Exploring Subsethood to Determine Firing Strength in Non-Singleton Fuzzy Logic Systems

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***Abstract*—Real world environments face a wide range of sources of noise and uncertainty. Thus, the ability to handle various uncertainties, including noise, becomes an indispensable element of automated decision making. Non-Singleton Fuzzy Logic Systems (NSFLSs) have the potential to tackle uncertainty within the design of fuzzy systems. The firing strength has a significant role in the accuracy of FLSs, being based on the interaction of the input and antecedent fuzzy sets. Recent studies have shown that the standard technique for determining firing strengths risks substantial information loss in terms of the interaction of the input and antecedents. Recently, this issue has been addressed through exploration of alternative approaches which employ the centroid of the intersection (*cen*-NS) and the similarity (*sim*-NS) between input and antecedent fuzzy sets. This paper identifies potential shortcomings in respect to the previously introduced similarity-based NSFLSs in which firing strength is defined as the similarity between an input FS and an antecedent. To address these shortcomings, this paper explores the potential of the subsethood measure to generate a more suitable firing level (*sub*-NS) in NSFLSs featuring various noise levels. In the experiment, the basic waiter tipping fuzzy logic system is used to examine the behaviour of *sub*-NS in comparison with the current approaches. Analysis of the results shows that the *sub*-NS approach can lead to more stable behaviour in real world applications.**

***Index Terms*—Inference based, Firing strength, Subsethood measure, Non-singleton Fuzzy Logic System, Uncertainty**

I. Introduction

In broad terms, uncertainty can be interpreted as information deficiencies in problem solving situations and it is an inseparable component of most real world applications as it depends on the variety of different circumstances [1]. Thus, the ability to handle uncertainties becomes an indispensable element of decision making. Fuzzy set (FS) theory was first introduced by Zadeh [2] and provided the basis for Fuzzy Logic Systems (FLSs) which are considered as robust systems to handle uncertainty in decision making [3]. FLSs have been successfully applied in a variety of areas, including data mining, pattern recognition and time series predictions [4]–[6].

FLSs processes are completed in three essential steps; fuzzification, inferencing and defuzzification. In fuzzification, crisp input values are transformed into FSs and this transformation can be implemented as a singleton (SFLSs) or non-singleton (NSFLSs). Due to simplicity and lower computational cost of SFLSs, singleton fuzzification is the most commonly used design in literature. However, due to the fact that inputs are commonly corrupted by noise, non-singleton fuzzy sets have the potential to specifically capture the noise in input data, and so may provide better results than SFLSs for the same number of rules [7]–[12].

In the inferencing step of FLSs, inputs are processed with respect to the system rules through interaction between the input and antecedent membership functions (MFs), resulting in rule firing strengths which in turn determine the degree of truth of the consequents of individual rules. In the most common *standard* NSFLS technique, the maximum membership degree grade of the intersection between the input and antecedent MF determines the firing strength. However, adopting the maximum point of the intersection to determine the firing strength risks substantial information loss in terms of the interaction of the input and antecedent MFs. For example, different input MFs (e.g. with different standard deviations I_1 and I_2 in Fig. 1) may intersect an antecedent at the same membership grade, resulting in the same firing level, despite the fact that these input MFs are clearly different.

Recent work, including Pourabdollah et al. ([13,14]) and Wagner et al. ([11]) have attempted to address this issue by introducing alternatives which employ the centroid of the intersection (*cen*-NS) and similarity measures (*sim*-NS) between input and antecedent FSs, respectively. In the case of the *cen*-NS method, consider Fig. 2, in which two different input MFs are shown, lying at the same point x on the universe — I_3

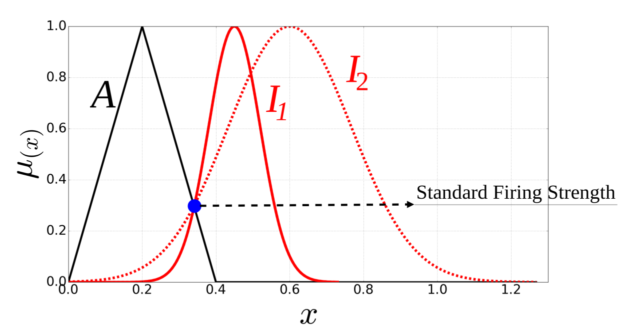


Fig. 1. An illustration of two distinct fuzzy sets having the same intersection level with A

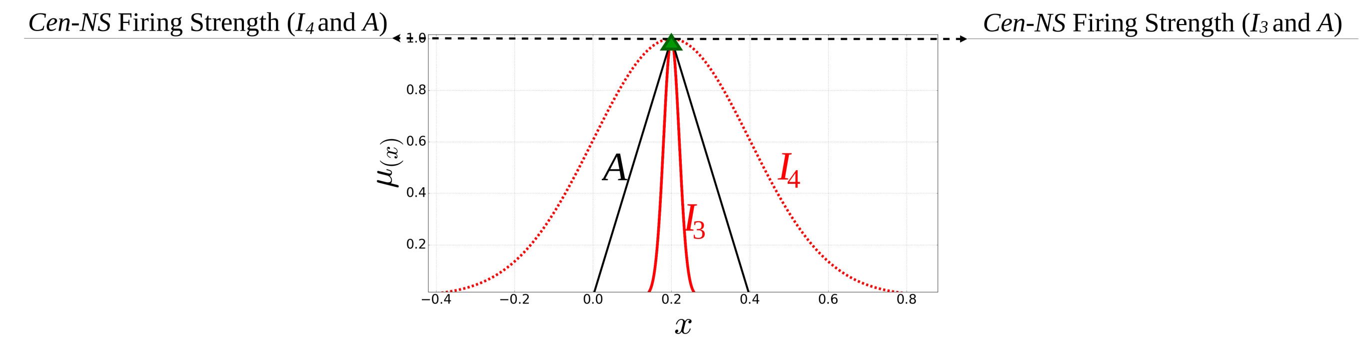


Fig. 2. An illustration of firing level obtained using the *cen*-NS method for two different levels of uncertainty -I_3 low and I_4 high on the antecedent A.

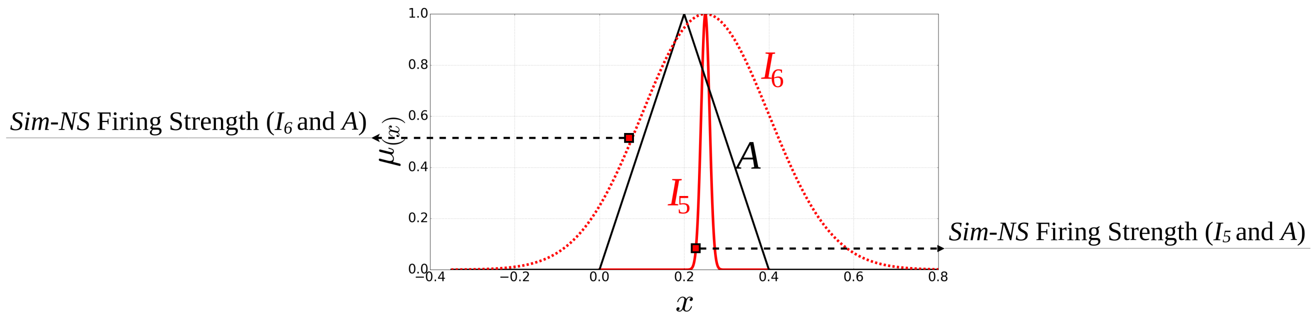


Fig. 3. An illustration of firing level obtained using the *sim*-NS method, of input x, in the case of two different levels of uncertainty, I_3 (low) and I_4 (high) on the antecedent A.

having low uncertainty (expressed as the standard deviation, SD, of the input MF) and I_4 having more uncertainty (larger SD). Even though the uncertainty in the input in the two cases is quite different, they both result in the same firing strength (one). Indeed, regardless of the level of uncertainty in x, a firing strength of one is always obtained - this seems counter-intuitive, perhaps. Consider now the case of the *sim*-NS method. A similar situation featuring an input x with two different levels of uncertainty, depicted by I_5 and I_6, is shown in Fig. 3. In this situation, the *sim*-NS method gives a firing strength of almost zero in the case of I_5. Indeed, as the uncertainty in x reduced further to the extreme situation of a singleton input at x, the firing strength given by *sim*-NS reduces to zero. Again, this seems counter-intuitive. Therefore it can be said that the *sim*-NS may produce a non-intuitive firing strengths when the input uncertainty levels are low.

In summary, adopting either the centroid of intersection or similarity measure based firing, although having some advantages over the *standard* approach, may not be the best option to define firing strength levels. More details of these limitations will be given in Section IV. This paper seeks to address these in-sensitivities by means of a subsethood measure (named *sub*-NS) which enables the more comprehensive capture of interactions between uncertain inputs and antecedents. To the best of our knowledge, this is the first time that a subsethood measure has been applied in the context of defining firing levels in NSFLSs. In order to enable a systematic comparison with the various alternative previously introduced NSFLS approaches, the paper follows the experimental strategy exploring the well-known waiter tipping problem, showing the performance obtained for all the methods.

The structure of this paper is as follows. Section II provides background information on the previous approaches (*standard*, *cen*-NS and *sim*-NS). In Section III, the subsethood measure is introduced in the context of defining firing strength. In Section IV, the behaviour of the introduced subsethood measure is explored by comparing current NSFLSs approaches in some specifically constructed cases. Section V provides details of the waiter tipping problem as a practical decision making context, presenting the experimental environment, the results and discussion. The conclusions of these explorations, with possible future work directions, are given in Section VI.

II. Background

In this section, an overview of the previous firing strength approaches (*standard*, *cen*-NS and *sim*-NS) will be provided.

In most real case scenarios, input data is usually corrupted by noise. Therefore capturing the noise (uncertainty) becomes critical, and this can be done by transforming input data (x) into non-singleton MFs. Let us assume there are two given fuzzy sets, A (for antecedent) and I (for input), on a universe of X, i.e.:

$$\begin{aligned} A &= (x, \mu_A(x) | x \in X) \\ I &= (x, \mu_I(x) | x \in X) \end{aligned} \tag{1}$$

A. Standard Firing Strength Definition

As the most common composition-based technique, in the standard Mamdani inference method [15], the maximum membership degree grade of the intersection between the input MF and antecedent MF is determined as the firing strength. However, recent works [11,14], have shown that adopting the maximum point of the intersection to determine the firing strength risks substantial information loss in terms of the interaction of the input and antecedent MFs. To address this issue, the authors introduced alternatives which employ the centroid of the intersection and similarity measures, between input and antecedent MFs, respectively.

B. Centroid Based Firing Strength Definition

The centroid-based inferencing approach, known as *cen*-NS, focuses on the area of intersection between input and antecedent MFs [14]. Firstly, the centroid of intersection between input MF (I) and antecedent MF (A) is calculated:

$$x_{cen}(I \cap A) = \frac{\int_{x \in X} x\mu(x)}{\int_{x \in X} \mu(x)} \tag{2}$$

where k is the number of discretisation levels in the intersection between the input FS (I) and the antecedent FS (A)

Then, the corresponding membership degree of the position of the centroid ($x_{cen}(I \cap A)$) on the membership function of the intersection is defined to be the firing strength:

$$\mu_{I \cap A}(x_{cen}(I \cap A)) \tag{3}$$

C. Similarity Based Firing Strength Definition

A similarity measure on fuzzy sets is a function that determines to what degree (in the interval of [0,1]) two fuzzy sets contain the same values with the same degree of membership [16]. Wagner et al. [11] suggested that any similarity measure between input MF and antecedent MF can be used to define the firing strength. In the initial work [11], they focused on the Jaccard similarity measure:

$$S(I, A) = \frac{\int_{x \in X} min(\mu_A(x), \mu_I(x))}{\int_{x \in X} max(\mu_A(x), \mu_I(x))} \tag{4}$$

where the input FS is (I) and the antecedent FS is (A).

III. Subsethood Measure

The subsethood measure [2] determines a ratio degree to which a fuzzy set is a subset of another fuzzy set. Various subsethood measures have been extensively studied by researchers over the years [17]–[20]. This paper will focus on one of the early definitions of subsethood measure as given by Kosko [17].

Perhaps the simplest way to express subsethood of set I in set A can be expressed as the ration of the cardinality of the intersection of the two sets over that of set I, i.e.:

$$s_{SH}(I, A) = \frac{|A \cap I|}{|I|} \tag{5}$$

where || refers to cardinality. This ratio can be formulated as follows:

$$s_{SH}(I, A) = \frac{\int_{x \in X} min(\mu_A(x_i), \mu_I(x_i))d_x}{\int_{x \in X} \mu_I(x_i)d_x} \tag{6}$$

Essentially, the subsethood ratio fits the following criteria:

- it is bounded between 0 and 1, i.e. $s_{SH} \in [0, 1]$;
- the ratio is equal to 1 if and only if I is a proper subset of A.

As the number of elements from the set I, which are part of the intersection, increases, the subsethood ratio will rise and eventually reach one when I is covered by A entirely. Likewise, as the set I moves further from A, meaning the non intersecting number of elements increases, the subsethood ratio decreases, and reaches zero when there is no intersection between the two sets at all. Based on above, in this paper, this subsethood measure (*sub*-NS) is utilised in definition of firing strength of fuzzy systems. The subsethood ratio between antecedent and input MFs is directly taken as the firing level of these two MFs.

IV. Exploration of Subsethood as a Definition of Firing Strength

In this Section, the previous firing strength definition approach will be examined and possible limitations will be highlighted along with the advantages of using subsethood in defining firing strength.

As mentioned in Section II, when systems contain different levels of noise, inputs are fuzzified into non-singleton MFs to capture uncertainty [5,11,12,14,21]–[26]. In this regard, it is generally assumed that the received input x is the value which is centred in the non-singleton MFs. Thereafter, the width of the input MFs are determined based on the noise level of systems. Based on this observation, there is usually a direct correlation between the assumed (or known) noise level and the width of the input MF used. As the Gaussian MF is one of the mostly used designs in the literature it has been chosen to be used in our investigation. Correspondingly, the standard deviation of the Gaussian MFs is used to imply different levels of noise.

In order to highlight limitations in the previous techniques and the potential advantages of the subsethood approach, we opted to select two different cases for analysis. In the first case, an input value x is fixed and different noise levels (SDs) are investigated in the firing strength definition aspect. In the second analysis, the standard deviation is fixed and different input values are processed over an antecedent to compare different measures.

A. Analysis 1

In this analysis, the input value x is fuzzified to the non-singleton Gaussian MF (I) and different levels of noise are projected to the value x by means of adjusting standard deviations (SDs). Throughout the analysis, different values of standard deviation (uncertainty levels) in I are defined to examine firing strengths from different approaches. While

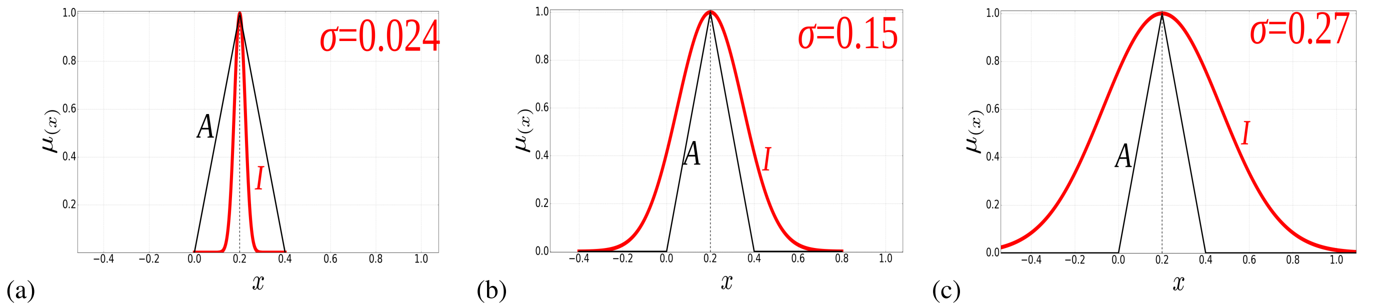


Fig. 4. An illustration of increased SDs in the input MFs (I) over the defined antecedent A.

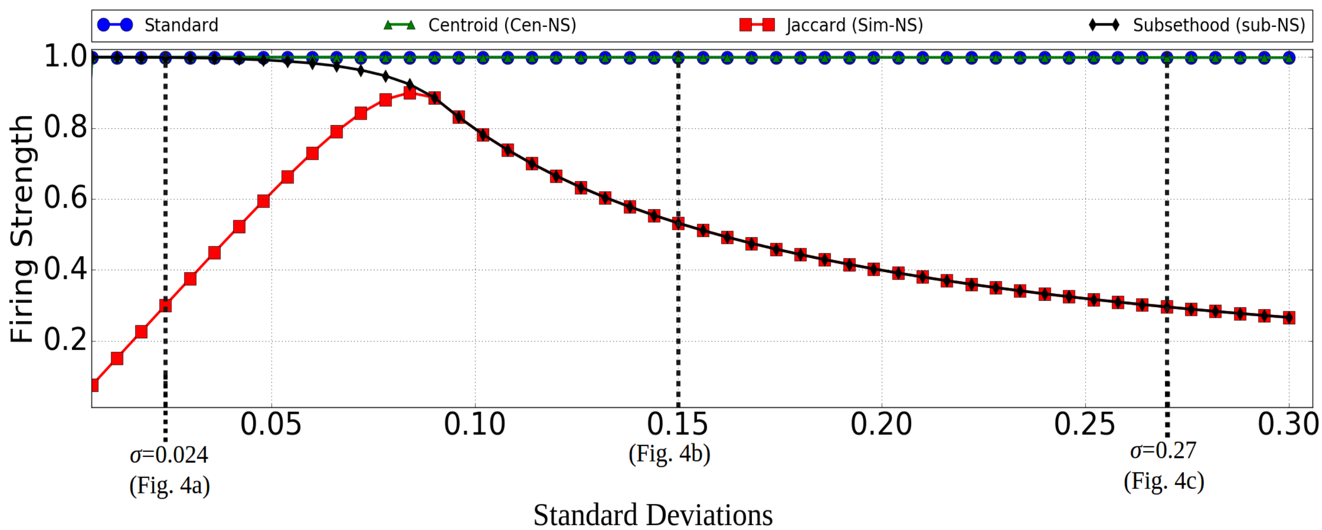


Fig. 5. Comparison of different firing strength determining approaches in each intersection of I-A based on changes in standard deviations of inputs I.

maintaining the value x, 50 different standard deviations (in the range $0.006 \ldots 0.300$ in steps of 0.006) are tested on the same antecedent (A).

1) Experiment 1: Firstly, the input value x is chosen as the same value of the mean of A (0.20). In this way, this experiment allows us to observe the behaviours of different firing strength approaches, under different noise levels, when the system input and antecedent are coincidental on the centre. As mentioned above, 50 different SDs (from 0.005 to 0.030) values are explored in the experiment and three samples of these inputs I (with SD 0.024, 0.150 and 0.270) are illustrated, together with the antecedent A, in Fig. 4. The firing strengths from each of the four proposed methods (*standard*, centroid-based, similarity-based and subsethood-based) is calculated for each level of SD and illustrated in Fig. 5. For instance, the dashed vertical line on the left-hand side of Fig. 5 represents the produced firing strengths in the case when I has the SD value of 0.024 as in Fig. 4a. Also, the middle and right-hand side dashed vertical lines show the produced firing strengths values for each approach from Fig. 4b and 4c, respectively.

Fig. 5 shows that as the uncertainty level increase (from the left to right-hand side), the *standard* and *cen*-NS approaches always produce the single firing strength (of one), regardless of the standard deviation in the input I. From this, it can be observed that the *standard* and *cen*-NS methods fail to take into consideration noise levels in the systems.

When the produced firing strengths by *sim*-NS are scrutinised, we observe from Fig. 4 that even though the input has a comparatively low uncertainty level, the produced firing strengths are close to zero and as the uncertainty level increases (towards the right), the firing strength gradually increases to reach a maximum of around 0.9 around the SD value of 0.084. Below this value, then, the firing strength gradually decreases as the uncertainty in the input I *decreases*; for example, when the input I has an SD of 0.024 (the dashed line on the left), the *sim*-NS generates a firing strength around 0.3. We argue, therefore, that *sim*-NS does not produce intuitive firing strengths, especially under low levels of noise.

2) Experiment 2: As a further experiment, another input value x was determined arbitrarily ($x = 0.3$) and 50 different standard deviations (in the same range $0.006 \ldots 0.300$ in steps of 0.006) were explored on the same antecedent A whilst maintaining the chosen input value x. Again, three arbitrary samples from those standard deviations of 0.024, 0.150 and 0.270 are illustrated in Fig. 6 and for each standard deviation

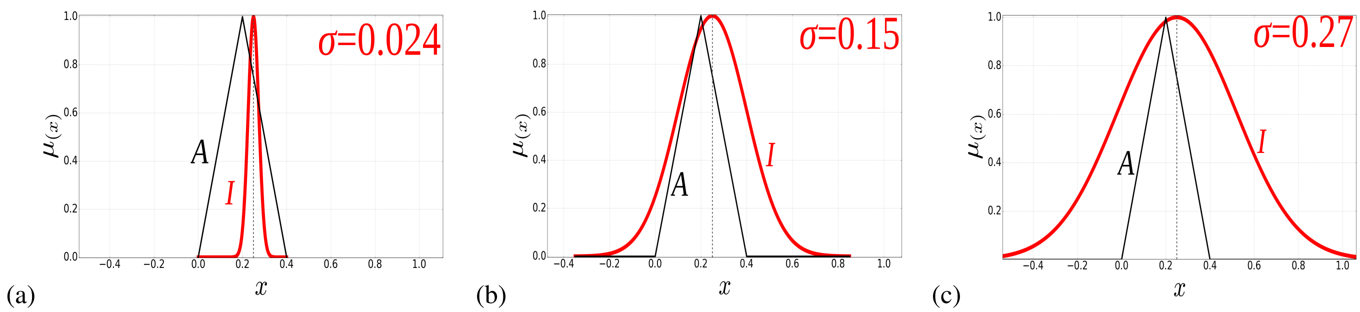


Fig. 6. The illustration of increased SDs in the input MFs (I) over the defined antecedent A.

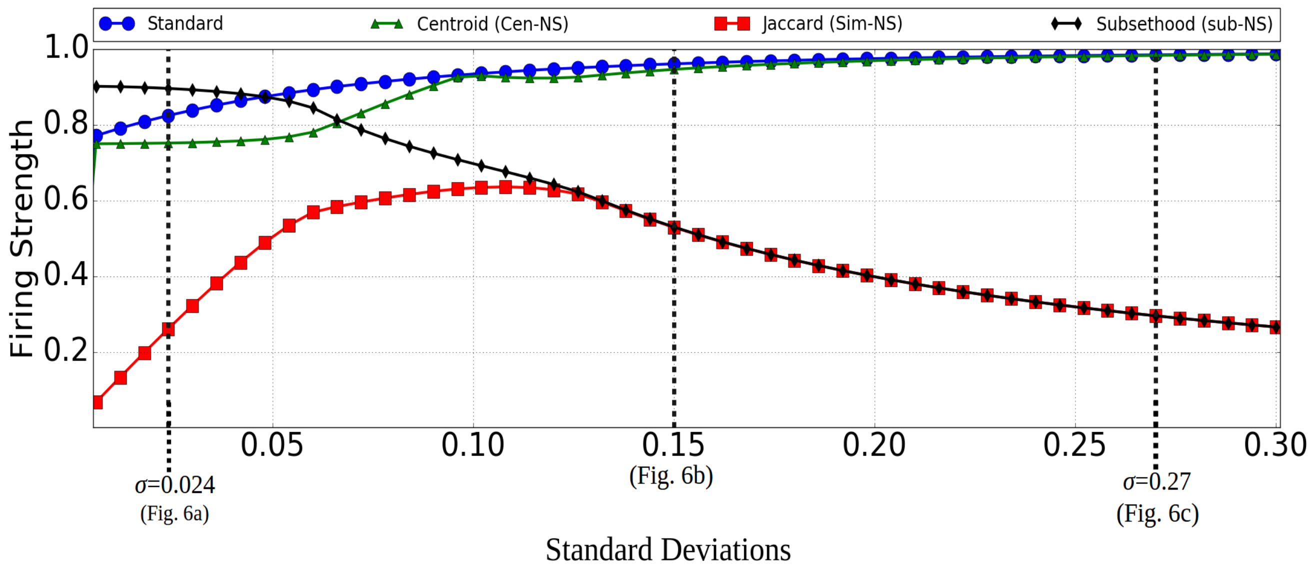


Fig. 7. Comparison of different firing strength determining approaches in each intersection of I-A based on changes in standard deviations of inputs I.

level in I, the firing strengths are calculated using the four different approaches. All the produced firing strengths can be seen in Fig. 7.

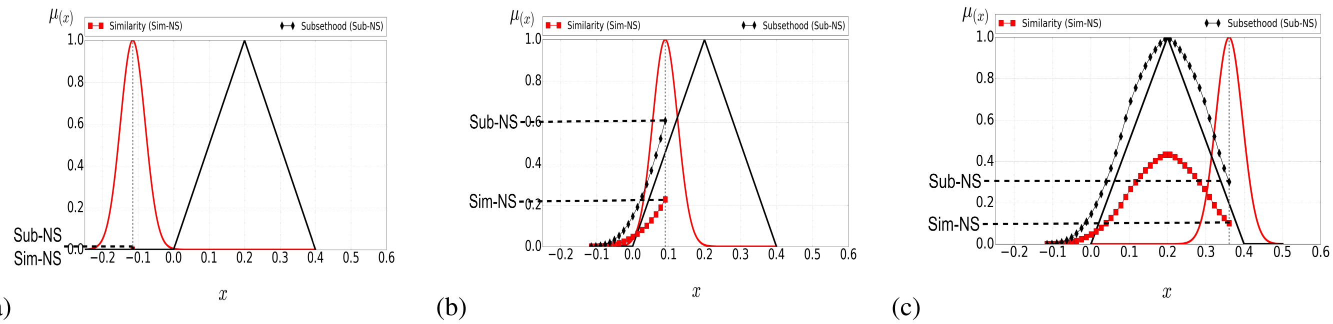
Fig. 7 shows a clear trend for the *standard* and *cen*-NS that the firing strengths for both approaches gradually increases, as the uncertainty levels in the inputs increases. These trends further strengthen the conviction that these approaches may tend to neglect the uncertainty levels under some circumstances. The firing strengths for the *sim*-NS shows that the lower uncertainty levels generate a relatively low firing strength which may also not be intuitively correct. For instance, the dashed vertical line on the left-hand of Fig. 7 represents the produced firing strengths in the case when the SD is 0.024 as shown in Fig. 6a. Also, the middle and right-hand dashed vertical lines show the firing strengths values for Fig. 6b and 6c, respectively.

To recapitulate, as illustrated in Fig. 5 and Fig. 7, on the left-hand sides, the input x has a low level of noise and as move towards to the right, the uncertainty for the x is increased. Therefore, it would be reasonable to expect that the firing strengths would decrease when the system uncertainty is increased. However, when Fig. 5 and Fig. 7 are examined, a clear trend can be seen that firing strengths of all *standard* (circle) and *cen*-NS (triangle) are increased. In addition, it can be seen that *sim*-NS tend to produce lower firing strength under low level of noise, which is not reasonably to be expected. On the other hand, the subsethood measures (diamond) are reduced as the uncertainty level increases in the fixed input x values. Therefore it may be concluded from Analysis 1 that this is compelling evidence for the usefulness of the subsethood measure in defining firing strength.

B. Analysis 2

In this analysis, the standard deviation of input MFs are fixed and different input values are examined over the antecedent A. In order to demonstrate the behaviours of *sim*-NS and *sub*-NS, the term *Firing Strength Map* is introduced and comparison between *sim*-NS and *sub*-NS is provided by means of the introduced *Firing Strength Map*. The generation of a *Firing Strength Map* is implemented as follows:

- Firstly, the input (x_1) is defined as Gaussian MF (I_1) and the firing strength (which is 0 in this case) of the



(a) (b) (c)

Fig. 8. The implementation procedures of the *Firing Strength Map* by using *sim*-NS and *sub*-NS over the defined antecedent (A)

input-antecedent is calculated by using *sim*-NS, *sub*-NS (See Fig. 8a). The produced firing strengths are marked on the figure as the shape of square for *sim*-NS and the diamond for the *sub*-NS (see Fig. 8a).

- Then the following input x_2 is fuzzified into I_2 and the firing strength calculations are implemented and marked on the figure as well. For each x_t, (the input MF toward to the right) the firing strength of the input-antecedent MFs are marked each time (while the previous firing strengths marks remain on the Figure). For instance as samples, I_{18}, I_{35} can be seen in Fig. 8b and 8c respectively.
- The iteration procedure is continued for all t by marking each firing strengths for *sim*-NS and the *sub*-NS. In this manner, all the possible firing strengths, for the inputs and the antecedent, are visualised.

All the possible firing strengths, for *sim*-NS and *sub*-NS, between the defined inputs I and the antecedent A are illustrated in Fig. 9. It is apparent that the *sim*-NS (square) never produce the firing strength value 1, in fact, it only increases to reach a maximum around 0.4 when the x value is equal to the mean of A.

Based on the analyses and the experiments above, it may be concluded that the main weakness in the current approaches is that they make no attempt to consider different uncertainty levels in the inputs which may result in misleading firing strength values under some circumstances. Therefore, the current approaches may not be the best option to be used because of the lack of sensitivity to width of input MFs in NSFLSs. Considering the fact that firing strength has a significant role in the system accuracy, a viable solution can enhance uncertainty capturing in NSFLSs. Hence, as subsethood measure is utilised to define firing strength, as can be seen in Fig. 5, 7 and 9, the changes in input values or uncertainty levels can be captured in a more sensitive and reasonable way.

V. Experiments and Results

In this section, the previous *sim*-NS and the proposed *sub*-NS approaches are evaluated on a well known waiter-tipping problem and the result comparison will be discussed.

A. Experimental Design and Results

In the experiment, the NSFLS consists of two inputs variables (*Food* and *Service*) and one output variable (*Tip*). Both

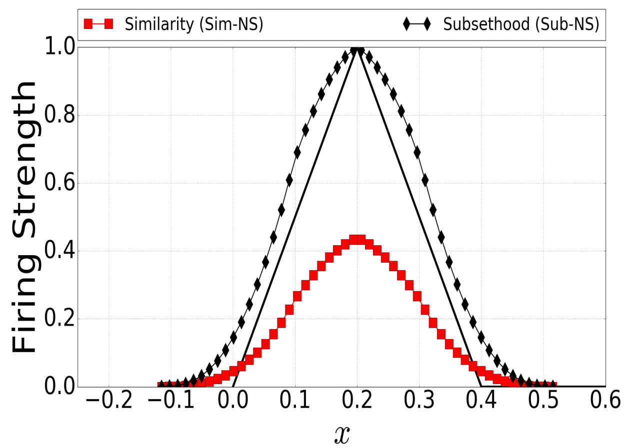


Fig. 9. The comparison of *sim*-NS and *sub*-NS by means of the generated *Firing Strength Map*

the *Food* and *Service* inputs are composed of two shoulder antecedents: while *Food* comprises the terms *Bad* and *Good*, the *Service* comprises as *Slow* and *Fast*. The output variable (*Tip*) of the system is mapped as two shoulder MFs *Low* (left shoulder) and *High* (right shoulder). On account of simplicity, the following two rules are created:

1) *IF* Food is Good *AND* Service is Fast *THEN* Tip is High
2) *IF* Food is Bad *AND* Service is Slow *THEN* Tip is Low

As a result of the design choices above, the experimental FLS is formed as shown in the Fig. 10.

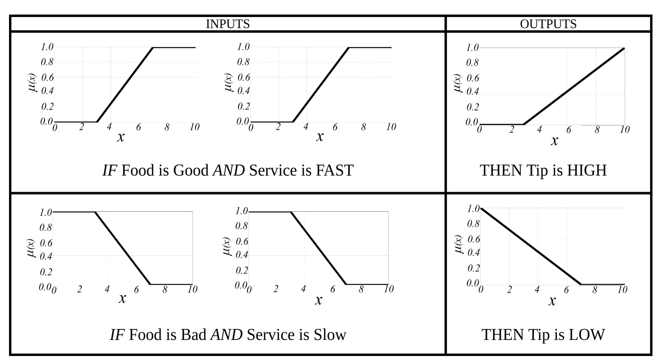


Fig. 10. The basic waiter tipping Fuzzy Logic systems with two defined rules.

For further evaluation, we construct the synthetic input values, with different uncertainty levels, which are fuzzified into Non-Singleton Gaussian MFs. Let us assume that a customer opinion is *7* out of *10* for both the (*Food*) and (*Service*) and also we assume that as a first step of the experiment, the customer has a relatively high confidence about these decisions. Therefore, the input values are fuzzified with a relatively low standard deviation value which is set as *0.1*. The customer decision on the FLS is illustrated as in Fig. 11. When *sim*-NS is implemented, the *Tip* is obtained as 7.5 and when *sub*-NS is used, the *Tip* is calculated as 6.5 approximately.

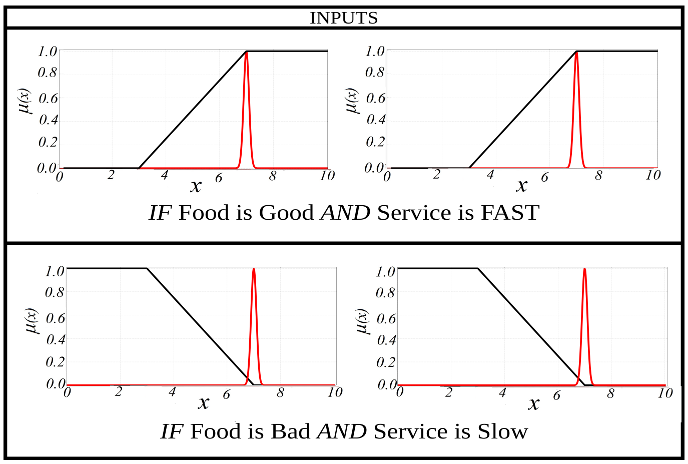


Fig. 11. Customer input with low level of uncertainty on the waiter tipping Fuzzy Logic System with the two defined rules.

The experiments are then repeated with the same customer value (*7* out of *10*) by the uncertainty of the customer (standard deviation) is gradually increased from *0.1* to *1*. Corresponding to different uncertainty levels, a subset of these input MFs is illustrated in Fig. 12. For each uncertainty level, the Tip is calculated for both *sim*-NS and *sub*-NS and for the each corresponding *Tip* output is shown in Fig. 13.

Fig. 12. Different uncertainty level input MSs on the waiter tipping Fuzzy Logic Systems with the two defined rules.

As can be seen in Fig. 12, when the customer uncertainty is increased, the input value covers/interacts more with the second rule which captures *Bad* Food and *Slow* Service. However, contrary to the intuitive expectation, based on *sim*-NS, the *Tip* is getting higher and higher as the uncertainty of the customer is increased (Fig. 13). This underlines the fact that *sim*-NS approach may produce unintuitive firing strengths which may mislead the NSFLSs results. On the other hand, when the proposed *sub*-NS is utilised, it is observed that the *Tip* is decreased as it would be expected. Therefore, in some applications - such as the waiter tipping example above, the *sub*-NS approach can be considered to be a more appropriate technique to define firing strengths in NSFLSs.

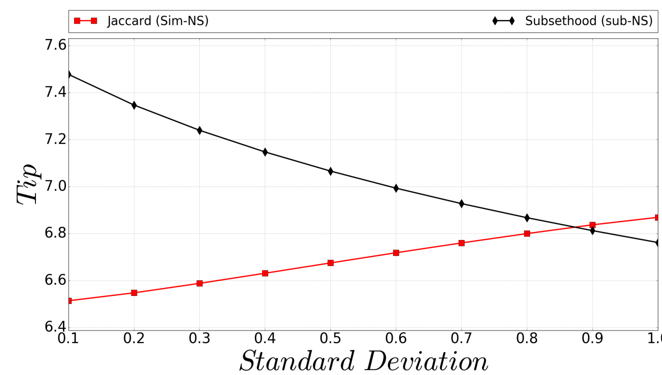


Fig. 13. The generated *Tip* values with both *sim*-NS and *sub*-NS under different uncertainty levels.

VI. Conclusion

In the paper we have considered the behaviour of the NSFLSs with different firing strength determining approaches (*standard*, *cen*-NS, *sim*-NS using the Jaccard similarity ratio and *sub*-NS) which shows that in some cases one may be less intuitive than the another. To the best of our knowledge, this is the first time that a subsethood measure has been proposed to be used as the mechanism for determining firing strength. The evidence from this study points towards the idea that the *sub*-NS could be a suitable approach to be used in FLSs. However, it should be noted that it is an initial evaluation of the approach which explores the value and utility of the subsethood measure in firing strength definition.

Future work will explore the *sub*-NS applicability and utility in real-world experiments, for using on robotic case studies. Further, due to the increased modelling capabilities of type-2 fuzzy logic in handling uncertainty, the application of subsethood in Type-2 fuzzy systems will be explored.

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