Dynamic cooperative advertising under manufacturer and retailer level competition

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Abstract

We study dynamic cooperative advertising decisions in a market that consists of a finite number of independent manufacturers and retailers. Each manufacturer sells its product through all retailers and can offer different levels of advertising support to the retailers. Each retailer sells every manufacturer's product and may choose to carry out a different amount of local advertising effort to promote the products. A manufacturer may offer to subsidize a fraction of the local advertising expense carried out by a retailer for its product, and this fraction is termed as that manufacturer's subsidy rate for that retailer. We model a Stackelberg differential game with manufacturers as leaders and retailers as followers. A Nash game between the manufacturers determines their subsidy rates for the retailers and another Nash game between the retailers determines their optimal advertising efforts for the products they sell in response to manufacturers' decisions. We obtain optimal policies in feedback form. In some special cases, we explicitly write the incentives for coop advertising as functions of different model parameters including the number of manufacturers and retailers, and study the impact of the competition at the manufacturer and the retailer levels. We analyse the profits of the players and find the model parameters under which a manufacturer benefits from a coop advertising program. Furthermore, in the case of two manufacturers and two retailers, we study the effect of various model parameters on all four subsidy rates. We also extend our model to include national level advertising by the manufacturer.

Keywords: OR in marketing, cooperative advertising, differential games, feedback Stackelberg equilibrium

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1 Introduction

Cooperative advertising (or coop advertising) is a powerful tool commonly used in marketing channels where one party in the channel agrees to subsidize the advertising expenditure of the other. Quite often, the local advertising (such as the advertising in local TV channels, radio stations, newspapers, magazines, pamphlets, mailing lists, billboards, etc.) is managed by a local retailer since the retailer is usually better aware of the effectiveness of local advertising media and has a better understanding of the local preferences and demographics of the consumers. However, the retailer's advertising effort might not be enough from the perspective of the manufacturer due to reasons such as limited budget or a lack of sufficient incentive for the retailer. The manufacturer may then offer to share a part of retailer's advertising expenditure in order to incentivize the retailer to advertise more and both the parties could benefit from this arrangement due to higher sales. In a typical arrangement, a manufacturer agrees to share a portion of the local advertising cost of a retailer. In such a situation, the fraction of the retailer's advertising cost that the manufacturer agrees to reimburse is commonly known as the participation rate or the subsidy rate.

The practice of cooperative advertising has seen a significant increase in recent years. Nagler (2006) estimated that the total expenditure on coop advertising in US was about \$ 0.9 billion in 1970 and had grown to about \$ 15 billion in 2000. More recently, some estimates report that total pool of money available for coop advertising programs in US exceeds \$ 50 billion annually[1]. Dant and Berger (1996) report that approximately 25-40% of local advertisements and promotion are funded on a cooperative basis. Karray et al. (2017) note that about \$36 billion are being paid by manufacturers to retailers in cooperative advertising funds, which represents about 12% of total advertising costs. Dutta et al. (1995) analyzed data for 2,156 coop advertising plans across 49 product categories and found the average subsidy rate to be 74.6%. More specifically, they found that average subsidy rate was 74.44% for consumer products, 69.02% for industrial products, 68.95% for consumer inconvenience products, and as high as 88.38% for consumer convenience products.

In supply chain and marketing problems, a key question for manufacturers in this context is to determine what should be its optimal subsidy rate policy for its retailers. Accordingly, the related question for a retailer is to decide its optimal advertising policy, given the level of support offered by a manufacturer. These questions become even more relevant and challenging when *a*) these decision makers are in a dynamic market where the sales and market share evolves with time and its evolution depends on these decisions, and *b*) when these decision makers face competition. We attempt to address these issues in this paper.

[1] http://marketingland.com/report-billions-in-co-op-advertising-funds-left-unspent-each-year-138671

We study cooperative advertising in a market that consists of a finite number of independent manufacturers and retailers. There is, thus, competition at manufacturer as well as retailer level. We formulate the problem as a Stackelberg differential game with the manufacturers as leaders and retailers as followers. First, a Nash game ensues between the manufacturers where each manufacturer determines its level of support (subsidy rate) for all the competing retailers. This is followed by a Nash game between competing retailers where each retailer responds to all the manufacturers' decisions to decide the local advertising effort for each manufacturer's product. We obtain optimal subsidy rates policy for all the manufacturers and optimal advertising efforts for all the retailers. In an extension, we also study the case when the manufacturer also carries national level advertising in addition to the local advertising by the retailer.

The rest of the paper is organized as follows. In Section 2, we discuss the background literature on game theoretic models in cooperative advertising and also highlight our contribution vis-à-vis the extant literature. In Section 3, we describe the general model and solve it to obtain the underlying set of algebraic equations characterizing a feedback Stackelberg equilibrium. We also investigate the incentives for cooperation for the manufacturers and in some special cases, investigate the role of different model inputs including the level of competition. In Section 4, in a case of two manufacturers and two retailers, we investigate the sensitivity of subsidy rates by the manufacturers for all retailers with respect to different model parameters. In Section 5, we examine the profit of each player with and without a cooperative advertising program. In Section 6, we extend our model to incorporate national level advertising by the manufacturers in addition to the local advertising by the retailers and obtain the optimal national and local advertising policies. We finally conclude the paper in Section 7. All figures (except Figure 1) that depict analytical insights are relegated to the end for ease of reading.

2 Background Literature

For a wider and detailed review of literature on cooperative advertising models of all types, the readers are referred to the recent survey papers by Aust and Buscher (2014 a) and Jørgensen and Zaccour (2014). In this segment however, our discussion is focussed on cooperative advertising game models in supply chain with competition. While the game theoretic modelling of coop advertising is a widely researched area which goes back to Berger (1972), the focus on coop advertising models under the presence of competition is a relatively new development. Examples of static models with one manufacturer and multiple retailers include Wang et al. (2011), Zhang and Xie (2012), Ghadimi et el. (2013), Aust and Buscher

(2014 b), and Karray and Amin (2015). Dynamic models with one manufacturer and multiple retailers include Sigué and Chintagunta (2009), He et al. (2011), He et al. (2012), Chutani and Sethi (2012a, 2012b), and Chutani and Sethi (2014). While there are a few models with retail level competition as listed above, it is noticeable that there are even fewer ones which account for manufacturer level competition. There are some examples of static models which account for competition between two manufacturers. Bergen and John (1997) considered a model with two manufacturers and two retailers where each product consists of two attributes (dimensions): one representing attributes from the two retailers and the other representing manufacturers. Thus, there are four possible customer offerings located at the four corners of the product space, which is a two dimensional square; and the customers are located uniformly throughout the square. They focus on studying the impacts on the optimal subsidy rates. Kim and Staelin (1999) modelled a Stackelberg game with two manufacturers (leaders) and two retailers (followers). In their paper, the cooperative advertising appears in terms of direct allowances paid by the manufacturers to the retailers. They focus on the retailer "pass-through" rates which denote the fraction of manufacturer allowances that the retailers pass on to the consumers in the form of more features, more shelf space, price cuts etc. Karray and Zaccour (2007) study a two manufacturer - two retailer Stackelberg game and account for brand and store substitution effects due to the retailers' advertising. They also study the impact of coop advertising programs on the payoffs of the manufacturers and retailers. In the case of homogeneous manufacturers and homogeneous retailers, they find that if the brand substitution rate is high enough the coop advertising programs are examples of prisoner's dilemma and that coop advertising programs lead to an increase in retailers' profits only when the store substitution rate is not too high. There are papers on coop advertising with competition in the case of dual exclusive supply chains (see, for e.g., Yan et al. (2006), Chutani and Sethi (2012 b), Liu et al. (2014), Chen (2015), Karray (2015), and Karray et al. (2017)). However, in these dual-channel settings, each of the two manufacturers sells exclusively through its retailer and each of the two retailers carries only one manufacturer's product. As mentioned earlier, all the above models with two manufacturers, however, are static in nature.

In this paper, we study dynamic cooperative advertising decisions in a market with a finite number of independent manufacturers (N_s) and retailers (N_r). Every manufacturer can sell its product through all the retailers and can offer advertising support to all of them. Similarly, every retailer can sell the products of all the manufacturers and can spend on local advertising for every manufacturer's product. This aspect is absent in the dual exclusive supply chain papers with two manufacturers and two retailers, as mentioned above. By considering such a generalized setting in a dynamic environment, we make some key contributions to the existing literature on cooperative advertising, as highlighted below. As discussed previously and also noted in the extensive survey by Jørgensen and Zaccour (2014), while there are some coop advertising papers involoving static game-theoretic models with two manufacturers and two retailers and differential game models with a single manufacturer and multiple retailers; to the best of our knowledge, ours is the only paper analysing a dynamic game with multiple manufacturers and multiple retailers at the same time. We are therefore able to assess the relative impacts of competition at brand as well as at retail level on coop advertising decisions. In some special cases, by considering different values of N_s, N_r, and other model parameters, we are able to obtain useful insights regarding coop advertising decisions and incentives as the level of competition changes at both manufacturer (N_s) and retailer (N_r) level. We also study the impacts of a coop advertising program on the profits of all the players. As discussed previously, Karray and Zaccour (2007) consider a static model with 2 manufacturers and show that coop advertising programs may be due to a prisoner's dilemma situation for the manufacturers. In their survey paper, Jørgensen and Zaccour (2014) highlight the need for "research efforts that extend existing models to include horizontal competition between manufacturers and retailers" and ask "if coop programs are simply prisoner's dilemma trap for competing manufacturers?" Adding to the existing knowledge on this issue, in a case of two manufacturers and two retailers, we find that a manufacturer may or may not benefit from a coop advertising program and it is possible that one manufacturer benefits whereas the other does not; and this depends on various model parameters. Furthermore, in the case of two manufactures and two retailers, through numerical study we also obtain useful managerial insights on the impacts of different parameters on the subsidy rates offered by the manufacturers for all retailers. We also extend our original model to include national level advertising by manufacturer as well. There are some coop advertising papers which consider national level advertising as well, examples of which include Huang et al. (2002), Karray and Zaccour (2005), Ezimadu and Nwozo (2017), Zhang et al. (2017). However, once again to the best of our knowledge, ours is the only paper that considers national level advertising with coop local advertising, and also accounts for competition at both levels. We obtain some interesting insights on how both types of advertising decisions change with competition and other model parameters. Finally, despite the complexity of a network type supply chain structure with multiple players at each echelon with a complete bipartite nature, we are still able to obtain feedback Stackelberg strategies for the manufacturers and the retailers.

3 General formulation with multiple manufacturers and multiple retailers

We formulate the problem for a general case with N_s manufacturers and N_r retailers. The sequence of events is as follows. Each manufacturer first announces its subsidy rate for each retailer, and the retailers in response decide their advertising efforts over time for the products of all the manufacturers. The advertising effort for any retailer for any particular brand boosts the sales of that product brand by the concerned retailer and increases its proportional market share. A Stackelberg differential game is thus played between the manufacturers and the retailers with manufacturers as the leaders and the retailers following their decisions to decide on their advertising efforts over time. Furthermore, two Nash differential games are also being played out in our model. The first between the manufacturers to decide on their subsidy rates for the retailers, and the second between the retailers to decide their advertising effort for each brand they sell. Figure 1 depicts the problem structure and the types of games being played by the parties. We introduce the key notations below.

t	Time $t \in [0, \infty)$
N_s	number of manufacturers
N_r	number of retailers
i, k, g	subscripts used to tag manufacturers, $i, k, g = 1, 2, \cdots, N_s$
j, l, h	subscripts used to tag retailers, $j, l, h = 1, 2, \cdots, N_r$
$x_{ij}(t) \in [0, 1]$	proportional market share of sales of manufacturer i's product via retailer j
$X \in [0, 1]$	$= \sum_{i=1}^{Ns} \sum_{j=1}^{Nr} x_{ij}(t)$, the proportion of the total market captured
M_{ij}	manufacturer i's unit margin from sales of manufacturer i's product by retailer j
m_{ij}	retailer j's unit margin from sales of manufacturer i's product by retailer j
$\rho_{ij} > 0$	advertising effectiveness parameter of retailer j for manufacturer i's product
$\delta_{ij} \geq 0$	market share decay parameter corresponding x_{ij}
$r > 0$	discount rate
$u_{ij}(t)$	retailer j's advertising effort for manufacturer i's product at time t
$\theta_{ij}(t) \geq 0$	manufacturer i's subsidy rate for retailer j at time t
V_{si}	value function of manufacturer i
V_{rj}	value function of retailer j

$$\hat{\theta}_{ij} = \frac{2\frac{\partial V_{si}}{\partial x_{ij}} - \frac{\partial V_{rj}}{\partial x_{ij}}}{2\frac{\partial V_{si}}{\partial x_{ij}} + \frac{\partial V_{rj}}{\partial x_{ij}}}$$

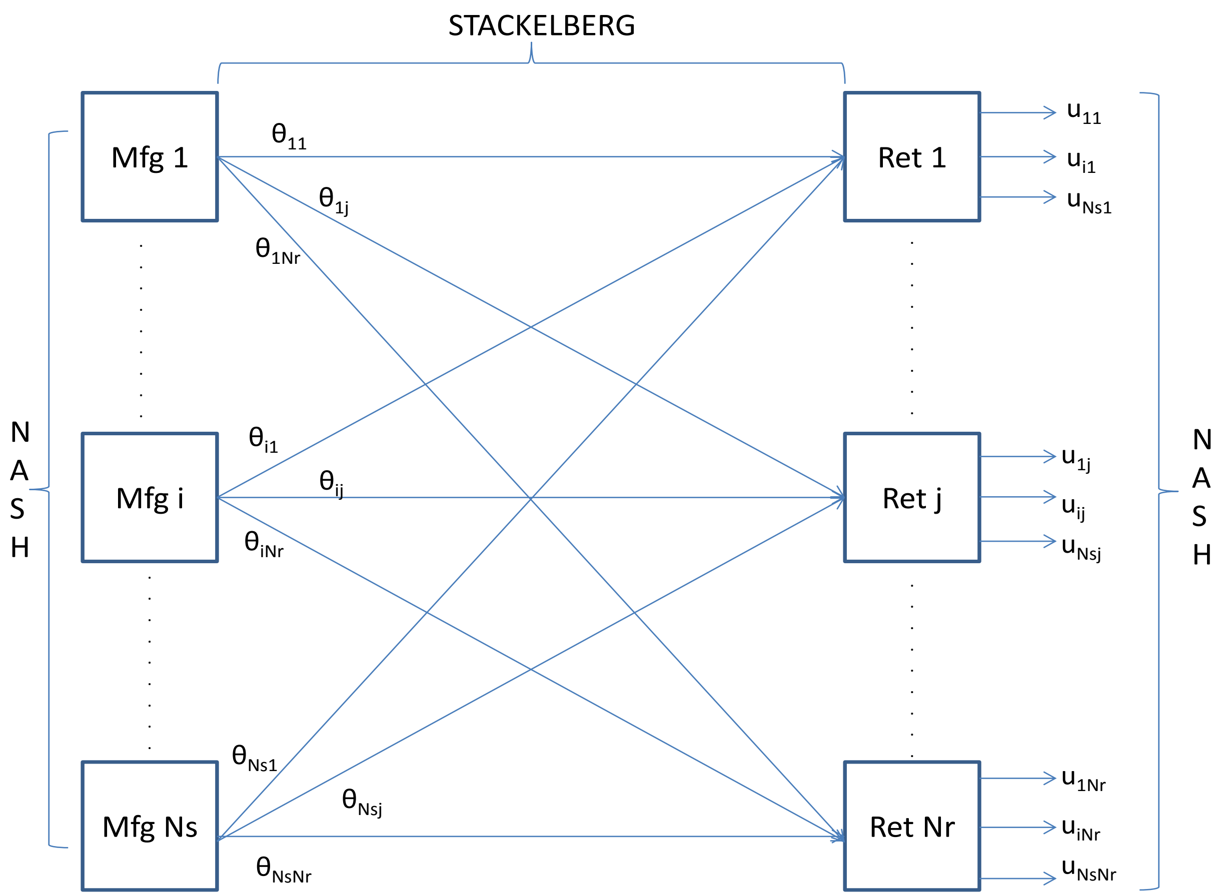


Figure 1: model

To model the impact of advertising efforts by the retailers on the evolution of market shares, we take inspiration from the Sethi (1983) advertising model and its extensions in Erickson (2009 a, b). Erickson (2009 a) presented an oligopolistic extension of the monopoly model of Sethi (1983) and used it to study advertising competition between different brands in the American beer industry. Erickson (2009 b) further extended the Erickson (2009 a) model to incorporate multiple brands sold by each competitor, and applied it to the carbonated soft drinks market. Our framework is based on these papers and includes the advertising competition at two levels, i.e., multiple brands advertised by the same retailer and multiple retailers advertising each brand. We write down the state dynamics for the market shares as follows

$$\dot{x}_{ij} = \frac{dx_{ij}}{dt} = \rho_{ij} u_{ij} \sqrt{1-X} - \delta_{ij} x_{ij}, \quad \forall i = 1, 2, \cdots, N_s,\ j = 1, 2, \cdots, N_r, \tag{1}$$

where x_{ij} represents the proportional market share of the sales of manufacturer i's product by retailer j w.r.t. the total market potential which is normalized to one. Clearly, the total market share of manufacturer i is $\sum_{j=1}^{N_r} x_{ij}$, and the same for retailer j is $\sum_{i=1}^{N_s} x_{ij}$. The parameter ρ_{ij} represents the advertising effectiveness of local advertising done by retailer j for the product of manufacturer i. This coefficient be a function of the medium of advertising chosen by the retailer. δ_{ij} represents the decay of the market share x_{ij} due to various factors such as competition (retailer or brand level), changing consumer preferences etc. The cost of advertising is written as the square of advertising effort. This cost structure is very common in literature and captures the marginally diminishing returns of advertising. We can now write, for $i = 1, 2, \cdots, N_s$, manufacturer i's optimal control problem

$$\max_{\substack{\theta_{ij}(t) \in [0,1] \\ j=1,2,\cdots,N_r}} \left\{ J_{si} = \int_{t=0}^{\infty} e^{-rt} \left[\sum_{j=1}^{Nr} \left[M_{ij} x_{ij}(t) - \theta_{ij}(t) u_{ij}^2(t) \right] dt \right\} , \tag{2}$$

and for $j = 1, 2, \cdots, N_r$, retailer j's optimal control problem

$$\max_{\substack{u_{ij}(t \,|\, \theta_{kl}(t), \forall k,l) \geq 0 \\ i=1,2,\cdots,N_r}} \left\{ J_{rj} = \int_{t=0}^{\infty} e^{-rt} \left[\sum_{i=1}^{Ns} \left[m_{ij} x_{ij}(t) - (1-\theta_{ij}(t)) u_{ij}^2(t) \right] dt \right\} , \tag{3}$$

both subject to the state dynamics in (1).

We focus on the optimal feedback policies of all manufacturers and retailers. This implies that we will look $\forall i = 1, 2, \cdots, N_s$ and $j = 1, 2, \cdots, N_r$, the optimal subsidy rates in the form $\theta_{ij}\big(x_{kl}(t), k = 1, 2, \cdots, N_s, l = 1, 2, \cdots, N_r\big)$, and the optimal advertising efforts in the form $u_{ij}\big(x_{kl}(t), k = 1, 2, \cdots, N_s, l = 1, 2, \cdots, N_r \mid \theta_{gh}, g = 1, 2, \cdots, N_s, h = 1, 2, \cdots, N_r\big)$. However, to simplify the notation and presentation, we will simply refer to them as θ_{ij}, and u_{ij}, respectively. We now write the Hamilton-Jacobi-Bellman (HJB) equation for manufacturer i as

$$rV_{si} = \max_{\substack{\theta_{ij} \\ j=1,2,\cdots N_r}} \left[\sum_{j=1}^{Nr} \left(M_{ij} x_{ij} - \theta_{ij} u_{ij}^2 \right) + \sum_{k=1}^{Ns} \sum_{j=1}^{Nr} \frac{\partial V_{si}}{\partial x_{kj}} \left(\rho_{kj} u_{kj} \sqrt{1-X} - \delta_{kj} x_{kj} \right) \right], \tag{4}$$

and the HJB equation for retailer j as

$$rV_{rj} = \max_{\substack{u_{ij} \\ i=1,2,\cdots N_s}} \left[\sum_{i=1}^{Ns} \left(m_{ij}x_{ij} - (1-\theta_{ij})u_{ij}^2 \right) + \sum_{i=1}^{Ns}\sum_{l=1}^{Nr} \frac{\partial V_{rj}}{\partial x_{il}} \left(\rho_{il}u_{il}\sqrt{1-X} - \delta_{il}x_{il} \right) \right], \quad (5)$$

where $V_{si} \equiv V_{si}\big(x_{kl}(t), k = 1,2,\cdots,N_s, l = 1,2,\cdots,N_r\big)$, and $V_{rj} \equiv V_{rj}\big(x_{kl}(t), k = 1,2,\cdots,N_s, l = 1,2,\cdots,N_r\big)$, denote the value functions of manufacturer i, and retailer j, respectively. Using backward induction, we first solve the retailers' problems. Given the subsidy rates by the manufacturers, solving the first-order condition (f.o.c.) w.r.t. u_{ij} yield the retailers' optimal advertising efforts

$$u_{ij}^* = \frac{\frac{\partial V_{rj}}{\partial x_{ij}}\rho_{ij}\sqrt{1-X}}{2(1-\theta_{ij})}, \qquad i = 1,2,\cdots N_s; j = 1,2,\cdots N_r. \quad (6)$$

It can be easily verified that the second-order-conditions are also satisfied.[2] Using (6), we rewrite equations (5) and (4), respectively, as

$$rV_{rj} = \sum_{i=1}^{Ns} \left(m_{ij}x_{ij} - \frac{(\frac{\partial V_{rj}}{\partial x_{ij}})^2\rho_{ij}^2(1-X)}{4(1-\theta_{ij})} \right) + \sum_{i=1}^{Ns}\sum_{l=1}^{Nr} \frac{\partial V_{rj}}{\partial x_{il}} \left(\frac{\frac{\partial V_{rl}}{\partial x_{il}}\rho_l^2(1-X)}{2(1-\theta_{il})} - \delta_{il}x_{il} \right), \quad (7)$$

$$rV_{si} = \max_{\substack{\theta_{ij} \\ j=1,2,\cdots N_r}} \left[\sum_{j=1}^{Nr} \left(M_{ij}x_{ij} - \frac{\theta_{ij}(\frac{\partial V_{rj}}{\partial x_{ij}})^2\rho_j^2(1-X)}{4(1-\theta_{ij})^2} \right) + \sum_{k=1}^{Ns}\sum_{j=1}^{Nr} \frac{\partial V_{si}}{\partial x_{kj}} \left(\frac{\frac{\partial V_{rj}}{\partial x_{kj}}\rho_j^2(1-X)}{2(1-\theta_{kj})} - \delta_{kj}x_{kj} \right) \right]. \quad (8)$$

We now apply the f.o.c. in (8) w.r.t. θ_{ij} and get the optimal subsidy rates as

$$\theta_{ij}^* = Max\left[\hat{\theta}_{ij}, 0\right], \qquad \hat{\theta}_{ij} = \frac{2\frac{\partial V_{si}}{\partial x_{ij}} - \frac{\partial V_{rj}}{\partial x_{ij}}}{2\frac{\partial V_{si}}{\partial x_{ij}} + \frac{\partial V_{rj}}{\partial x_{ij}}}, \qquad i = 1,2,\cdots N_s,\ j = 1,2,\cdots N_r. \quad (9)$$

It is easy to note that the solution of the first order conditions in (8) w.r.t. θ_{ij} is equal to

[2] To verify the second order conditions, we look at the Hessian matrix for every retailer's problem. For retailer j, we can see that the second order derivatives of the right hand side in equation (5) are: $\frac{\partial^2(.)}{\partial u_{ij}^2} = -2(1-\theta ij) < 0$ since it is expected to have $\theta_{ij} < 1$, as $\theta_{ij} \geq 1$ would imply a very large advertising effort by the retailer with all the cost borne by the manufacturer which will not be optimal for the manufacturer. We can also see that $\frac{\partial^2(.)}{\partial u_{ij}\partial u_{kj}} = 0, \forall i \neq k$. With these observations, we can conclude that the Hessian matrix corresponding to the maximization problem for every retailer is negative definite, thereby satisfying the second partial derivative test.

$\hat{\theta}_{ij}$. The second-order conditions for the manufacturers' problems are also satisfied.[3] Taking a cue from Sethi (1983), we conjecture value functions in linear forms as follows:

$$V_{si} = A_i + \sum_{k=1}^{Ns} \sum_{j=1}^{Nr} B_{ikj} x_{kj}, \qquad i = 1, 2, \cdots, N_s, \tag{10}$$

$$V_{rj} = \alpha_j + \sum_{i=1}^{Ns} \sum_{l=1}^{Nr} \beta_{jil} x_{il}, \qquad j = 1, 2, \cdots, N_r. \tag{11}$$

With these forms of value functions, it is clear that

$$B_{ikj} = \frac{\partial V_{si}}{\partial x_{kj}}, \quad \forall i, k, j; \quad i, k = 1, 2, \cdots, N_s, \quad j = 1, 2, \cdots, N_r \tag{12}$$

$$\beta_{jil} = \frac{\partial V_{rj}}{\partial x_{il}}, \quad \forall j, i, l; \quad j, l = 1, 2, \cdots, N_r, \quad i = 1, 2, \cdots, N_s. \tag{13}$$

Using (12)-(13), retailers' optimal advertising efforts in (6) can now be rewritten as:

$$u^*_{ij} = \frac{\beta_{jij}\rho_{ij}\sqrt{1-X}}{2(1-\theta_{ij})}, \qquad i = 1, 2, \cdots N_s; j = 1, 2, \cdots N_r. \tag{14}$$

We should note that in view of (10), $\forall i, j, \hat{\theta}_{ij}$ will be constant, even though in general they can be functions of the state variables. We use (12)-(13) in (7)-(8), and then compare the coefficients of $x_{ij}, \forall i, j$, and the constant terms in value functions given by (7)-(8) with those in (10)-(11). As a result we get the following set of $(N_s + N_r)(1 + N_s N_r)$ non-linear algebraic equations

$$rA_i = -\sum_{h=1}^{Nr} \frac{\theta^*_{ih}\beta^2_{hih}\rho^2_{ih}}{4(1-\theta^*_{ih})^2} + \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} B_{igh}\frac{\beta_{hgh}\rho^2_{gh}}{2(1-\theta^*_{gh})} \tag{15}$$

$$(r + \delta_{il})B_{iil} = M_{il} + \sum_{h=1}^{Nr} \frac{\theta^*_{ih}\beta^2_{hih}\rho^2_{ih}}{4(1-\theta^*_{ih})^2} - \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} B_{igh}\frac{\beta_{hgh}\rho^2_{gh}}{2(1-\theta^*_{gh})} \tag{16}$$

$$(r + \delta_{kl})B_{ikl} = \sum_{h=1}^{Nr} \frac{\theta^*_{ih}\beta^2_{hih}\rho^2_{ih}}{4(1-\theta^*_{ih})^2} - \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} B_{igh}\frac{\beta_{hgh}\rho^2_{gh}}{2(1-\theta^*_{gh})} \quad k \neq i \tag{17}$$

$$r\alpha_j = -\sum_{g=1}^{Ns} \frac{\beta^2_{jgj}\rho^2_{gj}}{4(1-\theta^*_{gj})} + \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \beta_{jgh}\frac{\beta_{hgh}\rho^2_{gh}}{2(1-\theta^*_{gh})} \tag{18}$$

[3] Similar to a retailer's problem we consider the Hessian matrix for manufacturer i's maximization problem, and can see that $\frac{\partial^2(.)}{\partial\theta^2_{ij}} < 0$ and $\frac{\partial^2(.)}{\partial\theta_{ij}\partial\theta_{il}} = 0, \forall j \neq l$. The Hessian matrix will therefore be negative definite and hence the second-order conditions for all the manufacturers' problems are also satisfied.

$$(r+\delta_{kj})\beta_{jkj} = m_{kj} + \sum_{g=1}^{Ns} \frac{\beta_{jgj}^2 \rho_{gj}^2}{4(1-\theta_{gj}^*)} - \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \beta_{jgh} \frac{\beta_{hgh}\rho_{gh}^2}{2(1-\theta_{gh}^*)} \tag{19}$$

$$(r+\delta_{kl})\beta_{jkl} = \sum_{g=1}^{Ns} \frac{\beta_{jgj}^2 \rho_{gj}^2}{4(1-\theta_{gj}^*)} - \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \beta_{jgh} \frac{\beta_{hgh}\rho_{gh}^2}{2(1-\theta_{gh}^*)} \quad \forall l \neq j \tag{20}$$

$$\theta_{ij}^* = Max\left[\hat{\theta}_{ij}, 0\right], \; \hat{\theta}_{ij} = \frac{2B_{iij} - \beta_{jij}}{2B_{iij} + \beta_{jij}}, \; i = 1, 2, \cdots N_s, \; j = 1, 2, \cdots N_r \tag{21}$$

Referring to the verification theorem in Bensoussan et al. (2014), we can now conclude the following result.

Proposition 1: *For the manufacturers' and retailers' optimal control problems, given by (2) and (3), respectively, both subject to (1), the feedback Stackelberg equilibrium is obtained by solving the set of non-linear algebraic equations in (15)-(21). The manufacturers' optimal subsidy rates policies are given by (21), and the retailers' optimal responses, i.e., their advertising efforts are given in (14). The value functions of all the players are linear in state variables, and are obtained using (10) and (11).*

In general, it is extremely difficult to get an explicit analytical solution to the system of equations given by (15)-(21). However, a numerical analysis of this system of equations can be carried out to get further insights in a cooperative solution. In subsection 3.1, we consider a scenario of a non-cooperative equilibrium and obtain explicit solutions to these equations in some special cases to understand the incentives for cooperation and get some useful insights.

3.1 Non-cooperative equilibrium and incentives for cooperation

We investigate a complete non-cooperative equilibrium in which no manufacturer supports any of its retailers, i.e., $\theta_{ij} = 0, \forall i, j$. We look to understand the incentives for such complete non-cooperation by understanding the conditions under which such an equilibrium will be optimal. To this end, we solve the Stackelberg game while enforcing $\theta_{ij}^* = 0, \forall i, j$, in equations (15)-(20). We then use the resulting values of coefficients from this solution and obtain the values of $\hat{\theta}_{ij}, \forall i, j$, using (21). Recall that mathematically, $\hat{\theta}_{ij}$ is allowed to take negative values. A complete non-cooperative equilibrium will thus be optimal if we have the resulting $\hat{\theta}_{ij} \leq 0, \forall i, j$, which would then validate the starting premise that $\theta_{ij}^* = 0, \forall i, j$. It can be seen that the system of equations (15)-(20) is too complicated to get a closed form solution even when we set $\theta_{ij} = 0, \forall i, j$. Therefore, we apply some simplifications and consider some special cases to understand the incentives for such an equilibrium (and the incentives to shift towards advertising cooperation), and in particular, the role of competition towards these incentives.

3.1.1 Special case: symmetric retailers and symmetric manufacturers

In the special case when $m_{ij} = m$, $M_{ij} = M$, $\rho_{ij} = \rho$, and $\delta_{ij} = \delta$, $\forall i, j$, we can say that $\forall i, k = 1, 2, \cdots, N_s$, $k \neq i$, and $\forall j, l, = 1, 2, \cdots, N_r$, $j \neq l$,

$$\beta_{jij} = \beta, \quad \beta_{jil} = \gamma, \quad B_{iij} = B, \quad B_{ikj} = G, \quad \text{and} \quad \hat{\theta_{ij}} = \hat{\theta}.$$

Under a non-cooperative equilibrium, we use $\theta^*_{ij} = 0, \forall i, j$, in equations (15)-(20) and obtain the following

$$\beta = \frac{m(N_r - 1)N_s\rho^2 + (r+\delta)\left[-2(r+\delta) + \sqrt{4(r+\delta)^2 + 4mN_rN_s\rho^2 + \frac{m^2(N_r-1)^2N_s^2\rho^4}{(r+\delta)^2}}\right]}{(2N_r - 1)N_s(r+\delta)\rho^2} \tag{22}$$

$$B = M\frac{2(N_s - N_r)(r+\delta)^2 + mN_rN_s\rho^2(2N_s - N_r - 1) + N_r(r+\delta)\sqrt{4(r+\delta)^2 + 4mN_rN_s\rho^2 + \frac{m^2(N_r-1)^2N_s^2\rho^4}{(r+\delta)^2}}}{2N_s(r+\delta)((r+\delta)^2 + mN_rN_s\rho^2)}. \tag{23}$$

From equation (22), we can observe the following.

Corollary 1: $\partial\beta/\partial N_s < 0$.

Corollary 1 can be obtained by taking the derivative w.r.t. N_s and after a few steps of algebra. It implies that in a non-cooperative equilibrium, with all other parameters unchanged, an increase in the number of products (brands) decreases the marginal utility for the retailers w.r.t. an increment in sales. In other words, the benefit for a retailer due to a small increase in market share decreases as the brand level competition increases. This has implications for the advertising efforts by the retailers. With higher brand level competition, the incentive for a retailer to advertise more and increase its market share, therefore, decreases. One might then argue that for higher N_s, given decreasing incentives for retailers, the manufacturers would be more inclined to provide a positive (i.e., non-zero) subsidy rate to advertise more. This in a nutshell would imply that $\hat{\theta}$ increases with N_s and therefore we move towards a cooperative equilibrium. However, it is difficult to prove the above statement through analytic means with equation (23) in its current form. We can however further investigate towards this insight through analytic means in a special case ($(r+\delta) \approx 0$), and numerical means more generally ($(r+\delta) \geq 0$). This is discussed below in the rest of this subsection, and in the subsection 3.1.2.

Scenario A: $(r+\delta) \approx 0$

To get further insights, we consider the case when $(r+\delta) \approx 0$ Using (22)-(23) in (21), and then applying $(r+\delta) \approx 0$, we obtain

$$\hat{\theta}_{ij} = \frac{M(2N_r - 1)(N_s - 1) - m(N_r - 1)N_s}{M(2N_r - 1)(N_s - 1) + m(N_r - 1)N_s} \tag{24}$$

When $\hat{\theta}_{ij} < 0$, $\forall i, j$ in equation (24), a complete non-cooperative equilibrium will be optimal where no manufacturer supports any of its retailer in its advertising efforts.

Equation (24) yields some interesting insights upon observation.

i. It is clear that $\hat{\theta}_{ij}$ increases with M and decreases with m, i.e., the incentives to shift towards a cooperative equilibrium increase as the manufacturers get more margins from sale and decrease as the retailers' margins increase. This can be explained as follows: the tendency of the manufacturers to support advertising efforts of retailers will increase if they benefit more from a higher sale, and this tendency will decrease if the retailers themselves have high incentives to advertise sufficiently due to their own higher margins.

ii. $\hat{\theta}_{ij} > 0$ always when $N_r = 1$ and $N_s > 1$. Thus, a non-cooperative equilibrium will never be optimal when there is only one retailer and more than one manufacturers selling their products, regardless of the number of manufacturers, margins, and other model parameters.

iii. $\hat{\theta}_{ij} < 0$ always when $N_s = 1$ and $N_r > 1$. Thus, a non-cooperative equilibrium will always be optimal when there is only one manufacturer and more than one retailers selling that manufacturer's product, regardless of the number of retailers, margins, and other model parameters.

iv. Figure 2 shows the changes in $\hat{\theta}_{ij}$ for different values of N_s and N_r. Observations ii. and iii. along with Figure 2 indicate that the likelihood of advertising cooperation increases as N_s increases, and decreases as N_r increases.

Scenario B: More generally $(r+\delta) \geq 0$

In a more general scenario when $(r+\delta) \geq 0$, we used equations (22)-(23) to compute the value of $\hat{\theta}_{ij}$ for a varied set of model parameters. In this case, Figure 3 shows the typical pattern of changes in $\hat{\theta}_{ij}$ w.r.t. N_s and N_r, and Figures 4-11 show changes in $\hat{\theta}_{ij}$ w.r.t. model parameters such as M, m, ρ, r, δ, for different values of N_s and N_r.

All these observations from both the cases, i.e., $(r+\delta) \approx 0$ and $(r+\delta) > 0$, point towards a common crucial insight on the role of competition towards the incentives for cooperation. It can be seen that the likelihood of advertising cooperation between the upstream and downstream partners in a supply chain increases as the manufacturer level competition increases, and decreases as the retail level competition increases. As more retailers compete for the market share, the retailers themselves have enough incentives on their own to advertise more, and therefore the manufacturers can afford to take a back-seat and offer less (or no) support for the local advertising. On the other hand, for a higher manufacturer level competition each manufacturer has a greater incentive to advertise its product more, and therefore, there is a greater possibility of positive subsidy rates (or in other words a non-zero subsidy rate) offered to its retailers.

3.1.2 Special case: single retailer, non-symmetric manufacturers

In this case, we assume that $\rho_{i1} = \rho_1$ and $\delta_{i1} = \delta_1$, $\forall i = 1, 2, \cdots, N_s$. However, the manufacturers need not be symmetric as we can have different margins, i.e., $M_{i1} \neq M_{k1}, i \neq k, i, k = 1, 2, \cdots, N_s$. We can solve equations (15)-(21) in a non-cooperative case to obtain

$$\beta_{1i1} = 2\frac{-(r+\delta_1) + \sqrt{(r+\delta_1)^2 + N_s\rho_1^2 C_k}}{N_s\rho_1^2},$$

where,

$$C_i = m_{i1} + \sum_{\substack{g=1 \\ g\neq i}}^{N_s} \left[2\left(\frac{m_{i1} - m_{g1}}{r+\delta_1}\right) - \left(\frac{m_{i1} - m_{g1}}{r+\delta_1}\right)^2 \right],$$

and

$$B_{ii1} = \frac{2M_{i1} + \frac{\rho_1^2 M_{i1}}{(r+\delta_1)} \sum_{\substack{g=1 \\ g\neq i}}^{N_s} \beta_{1g1}}{2(r+\delta_1) + \rho_1^2 \sum_{g=1}^{N_s} \beta_{1g1}}.$$

Furthermore, for a special case with $m_{i1} = m$ the above expressions can be further simplified as follows.

$$\beta_{1i1} = 2\frac{-(r+\delta) + \sqrt{(r+\delta)^2 + N_s m\rho^2}}{N_s\rho^2} \quad \forall i = 1, 2, \cdots, N_s, \tag{25}$$

$$B_{ii1} = \frac{M_{i1}}{N_s}\left(\frac{N_s - 1}{(r+\delta)} + \frac{1}{\sqrt{(r+\delta)^2 + N_s m\rho^2}}\right) \quad \forall i = 1, 2, \cdots, N_s. \tag{26}$$

Using (25) and (26) in (21), we also get $\forall i = 1, 2, \cdots, N_s$,

$$\hat{\theta_{i1}} = \frac{-(r+\delta_1)^3 + (M_{i1} - mNs)(r+\delta_1)\rho_1^2 + (r+\delta_1)^2\sqrt{(r+\delta_1)^2 + mNs\rho_1^2} + M_{i1}(Ns-1)\rho_1^2\sqrt{(r+\delta_1)^2 + mNs\rho^2}}{(r+\delta_1)^3 + (M_{i1} + mNs)(r+\delta_1)\rho_1^2 - (r+\delta_1)^2\sqrt{(r+\delta_1)^2 + mNs\rho_1^2} + M_{i1}(Ns-1)\rho_1^2\sqrt{(r+\delta_1)^2 + mNs\rho^2}} \tag{27}$$

From (26) and (27), we can obtain the following result.

Corollary 2: $\partial B_{ii1}/\partial N_s > 0$ *and* $\partial \hat{\theta_{i1}}/\partial N_s > 0$. *Furthermore,* $\lim_{N_s \to \infty} \hat{\theta_{i1}} = 1$.

Corollary 2 can be obtained by taking derivatives and limit w.r.t. N_s and after a few steps of algebra. It implies that in this particular case, under a non-cooperative equilibrium, the marginal benefit for manufacturers w.r.t. market share increases as manufacturer level competition increases. This is contrary to the result for retailers as shown in Corollary 1. Moreover, with the fact that $\partial \hat{\theta_{i1}}/\partial N_s > 0$, it is clear that as manufacturer level competition increases the tendency for the manufacturers to support the retailers increases and we move towards a cooperative equilibrium. Note that this result is for all values of r and δ, and even when the margins for the manufacturers are different. Finally, we find that if the number of manufacturers (brands) is very large, we are guaranteed to have a cooperative equilibrium. Under extremely high manufacturer level competition, the manufacturers will not only support the retailer, but will also tend towards taking the entire burden of local advertising.

4 Sensitivity analysis of subsidy rates: a case of two manufacturers and two retailers

In this section we study the impact of various model parameters on all the subsidy rates decisions of manufacturers in a two manufacturer two retailer channel. As it was mentioned previously, it is extremely difficult to obtain an explicit solution to the system of equations (15)-(21) in the general case. We therefore, resort to numerical analysis of these equations to get some insights. We performed numerical analysis for a large set of model parameters and in this section present some representative examples, depicting key insights obtained from all the instances that we studied. The numerical analysis was done using Mathematica 10.4. To study the impact of a single parameter, we focus on changing that parameter while keeping others constant. For illustration, we present the results from numerical analysis for two sets of model parameters. Figures 12-15 show results from one set of model parameters and Figures 16-19 from a second set of model parameters. We would like to highlight that for the second set of parameters (Figures 16-19), we take a cue from Erickson (2009 a). As mentioned earlier, Erickson (2009 a) studied an oligopolistic extension of the Sethi (1983) model and used it to study advertising competition between different brands in the American beer industry. Using real data on advertising expenditures by Anheuser-Busch, Miller, and Coors, their margins etc., Erickson (2009 a) estimated various model parameters for the sales dynamics such as advertising effectiveness and decay rate. The values of parameters in data set 2 follow the corresponding values reported in Erickson (2009 a). We report our key findings and insights below. It is noticeable that compared to data set 1, the magnitude of changes in subsidy rates are much less in data set 2. However, prominent trends are consistent in both examples.

Impact of M_{11} (Figures 12, 16): As M_{11} increases, manufacturer 1's incentive for an increase in sales via retailer 1 increases, and therefore, manufacturer 1 increases its subsidy rate for retailer 1. Manufacturer 1 also decreases its subsidy rate for retailer 2 as retailer 2's advertising yields comparatively less marginal returns. The changes in the subsidy rates by the competing manufacturer are comparatively much less pronounced. We find that manufacturer 2 slightly increases his subsidy rate for retailer 1 to counter greater advertising efforts by retailer 1 for manufacturer 1 due to higher θ_{11}. θ_{22} depicts a much more stable behaviour and does not change much, however the slight change (if any) appears to be in the downward direction.

Impact of m_{11} (Figures 13, 17): As m_{11} changes, major impacts are mainly observed in θ_{11} and θ_{21}, i.e., subsidy rates for retailer 1. As m_{11} increases, retailer 1 itself has a higher incentive of its own to increase advertising of product 1 and sell it more. Manufacturer 1 anticipates this and reduces the subsidy rate for retailer 1. To compensate for possible higher levels of u_{11} (due to higher m_{11}), the competition forces manufacturer 2 to incentivize retailer 1 to increase u_{21}, and therefore manufacturer 2 increases his subsidy rate. Hence, by and large, changes in m_{11} leads to somewhat opposite effect on subsidy rates by the two competing manufacturers. We also observe a small downward trend in θ_{12}, whereas θ_{22} remains comparatively stable

Impact of ρ_{11} (Figures 14, 18): As ρ_{11} increases, we find that θ_{21} increases. It can be argued that as retailer 1's advertising for manufacturer 1 becomes more effective, manufacturer 2 counters this higher "quality" of advertising for product 1 by a higher "quantity" of advertising of its own product through retailer 1, and therefore increases his subsidy rate. We also find that manufacturer 1 decreases its subsidy rate for retailer 2 as effectiveness of retailer 1's effort increases.

Impact of δ_{11} (Figures 15, 19): From the state dynamics, it is clear that for same levels of advertising, an increase in δ_{11} has a negative impact on x_{11} as opposed to the positive impact due to increase in ρ_{11}. This explains the fact that the effects of changes in δ_{11} on subsidy rates appear to be opposite of that of changes in ρ_{11}. We find a clear trend of decrease in θ_{21} and increase in θ_{12}.

5 Impact of coop advertising programs on payoffs and the role of competition

The decision to offer advertising support to a retailer is obviously dependent upon the benefit it may bring to the manufacturer towards his profit function. It is easy to say that when there is only one manufacturer, positive subsidy rate(s) clearly implies that the manufacturer is better off with a cooperative advertising arrangement than without it. In the presence of manufacturer level competition though, this decision is clearly a function of the actions taken by other manufacturers as well. To further investigate this question and examine the role of competition (at both the manufacturer as well as the retailer level) in our dynamic setting, we studied and compared the value functions of the players in a cooperative equilibrium (with positive optimal subsidy rates) and those in a non cooperative scenario.

We define V_{siC} as manufacturer i's value function with cooperative advertising and V_{siN} as their value functions when there is no coop advertising arrangement. Similarly, we define V_{rjC} and V_{rjN} as retailer j's value function with and without a coop advertising arrangement, respectively. V_{siC} and V_{riC} are obtained by solving equations (15)-(21) and then using (10)-(11). To obtain V_{siN} and V_{riN}, we solve (15)-(21) while enforcing using $\theta_{ij} = 0, \forall i, j.$ and then use the resulting solution in (10)-(11).

5.1 Impact of manufacturer and retail level competition in the case of symmetric manufacturers and symmetric retailers

To gauge the impact of competition, we compared value functions in coop and non coop equilibrium for several values of N_s and N_r, while keeping all the model parameters at a symmetric level for manufacturers and retailers, i.e., $M_{ij} = M, m_{ij} = m, \rho_{ij} = \rho, \delta_{ij} = \delta$, $\forall i, j, i = 1, 2, \cdots, N_s, j = 1, 2, \cdots, N_r$. To focus on N_s and N_r, we also assumed equal initial state for all the state variables, i.e., $x_{ij} = x$, $\forall i, j, i = 1, 2, \cdots, N_s, j = 1, 2, \cdots, N_r$. It is to be noted that under these circumstances, the value functions of all the manufacturers are equal and similarly those of all the retailers are equal as well.

In this case of symmetric manufacturers and symmetric retailers, a non cooperative equilibrium can be easily obtained in an explicit form using the results in Section 3.1.1. For a cooperative equilibrium, however, we once again have to turn to numerical analysis of equations (15)-(21), which we performed for a wide array of parameters. In all the instances we found that the manufacturers are always worse off with a coop advertising program than without it. The % decrease in manufacturer's profit due in a coop advertising program, i.e., $100 * \frac{(V_{siN} - V_{siC})}{V_{siN}}$ depends not only on the level of competition at two levels, but also on the way competition impacts the total captured market. In this regard, we considered two cases. In the first case, the entry of say a new competitor proportionally reduces the market share of all the previous brands, while keeping the overall captured market share, i.e., $X = \sum_{i=1}^{Ns} \sum_{j=1}^{Nr} x_{ij}$, remained constant. E.g., if there are two manufacturers and two retailers, and if the initial state variable has the value of 0.15 for all sales streams, i.e., $x_{ij} = 0.15, \forall i, j = 1, 2, \implies X = 0.6$, an entry of a third manufacture will mean: $X = 0.6 \implies x_{ij} = 0.1, \forall i = 1, 2, 3,\ j = 1, 2$. Figures (20) and (21) show typical behavior of the loss in the value functions of all the manufactures in this case when the total captured market is "high" ($X = 0.8$), and "low" ($X = 0.2$), respectively. We find that as the level of manufacturer competition increases, the loss to each manufacturer due to a coop advertising program increases. The second case is when the entry of a new brand (competitor) does not affect the market shares of the existing manufacturers, but captures a new market segment, thereby increasing the total captured market. Here we assume that the initial values of all the state variables are equal, including that of the new entrant. E.g., say if there are two manufacturers and two retailers and the initial state variable has the value of 0.1 for all sales streams, i.e., $x_{ij} = 0.1, \forall i, j = 1, 2, \implies X = 0.4$, an entry of a third manufacture will mean: $x_{ij} = 0.1, \forall i = 1, 2, 3,\ j = 1, 2, \implies X = 0.6$. Figure (22) shows a typical observed pattern in this case. The % loss for each manufacturer increases with competition at low values of N_s, but at high levels of competition the loss may actually start to decrease, which could be attributed to the increase in the total captured market due to a new competitor. Regarding the retailers' value functions, we find that the retailers benefit in all instances when there is a manufacturer level competition, i.e., $Ns \geq 2$.

5.2 A case of two manufacturers and two retailers, asymmetric players

In the case of two manufacturers and two retailers, we compared the value functions with and without coop advertising while changing different model parameters. To focus on the impact of a particular model parameter, say, e.g. M_{11}, we start with a base case of symmetric players, i.e., $M_{ij} = M, m_{ij} = m, \rho_{ij} = \rho, \delta_{ij} = \delta, x_{ij} = x,\ \forall i, j = 1, 2$, and then change the value of that particular parameter to consider asymmetric nature of players. Again, we considered a wide array variety of parameters to solve equations (15)-(21) in coop and non coop scenarios, and summarize our key observations through some representative results shown in Figures 23-30, which depict the ratio of coop to non coop value functions. The impact of a parameter on the benefit of a coop advertising program for different players can be related to its impact on the optimal subsidy rates in a coop advertising programs (Section 4).

Impact of M_{11} (Figures 23, 24): We find that both manufactures are worse off with coop advertising as long as their margins are "not too different". However, for larger differences in the margins, we find that the manufacturer who earns a substantially higher margin as compared to its competitor may actually benefit from coop advertising while its competitor loses. At high values of M_{11}, manufacturer 1 is better off with coop advertising while manufacturer 2 is worse off. We find that if M_{11} is large (small) enough, retailer 2(1) might start to lose from a coop advertising arrangement. This could be related to our observations in Section 4 (Figure 12). As M_{11} increases, there is overall more support for retailer 1 while less for retailer 2, and so retailer 1 gains more from a coop advertising program and retailer 2 might actually lose from it.

Impact of m_{11} (Figures 25, 26): As m_{11} increases and gets large enough, manufacturer 1 might lose from coop advertising and manufacturer 2 might start gaining from it. As has been discussed previously, higher m_{11} implies a higher incentive for retailer 1 to advertise for manufacturer 1, and therefore the incentive for the manufacturer 1 to offer a positive subsidy rate decreases. Interestingly, a high margin for one of the products (m_{11}) for retailer 1 might mean that the retailer is better off without coop advertising. It should be noted that as m_{11} increases, the profit for retailer 1 increases in both the scenarios, i.e., with and without coop advertising.

Impact of ρ_{11} (Figures 27, 28) and δ_{11} (Figures 29, 30): Similar to our observations regarding the subsidy rates, changes in ρ_{11} and δ_{11} appear to have opposite effects on the ratio of coop to non-coop value functions. These observations could be tied to the fact that ρ_{11} has a positive impact on the market share x_{11} and δ_{11} has a negative impact on it. At low values of ρ_{11}, it is possible that manufacturer 1 is better off with coop advertising while manufacturer 2 is worse off, and vice versa. As ρ_{11} increases, the benefit from a coop advertising program increases for retailer 1 and decreases for retailer 2. Opposite trends are obtained w.r.t. δ_{11}. The larger insight appears to be the following. When there is a greater 'need' for higher advertising (due to low effectiveness or high decay rate) for manufacturer 1 through retailer 1, then manufacturer 1 might be better off with a coop advertising arrangement as such program will lead to overall higher advertising efforts thereby satisfying its 'need'. Furthermore, in that case, an overall low level of advertising might be better for manufacturer 2 (as low advertising levels will hurt manufacturer 1), and therefore manufacturer 2 might be worse off with a coop advertising program.

6 An Extension: Model with National advertising

In this segment we extend our model to include national advertising by the manufacturers in addition to local level advertising carried by the retailers. We assume that each manufacturer, say manufacturer i, carries a national advertising effort $v_i(t)$ which positively impacts its sales through all the retailers with an appropriate national advertising effectiveness parameter σ_i. The cost of this national advertising effort by manufacturer i is equal to $v_i^2(t)$. Each manufacturer now announces its national advertising effort and its subsidy rate for each retailer, and the retailers in response decide their advertising efforts over time for the products of all the manufacturers. We re-write the state dynamics (1) as,

$$\dot{x}_{ij} = \frac{dx_{ij}}{dt} = \left(\rho_{ij} u_{ij(t)} + \sigma_i v_i(t)\right) \sqrt{1-X} - \delta_{ij} x_{ij},$$

and the manufacturer i's objective function (2) as.

$$\max_{\substack{\theta_{ij}(t)\in[0,1] \\ j=1,2,\cdots,N_r}} \left\{ J_{si} = \int_{t=0}^{\infty} e^{-rt} \left[\sum_{j=1}^{Nr} \left[M_{ij} x_{ij}(t) - \theta_{ij}(t) u_{ij}^2(t) \right] - v_i^2(t) \right] dt \right\},$$

Retailer j's objective function is same as (3). Similar to previous sections, to simplify the analysis and presentation, we will simplify the notation and write the decision variables as simply θ_{ij}, v_i, and u_i in the rest of the section. We now re-write the Hamilton-Jacobi-Bellman (HJB) equations for manufacturer i (4) and retailer j (5), respectively, as

$$rV_{si} = \max_{\substack{\theta_{ij} \\ j=1,2,\cdots N_r}} \left[\sum_{j=1}^{Nr} \left(M_{ij} x_{ij} - \theta_{ij} u_{ij}^2 \right) - v_i^2 + \sum_{k=1}^{Ns} \sum_{j=1}^{Nr} \frac{\partial V_{si}}{\partial x_{kj}} \left((\rho_{kj} u_{kj} + \sigma_k v_k)\sqrt{1-X} - \delta_{kj} x_{kj} \right) \right], \tag{28}$$

$$rV_{rj} = \max_{\substack{u_{ij} \\ i=1,2,\cdots N_s}} \left[\sum_{i=1}^{Ns} \left(m_{ij} x_{ij} - (1-\theta_{ij}) u_{ij}^2 \right) + \sum_{i=1}^{Ns} \sum_{l=1}^{Nr} \frac{\partial V_{rj}}{\partial x_{il}} \left((\rho_{il} u_{il} + \sigma_i v_i)\sqrt{1-X} - \delta_{il} x_{il} \right) \right]. \tag{29}$$

Following same procedure as in Section 3, we use backward induction to first obtain optimal advertising efforts for all the retailers. We then use the optimal local advertising efforts to rewrite HJB equations in (28) and (29), and then obtain optimal subsidy rates and national advertising effort for each manufacturer. The expressions for the optimal local advertising efforts, and that of optimal subsidy rates are same as in the original model and are given by equations (6) and (9), respectively. The optimal national advertising effort for manufacturer i, $\forall i, i = 1, 2, \cdots, N_s$, in feedback form is given by

$$v_i^* = \frac{\sigma_i \sqrt{1-X}}{2} \sum_{q=1}^{Nr} \frac{\partial V_{si}}{\partial x_{iq}}. \tag{30}$$

In equation (30), the term $\sum_{q=1}^{Nr} \frac{\partial V_{si}}{\partial x_{iq}}$ is the marginal benefit for manufacturer i w.r.t. total market share through all the retailers, and manufacturer's advertising effort increases as this term increases. The national advertising effort does not depend on the impact of changes in the market share through any single retailer, but on that of cumulative market share over all the retailers.

Similar to Section 3, we consider linear value functions of the form (10)-(11). Using this form of linear value functions and (12)-(13), we can write the national advertising efforts by the manufacturers in (30) as

$$v_i^* = \frac{\sigma_i \sqrt{1-X}}{2} \sum_{q=1}^{Nr} B_{iiq}. \tag{31}$$

We then obtain the following set of non-linear equations in the value function coefficients, similar to (15)-(21).

$$\begin{aligned} rA_i = &- \sum_{h=1}^{Nr} \frac{\theta_{ih}^* \beta_{hih}^2 \rho_{ih}^2}{4(1-\theta_{ih}^*)^2} + \sum_{g=1}^{Ns} \sum_{h=1}^{Nr} B_{igh} \frac{\beta_{hgh} \rho_{gh}^2}{2(1-\theta_{gh}^*)} \\ &- \frac{\sigma_i^2}{4} \left(\sum_{q=1}^{N_r} B_{iiq} \right)^2 + \sum_{g=1}^{Ns} \sum_{h=1}^{Nr} \frac{B_{igh} \sigma_g^2}{2} \left(\sum_{q=1}^{N_r} B_{ggq} \right) \end{aligned} \tag{32}$$

$$\begin{aligned} (r+\delta_{il}) B_{iil} = &M_{il} + \sum_{h=1}^{Nr} \frac{\theta_{ih}^* \beta_{hih}^2 \rho_{ih}^2}{4(1-\theta_{ih}^*)^2} - \sum_{g=1}^{Ns} \sum_{h=1}^{Nr} B_{igh} \frac{\beta_{hgh} \rho_{gh}^2}{2(1-\theta_{gh}^*)} \\ &+ \frac{\sigma_i^2}{4} \left(\sum_{q=1}^{N_r} B_{iiq} \right)^2 - \sum_{g=1}^{Ns} \sum_{h=1}^{Nr} \frac{B_{igh} \sigma_g^2}{2} \left(\sum_{q=1}^{N_r} B_{ggq} \right) \end{aligned} \tag{33}$$

$$\begin{aligned} (r+\delta_{kl}) B_{ikl} = &\sum_{h=1}^{Nr} \frac{\theta_{ih}^* \beta_{hih}^2 \rho_{ih}^2}{4(1-\theta_{ih}^*)^2} - \sum_{g=1}^{Ns} \sum_{h=1}^{Nr} B_{igh} \frac{\beta_{hgh} \rho_{gh}^2}{2(1-\theta_{gh}^*)} \\ &+ \frac{\sigma_i^2}{4} \left(\sum_{q=1}^{N_r} B_{iiq} \right)^2 - \sum_{g=1}^{Ns} \sum_{h=1}^{Nr} \frac{B_{igh} \sigma_g^2}{2} \left(\sum_{q=1}^{N_r} B_{ggq} \right) \quad k \neq i \end{aligned} \tag{34}$$

$$
\begin{aligned}
r\alpha_j =& -\sum_{g=1}^{Ns} \frac{\beta_{jgj}^2 \rho_{gj}^2}{4(1-\theta_{gj}^*)} + \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \beta_{jgh} \frac{\beta_{hgh}\rho_{gh}^2}{2(1-\theta_{gh}^*)} \\
&+ \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \frac{\beta_{jgh}\sigma_g^2}{2} \left(\sum_{q=1}^{N_r} B_{ggq} \right) \qquad (35) \\
(r+\delta_{kj})\beta_{jkj} =& m_{kj} + \sum_{g=1}^{Ns} \frac{\beta_{jgj}^2 \rho_{gj}^2}{4(1-\theta_{gj}^*)} - \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \beta_{jgh} \frac{\beta_{hgh}\rho_{gh}^2}{2(1-\theta_{gh}^*)} \\
&- \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \frac{\beta_{jgh}\sigma_g^2}{2} \left(\sum_{q=1}^{N_r} B_{ggq} \right) \qquad (36) \\
(r+\delta_{kl})\beta_{jkl} =& \sum_{g=1}^{Ns} \frac{\beta_{jgj}^2 \rho_{gj}^2}{4(1-\theta_{gj}^*)} - \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \beta_{jgh} \frac{\beta_{hgh}\rho_{gh}^2}{2(1-\theta_{gh}^*)} \\
&- \sum_{g=1}^{Ns}\sum_{h=1}^{Nr} \frac{\beta_{jgh}\sigma_g^2}{2} \left(\sum_{q=1}^{N_r} B_{ggq} \right) \quad \forall l \neq j \qquad (37) \\
\theta_{ij}^* =& Max\left[\hat{\theta}_{ij}, 0\right],\ \hat{\theta}_{ij} = \frac{2B_{iij} - \beta_{jij}}{2B_{iij} + \beta_{jij}},\ i = 1,2,\cdots N_s,\ j = 1,2,\cdots N_r \qquad (38)
\end{aligned}
$$

We can then conclude that a feedback Stackelberg equilibrium is obtained by solving the non-linear algebraic equations in (32)-(38). The manufacturer's optimal subsidy rates are given by (38), each manufacturer's optimal national advertising effort is given by (31), and the retailers' optimal responses, i.e., their local advertising efforts can be obtained by (14). It can be seen that the above system of equations (32)-(38) is far more complicated than (15)-(21) with more complex terms involving interaction between manufacturers' and retailers' value function coefficients. It is very difficult to obtain a closed form analytical solution for the above system of equations, even for a special case of identical retailers and identical manufacturers as was done in Section 3. However, we can obtain useful insights by numerically solving these equations. We discuss some of these insights in the following segment.

6.1 Numerical Analysis

We performed numerical analysis for a a wide set of model parameters and present here some representative results depicting the wider insights that we obtained across all experiments. We look at the impact of different model parameters on the local and national advertising efforts and on the subsidy rates. Since the local and national advertising efforts depend on overall uncaptured market share ($\sqrt{1-X}$), we analyze advertising efforts for unit value of uncaptured market share, i.e., $v^*/\sqrt{1-X}$, and $u^*/\sqrt{1-X}$. In the case of identical retailers and identical manufacturers, we also look at the ratios of the two advertising efforts and the ratio of local and national advertising expenses. One can easily see that in these rations the term that depends on X gets cancelled out.

(i) Impact of competition, a case of identical retailers and identical manufacturers (Figures 31 - 36).

To focus on the impact of changes in N_s and N_r, we analyze a case of identical manufactures and identical retailers with $M_{ij} = M, m_{ij} = m, \rho_{ij} = \rho, \sigma_i = \sigma, \delta_{ij} = \delta, \forall i, j$. We find that as N_s or N_r increase, the local advertising effort by each retailer decreases and national advertising effort by each manufacturer increases. However, we find that the impact of changes in N_s is much less in magnitude than the impact of changes in N_r. Figure 34 shows $N_r * u$ which is the total local advertising effort by all retailers for any single manufacturer's product, for a unit uncaptured market. We find that as retailer level competition increases the total local advertising for a particular brand increases, however, as the manufacturer level competition increases the this total local advertising effort decreases at comparatively very small rate. Figure (31) shows that as the manufacturer level competition increases, the common subsidy rate increases. The impact of N_r however is not straightforward. In our numerical experiments we find that at low levels of manufacturer level competition, the subsidy rate decreases as the number of retailers increases, with all other parameters remaining constant. However, when the number of brands is relatively high, the subsidy rate increases as the number of retailers increases. To get grater insights on the relative dominance of national vs local advertising, we also look at the ratios of total local vs national advertising, and the ratio of total expenditure in local vs national advertising. Figure (35) shows the ratio $N_r * u/v$, which is the ratio of total local advertising for a single brand by all N_r retailers over the total national advertising by that manufacturer, and this is same for all brands as we are dealing with the case of identical players. If we multiply N_s in both the numerator and the denominator in this ratio, we can see that $N_r * u/v$ is also the ratio of total local advertising by all retailers over total national advertising by all manufactures in the market. Similarly, Figure (36) depicts $N_r * u^2/v^2$ which is the total local advertising expense by all the retailers for a single brand over the total national advertising for that brand. $N_r * u^2/v^2$ is also equal to the total local advertising expense over the total national advertising expense in the market. We find that the behaviour of the ratio of total local over national advertising effort is somewhat opposite of that of the common subsidy rate. This ratio marginally decreases as N_s increases when N_r is fixed. At low values of N_s this ratio increases with the number of retailers, whereas at relatively high values of N_s, it decreases with the number of retailers. We also find that the ratio of total local over national advertising expense decreases with the number of retailers and marginally decreases with the number of manufacturers.

(ii) Impact of margins and advertising effectiveness, a case of two non-identical manufacturers and two non-identical retailers (Figures 37 - 48).

(a) Impact of M_{11} (Figures 37-39): As M_{11} increases we observe that all subsidy rates by the manufacturers increase (at different rates) with the exception of θ_{12} which is the subsidy rate by manufacturer 1 to retailer 2. This pattern have been observed in the original model (Section 4) as well. Following this pattern of incentives offered by the manufacturers, the respective local advertising efforts also follow the same behaviour (Figure 38). Also, Figure 39 indicates that national advertising by both manufactures increase, as well as total local advertising for both brand by the two retailers (by and large). A slight contrary result is seen when M_{11} is too small for manufacturer 1 to offer any subsidy to retailer 1, and in this zone as M_{11} increases the total local advertising for manufacturer 1 decreases due to heavy decrease in u_{12} and a stable u_{11}. However, once the margin M_{11} is large enough to have a positive subsidy for retailer 1, the local advertising increases with M_{11} as significant increase u_{11} overtakes the decrease in u_{12}. We also observe that rate of increase of advertising for product 1 (both national and total local) is higher than that for product 2.

(b) Impact of m_{11} (Figures 40-42): The changes in subsidy rates as m_{11} changes are very similar to those in the original model (Section 4), i.e., θ_{11} decreases sharply, manufacturer 2 increases θ_{21} to fill the space left by manufacturer 1, and the subsidy rates for retailer 2 decrease at a small rate. As far as local advertising efforts are concerned, we see that u_{11} increases due to higher incentive for retailer 1 (higher m_{11}), and all the other local advertising efforts decrease. In Figure 42 we see that the total local advertising effort for product 1 increases (due to increase in u_{11}). The total local advertising for product 2 and the national advertising efforts by both the manufacturers decrease as m_{11} increases.

(c) Impact of ρ_{11} (Figures 43-45): The impact of changes in ρ_{11} on subsidy rates is very similar to the original model in section 3, as shown in Figure 43. As product 1's advertising becomes more effective for retailer 1, it increases its local advertising for product 1. Because of competition, manufacturer 2's support to retailer1 and hence $u21$ also increase. Local advertising by retailer 2 for both the product decreases due to lower incentives. Overall, however, we find that the total local advertising for both the products increases. National advertising by manufacturer 1 decreases but that by manufacturer 2 increases slightly.

(d) Impact of σ_1 (Figures 46-48): As σ_1 increases and therefore manufacturer 1's national advertising becomes more effective, he finds greater value in increasing his national advertising effort and reduces the support for the local advertising and the retailers follow by reducing their advertising for manufacturer 1. Manufacturer 2 on the other hand increases its subsidy rate for both the retailers and the retailers follow by increasing local advertising for retailer 2. Manufacturer 1 increases his national advertising effort as it becomes more effective and the competition makes manufacturer 2 to increase its national advertising as well. Manufacturer 1's national advertising however increases at a higher rate than that of manufacturer 2.

7 Concluding Remarks

Use of subsidy rates is a very useful and common mechanism used by manufacturers to support their retailers' advertising efforts and influence their advertising decisions. While there are several models to study subsidy rates decisions, relatively very few focus on competition between manufacturers, and to the best of our knowledge, none that account for competition at retailer and manufacturer levels simultaneously. We present a dynamic game involving an arbitrary number of manufacturers and retailers. We obtain the optimal subsidy rates for all the manufacturers and optimal advertising efforts for all the retailers in feedback form, where the state variables represent proportional market shares. We obtain several useful insights, sometimes by analytical and sometimes by numerical means. We investigate the role of competition at the manufacturer as well as at the retailer level. A broader insight appears that in a coop advertising arrangement, high competition at the manufacturer level pushes the manufactures towards optimal positive subsidy rates, while higher competition at the retailer level pushes towards lesser support. We find instances of cases when a co-operation (or no-cooperation) is always optimal. We analyse the impact of various model parameters on the optimal subsidy rates. We also study the benefit of a coop advertising program on manufacturers and retailers, and find that it depends on the level of competition and various other model parameters. Interestingly, our analysis indicates that as long as the manufacturers and retailers are "somewhat symmetric/homogeneous" with respect to their horizontal competition, the manufacturers always seem to lose from a coop advertising program while the retailers seem to benefit from it. This creates a prisoner's dilemma situation for the manufacturers in the case of symmetric manufacturers and symmetric retailers. The extent of loss for the manufacturers also depends on the level of competition. However, if there are sufficient differences between the manufacturers or retailers, it is possible that one manufacturer benefits while the other loses, and similarly one retailer benefits while the other loses from a cooperative advertising arrangement. This insight can create interesting incentives for different players vis-à-vis their parameters. For example, we find that manufacturer 1 may be better off with a coop advertising program and manufacturer 2 worse off if M_{11} is high enough. If manufacturer 1 is aware of this, it is in its interest to push for a coop advertising arrangement with its retailers. Manufacturer 2 might not like this arrangement but has to respond to it by following it as it loses more by not offering cooperation to its retailers than by offering it. Finally, we extend our model to include national level advertising in addition to local advertising carried by the retailers and obtain insights on how the two types of advertising decisions change with different model parameters.

Figures

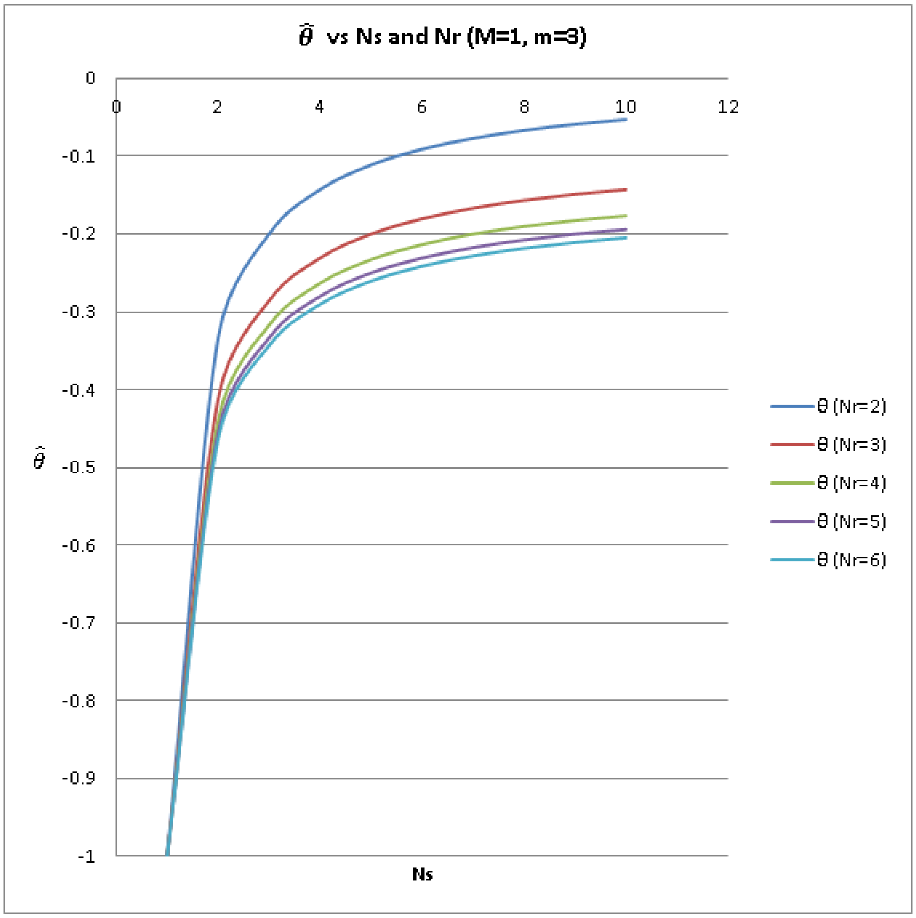


Figure 2: $\hat{\theta}_{ij}$ vs N_s, N_r in non-cooperative equilibrium, symmetric players with $(r + \delta) \approx 0$

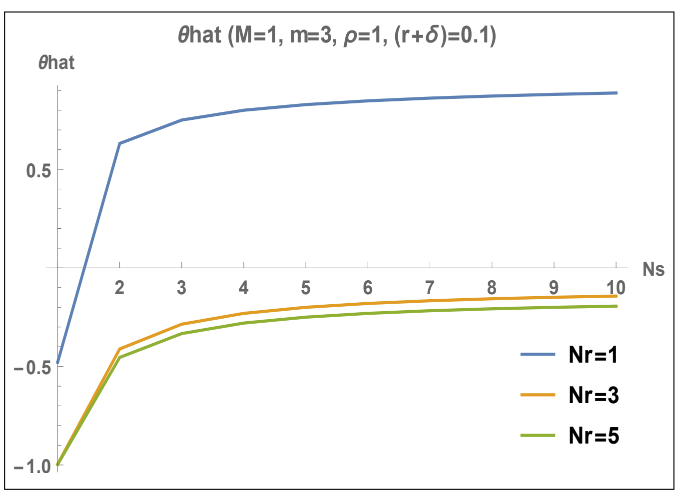


Figure 3: $\hat{\theta}_{ij}$ vs N_s

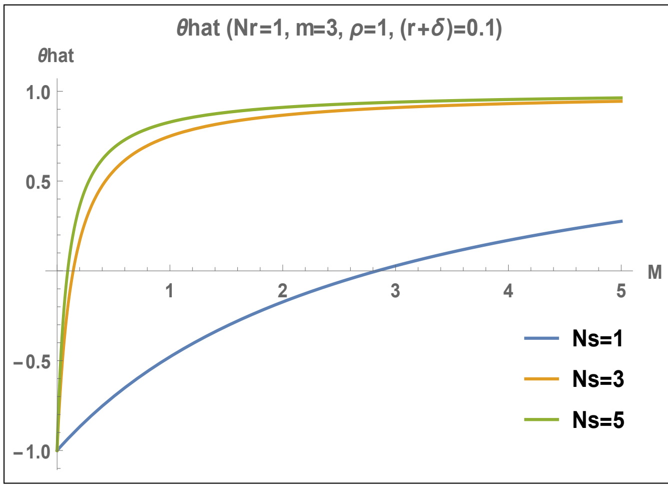


Figure 4: $\hat{\theta}_{ij}$ vs $M(N_r = 1)$

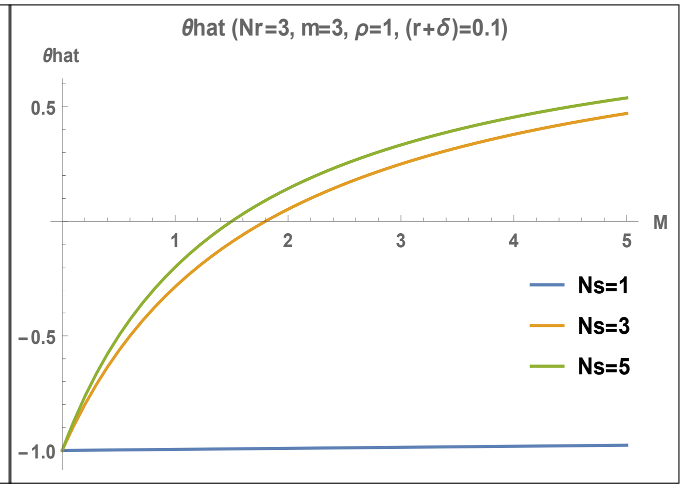


Figure 5: $\hat{\theta}_{ij}$ vs $M(N_r = 3)$

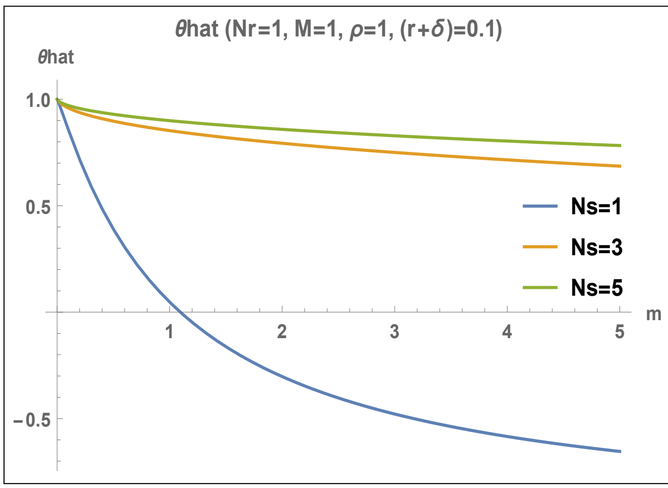


Figure 6: $\hat{\theta}_{ij}$ vs $m(N_r = 1)$

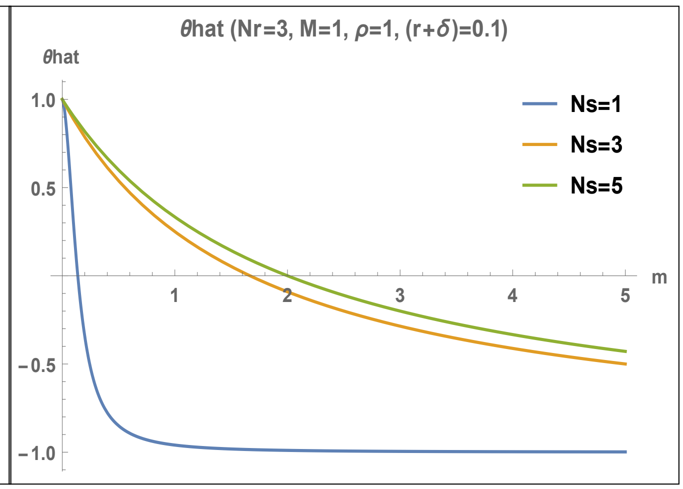


Figure 7: $\hat{\theta}_{ij}$ vs $m(N_r = 3)$

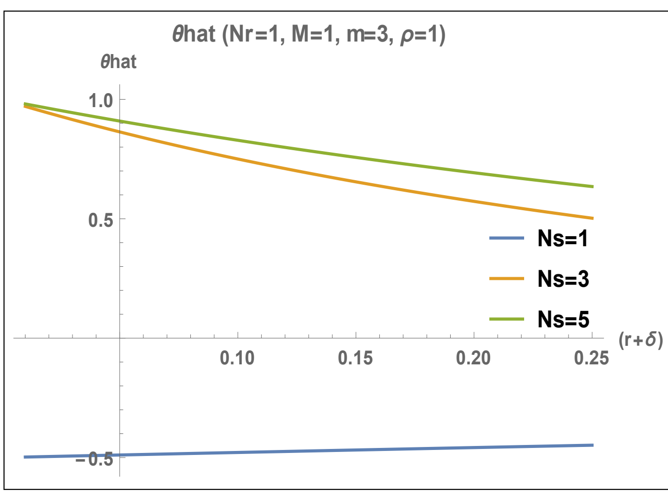


Figure 8: $\hat{\theta}_{ij}$ vs $r + \delta(N_r = 1)$

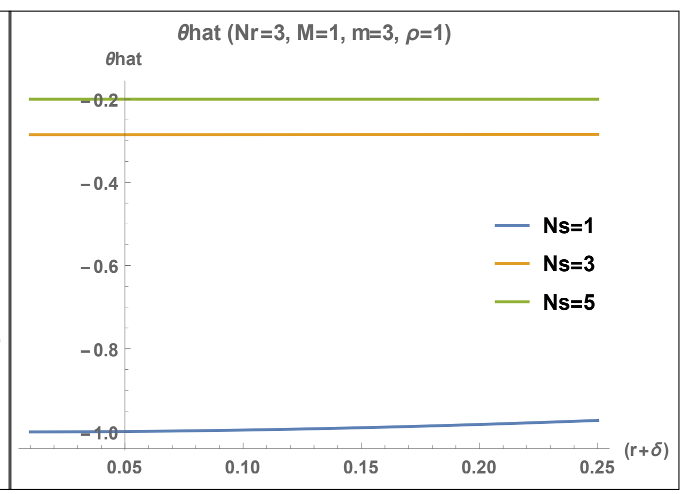
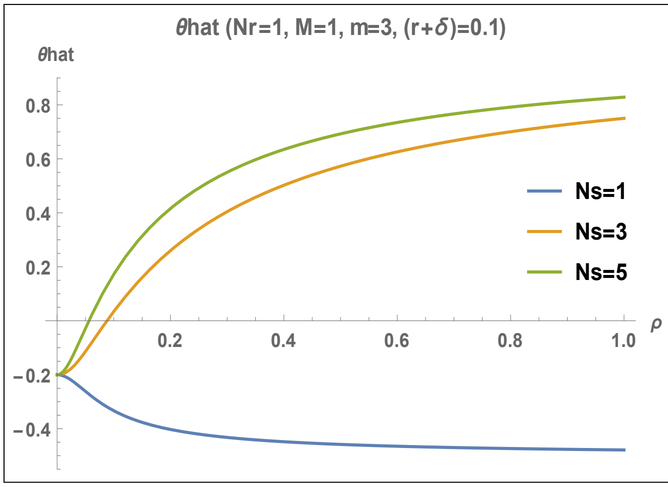


Figure 9: $\hat{\theta}_{ij}$ vs $r + \delta(N_r = 3)$



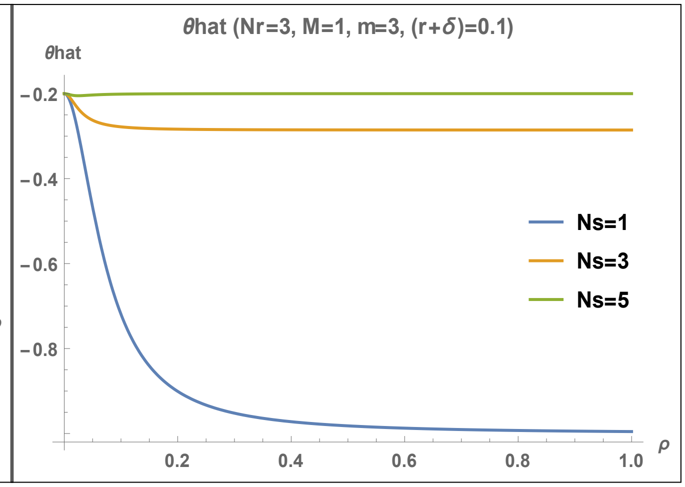
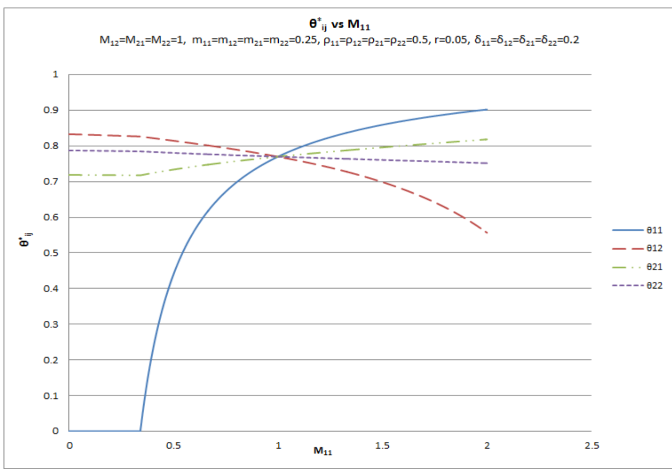


Figure 10: $\hat{\theta}_{ij}$ vs $\rho(N_r = 1)$

Figure 11: $\hat{\theta}_{ij}$ vs $\rho(N_r = 3)$



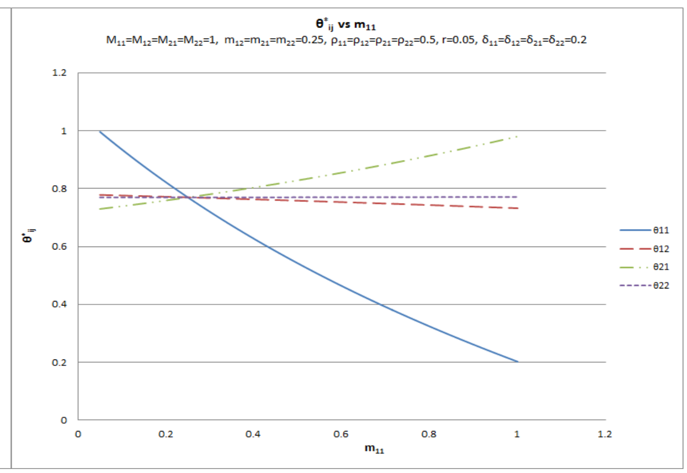
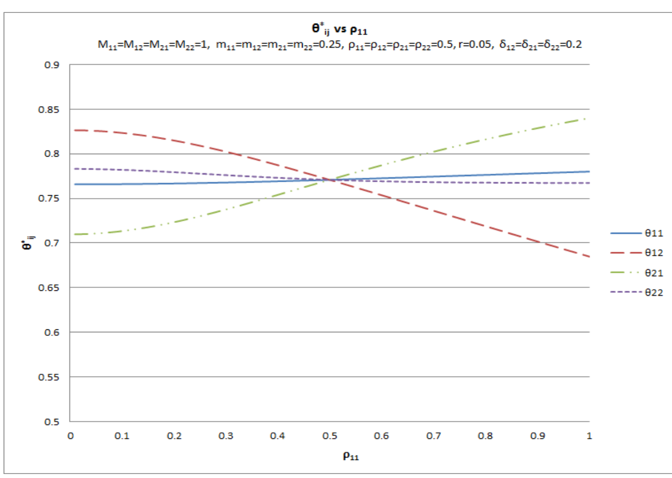


Figure 12: θ^*_{ij} vs M_{11}

Figure 13: θ^*_{ij} vs m_{11}



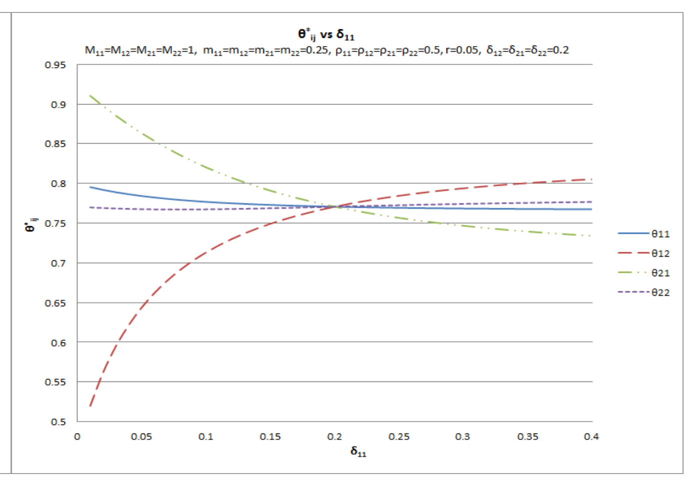


Figure 14: θ^*_{ij} vs ρ_{11}

Figure 15: θ^*_{ij} vs δ_{11}

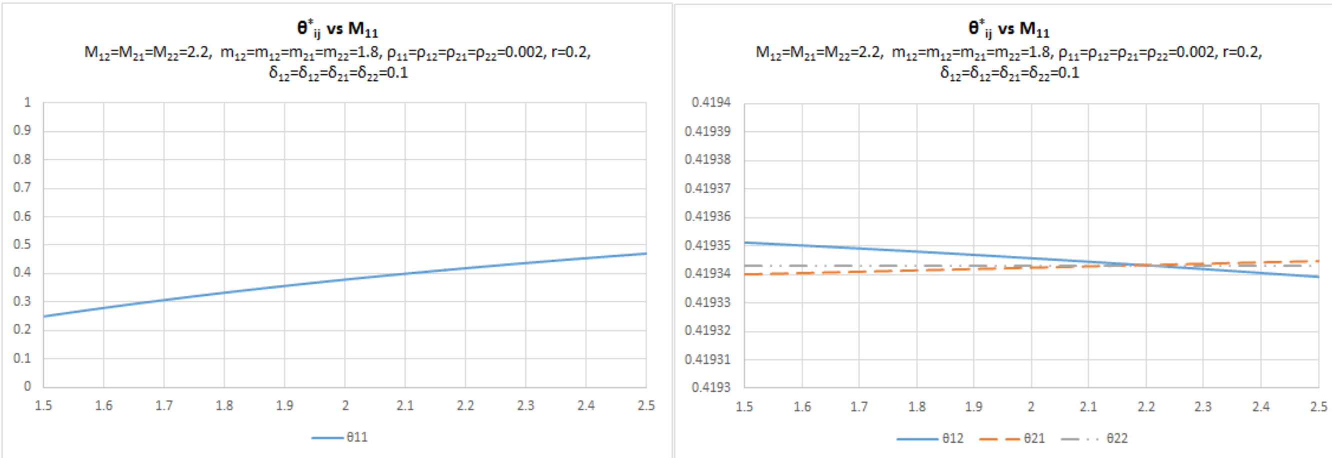


Figure 16: θ^*_{ij} vs M_{11} (dataset 2)

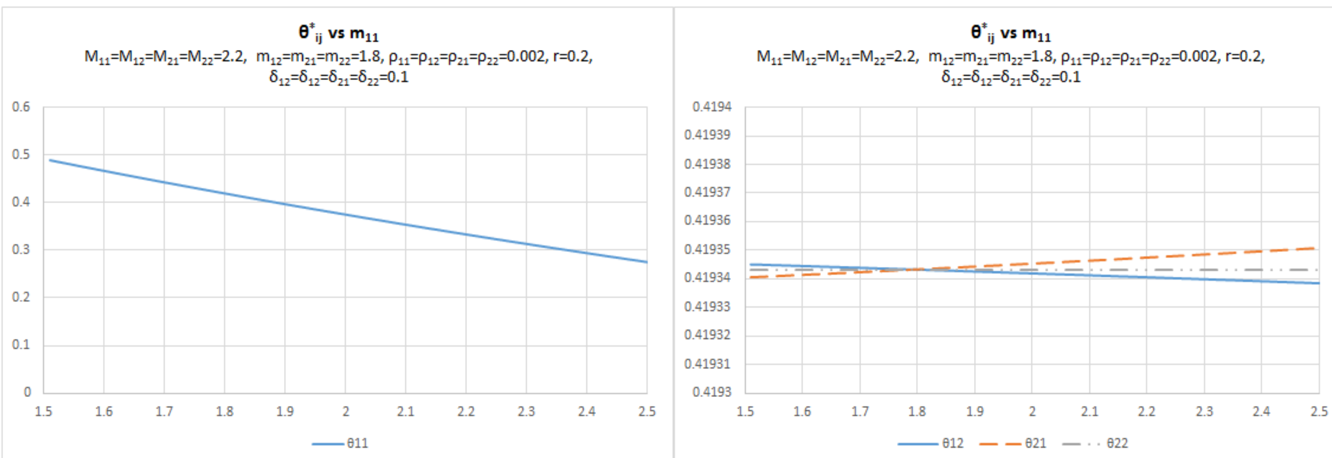


Figure 17: θ^*_{ij} vs m_{11} (dataset 2)

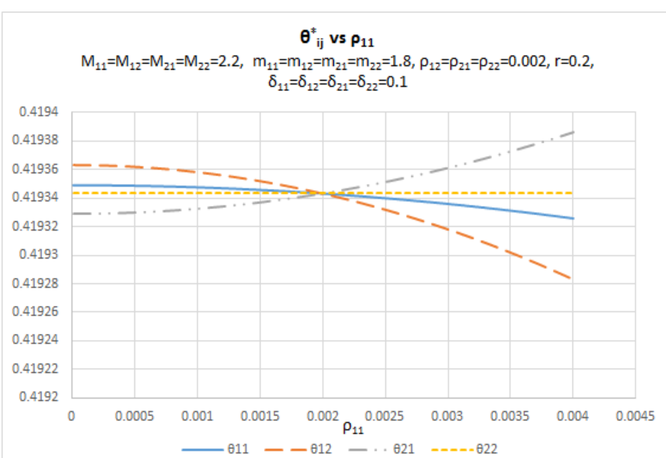


Figure 18: θ^*_{ij} vs ρ_{11} (dataset 2)

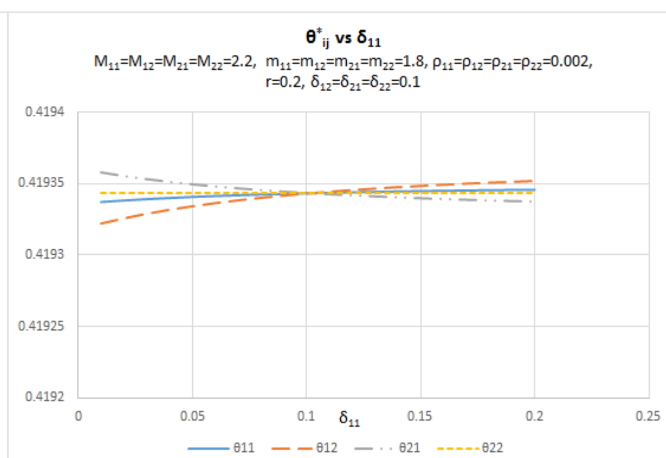


Figure 19: θ^*_{ij} vs δ_{11} (dataset 2)

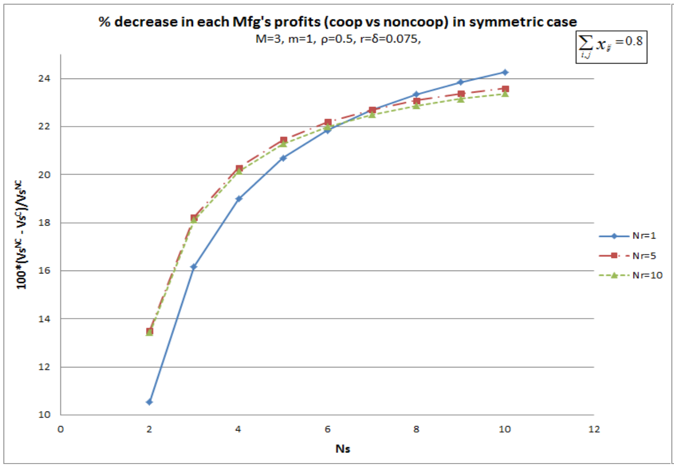


Figure 20: % decrease in each mfg's profit (coop vs noncoop) vs Ns, $\sum_{i,j} x_{ij} = 0.8$

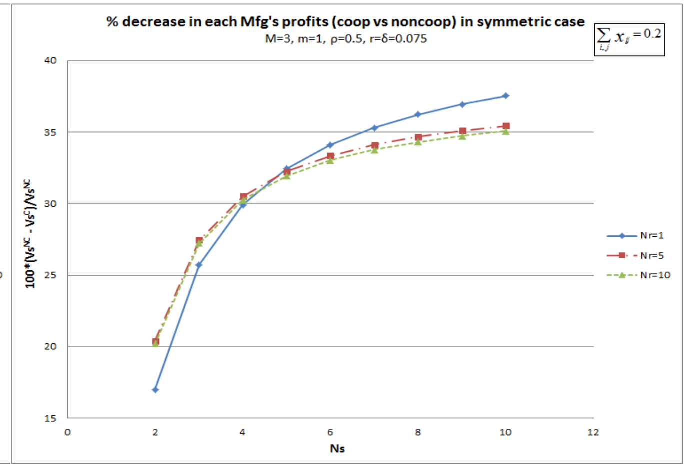


Figure 21: % decrease in each mfg's profit (coop vs noncoop) vs Ns, $\sum_{i,j} x_{ij} = 0.2$

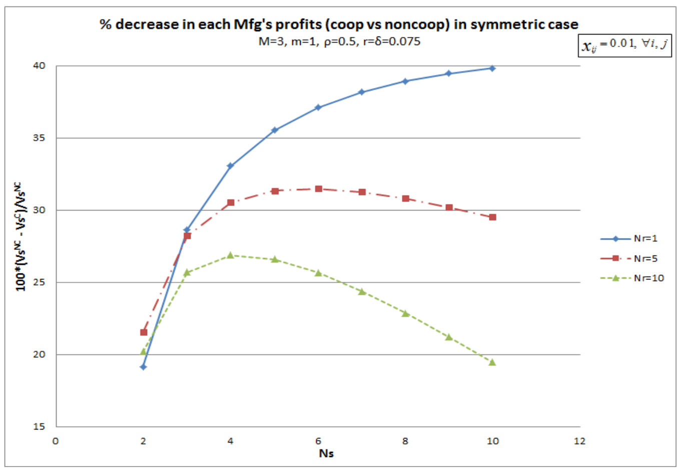


Figure 22: % decrease in each mfg's profit (coop vs noncoop) vs Ns, $x_{ij} = 0.01, \forall i, j$

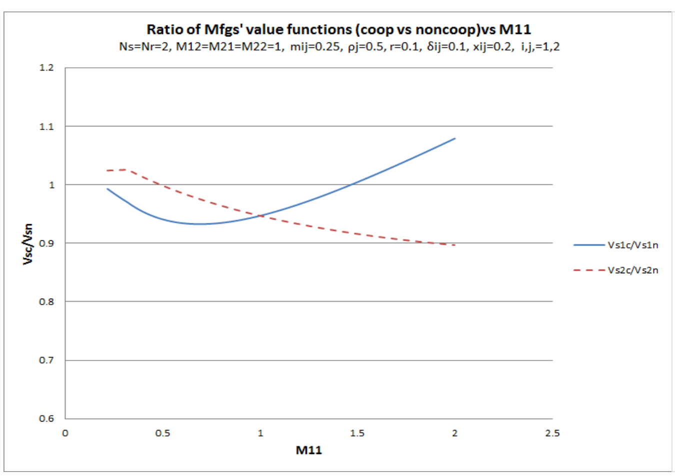


Figure 23: Ratio of Mfgs' value functions (coop/noncoop) vs M_{11}

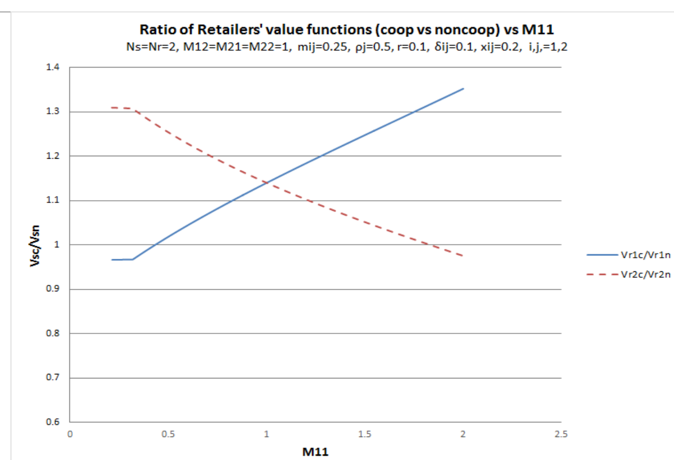


Figure 24: Ratio of Retailers' value functions (coop/noncoop) vs M_{11}

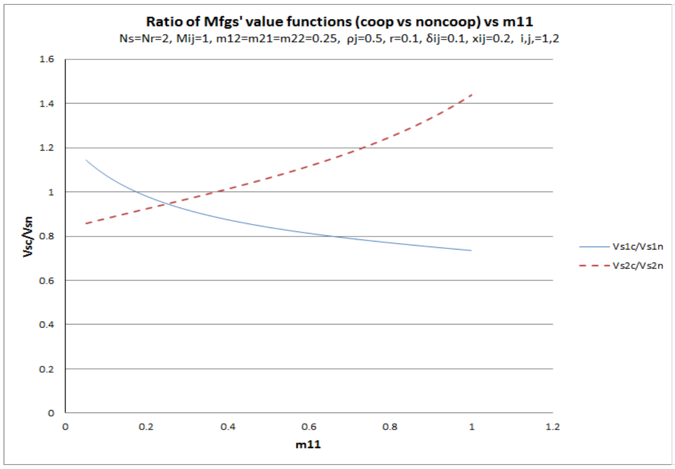


Figure 25: Ratio of Mfgs' value functions (coop/noncoop) vs m_{11}

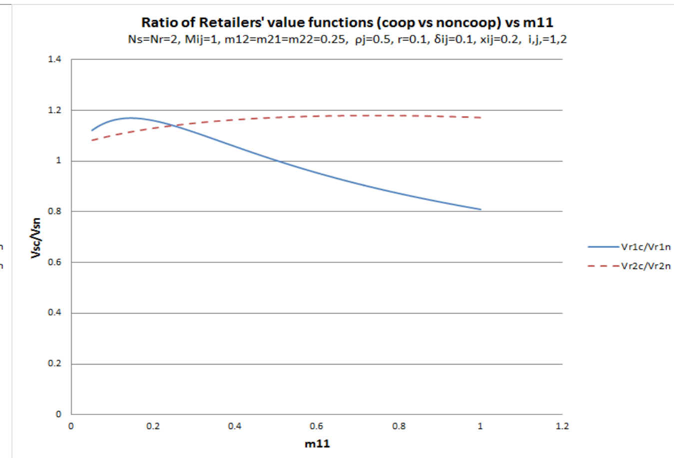


Figure 26: Ratio of Retailers' value functions (coop/noncoop) vs m_{11}

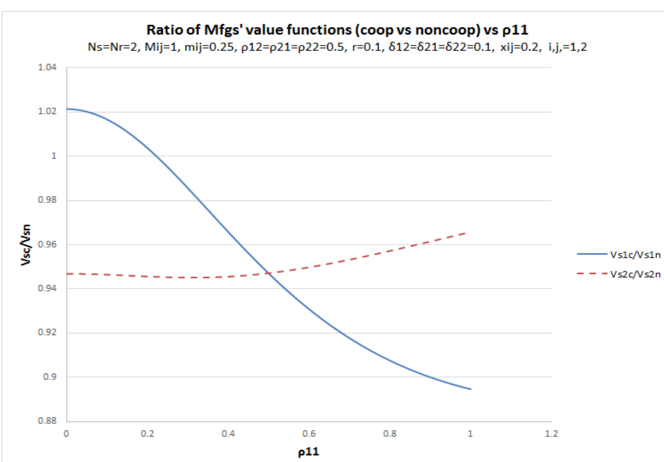


Figure 27: Ratio of Mfgs' value functions (coop/noncoop) vs ρ_{11}

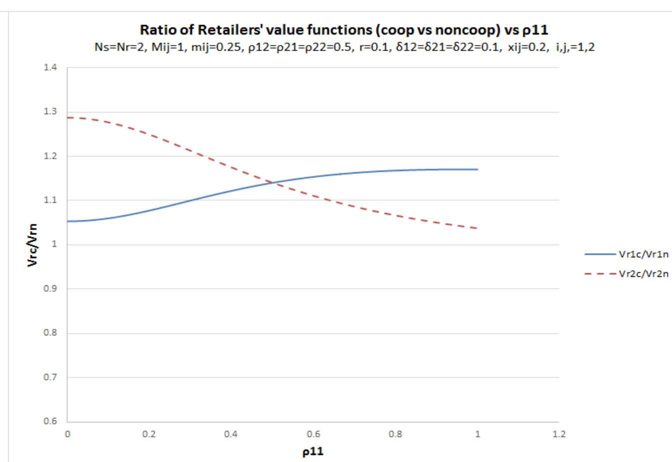


Figure 28: Ratio of Retailers' value functions (coop/noncoop) vs ρ_{11}

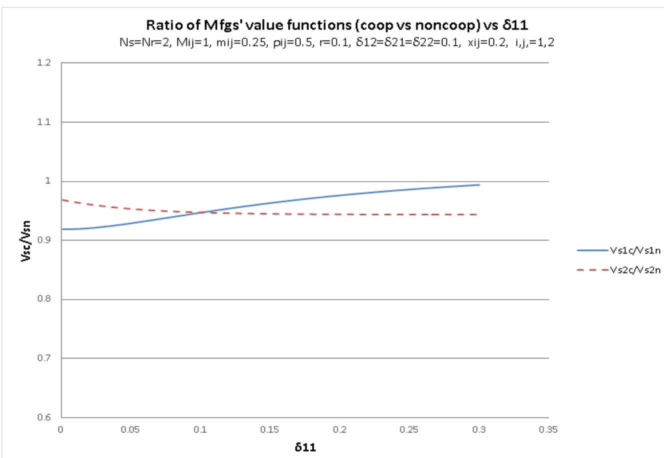


Figure 29: Ratio of Mfgs' value functions (coop/noncoop) vs δ_{11}

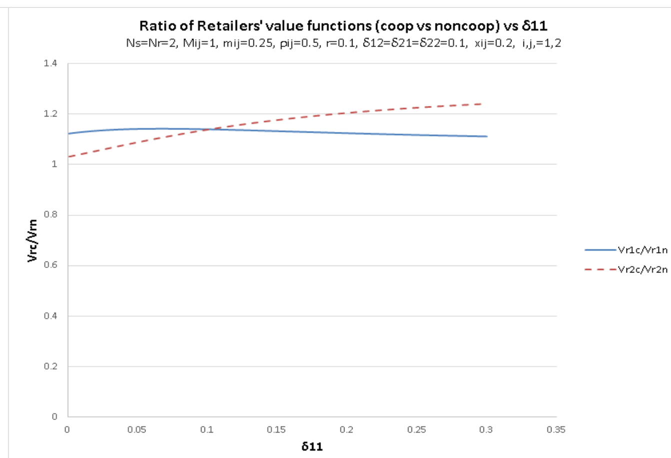


Figure 30: Ratio of Retailers' value functions (coop/noncoop) vs δ_{11}

Figures for Section 6

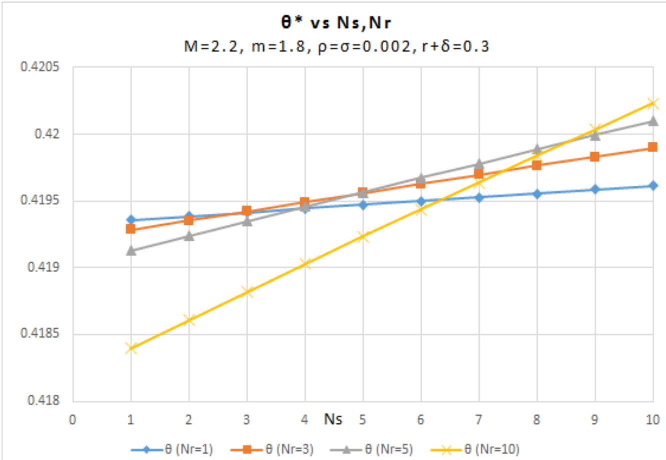


Figure 31: Subsidy rates for different N_s, N_r

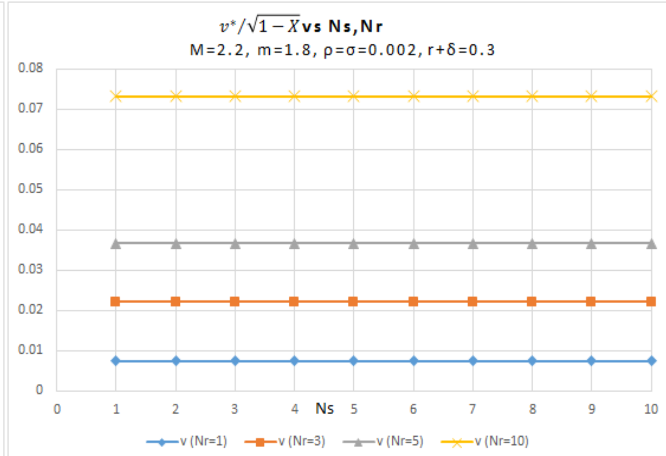


Figure 32: National advertising for a product per unit uncaptured market share for different N_s, N_r

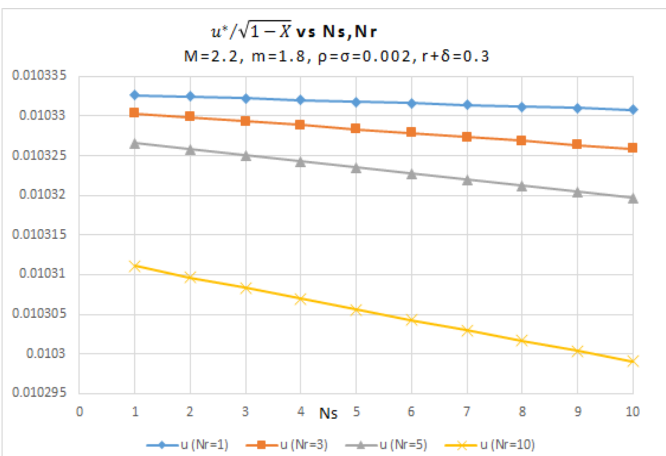


Figure 33: Local advertising efforts per unit uncaptured market share for different N_s, N_r

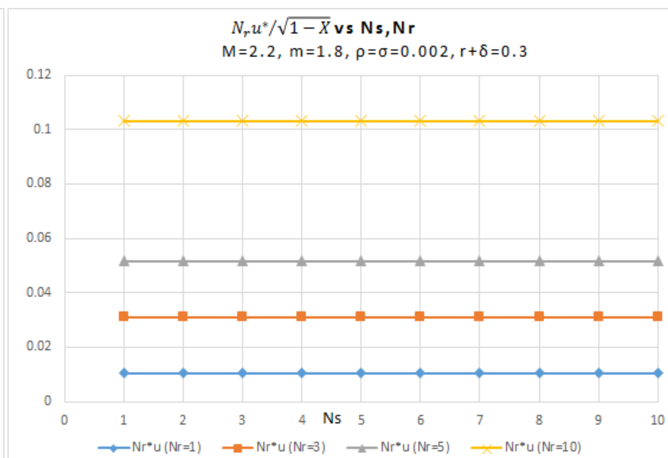


Figure 34: Total local advertising for a product per unit uncaptured market share for different N_s, N_r

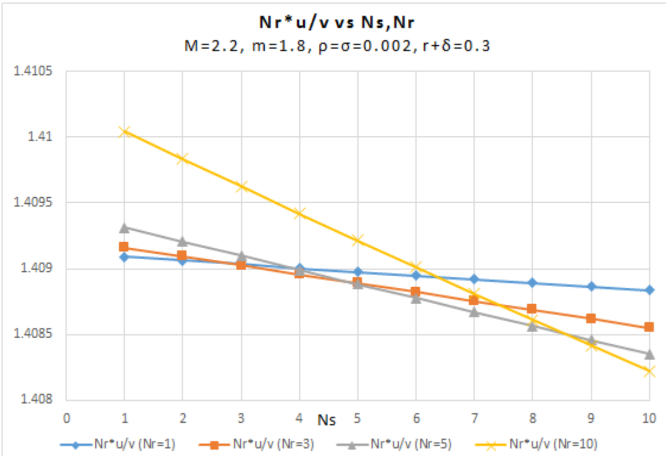


Figure 35: Ratio of total local advertising over national advertising for a product for different N_s, N_r

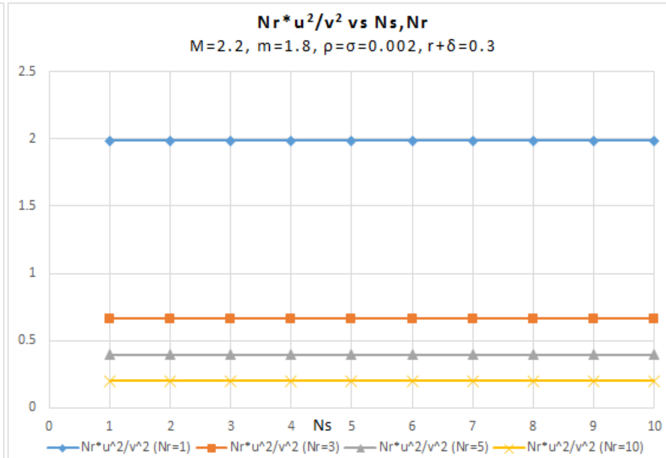


Figure 36: Ratio of total local advertising expense over national advertising expense for a product for different N_s, N_r

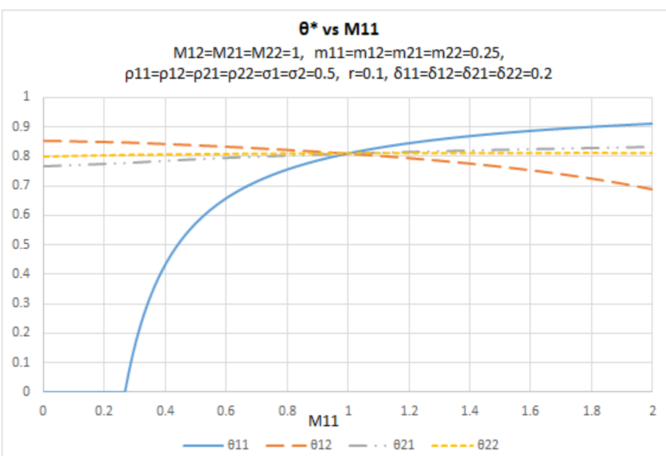


Figure 37: Subsidy rates vs M_{11}

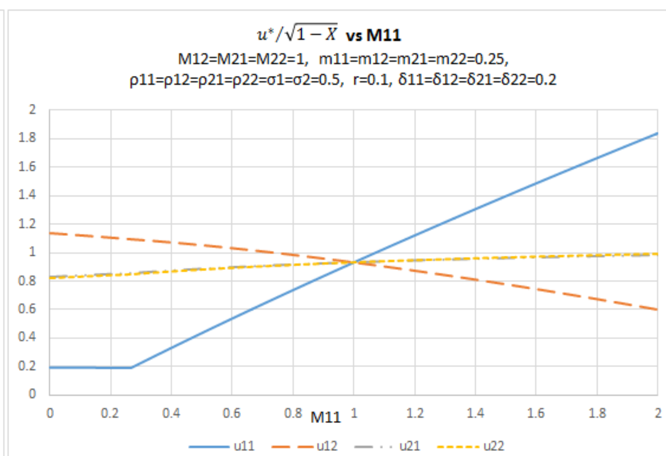


Figure 38: Local advertising efforts per unit uncapturerd market share vs M_{11}

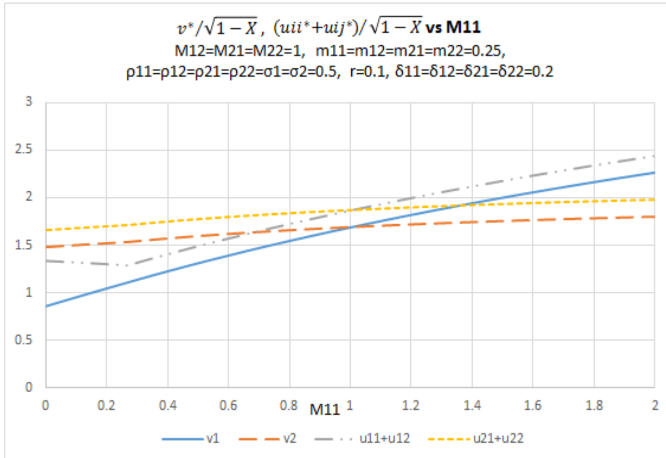


Figure 39: National and total local advertising for a product vs M_{11}

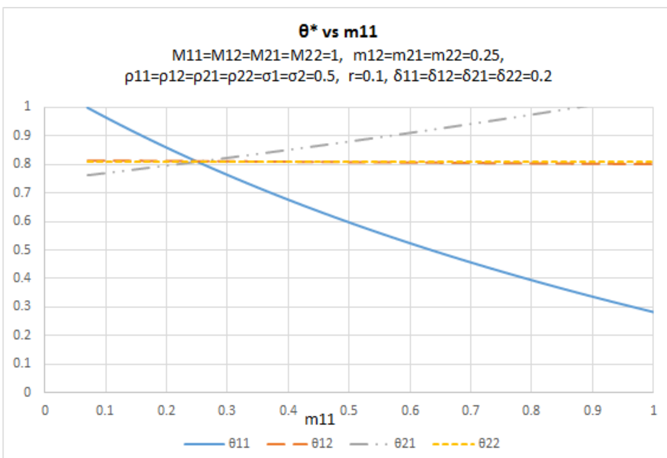


Figure 40: Subsidy rates vs m_{11}

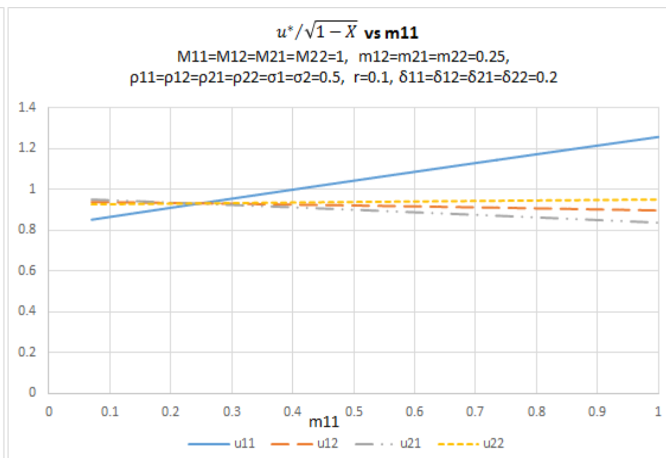


Figure 41: Local advertising efforts per unit uncapturerd market share vs m_{11}

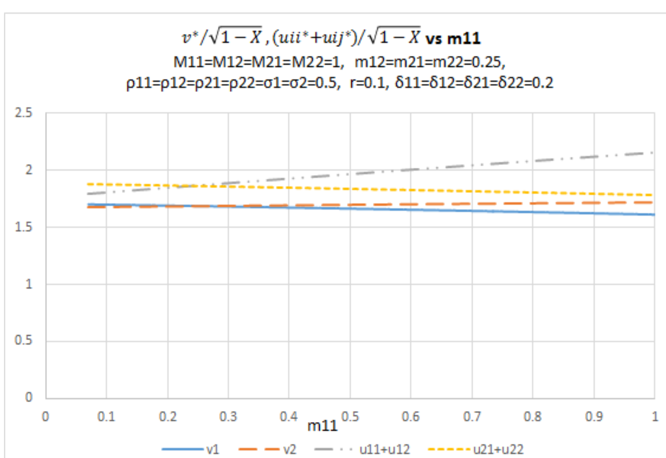


Figure 42: National and total local advertising for a product vs m_{11}

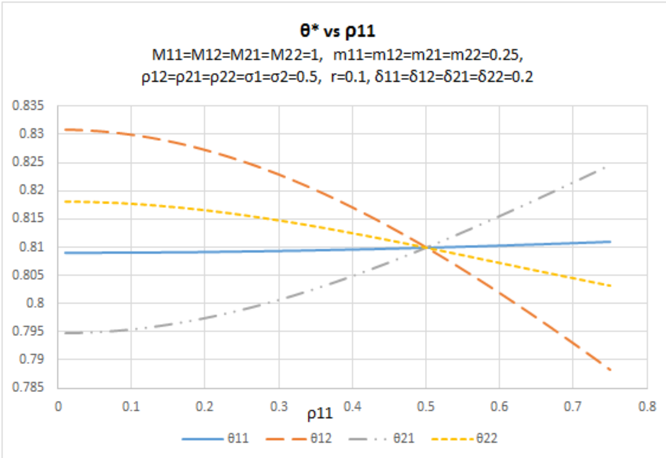


Figure 43: Subsidy rates vs ρ_{11}

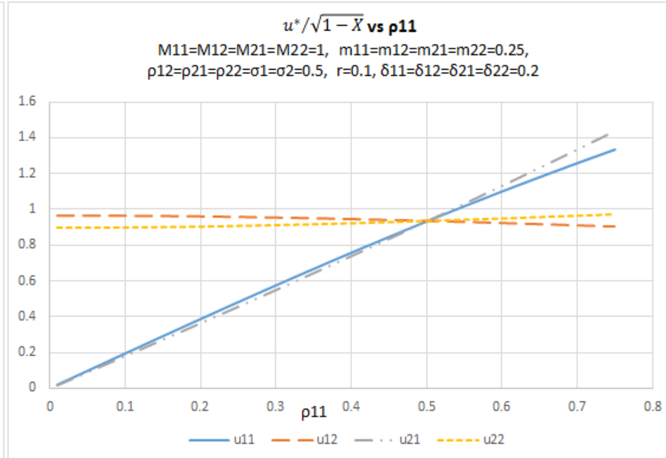


Figure 44: Local advertising efforts per unit uncapturerd market share vs ρ_{11}

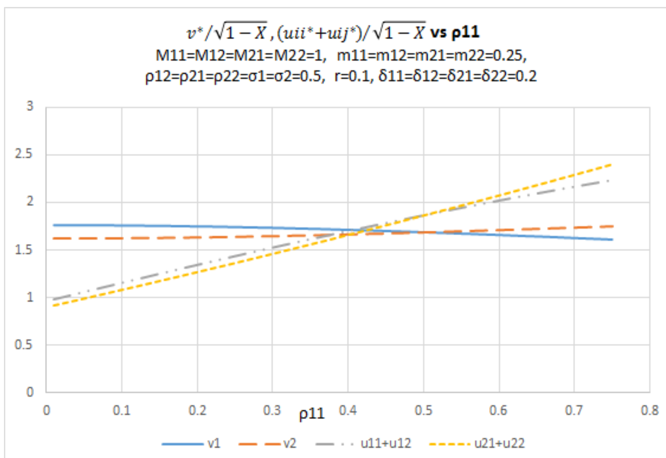


Figure 45: National and total local advertising for a product vs ρ_{11}

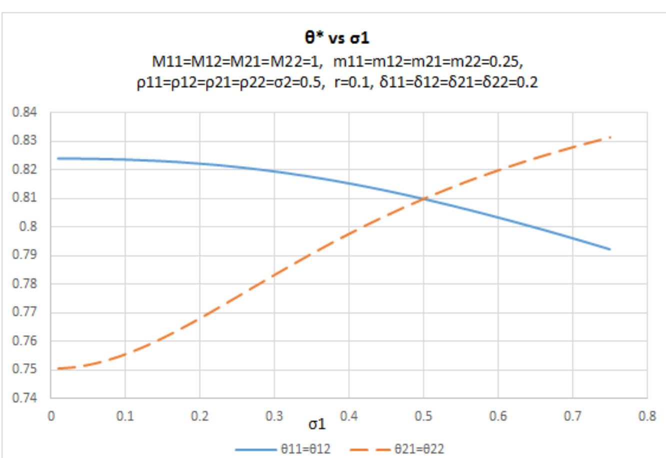


Figure 46: Subsidy rates vs σ_1

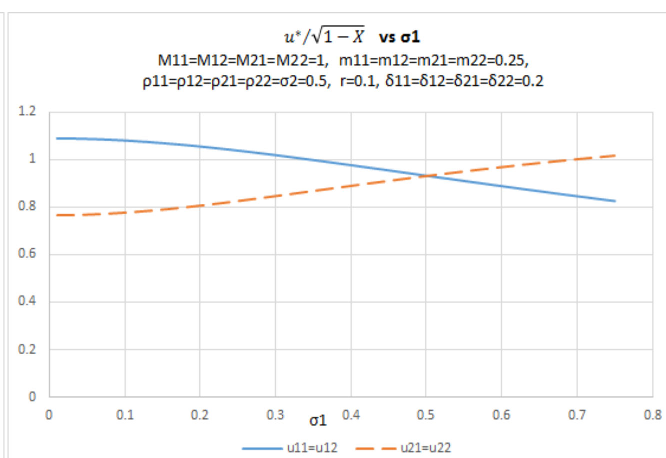


Figure 47: Local advertising efforts per unit uncapturerd market share vs σ_1

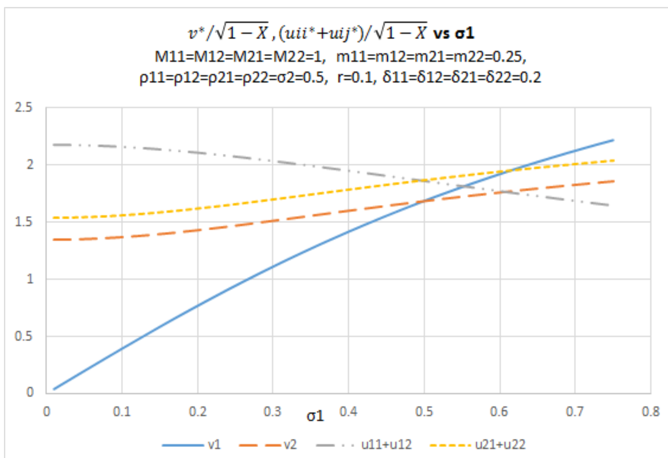


Figure 48: National and total local advertising for a product vs σ_1

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