsilon}, \frac{(e^{6B\epsilon^2 t} - 1)}{6B}\right). \tag{2.32}$$

2.6. **Case IIc:** $\beta = O(\epsilon^2)$, $k = O(\epsilon^{-2})$

With $\beta = \epsilon^2 B$ and $k = K/\epsilon^2$ and $K, B = O(1)$ we have a slightly modified mKdV equation at $O(\epsilon^2)$, namely

$$Q_{T_1} = KQ_{yyy} - 6KQ^2Q_y + 2Q_y, \tag{2.33}$$

which gives dynamics on the T_1 timescale. This can be converted to the standard mKdV equation by the simple transformation $z = y + 2T_1$.

2.7. **Case IIIa:** $\beta = O(\epsilon^1)$, $k = O(1)$

For this case we define $\beta = \epsilon B$ and $k = K$ with $B, K = O(1)$, which in (2.2)–(2.5) yields

$$\begin{aligned} \epsilon^2 Q_{T_1} + \epsilon^3 Q_{T_2} + \epsilon^4 Q_{T_3} \;=\; & \epsilon^4(K + \tfrac{1}{3})(Q_{yyy} - 6Q^2Q_y) + 2\epsilon^2 Q_y - 2\epsilon^3 BQ_y + 2\epsilon^2 B(yQ)_y \\ & + \tfrac{1}{3}B\epsilon^4(yQ)_{yyy} - 2\epsilon^4 BQ^2(yQ)_y. \end{aligned} \tag{2.34}$$

This equation can be simplified by transforming to the moving coordinate frame $z = y + s(T_1)$, where $s(T_1)$ is chosen to satisfy $s'(T_1) + 2Bs(T_1) = 2(1 - \epsilon B)$, which implies

$$s(T_1) = s_0 e^{-2BT_1} + \frac{(1 - \epsilon B)}{B}(1 - e^{-2BT_1}). \tag{2.35}$$

This leaves the $O(\epsilon^2)$ terms $Q_{T_1} = 2B(zQ)_z$ in (2.34). Hence we make the further transformation

$$Q(z, T_1, T_2, T_3) = e^{2BT_1}\widetilde{Q}(\zeta, T_3), \qquad \zeta = ze^{2BT_1}. \tag{2.36}$$

The only term of $O(\epsilon^3)$ in (2.34) is then Q_{T_2}, hence we assume there is no T_2 dependence in the solution. At the next order we have the terms

$$e^{-6BT_1}\widetilde{Q}_{T_3} = \left(K + \tfrac{1}{3} - \tfrac{1}{3}Bs(T_1)\right)(\widetilde{Q}_{\zeta\zeta\zeta} - 6\widetilde{Q}^2\widetilde{Q}_\zeta) + \tfrac{1}{3}Be^{-2BT_1}\left((\zeta\widetilde{Q})_{\zeta\zeta\zeta} - 6\widetilde{Q}^2(\zeta\widetilde{Q})_\zeta\right). \tag{2.37}$$

Even if this could be transformed onto mKdV, the resulting solution would not be useful, since when converted to the original variables, $T_3 = O(1)$ implies $T_1 \gg 1$, and so some of the terms in (2.37) would be vanishingly small and lost from the balance.

2.8. **Case IIIb:** $\beta = O(\epsilon^1)$, $k = O(\epsilon^{-1})$

In this case we write $\beta = \epsilon B$, $k = \epsilon^{-1}K$ with $B, K = O(1)$, we obtain

$$\epsilon^2 Q_{T_1} + \epsilon^3 Q_{T_2} = K\epsilon^3 Q_{yyy} - 6K\epsilon^3 Q^2 Q_y + 2\epsilon^2 Q_y + 2\epsilon^2 B(yQ)_y - 2\epsilon^3 BQ_y + O(\epsilon^4) \tag{2.38}$$

with the transformation to a moving coordinate frame via $z = y + s(T_1)$, we obtain

$$Q_{T_1} - 2B(zQ)_z + Q_z\left[s'(T_1) - 2 + 2B\epsilon + 2Bs(T_1)\right] = \epsilon K(Q_{zzz} - 6Q^2 Q_z) - \epsilon Q_{T_2} + O(\epsilon^2). \tag{2.39}$$

To make the term in square brackets equal zero, we choose

$$s(T_1) = s_0 \mathrm{e}^{-2BT_1} + \tfrac{1}{B}(1 - \epsilon B)(1 - \mathrm{e}^{-2BT_1}), \tag{2.40}$$

which leaves the terms $Q_{T_1} = 2B(zQ)_z$ at leading order. This equation is solved by

$$Q(z, T_1, T_2) = \mathrm{e}^{2BT_1}\widetilde{Q}(\zeta, T_2), \qquad \zeta = z\mathrm{e}^{2BT_1}, \tag{2.41}$$

where $\widetilde{Q}$ satisfies

$$\epsilon^{-6BT_1}\widetilde{Q}_{T_2} = K(\widetilde{Q}_{\zeta\zeta\zeta} - 6\widetilde{Q}^2\widetilde{Q}_\zeta). \tag{2.42}$$

A further timescale can be defined by $T = T_2\mathrm{e}^{6BT_1}$, with $\widehat{Q}(\zeta, T) = \widetilde{Q}(\zeta, T_2)$ satisfying the mKdV equation $\widehat{Q}_T = K(\widehat{Q}_{\zeta\zeta\zeta} - \widehat{Q}^2\widehat{Q}_\zeta)$. In terms of the original variables, we have

$$q(x, t) = \epsilon\mathrm{e}^{2B\epsilon t}\widehat{Q}(\epsilon x\mathrm{e}^{2B\epsilon t} + s_0 + \tfrac{1}{B}(1 - \epsilon B)(\mathrm{e}^{2B\epsilon t} - 1), \epsilon^2 t\mathrm{e}^{6B\epsilon t}). \tag{2.43}$$

2.9. **Case IIIc:** $\beta = O(\epsilon^1)$, $k = O(\epsilon^{-2})$

In this case, we define $\beta = \epsilon B$, $k = \epsilon^{-2}K$, with $B, K = O(1)$. The large amplitude of the k terms bring the mKdV into the leading order balance at $O(\epsilon^2)$, and so the mKdV dynamics occur on the T_1 timescale, which is coincident with the timescale of the most significant of the β terms. Thus we have

$$Q_{T_1} = KQ_{yyy} - 6KQ^2 Q_y + 2Q_y + 2B(yQ)_y - 2\epsilon BQ_y + O(\epsilon^2). \tag{2.44}$$

Transforming to a moving coordinate frame $z = y + s(T_1)$, leads to

$$Q_{T_1} + Q_z\left[s'(T_1) - 2 + 2Bs(T_1) + 2B\epsilon\right] = KQ_{zzz} - 6KQ^2 Q_z + 2B(zQ)_z. \tag{2.45}$$

The translation coordinate $s(T_1)$ is determined by setting the term in square brackets to zero, and solving the resulting ODE which yields

$$s(T_1) = s_0\mathrm{e}^{-2BT_1} + \tfrac{1}{B}(1 - B\epsilon)(1 - \mathrm{e}^{-2BT_1}). \tag{2.46}$$

The remaining terms in (2.45) can then be transformed by

$$Q(z, T_1) = \mathrm{e}^{2BT_1}\widetilde{Q}(\zeta, T), \qquad \zeta = z\mathrm{e}^{2BT_1}, \qquad T = \frac{\mathrm{e}^{6BT_1} - 1}{6B}, \tag{2.47}$$

whereupon $\widetilde{Q}$ satisfies the mKdV equation $\widetilde{Q}_T = K\widetilde{Q}_{\zeta\zeta\zeta} - 6K\widetilde{Q}^2\widetilde{Q}_\zeta$. Thus this case exhibits mKdV dynamics, albeit on dynamically transformed space and timescales

$$q(x, t) = \epsilon\mathrm{e}^{2B\epsilon t}\widetilde{Q}\left(\epsilon x\mathrm{e}^{2B\epsilon t} + s_0 + \tfrac{1}{B}(1 - \epsilon B)(\mathrm{e}^{2B\epsilon t} - 1), \frac{\mathrm{e}^{6B\epsilon t} - 1}{6B}\right). \tag{2.48}$$

2.10. **Case IVa:** $\beta = O(1)$, $k = O(1)$

For consistency with earlier derivations, we write $\beta = B$ and $k = K$ where both are $O(1)$, then obtain

$$\begin{aligned}\epsilon\left[Q_{T_0} - 2B(yQ)_y\right] &= \epsilon^2\left[-Q_{T_1} + 2Q_y(1-B)\right] + \epsilon^3\left[-Q_{T_2} - 2BQ^2(yQ)_y + \tfrac{1}{3}B(yQ)_{yyy}\right] \\ &\quad + \epsilon^4\left[-Q_{T_3} + KQ_{yyy} - 6KQ^2Q_y + \tfrac{1}{3}(1-B)Q_{yyy} - 2(1-B)Q^2Q_y\right] \end{aligned} \tag{2.49}$$

so we see the mKdV terms entering at $O(\epsilon^4)$; however, there are many other effects which enter earlier in the expansion. The leading order terms imply that $Q(y, T_j)$ has the form of a similarity solution with

$$Q(y, T_j) = e^{2BT_0}\widetilde{Q}(\xi, T_1, T_2), \qquad \text{where} \quad \xi = ye^{2BT_0}, \tag{2.50}$$

and the shape, $\widetilde{Q}$, is undetermined. At $O(\epsilon^2)$ we find that on the slower (T_1) timescale, $\widetilde{Q}$ is a travelling wave, with the form

$$\widetilde{Q}(\xi, T_1, T_2) = \widehat{Q}(\zeta, T_2), \qquad \text{where} \;\; s = 2(1-B)e^{2BT_0}T_1, \;\; \zeta = \xi + s = e^{2BT_0}(y + 2(1-B)T_1), \tag{2.51}$$

and the shape, $\widehat{Q}$, is still undetermined. Finally, the $O(\epsilon^3)$ terms yield a nonlinear equation for the shape

$$e^{-4BT_0}\widehat{Q}_{T_2} + 2B\widehat{Q}^2(\zeta\widehat{Q})_\zeta - 2Bs\widehat{Q}^2\widehat{Q}_\zeta = \tfrac{1}{3}B(\zeta\widehat{Q})_{\zeta\zeta\zeta} - \tfrac{1}{3}Bs\widehat{Q}_{\zeta\zeta\zeta}. \tag{2.52}$$

However, this results may not be of great physical relevance since taking $\zeta = O(1)$ implies $y + 2BT_1 \sim e^{-2BT_0}$, which is either asymptotically small or large, depending on whether $B > 0$ or $B < 0$.

2.11. **Case IVb:** $\beta = O(1)$, $k = O(\epsilon^{-1})$

We now write $\beta = B$ and $k = K\epsilon^{-1}$, whereupon (2.2)–(2.5) yield

$$\begin{aligned}&\epsilon\left[Q_{T_0} - 2B(yQ)_y\right] + \epsilon^2\left[Q_{T_1} - 2Q_y(1-B)\right] \\ &\quad = \epsilon^3\left[KQ_{yyy} - 6KQ^2Q_y - 2BQ^2(yQ)_y + \tfrac{1}{3}B(yQ)_{yyy} - Q_{T_2}\right]. \end{aligned} \tag{2.53}$$

As in Case IVa, the leading order terms imply the solution has the form (2.50). At the next order, we have a first-order travelling wave equation with solution (2.51). Finally, at the highest order in (2.53), we find a generalised modified KdV equation

$$\begin{aligned} e^{-6BT_0}\widetilde{Q}_{T_2} &= K\widetilde{Q}_{\zeta\zeta\zeta} - 6K\widetilde{Q}^2\widetilde{Q}_\zeta + \tfrac{1}{3}Be^{-2BT_0}(\zeta\widetilde{Q})_{\zeta\zeta\zeta} - 2Be^{-2BT_0}\widetilde{Q}^2(\zeta\widetilde{Q})_\zeta \\ &\quad + 2Bse^{-2BT_0}\widetilde{Q}^2\widetilde{Q}_\zeta - \tfrac{1}{3}Bse^{-2BT_0}\widetilde{Q}_{\zeta\zeta\zeta}. \end{aligned} \tag{2.54}$$

Due to the presence of terms such as e^{-2BT_0} in this equation, this result may not be of any physical relevance since any dynamics on the T_2 timescale will correspond to the limit $T_0 \to \infty$, leading to the terms in the above equation either becoming vanishingly small or asymptotically large.

2.12. **Case IVc:** $\beta = O(1)$, $k = O(\epsilon^{-2})$

With $\beta = B$ and $k = K\epsilon^{-2}$, we find

$$\epsilon\left[Q_{T_0} - 2B(yQ)_y\right] = \epsilon^2\left[-Q_{T_1} + 2Q_y(1-B) + KQ_{yyy} - 6KQ^2Q_y\right], \tag{2.55}$$

As in Cases IVa and IVb, the leading order terms imply the solution has the form (2.50). However, at the next order, we now have all the mKdV terms present and with the only extra term being the Q_y which is easily removed by the transformation to a travelling wave coordinate via

$$\zeta = \xi + 2(1-B)e^{2BT_0}T_1, \qquad T = Ke^{6BT_0}T_1, \qquad \widetilde{Q}(\xi, T_1) = \widehat{Q}(\zeta, T), \tag{2.56}$$

which gives $\widehat{Q}_T = \widehat{Q}_{\zeta\zeta\zeta} - 6\widehat{Q}^2\widehat{Q}_\zeta$. Hence the final solution is given by

$$q(x,t) \sim \epsilon e^{2Bt}\widehat{Q}\left(\epsilon e^{2Bt}(x + 2(1-B)t), Ke^{6Bt}\epsilon t\right). \tag{2.57}$$

3. Reductions to generalisations of NLS

As in Section 2, we take $\beta_1 = -4$ and $\alpha = 1$, rewrite β_0 as β which, alongside k, will be taken to have various magnitudes in the cases considered in subsections below. We write the full eq (1.1) as $q_t = R_1 + R_2 + R_3$ where, retaining only those terms of orders $O(\epsilon e^{i\theta})$, $O(\epsilon^2 e^{0i\theta})$, $O(\epsilon^2 e^{i\theta})$, $O(\epsilon^2 e^{2i\theta})$, and $O(\epsilon^3 e^{i\theta})$, the expression for q_t is

$$q_t = -i\omega\epsilon e^{i\theta} F - 2i\omega\epsilon^2 e^{2i\theta} G_2 + \epsilon^2 e^{i\theta}\left(F_{T_1} - i\omega G_1\right) + \epsilon^3 e^{i\theta}\left(F_{T_2} + G_{1,T_1} - i\omega H_1\right). \tag{3.1}$$

The differential-difference operators R_j are defined by

$$R_1 = (1-q^2)\mathcal{A}\frac{kq_x}{1-q^2}, \qquad \mathcal{A} = (E - E^{-1})(1+E)^{-1}(E-1), \tag{3.2}$$

$$R_2 = (1-q^2)(E - E^{-1})q(1-E)^{-1}(1+E)\frac{kqq_x}{1-q^2}, \tag{3.3}$$

$$R_3 = (1-q^2)(E - E^{-1})(q - \beta q + \beta x q). \tag{3.4}$$

We consider the effect of the expressions, R_j when the ansatz (1.10) is used. Since we will typically assume β to be $O(\epsilon)$ or smaller, we rewrite the explicit x in R_3 as y/ϵ.

First we first note the effect of the operators E and E^{-1} on arbitrary functions of the form $e^{ibx} f(y)$ where $y = \epsilon x$, that is

$$E e^{ibx} f(y) = e^{ibx+ib} f(y+\epsilon) \sim e^{ib} e^{ibx} f(y) + \epsilon e^{ib} e^{ibx} f'(y) + \tfrac{1}{2}\epsilon^2 e^{ib} e^{ibx} f''(y) + \ldots, \tag{3.5}$$

$$E^{-1} e^{ibx} f(y) = e^{ibx-ib} f(y-\epsilon) \sim e^{-ib} e^{ibx} f(y) - \epsilon e^{-ib} e^{ibx} f'(y) + \tfrac{1}{2}\epsilon^2 e^{-ib} e^{ibx} f''(y) + \ldots. \tag{3.6}$$

so that

$$\begin{aligned}(E - E^{-1}) e^{ibx} f(y) &= e^{ibx+ib} f(y+\epsilon) - e^{ibx-ib} f(y-\epsilon) \\ &\sim 2i\sin(b) e^{ibx} f(y) + 2\epsilon\cos(b) e^{ibx} f'(y) + i\sin(b)\epsilon^2 e^{ibx} f''(y) + \ldots.\end{aligned} \tag{3.7}$$

Next, we need to approximate the inverse operators $(1+E)^{-1}$ and $(1-E)^{-1}$; first we note that

$$(1+E) e^{ibx} f(y) = e^{ibx} f(y)(1 + e^{ib}) + \epsilon e^{ibx} e^{ib} f'(y) + \tfrac{1}{2}\epsilon^2 e^{ibx} f''(y), \tag{3.8}$$

$$(1-E) e^{ibx} f(y) = e^{ibx} f(y)(1 - e^{ib}) - \epsilon e^{ibx} e^{ib} f'(y) - \tfrac{1}{2}\epsilon^2 e^{ibx} f''(y). \tag{3.9}$$

Using the small asymptotic size of ϵ, we invert these expressions, obtaining

$$(1+E)^{-1} e^{ibx} f(y) = \frac{e^{ibx} f(y)}{(1+e^{ib})} - \frac{\epsilon e^{ib} e^{ibx} f'(y)}{(1+e^{ib})^2} + \frac{\epsilon^2 e^{ib}(e^{ib}-1) e^{ibx} f''(y)}{2(1+e^{ib})^3}, \tag{3.10}$$

$$(1-E)^{-1} e^{ibx} f(y) = \frac{e^{ibx} f(y)}{(1-e^{ib})} + \frac{\epsilon e^{ib} e^{ibx} f'(y)}{(1-e^{ib})^2} + \frac{\epsilon^2 e^{ib}(e^{ib}+1) e^{ibx} f''(y)}{2(1-e^{ib})^3}. \tag{3.11}$$

We split the term R_1 (3.2) into three components

$$R_1 = (1-q^2)\mathcal{A}kq_x/(1-q^2) \sim R_{10} + R_{11} - R_{12}, \tag{3.12}$$

$$\text{where} \quad R_{10} = k\mathcal{A}q_x, \qquad R_{11} = k\mathcal{A}q^2 q_x, \qquad R_{12} = kq^2\mathcal{A}q_x. \tag{3.13}$$

Since $q \sim \epsilon \ll 1$, we have $R_{11}, R_{12} \ll R_{10}$, and it will be sufficient to calculate the leading order terms of R_{11}, R_{12} and leading and first correction terms of R_{10}. From (3.2) the leading order effect of $\mathcal{A}$ is $\mathcal{A} e^{ibx} f(y) = -4\sin^2(\tfrac{1}{2}b) e^{ibx} f(y)$. For the higher order quantities R_{11} and R_{12} we have

$$R_{11} = -4\sin^2(\tfrac{1}{2}p) kip\epsilon^3 e^{i\theta}|F|^2 F, \qquad \text{and} \quad R_{12} = -4\sin^2(\tfrac{1}{2}p) kip\epsilon^3 e^{i\theta}|F|^2 F, \tag{3.14}$$

so $R_{11} = R_{12}$ and the only relevant contribution to R_1 is from R_{10}. Including first and second correction terms, the asymptotic expansion for $\mathcal{A}$ is given by

$$\mathcal{A}\mathrm{e}^{ipx} f(y) \sim -4\mathrm{e}^{ipx} f(y)\sin^2(\tfrac{1}{2}p) + 2i\epsilon\mathrm{e}^{ipx} f'(y)\sin(p) + \epsilon^2\mathrm{e}^{ipx} f''(y)\cos(p). \tag{3.15}$$

Recalling that $R_1 \sim R_{10} = k\mathcal{A}q_x$, we have

$$\begin{aligned} R_1 &\sim k\mathcal{A}\left(ip\epsilon\mathrm{e}^{i\theta}F + 2ip\epsilon^2\mathrm{e}^{2i\theta}G_2 + ip\epsilon^2\mathrm{e}^{i\theta}G_1 + ip\epsilon^3\mathrm{e}^{i\theta}H_1\right) \\ &\sim -4ipk\epsilon\mathrm{e}^{i\theta}F\sin^2(\tfrac{1}{2}p) - 2pk\epsilon^2\mathrm{e}^{i\theta}F_y\sin(p) + ipk\epsilon^3\mathrm{e}^{i\theta}F_{yy}\cos(p) - 8ipk\epsilon^2\mathrm{e}^{2i\theta}G_2\sin^2(\tfrac{1}{2}p) \\ &\quad -4ipk\epsilon^2\mathrm{e}^{i\theta}G_1\sin^2(\tfrac{1}{2}p) - 2pk\epsilon^3\mathrm{e}^{i\theta}G_{1,y}\sin(p) - 4ipk\epsilon^3\mathrm{e}^{i\theta}H_1\sin^2(\tfrac{1}{2}p). \end{aligned} \tag{3.16}$$

For R_2 we require only the leading order term; after noting $qq_x \sim ip\epsilon^2\mathrm{e}^{2i\theta}F^2 + c.c.$, we obtain

$$R_2 \sim (E - E^{-1})q(1 - E)^{-1}(1 + E)kqq_x \sim -4ikp\cos^2(\tfrac{1}{2}p)\epsilon^3\mathrm{e}^{i\theta}|F|^2F. \tag{3.17}$$

We find the expansion for R_3 by splitting the term into two parts: $R_3 = R_{31} - R_{32}$ where

$$R_{32} = q^2(E - E^{-1})(q - \beta q + \beta y/\epsilon), \quad \text{and} \quad R_{31} = (E - E^{-1})(q - \beta q + \beta y/\epsilon). \tag{3.18}$$

It is sufficient to find the leading order term for R_{32} but more terms are needed in the expression for R_{31}. Combining the results yields

$$\begin{aligned} R_3 &\sim 2i\sin(2p)\mathrm{e}^{2i\theta}G_2\left(\epsilon^2(1-\beta) + \epsilon\beta y\right) + \mathrm{e}^{i\theta}\left[2i\sin(p)(1-\beta)\epsilon F + 2i\sin(p)\beta yF\right. \\ &\quad +2i\sin(p)(1-\beta)\epsilon^2G_1 + 2\cos(p)(1-\beta)\epsilon^2F_y + 2\cos(p)\beta\epsilon(yF)_y + 2i\sin(p)\beta\epsilon yG_1 \\ &\quad +2i\sin(p)(1-\beta)\epsilon^3H_1 + i\sin(p)\epsilon^3(1-\beta)F_{yy} + 2\cos(p)\epsilon^3(1-\beta)G_{1,y} + 2i\sin(p)\beta\epsilon^2yH_1 \\ &\quad \left.+i\sin(p)\beta\epsilon^2(yF)_{yy} + 2\cos(p)\beta\epsilon^2(yG_1)_y - 2i\sin(p)|F|^2F(\epsilon^3-\beta\epsilon^3+\beta\epsilon^2y)\right]. \end{aligned} \tag{3.19}$$

In the following sections, we combine the expressions for q_t (3.1) with the expressions for R_j (3.16), (3.17) and (3.19) for specific choices for the magnitudes of the parameters β, k.

3.1. Case A: $k = \mathcal{O}(\epsilon^0), \beta = \mathcal{O}(\epsilon^4)$

Here we write $\beta = \epsilon^4B$ and $k = K$ with $B, K = \mathcal{O}(1)$, which removes β from the problem and leaves a system slightly modified from §1.3 only by the presence of K. The expressions R_1, R_2, R_3 are given by

$$\begin{aligned} R_1+R_2 &\sim -4ipk\epsilon\mathrm{e}^{i\theta}F\sin^2(\tfrac{1}{2}p) - 2pk\epsilon^2\mathrm{e}^{i\theta}F_y\sin(p) - 8ipk\epsilon^2\mathrm{e}^{2i\theta}G_2\sin^2(\tfrac{1}{2}p) \\ &\quad -4ipk\epsilon^2\mathrm{e}^{i\theta}G_1\sin^2(\tfrac{1}{2}p) - 2pk\epsilon^3\mathrm{e}^{i\theta}G_{1,y}\sin(p) - 4ipk\epsilon^3\mathrm{e}^{i\theta}H_1\sin^2(\tfrac{1}{2}p) \\ &\quad +ipk\epsilon^3\mathrm{e}^{i\theta}F_{yy}\cos(p) - 4ikp\cos^2(\tfrac{1}{2}p)\epsilon^3\mathrm{e}^{i\theta}|F|^2F, \end{aligned} \tag{3.20}$$

$$\begin{aligned} R_3 &\sim 2i\sin(2p)\mathrm{e}^{2i\theta}G_2\epsilon^2 + \mathrm{e}^{i\theta}\left[2i\sin(p)\epsilon F + 2i\sin(p)\epsilon^2G_1 + 2\cos(p)\epsilon^2F_y\right. \\ &\quad \left.+2i\sin(p)\epsilon^3H_1 + i\sin(p)\epsilon^3F_{yy} + 2\cos(p)\epsilon^3G_{1,y} - 2i\sin(p)\epsilon^3|F|^2F\right] \end{aligned} \tag{3.21}$$

We equate the sum of these to the time derivative (3.1), and equate terms at each magnitude $\mathcal{O}(\epsilon^m)$ with $(m = 1, 2, 3)$ and each frequency $\mathrm{e}^{ij\theta}$ with $j = 1, 2, \ldots m$.

At $\mathcal{O}(\epsilon\mathrm{e}^{i\theta})$, we obtain the dispersion relation, which relates the frequency to the wavenumber

$$\omega = -2\sin(p) + 4kp\sin^2(\tfrac{1}{2}p). \tag{3.22}$$

We note that the presence of the parameter k changes the dispersion relation from (1.11), so that it is no longer periodic in p. In Figure 2 we plot both forms of $\omega(p)$ showing the change made by $k \neq 0$. At $\mathcal{O}(\epsilon^2\mathrm{e}^{2i\theta})$ we obtain an equation for the second harmonic, G_2

$$-2i\omega\epsilon^2\mathrm{e}^{2i\theta}G_2 = 2i\sin(2p)G_2 - 8ipkG_2\sin^2(p); \tag{3.23}$$

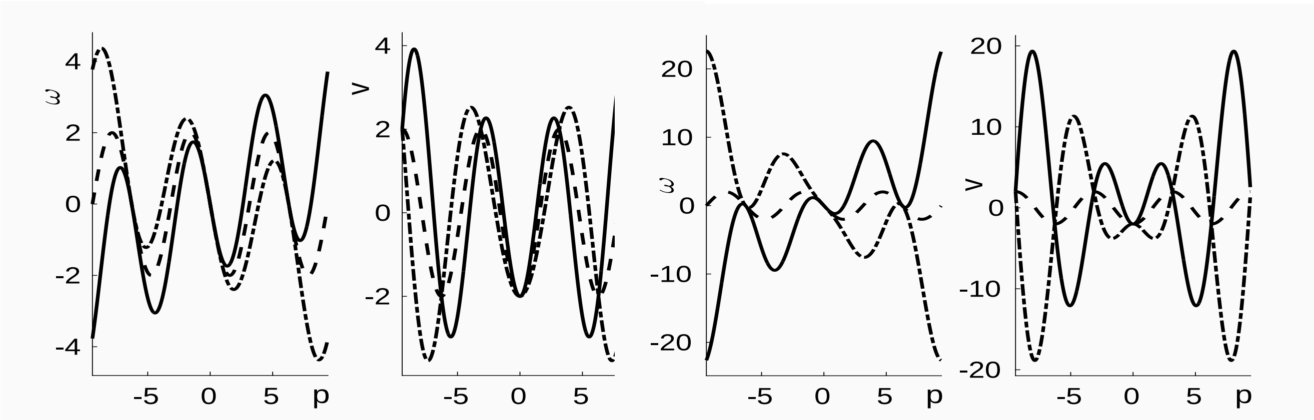

Figure 2: Plots of the dispersion relation $\omega(p)$ (left panel) and velocity $v(p)$ (second panel) against wavenumber, p $(-3\pi < p < 3\pi)$, from equations (1.11) and (1.12); solid lines represent the case $k = 0.2$, dash-dotted lines represent $k = -0.2$ and dashed lines, the case $k = 0$, which corresponds to the standard Volterra system. Third panel: as first panel, but for $k = 0, \pm 1.2$; fourth panel: as second panel, but for $k = 0, \pm 1.2$.

however, as the nonlinearity is cubic, no second harmonics are generated, so we have $G_2 = 0$.

Considering the equations at $O(\epsilon^2 e^{i\theta})$, we find

$$F_{T_1} - i\omega G_1 = -2pkF_y \sin(p) - 4ipkG_1 \sin^2(\tfrac{1}{2}p) + 2i\sin(p)G_1 + 2\cos(p)F_y. \tag{3.24}$$

Here, all the terms involving G_1 can be removed due to the definition of ω given by (3.22). This leaves a first order travelling wave equation for F whose solution has the form $F(y, T_1, T_2) = F(z, T_2)$ where $z = y - vT_1$, and the velocity is given by

$$v = 2pk\sin(p) - 2\cos(p). \tag{3.25}$$

This expression differs from the velocity in the standard modified Volterra equation, (1.12). In Figure 2 we plot the velocity v against wave number p for both cases, and as with the dispersion relation, we note that the presence of $k \neq 0$ makes v non-periodic in p.

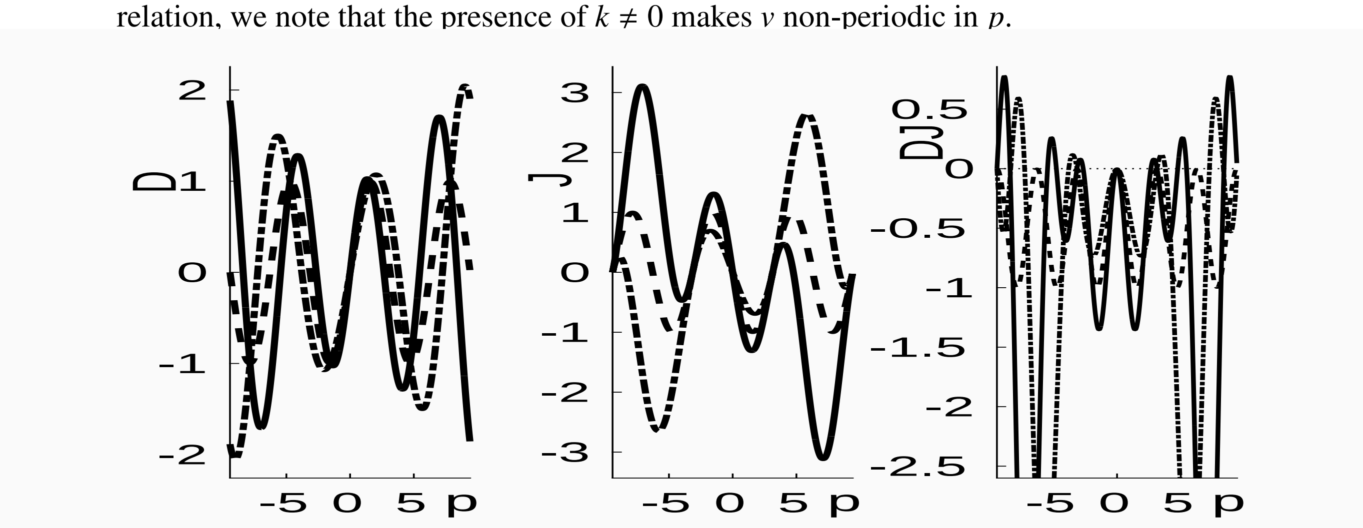

Figure 3: Plots of the parameters $D(p)$ (left panel), $J(p)$ (central panel) and the product $D(p)J(p)$ (right panel) against wavenumber, p for $(-3\pi < p < 3\pi)$. The solid lines correspond to (3.27) with $k = 0.2$, the dash-dotted lines to $k = -0.2$, and the dashed lines to the standard case, $k = 0$.

The final equation we need to consider comes from terms of the form $O(\epsilon^3 e^{i\theta})$, which yields

$$\begin{aligned} F_{T_2} + G_{1,T_1} - i\omega H_1 \quad = \quad & ipkF_{yy}\cos(p) - 4ikp\cos^2(\tfrac{1}{2}p)|F|^2F - 2pkG_{1,y}\sin(p) + i\sin(p)F_{yy} \\ & -4ipkH_1\sin^2(\tfrac{1}{2}p) + 2i\sin(p)H_1 + 2\cos(p)G_{1,y} - 2i\sin(p)|F|^2F, \end{aligned} \tag{3.26}$$

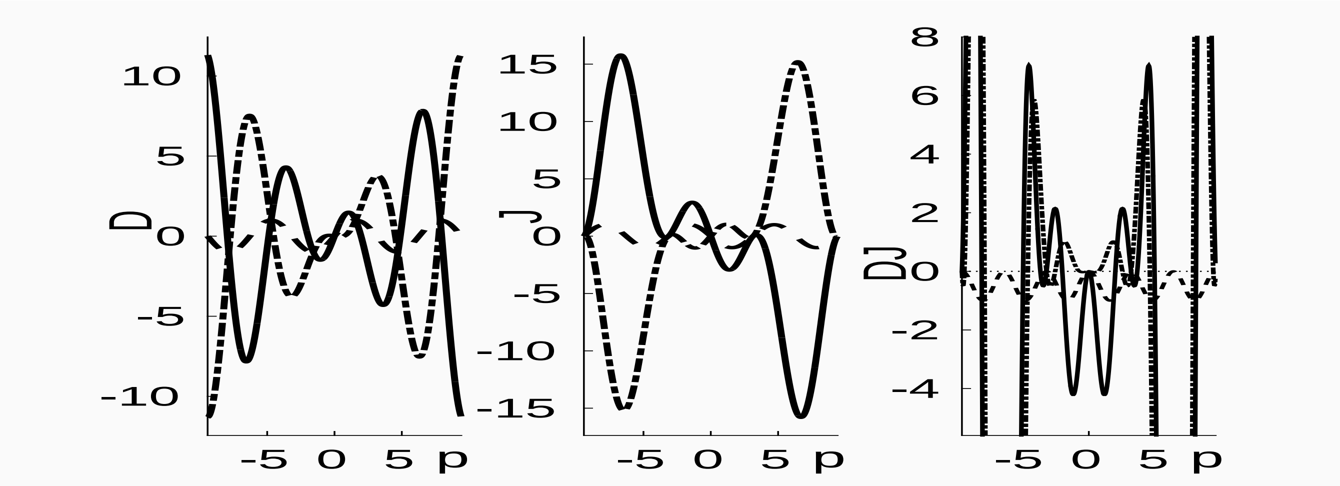

Figure 4: As figure 3, but for larger value of k. Plots of the parameters $D(p)$ (left panel), $J(p)$ (central panel) and the product $D(p)J(p)$ (right panel) against wavenumber, p for $(-3\pi < p < 3\pi)$. The solid lines correspond to (3.27) with $k = 1.2$, the dash-dotted lines to $k = -1.2$, and the dashed lines to the standard case, $k = 0$.

As with G_1 in the equation at $O(\epsilon^2 e^{i\theta})$, the definition of ω (3.22) means that all the terms involving H_1 in (3.26) cancel. We can also remove G_1 from this calculation by redefining $F + \epsilon G_1 \mapsto F$, hence it is sufficient to consider

$$-iF_{T_2} = DF_{zz} + J|F|^2F, \qquad D = (pk\cos(p) + \sin(p)), \qquad J = -(4kp\cos^2(\tfrac{1}{2}p) + 2\sin(p)). \tag{3.27}$$

We note that in the case $k = 0$ this reduces to (1.13) but that in the case $k \neq 0$ we have coefficients which are not periodic in p.

Furthermore, whilst (1.13) of the defocusing form for all $p \neq 0$, equation (3.27) is of the focusing form for some combinations of wavenumbers p and parameter k, that is when $JD > 0$. In Figures 3 and 4, we plot $J(p)$, $D(p)$ and the product $J(p)D(p)$ in the cases $k = \pm 0.2$, and $k = \pm 1.2$. to show the corresponding ranges of wavenumbers, p. Zeros of $D(k)$ occur when $\tan(p) = -kp$ and zeros of $J(k)$ occur when $\tan(p/2) = -kp$.

In this subsection, we have seen that for $k = O(1)$ and very small $\beta = \beta_0$, the analysis which reduces the standard modified Volterra equation to NLS also works for the extended equation, although, the presence of $k \neq 0$ means that both forms of NLS equation are relevant.

3.2. **Case B:** $k = O(\epsilon^0)$**,** $\beta = O(\epsilon^3)$

We now consider a larger magnitude for the parameter β so that its presence will influence the higher-order equations in the asymptotic expansion. For this case we write $\beta = \epsilon^3 B$ and $k = K$ with $B, K = O(1)$, giving

$$\begin{aligned} R_3 \sim\ & 2i\sin(2p)e^{2i\theta}G_2\left(\epsilon^2 + B\epsilon^3 y\right) + e^{i\theta}\Big[2i\sin(p)\epsilon F + 2i\sin(p)B\epsilon^3 yF \\ & +2i\sin(p)\epsilon^2 G_1 + 2\cos(p)\epsilon^2 F_y + 2i\sin(p)\epsilon^3 H_1 + i\sin(p)\epsilon^3 F_{yy} \\ & +2\cos(p)\epsilon^3 G_{1,y} - 2i\sin(p)\epsilon^3|F|^2F\Big], \end{aligned} \tag{3.28}$$

in addition to (3.20) and (3.1) for $R_1 + R_2$ and q_t.

Equating terms of equal magnitude and frequency, that is, $O(\epsilon^m e^{ij\theta})$, we obtain a similar hierarchy of equations for Case B as in Case A. At $O(\epsilon e^{i\theta})$, we find an identical dispersion relation as in Case A, namely (3.22). At $O(\epsilon^2 e^{2i\theta})$, we again obtain $G_2 = 0$. Also, at $O(\epsilon^2 e^{i\theta})$ we have the same travelling wave PDE, so we again assume the solution $F(y, T_1, T_2) = F(z, T_2)$ with $z = y - vT_1$, and v given by (3.25).

The equation at $\mathcal{O}(\epsilon^3 e^{i\theta})$ differs from Case A. In place of (3.26) we have

$$\begin{aligned} F_{T_2} + G_{1,T_1} - i\omega H_1 &= -2pkG_{1,y}\sin(p) - 4ipkH_1\sin^2(\tfrac{1}{2}p) + ipkF_{yy}\cos(p) - 4ikp\cos^2(\tfrac{1}{2}p)|F|^2F \\ &\quad +2i\sin(p)ByF + 2i\sin(p)H_1 + i\sin(p)F_{yy} + 2\cos(p)G_{1,y} - 2i\sin(p)|F|^2F. \end{aligned} \tag{3.29}$$

The H_1 terms can be removed due to the definition of ω. Transforming to $z = y - vT_1$ and retaining T_1 so that $G_1 = G_1(z, T_1, T_2)$ we obtain

$$-iF_{T_2} - iG_{1,T_1} = DF_{zz} + J|F|^2F + 2B\sin(p)zF + 2B\sin(p)vT_1F, \tag{3.30}$$

with J, D as given in (3.27). Since the only terms in this equation which explicitly depend on T_1 are the last term and the G_1 term, we define G_1 to satisfy $-iG_{1,T_1} = 2Bv\sin(p)T_1F(z, T_2)$, hence $G_1(z, T_1, T_2) = -iBv\sin(p)T_1^2F(z, T_2)$. This leaves a generalised NLS equation for $F(z, T_2)$, namely

$$0 = iF_{T_2} + DF_{zz} + J|F|^2F + 2BzF\sin(p). \tag{3.31}$$

Following Calogero & Degasperis [17], equations of the form

$$0 = iu_T + Du_{zz} + 2D|u|^2u + 2B_0u + 2B_1zu + 2iB_2(zu)_z, \tag{3.32}$$

can be mapped onto the NLS

$$0 = ie^{4B_2T}\widetilde{u}_T + D\widetilde{u}_{yy} + 2D|\widetilde{u}|^2\widetilde{u}, \tag{3.33}$$

by the transformation

$$\begin{aligned} y &= z\,e^{-2B_2T} - \frac{DB_1}{2B_2^2}\left(1 - e^{-2B_2T}\right)^2, & (3.34)\\ \Omega(T) &= \left(2B_0 - \frac{DB_1^2}{B_2^2}\right)T + \frac{DB_1^2}{4B_2^3}\left(3 - e^{-2B_2T}\right)\left(1 - e^{-2B_2T}\right), & (3.35)\\ u(z, T) &= \widetilde{u}(y, T)\,e^{-2B_2T}\exp\left(iz\frac{B_1}{B_2}\left(1 - e^{-2B_2T}\right) + i\Omega(T)\right). & (3.36) \end{aligned}$$

The transformation $u = F\sqrt{J/2D}$ maps (3.31) onto (3.32) in the case $B_0 = 0 = B_2$ and $B_1 = B\sin(p)$, and hence (3.31) can be transformed to the NLS equation.

3.3. Case C: $k = \mathcal{O}(\epsilon^0), \beta = \mathcal{O}(\epsilon^2)$

Putting $\beta = \epsilon^2B$ and $k = K$ with $B, K = \mathcal{O}(1)$, we obtain

$$\begin{aligned} R_3 \sim\ & 2i\sin(2p)e^{2i\theta}G_2\left(\epsilon^2 + B\epsilon^3y\right) + e^{i\theta}\Big[2i\sin(p)(1 - B\epsilon^2)\epsilon F + 2i\sin(p)B\epsilon^2yF \\ & +2i\sin(p)\epsilon^2G_1 + 2\cos(p)\epsilon^2F_y + 2\cos(p)B\epsilon^3(yF)_y + 2i\sin(p)B\epsilon^3yG_1 + 2i\sin(p)\epsilon^3H_1 \\ & +i\sin(p)\epsilon^3F_{yy} + 2\cos(p)\epsilon^3G_{1,y} - 2i\sin(p)\epsilon^3|F|^2F\Big] \end{aligned} \tag{3.37}$$

together with (3.20) and (3.1). Combining these expressions and extracting the equations at each $\mathcal{O}(\epsilon^m e^{ij\theta})$, we obtain the dispersion relation (3.22) at $\mathcal{O}(\epsilon e^{i\theta})$ and $G_2 = 0$ from $\mathcal{O}(\epsilon^2 e^{2i\theta})$.

Due to the larger size of β, its effect is now apparent at $\mathcal{O}(\epsilon^2 e^{i\theta})$, where we obtain

$$F_{T_1} - i\omega G_1 = -2pkF_y\sin(p) - 4ipkG_1\sin^2(\tfrac{1}{2}p) + 2i\sin(p)ByF + 2i\sin(p)G_1 + 2\cos(p)F_y. \tag{3.38}$$

We still have cancellation of the G_1 terms due to the definition of ω (3.22). Retaining the definition of velocity v from (3.25) we are left with $F_{T_1} + vF_y = 2Bi\sin(p)yF$. Using $z = y - vT_1$, this PDE can be converted to $F_{T_1} = 2iB\sin(p)(z + vT_1)F$ and hence solved by

$$F(z, T_1, T_2) = \widetilde{F}(z, T_2)\mathrm{e}^{iB\sin(p)T_1(2z+vT_1)}, \tag{3.39}$$

and we consider the next higher order equation to find an equation for $\widetilde{F}$.

At $O(\epsilon^3 \mathrm{e}^{i\theta})$ we find the equation

$$\begin{aligned} F_{T_2} + G_{1,T_1} - i\omega H_1 &= -2pkG_{1,y}\sin(p) - 4ipkH_1\sin^2(\tfrac{1}{2}p) + ipkF_{yy}\cos(p) - 4ikp\cos^2(\tfrac{1}{2}p)|F|^2F \\ &\quad +2\cos(p)B(yF)_y + 2i\sin(p)ByG_1 - 2iB\sin(p)F + 2i\sin(p)H_1 \\ &\quad +i\sin(p)F_{yy} + 2\cos(p)G_{1,y} - 2i\sin(p)|F|^2F. \end{aligned} \tag{3.40}$$

As in the above cases, the definition of ω (3.22) means that the H_1 terms cancel. Using the definition of v (3.25), $z = y - vT_1$, J, D (3.27) and defining $\gamma = B\sin(p)$, we can rewrite the above PDE as

$$-iF_{T_2} - iG_{1,T_1} = DF_{zz} + J|F|^2F - 2\gamma F + 2\gamma yG_1 + ivG_{1,z} - 2iB\cos(p)(zF)_z - 2iBv\cos(p)T_1F_z. \tag{3.41}$$

Making use of (3.39) and writing $G_1(z, T_1, T_2) = \widetilde{G}_1(z, T_1, T_2)\mathrm{e}^{iB\sin(p)T_1(2z+vT_1)}$, the terms in the resulting equation can be separated into those that depend on T_1, which together form a linear PDE for G_1, and the remainder, which form a closed nonlinear PDE for $\widetilde{F}(z, T_2)$. These equations are

$$\begin{aligned} -i\widetilde{G}_{1,T_1} &= iv\widetilde{G}_{1,z} - 2\gamma vT_1\widetilde{G}_1 + 2iT_1\widetilde{F}_z[2\gamma D - vB\cos(p)] + 4B\gamma T_1\cos(p)z\widetilde{F} \\ &\quad -4D\gamma^2T_1^2\widetilde{F} + 4\gamma BvT_1^2\cos(p)\widetilde{F} \end{aligned} \tag{3.42}$$

$$-i\widetilde{F}_{T_2} = D\widetilde{F}_{zz} + J|\widetilde{F}|^2\widetilde{F} - 2B\sin(p)\widetilde{F} - 2iB\cos(p)(z\widetilde{F})_z, \tag{3.43}$$

Following $u = \widetilde{F}\sqrt{J/2D}$, the transformation (3.34)–(3.36) maps this equation onto (3.33) in the case $B_0 = -B\sin(p)$, $B_1 = 0$, $B_2 = -B\cos(p)$, and so this can be transformed to the standard NLS.

3.4. Summary

In the NLS reduction, k changes the nature of the NLS equation through altering the coefficients of the nonlinear term, the spatial derivative term and, earlier in the derivation, the dispersion relation and the velocity expression. The $\beta = \beta_0$ parameter introduces extra terms to the equations, both in the NLS equation and, when β is larger, into the equations at earlier stages of the derivation, leading to more complicated behaviour of the solution $q(x, t)$.

4. Conclusions

In this paper, we have used asymptotic techniques to reduce the modified Volterra system to the modified Korteweg-de Vries (mKdV) and the nonlinear Schrödinger (NLS) equations. We have then applied similar methods to the extended modified Volterra equation, generated from the nonisospectral hierarchy [4]. This equation, which has several extra parameters has, in some cases, generated the same reductions as the standard modified Volterra equation; in other cases, the reduction leads to the same equation (mKdV or NLS), but with coordinates which are modified, either by the conversion to a moving coordinate frame, or this combined with a dynamically stretched space scale similar to those seen in similarity solutions. In other cases, we have obtained generalised NLS and mKdV equations, which have extra terms present.

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