CONNECTION AND CURVATURE IN CRYSTALS WITH NON-CONSTANT DISLOCATION DENSITY

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Abstract. Given a smooth defective solid crystalline structure defined by linearly independent 'lattice' vector fields, the Burgers vector construction characterizes some aspect of the 'defectiveness' of the crystal by virtue of its interpretation in terms of the closure failure of appropriately defined paths in the material, and this construction partly determines the distribution of dislocations in the crystal. In the case that the topology of the body manifold M is trivial (e.g., a smooth crystal defined on an open set in $\mathbb{R}^2$), it would seem at first glance that there is no corresponding construction that leads to the notion of a distribution of disclinations, that is, defects with some kind of 'rotational' closure failure, even though the existence of such discrete defects seems to be accepted in the physical literature; see e.g. [7], [10]. For if one chooses to parallel transport a vector, given at some point P in the crystal, by requiring that the components of the transported vector on the lattice vector fields are constant, there is no change in the vector after parallel transport along any circuit based at P. So the corresponding curvature is zero.

However, we show that one can define a certain (generally non-zero) curvature in this context, in a natural way. In fact, we show (subject to some technical assumptions) that given a smooth solid crystalline structure, there is a Lie group acting on the body manifold M which has dimension greater or equal to that of M. When the dislocation density is non-constant in M the group generally has a non-trivial topology, and so there may be an associated curvature. Using standard geometric methods in this context, we show that there is a linear connection invariant with respect to the said Lie group, and give examples of structures where the corresponding torsion and curvature may be non-zero even when the topology of M is trivial. So we show that there is a 'rotational' closure failure associated with the group structure - however we do not claim, as yet, that this leads to the notion of a distribution of disclinations in the material, since we do not provide a physical interpretation of these ideas. We hope to provide a convincing interpretation in future work.

The theory of fibre bundles, in particular the theory of homogeneous spaces, is central to the discussion.

Introduction

This work is an effort to further develop Davini's proposal for a continuum theory of defective crystals [3] by studying the geometry of the continuous structures he introduced using modern mathematical methods, motivated by the assumption that an appropriately detailed description of the geometry of the crystal continuum in the current configuration informs us of the kinematic constitutive variables. In the next section, we summarize some previous work showing that certain geometric fields are elastically invariant and such that different crystals are (locally) elastically related only if those fields match (in a prescribed sense) in the two states. In other words, those fields provide a complete set of 'plastic strain variables', and there is only a finite number of such variables. These two facts suggest that those particular elastically invariant fields should be incorporated, together with a measure of elastic strain, in any general list of kinematic constitutive variables in a continuum mechanics context based on Davini's model.

To phrase these concepts in geometrical language we indicate in Section 2 that the 'plastic strain variables' can be rewritten as combinations of successive Lie brackets of the vector fields that define the crystalline structure. This reformulation effectively introduces *iterations* of the Burgers vector construction, and we make the assumption that there is a finite basis for the Lie algebra of all vector fields so formed. (This does not follow from the fact that there is a finite number of plastic strain variables). The utility of this assumption is the central idea in Elżanowski and Preston's analysis [6]. Then, the basis vector fields define a finite dimensional Lie algebra (with appropriate choice of Lie bracket) and there is generally a corresponding Lie group of dimension strictly greater than that of M, which acts on M. It would seem, therefore, that the topology of the group acting on M should play a role in the mechanics of a crystalline material with kinematic constitutive variables as specified, and we note that this topology can be non-trivial even if that of M is trivial.

In this presentation we focus on the mathematical apparatus required to give substance to the remarks above introducing such concepts as the isotropy group of the action of the Lie group on M, the principal bundle structure induced on the Lie group by the isotropy group, and the corresponding lattice canonical connection with covariantly constant measures of curvature and torsion. We also give a couple of explicit examples of lattice structures (M is an open set in $\mathbb{R}^2$) where curvature and torsion do not vanish. We present here only the mathematical foundations, but in future work we hope to provide more detailed physical interpretations of quantities/procedures employed in this paper, appropriate to the context and convincing from the point of view of engineering applications. Note that the theory of fibre bundles, which we employ here, has long been an integral part of the mathematical physicist's armory, and that insights deriving from the perspective afforded by this theory have been instrumental in understanding/interpreting solutions of field equations with certain symmetries.

1. Continuous elastic crystals

Given a body manifold M of dimension $n \leq 3$ (mathematically, the presentation is valid for any finite dimension), let the kinematic state of a continuous solid crystal body be defined by n linearly independent smooth vectors fields $l_i : M \to TM$, $i = 1, \dots, n$, where TM denotes the tangent space of M. In other words, the state of a continuous elastic solid crystal, called a *continuous lattice* or simply a *lattice*, is defined as a smooth (local) section $\mathbf{l} : M \to L(M)$ of the bundle of the linear frames, [12], of the body manifold M[1]. Subject to the choice of a local chart and invoking the Euclidean structure of $\mathbb{R}^n$, the lattice $\mathbf{l}(x)$, $x \in M$, induces a dual frame (*dual lattice*) $\mathbf{d} : M \to L(M)$ such that $d_i(x) \cdot l_j(x) = \delta_{ij}$, $i, j = 1, \dots, n$, $x \in M$, where δ_{ij} denotes the usual Kroneker delta. Some aspects of the "defectiveness" of the lattice $\mathbf{l}(x)$, $x \in M$, may be characterized in dimension three (as is traditional) by the *dislocation density tensor field* S_{ij} (ddt), the components of which are defined by the equations

$$n(x)S_{ij}(x) = \nabla \wedge d_i(x) \cdot d_j(x), \quad i, j = 1, \dots, \quad x \in M, \tag{1.1}$$

where $n(x)$ is the lattice volume element ($n(x)$ is the determinant of the dual lattice at x). Note that if the defining frame field $\mathbf{l}(x)$ is holonomic (integrable) the corresponding dislocation density tensor vanishes everywhere, and that the opposite is also true, [4]. In particular, the dislocation density tensor of the *ideal lattice* defined by the standard frame $l_i(x) = \mathbf{e}_i$, $i = 1, \dots, n$, vanishes identically. Alternatively, some aspects of the defectiveness can be characterized in any dimension by the (torsion) tensor T

$$T = \frac{1}{2} T^i_{jk} d_i \otimes \eta^j \wedge \eta^k, \tag{1.2}$$

of the linear connection induced by the given lattice frame, where η^l denote the corresponding coframe[2].

Two crystalline structures, say $\mathbf{l}(x)$ and $\widetilde{\mathbf{l}}(x)$, having the same domain of definition M, are called *elastically related* if there exists a diffeomorphism $\phi : M \to M$ such that

$$\widetilde{l}_i(\phi(x)) = \phi_*(l_i(x)), \quad i = 1, \dots, n, \quad x \in M \tag{1.3}$$

where $\phi_* : TM \to TM$ denotes the tangent map of ϕ. Thus, any diffeomorphism of M, when applied to a continuous lattice via (1.3), induces an elastically related lattice structure. It is clear however that, in general, two (smooth) crystalline structures are not

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[1]In general, a differentiable manifold may not admit a global section of its frame bundle. As our approach is local, we shall only consider local section of $L(M)$. So the reader may think about the manifold M as an open neighborhood in $\mathbb{R}^n$

[2]For the relation between the components of the ddt S^{ij} and the tensor T^i_{jk} see [5].

necessarily elastically related, see [4]. Indeed, given a diffeomorphism $\phi : M \to M$, the lattice $\mathbf{l}(x)$, and the elastically related lattice $\widetilde{\mathbf{l}}(\phi(x)) = \phi_*(\mathbf{l}(x))$, one may show that

$$\widetilde{S}_{ij}(\phi(x)) = S_{ij}(x), \quad i,j = 1,\ldots,n, \quad x \in M \tag{1.4}$$

where $\widetilde{S}_{ij}(x)$ are the components of the dislocation density tensor of the new structure. So the set defined by

$$CM = \{S_{ij}(x) : \ x \in M\} \tag{1.5}$$

is an invariant of elastic deformation as it is unchanged by any diffeomorphism $\phi : M \to M$. Thus, a necessary condition that two continuous lattices $\mathbf{l}$ and $\widehat{\mathbf{l}}$ be elastically related is that

$$\widehat{CM} = CM \tag{1.6}$$

where $\widehat{CM}$ is the set corresponding to the section $\widehat{\mathbf{l}}$.

Although the ddt is an *elastic scalar invariant* in the sense that (1.4) holds, it is not the only scalar invariant. For instance, successive directional derivatives of the dislocation density tensor e.g., the first order directional derivatives $l_i \cdot \nabla S_{jk}$, are also unchanged under a diffeomorphism of M (we call these the invariants of 'first order'). In fact there is an infinite number of scalar invariants, satisfying equations analogous to (1.4) - however at most n of these scalar invariant functions can be independent, since n independent functions parameterize a local chart. Corresponding to each of the independent scalar invariants there is a necessary condition that two continuous lattices be elastically related, analogous to (1.6).

If there are n independent scalar invariants, they must occur amongst the first $(n-1)$ directional derivatives of the ddt : for if the first such invariant is some component of the ddt, and if no other component is independent of the first then a second invariant must be found amongst the first order directional derivatives of the ddt, and so on. Suppose that the independent scalar invariants occur amongst the first k directional derivatives of the ddt, where $k \leq (n-1)$. Then the scalar invariants of order $(k+1)$ may be expressed as functions of the n independent invariants, and given these functions it is straightforward to show by induction that any invariant of arbitrary finite order may be similarly expressed. The case where there are fewer than n independent scalar invariants may be treated analogously.

To progress, it is useful to generalize the definition of the set CM to incorporate all scalar invariants of order $\leq (k+1)$, not just the nine components of the ddt. This set represents the 'classifying manifold' corresponding to the crystal state, given certain regularity assumptions - this set is a fundamental construct in E.Cartan's 'equivalence method' (which allows one to decide if two coframes are mapped to each other by a diffeomorphism), [15]. The central fact which makes this definition important is the following:

if one constructs the classifying manifolds corresponding to two crystal states, and those manifolds overlap (in a precise sense, see [15]), then the two continuous lattices are *locally* elastically related to one another (i.e., the lattice vector fields in certain neighbourhoods of points determined by the overlap condition are elastically related). So the identity of classifying manifolds corresponding to two crystal states, generalizing (1.6), is *necessary* if the crystal states are to be elastically related to each other, whereas, as a kind of converse result, if the two classifying manifolds overlap, *then* the crystal states are *locally* elastically related. By virtue of this last fact one may regard the quantities that enter into the definition of the classifying manifold as the 'plastic strain variables' which determine whether or not different crystal states are locally elastically related to one another. (This overlap condition is 'local', so the topology of the classifying manifold plays no role in this context.) See [15],[18] for details.

Finally, in this section, we say that a continuous lattice is *uniformly defective* if its dislocation density tensor $S_{ij}(x)$ is constant in M, that is, if it is material point independent. From equation (1.4), if two uniformly defective lattices are elastically related they have the same dislocation density tensor. It can be shown that if two uniformly defective lattices have the same dislocation density tensor, then they are locally elastically related (but not necessarily elastically related). However, in the sequel we deal solely with non-uniformly defective structures.

2. Non-uniformly defective structures

Consider a continuous lattice defined by the frame field $\mathbf{l} : M \to L(M)$ and assume that the corresponding smooth vector fields $l_i(x)$, $i = 1, \dots, n$, induce an m-dimensional Lie subalgebra, say $\mathfrak{l}$, of the algebra $\mathcal{X}(M)$ of all smooth vector fields on M, where $n \leq m < \infty$. We shall call the subalgebra $\mathfrak{l}$ the *lattice algebra* and number its generating vector fields, say $\mathfrak{l}_1, \mathfrak{l}_2, \dots, \mathfrak{l}_m$, so that $\mathfrak{l}_i = l_i(x)$, $i = 1, \dots, n$, unless stated otherwise.

Our assumption that the lattice algebra $\mathfrak{l}$ is of finite dimension is motivated by the following two observations. First, as intimated in the previous section, the fields of scalar invariants of order less than or equal to n determine whether or not two continuous lattices are locally elastically related, and any scalar invariants of higher order are determined (via appropriate functional relations) by the lower order invariants. In fact, as scalar invariants are unchanged by elastic deformations, we may regard this finite set of scalar invariants as a rather general set of *inelastic constitutive variables*. Moreover, as shown in [18], [20], this set of inelastic variables may be expressed in terms of Lie brackets of the generating vector fields of order less than or equal to $(n+1)$ (We say that terms such as $[l_i(x), l_j(x)]$ are Lie brackets of second order, terms such as $[[l_i, l_j], l_k]$ are Lie brackets of third order, etc., and refer to l_i as a Lie bracket of first order, for convenience). We therefore ask what assumption guarantees that this set of Lie brackets determines all higher order brackets.

Clearly this is so if the smooth vector fields $l_i(x)$, $i = 1, \ldots, n$, induce a finite dimensional Lie subalgebra of $\mathcal{X}(M)$.

Finally, we assume also that all generators of the subalgebra $\mathfrak{l}$ are complete vector fields on the manifold M implying that the algebra $\mathfrak{l}$ consists entirely of complete[3], vector fields, [8]. Thus, there exists (see [8], [16],) an abstract Lie group, say, G acting on the body manifold M, the Lie algebra of which is isomorphic to the subalgebra $\mathfrak{l}$. That is:

Theorem 1. *Consider a continuous lattice defined by n linearly independent smooth vector fields $l_i : M \to TM$, $i = 1, \cdots, n$. Let $\mathfrak{l} \subset \mathcal{X}(M)$ denote the smallest algebra of vector fields containing the given lattice vector fields. Assume that $\mathfrak{l}$ is finite-dimensional and complete. Then, there exists a simply connected Lie group G contained in Diff(M) as an abstract subgroup[4] and such that the natural action $\Lambda : G \times M \to M$ of the group G on M is smooth and the algebra $\mathfrak{l}$ is homomorphic to the Lie algebra, say $\mathfrak{g}$, of the group G.*

Indeed, given the smooth left action $\Lambda : G \times M \to M$ of the group G on the body manifold M, there exists a homomorphism $\chi : G \to \mathrm{Diff}(M)$ from the group G into the group of all diffeomorphisms of M such that

$$\chi(g)(x) = \Lambda(g, x), \quad g \in G,\, x \in M. \tag{2.1}$$

If, in addition, the action Λ is effective[5] the homomorphism χ identifies the group G with a subgroup, say, $\chi(G) \subset \mathrm{Diff}(M)$. Correspondingly, there exists a relation between the Lie algebra $\mathfrak{g}$ of the group G and the algebra of all smooth vector fields $\mathcal{X}(M)$. To this end, given $x \in M$, consider the smooth mapping $\Lambda_x : G \to M$ such that

$$\Lambda_x(g) = \Lambda(g, x) \tag{2.2}$$

for any $g \in G$, i.e., Λ_x maps the group G onto the orbit $G(x)$ (under the action Λ) of the point x. The mapping Λ_x is a morphism (but not necessarily an isomorphism) of the action of G on itself (by left translations) into the action of Λ on M. Let $d\Lambda_x : TG \to TM$ be the tangent map of Λ_x, where $d\Lambda_x : T_gG \to T_{\Lambda(g,x)}M$ for any $g \in G$. Identifying the tangent space T_eG at the identity e of the group G with the Lie algebra $\mathfrak{g}$ of G, define

$$d\chi : \mathfrak{g} \to \mathcal{X}(M) \tag{2.3}$$

by requiring that

$$d\chi(\mathfrak{v})(x) = d_e\Lambda_x(\mathfrak{v}) \tag{2.4}$$

for any $\mathfrak{v} \in \mathfrak{g}$ and any $x \in M$. It can than be shown, [8], that:

[3]A vector field on M is complete if the corresponding flow on M is global.

[4]Note that although the set Diff(M) of all diffeomorphisms of M, is a group, it is not a Lie group.

[5]If for any $g \in G$ there exists $x \in M$ such that $\Lambda(g, x) \neq x$, the action of G on M is said to be effective.

Proposition 1. *The mapping $d\chi : \mathfrak{g} \to \mathcal{X}(M)$ is a homomorphism of Lie algebras. In fact, $d\chi(\mathfrak{g}) = \mathfrak{l}$.*

Given an m-parameter Lie group G acting on the left on the body manifold M, where the Lie algebra $\mathfrak{g}$ of G is homomorphic to the lattice algebra $\mathfrak{l}$, consider a point, say $x_0 \in M$, and let G_{x_0} be the *isotropy group* of the action Λ at x_0. That is, let

$$G_{x_0} := \{g \in G : \Lambda(g, x_0) = x_0\}. \tag{2.5}$$

If the action Λ is transitive[6] the orbit $\Lambda_{x_0}(G) = M$ and the rank of the projection Λ_{x_0} is constant, [8]. This, in turn, allows one to identify M with the quotient space G/G_{x_0}. Namely, consider the mapping $\widehat{\Lambda(x_0)} : G/G_{x_0} \to M$, called here a *realization*, defined by

$$\widehat{\Lambda(x_0)}(hG_{x_0}) = \Lambda_{x_0}(h) = \Lambda(h, x_0), \quad h \in G \tag{2.6}$$

where hG_{x_0} denotes the left co-set of G_{x_0} generated by h. It can be shown easily that $\widehat{\Lambda(x_0)}$ is a diffeomorphism commuting with the natural left action of G on G/G_{x_0}. Note, that in general a realization is base point dependent. That is, two realizations based at two different points, say, $\widehat{\Lambda(x_0)} : G/G_{x_0} \to M$ and $\widehat{\Lambda(y_0)} : G/G_{y_0} \to M$, where $y_0 = \Lambda(g, x_0)$ for some $g \in G$, are two different mappings with the corresponding isotropy groups being a conjugate of each other, i.e., $G_{y_0} = gG_{x_0}g^{-1}$. Indeed, let $g_0 \in G_{x_0}$ then,

$$\Lambda(gg_0g^{-1}, y_0) = \Lambda(gg_0g^{-1}, \Lambda(g, x_0)) = \Lambda(gg_0, x_0) = \Lambda(g, x_0) = y_0. \tag{2.7}$$

Summarizing what we have just discussed, we can state that:

Theorem 2. *Consider a continuous lattice defined by n linearly independent smooth vector fields $l_i : M \to TM$, $i = 1, \cdots, n$, where $\mathfrak{l} \subset \mathcal{X}(M)$ is the corresponding lattice algebra and where the induced action $\Lambda : G \times M \to M$ (Theorem 1) is transitive. Then, the underlying body manifold M can be identified with the homogeneous space[7] G/G_{x_0} where the subgroup $G_{x_0} \subset G$ is the isotropy group of the action Λ at the point $x_0 \in M$.*

In other words, the body manifold M with the lattice frame $\mathbf{l}$ may be viewed as the homogeneous space G/G_{x_0} on which the group G acts in the natural way on the left. This generalizes the uniformly defective case where the body manifold M is identified with a Lie group acting on itself, [17].

In addition to M being identified with the homogeneous space G/G_{x_0}, the subgroup G_{x_0} (in general, any closed subgroup of G), introduces a principal bundle structure on the group G with the bundle projection $\pi : G \to G/G_{x_0}$ such that $\pi(g) = gG_{x_0}$, for any $g \in G$, and the natural right action of G_{x_0} on G. Moreover, as the tangent map

$$d_e\pi : T_eG := \mathfrak{g} \to T_{G_{x_0}}G/G_{x_0} \tag{2.8}$$

[6]The group action is transitive if there is only one orbit.

[7]A homogeneous space is the quotient space of a Lie group by a closed subgroup.

is surjective its kernel is the Lie algebra $\mathfrak{g}_0$ of the isotropy group G_{x_0}. This allows one to identify the tangent space $T_{G_{x_0}}G/G_{x_0}$ with the algebra quotient $\mathfrak{g}/\mathfrak{g}_0$, [8]. Furthermore, the specific realization $\widehat{\Lambda(x_0)} : G/G_{x_0} \to M$ induces a bundle isomorphism between the principal bundle $G(G/G_{x_0}, G_{x_0})$[8] and the principal bundle $G(M, G_{x_0})$ with the projection $\pi_0 : G \to M$ such that

$$\pi_0(g) = \widehat{\Lambda(x_0)}(\pi(g)) = \widehat{\Lambda(x_0)}(gG_{x_0}) = \Lambda(g, x_0). \tag{2.9}$$

We shall explore this way of looking at the group G acting on the body manifold M in the next section. For now, note that due to the fact that the realization $\widehat{\Lambda(x_0)} : G/G_{x_0} \to M$ is a diffeomorphism, the kernel of the tangent map $d_e\pi_0 : \mathfrak{g} \to T_{x_0}M$ is again the Lie algebra $\mathfrak{g}_0$ of the isotropy group G_{x_0}.

Example 1. Consider a two-dimensional continuous lattice given by the frame $l_1 = (1, 0)$ and $l_2 = (0, -x)$. As the corresponding dislocation density tensor is not constant, the lattice is non-uniformly defective. In fact, it generates a three-dimensional Lie algebra spanned by

$$l_1 = (1, 0),\ l_2 = (0, -x),\ l_3 = (0, 1) \tag{2.10}$$

as $[l_1, l_2] = l_3$ and $[l_1, l_3] = [l_2, l_3] = 0$. Viewing the vector fields l_i, $i = 1, 2, 3$, as infinitesimal generators of one-parameter groups acting on $\mathbb{R}^2$ and using the exponential map construction to determine the three associated flows $exp(tl_i) : \mathbb{R}^2 \to \mathbb{R}^2$ we obtain: $(x, y) \mapsto (x + t, y)$, $(x, y) \mapsto (x, y - xt)$ and $(x, y) \mapsto (x, y + t)$. The composition of these flows generates the (left) action of a three-parameter group, say G,

$$\Lambda((a, b, c), (x, y)) = (x + a, y - b(x + a) + c) \tag{2.11}$$

for any $(a, b, c) \in G$ and $(x, y) \in \mathbb{R}^2$ where the group multiplication

$$g\overline{g} = (a + \overline{a}, b + \overline{b}, c + \overline{c} + \overline{b}a),\ \ g, \overline{g} \in G \tag{2.12}$$

can easily be determined from the equation

$$\Lambda(g\overline{g}, (x, y)) = \Lambda(g, \Lambda(\overline{g}, (x, y))) \tag{2.13}$$

for any two $g, \overline{g} \in G$. Obviously, the group G is connected (in fact, path connected) and its action Λ on $\mathbb{R}^2$ is transitive. Given an arbitrary point $(x, y) \in \mathbb{R}^2$, consider its orbit map $\Lambda_{(x,y)} : G \to \mathbb{R}^2$, (2.2). Its tangent map $d\Lambda_{(x,y)} : T_gG \to T_{\Lambda(g,(x,y))}\mathbb{R}^2$, where

[8]We use here the standard principal bundle notation $P(N, K)$, [12], where P denotes the total space of the bundle, K is its structure group, and N is its base.

$g = (a, b, c)$, is represented in the standard coordinate systems on $G = \mathbb{R}^3$ and $\mathbb{R}^2$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -b & -x & 1 \end{pmatrix} \tag{2.14}$$

inducing (at the identity of the group, $e = (0, 0, 0)$) our lattice algebra $\mathfrak{l}$. Moreover, analyzing the group multiplication of the group G, one can easily show that its Lie algebra $\mathfrak{g}$ is generated by

$$\mathfrak{l}_1 = (1, 0, 0),\ \mathfrak{l}_2 = (0, 1, a),\ \mathfrak{l}_3 = (0, 0, 1) \tag{2.15}$$

and that the algebras $\mathfrak{l}$ and $\mathfrak{g}$ are isomorphic. Finally, selecting a point, say $(x_0, y_0) \in \mathbb{R}^2$, the corresponding isotropy group of the action Λ at (x_0, y_0) is

$$G_{x_0} = \{(0, b, bx_0) :\ b \in \mathbb{R}\} \tag{2.16}$$

and its one-dimensional Lie algebra $\mathfrak{g}_0$ is spanned by $(0, 1, x_0)$.

3. The canonical connection on reductive homogeneous space G/G_{x_0}

As the Lie algebra $\mathfrak{g}_0$ of the isotropy group G_{x_0} is a subalgebra of the Lie algebra $\mathfrak{g}$, there exists a complementing vector space, say $\mathfrak{D}$, such that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{D}$. Using the realization $\widehat{\Lambda(x_0)}$ and utilizing the fact that the tangent $T_{x_0}G/G_{x_0}$ is identifiable with the algebra quotient $\mathfrak{g}/\mathfrak{g}_0$, one can easily show that the projection $d_e\pi_0|_{\mathfrak{D}}$ is a linear isomorphism onto $T_{x_0}M$. This allows one to lift the generators $l_i(x)$, $i = 1, \ldots, n$ of the lattice algebra $\mathfrak{l} \subset \mathcal{X}(M)$ to the Lie algebra $\mathfrak{g}$ of the group G by requiring that the lifted frame $\mathfrak{l}_i$, $i = 1, \ldots, n$ in $\mathfrak{g}$ be such that $d_e\Lambda_x(\mathfrak{l}_i) = d\chi(\mathfrak{l}_i)(x) = l_i(x)$, for every $x \in M$. Note that as the complementing vector space $\mathfrak{D}$ is not uniquely defined, neither is the lifting of the generators of the lattice algebra (see Remark 1). However, as the morphism $d\chi$, see (2.3), is of the maximum rank and as the Lie algebra $\mathfrak{g}$ is isomorphic to the space of all left invariant vector fields on the group G, the frame $\mathfrak{l}_i$, $i = 1, \ldots, n$ induces a left-invariant n-dimensional distribution, say, $\mathfrak{L} : G \to TG$, on the tangent space of the group G such that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{L}(e)$ and

$$T_gG = T_g gG_{x_0} \oplus \mathfrak{L}(g),\ \ g \in G, \tag{3.1}$$

where the cosets gG_{x_0} are regarded as smooth submanifolds of G[9]. Moreover, the distribution $\mathfrak{L}$ defines a left-invariant (by the left translations of G) *horizontal distribution*[10] on the principal bundle $G(M, G_{x_0})$. That is, for any $g \in G$, $\mathfrak{L}(g)$ is a vector subspace of T_gG, it depends smoothly on g, and $d_g\pi_0(\mathfrak{L}(g)) = T_{\pi_0(g)}M$. Although the distribution $\mathfrak{L}$

[9] Obviously, all tangent spaces $T_g gG_{x_0}$, $g \in G$, are isomorphic (as vector subspaces) to the subalgebra $\mathfrak{g}_0$.

[10] The distribution $\mathfrak{L}$ is horizontal in the sense that its projection $d\pi_0(\mathfrak{L}) = TM$.

is, by definition, left invariant under the action of the group G it is not, in general, right-invariant under the action of the isotropy group G_{x_0}, the structure group of $G(M, G_{x_0})$. Namely, in general, there is no guarantee that $\mathfrak{L}(gg_0) = R_{g_0 *}\mathfrak{L}(g)$ for every $g \in G$ and every $g_0 \in G_{x_0}$, where $R_{g_0} = gg_0$. This means that although horizontal, the distribution $\mathfrak{L}$ does not, in general, induce a principal connection on $G(M, G_{x_0})$. Yet, it is true that at every $g \in G$ the kernel of the tangent bundle projection $d_g\pi_0 : T_gG \to T_{\pi_0(g)}M$ is the vertical space $T_g gG$.

The construction of the horizontal distribution $\mathfrak{L}$ on the principal bundle $G(M, G_{x_0})$ can be mimicked on the bundle of linear frames of the base manifold M using the concept of the *linear isotropy representation* of the isotropy group G_{x_0}. To this end, given $g \in G$, let us consider the mapping $\Lambda_g : M \to M$ where $\Lambda_g(x) = \Lambda(g, x)$, $x \in M$. In particular, $\Lambda_{g_0}(x_0) = x_0$ for any $g_0 \in G_{x_0}$ and the tangent map

$$d_{x_0}\Lambda_{g_0} : T_{x_0}M \to T_{x_0}M \tag{3.2}$$

is a linear isomorphism corresponding, subject to the choice of a basis in $T_{x_0}M$, to an element of the general linear group $GL(n, \mathbb{R})$. That is, let $u_0 : \mathbb{R}^n \to T_{x_0}M$ be a linear frame (a linear isomorphism) at $x_0 \in M$ assigning to an n-tuple $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ a vector in $T_{x_0}M$ having $(\xi_1, \ldots, \xi_n)$ as its coordinates in the selected basis. By the *linear isotropy representation* of G_{x_0} we shall mean the homomorphism $\lambda : G_{x_0} \to GL(n, \mathbb{R})$ such that

$$\lambda(g_0) = u_0^{-1} \circ d_{x_0}\Lambda_{g_0} \circ u_0 : \mathbb{R}^n \to \mathbb{R}^n, \quad g_0 \in G_{x_0}. \tag{3.3}$$

Selecting a particular realization $\widehat{\Lambda(x_0)}$ identifying the tangent space of the homogeneous space G/G_{x_0} with TM and fixing the choice of the frame u_0 at $T_{x_0}M$[11], allows one to induce through the homomorphism λ a G-invariant G_{x_0}-structure (more specifically a $\lambda(G_{x_0})$-structure) on M, that is, a reduction of the bundle of linear frames of M, $L(M)$, to the subgroup $\lambda(G_{x_0})$. Namely, given the reference frame u_0 at x_0, a frame at any other point, say $x \in M$, (including x_0) can be represented as $d_{x_0}\Lambda_g \circ u_0$ for some $g \in G$ such that $\Lambda_g(x_0) = \Lambda(g, x_0) = x$; all due to the fact that the action Λ of the group G on M is transitive. Moreover, the group $\lambda(G_{x_0})$ acts on such a selection of frames of M on the right by

$$d_{x_0}\Lambda_g \circ u_0 \circ (u_0^{-1} \circ d_{x_0}\Lambda_{g_0} \circ u_0) = d_{x_0}\Lambda_{gg_0} \circ u_0. \tag{3.4}$$

This, in fact, shows that $G(M, G_{x_0})$ and the just constructed $\lambda(G_{x_0})$-structure, labelled here as $P(M, G_{x_0})$, are isomorphic via the mapping $g \mapsto d_{x_0}\Lambda_g \circ u_0$, $g \in G$. Also, as the bundle $G(M, G_{x_0})$ is left invariant under the action of the group G on the quotient G/G_{x_0}, so is the structure $\pi : P \to M$ where the left action of G on P is given by $gu \mapsto d_x\Lambda_g \circ u$, $g \in G$, $u \in P$, $\pi(u) = x$.

[11] These specific choices are maintained henceforward.

The horizontal distribution $\mathfrak{L}$ on $G(M, G_{x_0})$ can now be reconstructed on the isomorphic frame subbundle $P(M, G_{x_0})$. However, as the distribution $\mathfrak{L}$ is generally not invariant under the right action of the isotropy group, its $P(M, G_{x_0})$ counterpart is not invariant under the right action of the subgroup $\lambda(G_{x_0})$ and it does not correspond to a linear connection on M. Assume however that the homogeneous space G/G_{x_0} is *reductive*, that is, there exists a vector subspace, say, $\mathfrak{M} \subset \mathfrak{g}$ such that the Lie algebra $\mathfrak{g}$ is the direct sum of the isotropy subalgebra $\mathfrak{g}_0$ and the vector space $\mathfrak{M}$, and the subspace $\mathfrak{M}$ is invariant under the "action" of the subalgebra $\mathfrak{g}_0$ i.e., $[\mathfrak{g}_0, \mathfrak{M}] \subset \mathfrak{M}$, or equivalently, it is invariant under the adjoint action of the group G_{x_0}, i.e., $\mathrm{ad}_{G_{x_0}}(\mathfrak{M}) \subset \mathfrak{M}$[12]. Suppose now that the horizontal distribution $\mathfrak{L}$ is such that $\mathfrak{L}(e) = \mathfrak{M}$. As the distribution $\mathfrak{L}$ is left invariant under the action of the whole group G, the condition $\mathrm{ad}_{G_{x_0}}(\mathfrak{M}) \subset \mathfrak{M}$ implies its right invariance under the action of the isotropy group G_{x_0}.

Remark 1. Note that not every homogeneous space is reductive; see for example [21]. Note also that establishing whether or not a given homogeneous space is reductive may not be easy. Indeed, the definition of the reductive homogeneous space states that there exists a vector space $\mathfrak{M}$ complementing the subalgebra $\mathfrak{g}_0$ to the whole algebra $\mathfrak{g}$ such that $\mathfrak{M}$ is invariant under the adjoint action of the isotropy group. The subalgebra $\mathfrak{g}_0$ can be complemented to the whole algebra $\mathfrak{g}$ by a variety of different vector spaces and, in general, it is not clear how to identify a subspace invariant under the adjoint action, if one exists at all. Moreover, one may also ask if such a choice (if there is one) is unique.

Given the specific linear isotropy representation λ of the isotropy group G_{x_0} in the general linear group $GL(n, \mathbb{R})$ via (3.3), and having assumed that the homogeneous space $G/G_{x_0} \cong M$ is reductive, we are now ready to define a linear connection on $P(M, G_{x_0})$. To this end, let us define first an equivariant (as we shall prove next) linear mapping from the Lie algebra $\mathfrak{g}$ of the group G into the Lie algebra of the general linear group, i.e., $\Pi : \mathfrak{g} \to gl(n, \mathbb{R})$, such that

$$\Pi(X) = \begin{cases} \lambda(X), & X \in \mathfrak{g}_0, \\ 0, & X \in \mathfrak{M}, \end{cases} \tag{3.5}$$

where λ denotes the induced by the linear isotropy representation homomorphism of the corresponding Lie algebras, $\mathfrak{g}_0$ and $gl(n, \mathbb{R})$.

[12] The reductivity of a homogeneous space is usually defined by requiring the invariance of the vector space $\mathfrak{M}$ under the adjoint action of the subalgebra of the isotropy group, that is, $\mathrm{ad}_{G_{x_0}}(\mathfrak{M}) \subset \mathfrak{M}$. The condition $[\mathfrak{g}_0, \mathfrak{M}] \subset \mathfrak{M}$ implies the invariance of $\mathfrak{M}$ under the adjoint action of the isotropy group, but not vice versa. However, when the isotropy group is a connected Lie group both conditions are equivalent, [21].

Proposition 2. *The mapping* Π *is equivariant under the action of the isotropy group* G_{x_0}*, that is,*

$$\Pi(R_{g_0*}X) = ad(\lambda(g_0))\Pi(X) \tag{3.6}$$

for any $X \in \mathfrak{g}$ *and any* $g_0 \in G_{x_0}$*.*

Proof. Note first that as the algebra $\mathfrak{g}$ is a collection of left invariant vector fields $R_{g_0*}X = \mathrm{ad}(g_0)X$. Moreover, as the map Π is linear, it is enough to consider two separate cases. First, suppose that $X \in \mathfrak{M}$. Then the right-hand side vanishes from the definition of the mapping Π and the fact that the adjoint is an inner automorphism, while the left-hand side equals 0 due to the fact that the subspace $\mathfrak{M}$ is adjoint-invariant. On the other hand, when $X \in \mathfrak{g}_0$, $\lambda(\mathrm{ad}(g_0)X) = \mathrm{ad}(\lambda(g_0))\lambda(X)$ as λ is a group homomorphism and the adjoint is an algebra inner automorphism. □

We can now define a linear connection on $P(M, G_{x_0})$, called here the *lattice canonical connection*, by requiring that the corresponding $gl(n, \mathbb{R})$-valued one-form (a connection form) ω on P is such that

$$\Pi(X) = \omega(\widetilde{X}) \text{ for any } X \in \mathfrak{g}, \tag{3.7}$$

where $\widetilde{X}$ is the *natural lift* of X to the frame bundle $P(M, G_{x_0})$. Although the construction of the natural lift of a vector field is thoroughly discussed in, for example, [12], we recap some relevant parts for the readers' benefit. That is, given an element $X \in \mathfrak{g}$, consider the one-parameter group $g(t) = \exp tX \subset G$. Its action on the body manifold M induces a vector field X^* on TM by

$$X^*_x = \frac{d}{dt}\Big|_{t=0} \Lambda(g(t), x) = d_e\Lambda_x(X) \tag{3.8}$$

where $\Lambda_x(g) = \Lambda(g, x)$, $g \in G$, $x \in M$; see also (2.4). By the *natural lift* of $X \in \mathfrak{g}$ (or the corresponding X^*) we mean the vector field on $P(M, G_{x_0})$ such that

$$\widetilde{X}_u = \frac{d}{dt}\Big|_{t=0} d_{\pi(u)}\Lambda_{g(t)} \circ u, \; u \in P. \tag{3.9}$$

As the bundles $G(M, G_{x_0})$ and $P(M, G_{x_0})$ are isomorphic and both left-invariant under the action of the group G, the projection $\pi : P \to M$ "commutes" with the group action implying that the vector fields $\widetilde{X}$ and X^* are π-related, that is, $\pi_*(\widetilde{X}_u) = X^*_{\pi(u)}$. Consequently, given the *canonical form* on a frame bundle, that is, an $\mathbb{R}^n$-valued one-form θ on P such that

$$\theta(\widehat{X}_u) = u^{-1}(\pi_*(\widehat{X}_u)) \tag{3.10}$$

for any $u \in P$, and any $\widehat{X}_u \in T_uP$, we have that

$$u(\theta(\widetilde{X}_u)) = \pi_*(\widetilde{X}_u) = X^*_{\pi(u)}. \tag{3.11}$$

Moreover, as the natural lift is a Lie algebra homomorphism from the Lie algebra $\mathfrak{g}$ of the group G into the algebra of smooth vector fields on $P(M, G_{x_0})$, the natural lift of a Lie bracket is a Lie bracket of the natural lifts, i.e.,

$$\widetilde{[X,Y]} = -[\widetilde{X}, \widetilde{Y}] \tag{3.12}$$

for any $X, Y \in \mathfrak{g}$. Note also that if $X \in \mathfrak{g}_0$ the corresponding induced vector field $\widetilde{X}_u$ is vertical. Indeed, when $X \in \mathfrak{g}_0$ the one-parameter group $g(t) = \exp tX \in G_{x_0}$ and the vector $X^*_{x_0} = 0$. Thus, due to the left-invariance of $P(M, G_{x_0})$, $\pi_*(\widetilde{X}_u) = 0$ implying that the field $\widetilde{X}_u$ is vertical in P, i.e., $\widetilde{X}_u \in T_u\pi^{-1}(u)$ for every $u \in P$.

Given the canonical connection ω on the reductive homogeneous space $G/G_{x_0} \cong M$, where $\mathfrak{g}_0 \oplus \mathfrak{M}$ and $[\mathfrak{g}_0, \mathfrak{M}] \subset \mathfrak{M}$, the induced vector fields corresponding to the vector space $\mathfrak{M}$ form the horizontal distribution of ω as $\omega(\widetilde{X})$ vanishes whenever $X \in \mathfrak{M}$, (3.5), and the distribution is right-invariant under the action of G_{x_0} on $P(M, G_{x_0})$.

Let Θ and Ω denote the torsion and the curvature forms of the connection ω, respectively. Utilizing its standard structure equations, [12], we have

$$2\Theta(\widetilde{X}, \widetilde{Y}) = \theta([\widetilde{X}, \widetilde{Y}]) + \omega(\widetilde{X})\theta(\widetilde{Y}) - \omega(\widetilde{Y})\theta(\widetilde{X}), \tag{3.13}$$

and

$$2\Omega(\widetilde{X}, \widetilde{Y}) = \omega([\widetilde{X}, \widetilde{Y}]) + \omega(\widetilde{X})\omega(\widetilde{Y}) - \omega(\widetilde{Y})\omega(\widetilde{X}), \tag{3.14}$$

for any $X, Y \in \mathfrak{g}$. In particular, if $X, Y \in \mathfrak{M}$

$$2\Theta(\widetilde{X}, \widetilde{Y}) = \theta([\widetilde{X}, \widetilde{Y}]), \tag{3.15}$$

$$2\Omega(\widetilde{X}, \widetilde{Y}) = \omega([\widetilde{X}, \widetilde{Y}]) \tag{3.16}$$

as $\omega(\widetilde{X}) = \omega(\widetilde{Y}) = 0$. Moreover, recognizing the fact that, in general, the vector space $\mathfrak{M}$ is not a Lie algebra and that the algebra of the induced vector fields is homomorphic to the Lie algebra $\mathfrak{g}$, (3.12), we have

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{[X,Y]} = \widetilde{[X,Y]_{\mathfrak{M}}} + \widetilde{[X,Y]_{\mathfrak{g}_0}} \tag{3.17}$$

where $[X,Y]_{\mathfrak{M}}$ and $[X,Y]_{\mathfrak{g}_0}$ denote the $\mathfrak{M}$ and $\mathfrak{g}_0$ components of $[X,Y]$, respectively. Consequently,

$$2\Theta(\widetilde{X}, \widetilde{Y}) = -\theta(\widetilde{[X,Y]_{\mathfrak{M}}}) \tag{3.18}$$

and

$$2\Omega(\widetilde{X}, \widetilde{Y}) = \omega(\widetilde{[X,Y]_{\mathfrak{g}_0}}) = -\lambda([X,Y]_{\mathfrak{g}_0}) \tag{3.19}$$

for any pair $X, Y \in \mathfrak{M}$ as the fundamental form θ vanishes on the vertical subbundle of TP while the connection form ω vanishes on its horizontal space. Finally, consider the base point $x_0 \in M$ and let us identify the vector space $\mathfrak{M}$ with the tangent space $T_{x_0}M$ by identifying $X \in \mathfrak{M}$ with the corresponding vector $X^*_{x_0}$, (3.8). Moreover, let us identify $T_{x_0}M$ with $\mathbb{R}^n$ by means of the frame u_0 at x_0. Then, as $\theta(\widetilde{X}_u) = X^*_{\pi(u)}$ the *torsion tensor* at x_0

$$T(X,Y) = u_0(2\Theta(\widetilde{X}_{u_0}, \widetilde{Y}_{u_0})) = -u_0 \circ \theta(\widetilde{[X,Y]_{\mathfrak{M}}}) = -[X,Y]_{\mathfrak{M}} \tag{3.20}$$

for any $X, Y \in \mathfrak{M}$ viewed as elements of $\mathbb{R}^n$. Similarly, the *curvature tensor*

$$R(X,Y) = u_0(2\Omega(\widetilde{X}_{u_0}, \widetilde{Y}_{u_0})) = -u_0 \circ \lambda([X,Y]_{\mathfrak{g}_0}) = -[X,Y]_{\mathfrak{g}_0}. \tag{3.21}$$

This gives us the value of both tensors at any and all points of the body manifold M due to the fact that the canonical connection ω is left invariant. In summary (compare e.g., [2], [12]):

Theorem 3. *Let $\mathbf{l} : M \to L(M)$ be a continuous lattice defined on the body manifold M. Select a point $x_0 \in M$ and let $P(M, G_{x_0})$ be the G-invariant G_{x_0}-frame structure*[13] *generated by the lattice $\mathbf{l}$. Assume that the body manifold M, viewed as a homogeneous space G/G_{x_0}, is reductive with the decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{M}$. Then, there exists a unique (G-invariant) lattice canonical connection ω in P as defined by the relation (3.7). The connection ω is such that its torsion tensor T and the curvature tensor R are given at $x_0 \in M$ by:*

- *$T(X,Y)_{x_0} = -[X,Y]_{\mathfrak{M}}$, for $X, Y \in \mathfrak{M}$,*
- *$(R(X,Y)Z)_{x_0} = -[[X,Y]_{\mathfrak{g}_0}, Z]$, for $X, Y, Z \in \mathfrak{M}$.*

In addition, both tensors are covariantly constant.

Remark 2. Note that if a continuous lattice is uniformly defective, that is, $M \cong G$ as the body manifold is viewed as a Lie group acting on itself, the lattice canonical connection ω is identical to the linear connection induced on M by the lattice frame $\mathbf{l}$. Indeed, as the isotropy group G_{x_0} of such an action of M on itself is trivial, the curvature of the lattice canonical connection vanishes and the torsion is given by the Lie algebra constants of the subalgebra $\mathfrak{M} = \mathfrak{g}$. This is certainly consistent with the fact that the given lattice frame induces a long distance parallelism on M and the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{l}$; the subalgebra of smooth vector fields generated by the lattice frame. On the other hand, when the continuous lattice is non-uniformly defective, its lattice canonical connection is completely different from the linear connection induced on M by the lattice frame. Indeed, as clearly illustrated by the examples presented in the next section, the lattice canonical connection ω may have a non-vanishing curvature and its torsion seems to be in no relation to the torsion of the frame $\mathbf{l}$. This, in fact, begs the question of what is

[13] To avoid any notational confusion, by a G_{x_0}-frame structure we mean a reduction of the bundle of frames $L(M)$ to the subgroup G_{x_0}.

the relation between the flat linear connection induced on M by the lattice frame and the lattice canonical connection ω; a question we shall investigate in the forthcoming work.

4. Examples

Example 2. Here we develop Example 1 - for completeness and for the benefit of the reader, we first restate some facts. So, consider the three-parameter group $G = \mathbb{R}^3$ with the group multiplication

$$g\overline{g} = (a + \overline{a}, b + \overline{b}, c + \overline{c} + \overline{b}a), \;\; g, \overline{g} \in G \tag{4.1}$$

and assume that the group G acts on $\mathbb{R}^2$ (on the left) by

$$\Lambda((a, b, c), (x, y)) = (x + a, y - b(x + a) + c) \tag{4.2}$$

for any $(a, b, c) \in G$ and $(x, y) \in \mathbb{R}^2$. Given an arbitrary point $(x, y) \in \mathbb{R}^2$, consider its orbit map $\Lambda_{(x,y)} : G \to \mathbb{R}^2$, (2.2)and its tangent map $d\Lambda_{(x,y)} : T_gG \to T_{\Lambda(g,(x,y))}\mathbb{R}^2$, represented in the standard coordinate systems on $G = \mathbb{R}^3$ and $\mathbb{R}^2$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -b & -x & 1 \end{pmatrix}. \tag{4.3}$$

At the identity of the group, $e = (0, 0, 0))$ the tangent map induces the Lie algebra $\mathfrak{l}$ of vector fields on $\mathbb{R}^2$ generated by

$$l_1 = (1, 0),\; l_2 = (0, -x),\; l_3 = (0, 1). \tag{4.4}$$

Moreover, analyzing the group multiplication of the group G, one can easily show that its Lie algebra $\mathfrak{g}$ is generated by

$$\mathfrak{l}_1 = (1, 0, 0),\; \mathfrak{l}_2 = (0, 1, a),\; \mathfrak{l}_3 = (0, 0, 1) \tag{4.5}$$

and that the algebras $\mathfrak{l}$ and $\mathfrak{g}$ are isomorphic.

At the point $(x_0, y_0) \in \mathbb{R}^2$ the isotropy group of the action Λ is

$$G_{x_0} = \{(0, b, bx_0) :\; b \in \mathbb{R}\} \tag{4.6}$$

and its one-dimensional Lie algebra $\mathfrak{g}_0$ is spanned by $(0, 1, x_0)$. To determine if the homogeneous space G/G_{x_0} is reductive, select now a Lie algebra $\widehat{\mathfrak{g}}$ of vector fields on the group G generated by

$$\widehat{\mathfrak{l}}_1 = (1, 0, b),\; \widehat{\mathfrak{l}}_2 = (0, 1, x_0),\; \widehat{\mathfrak{l}}_3 = (0, 0, 1) \tag{4.7}$$

to realize that it is isomorphic to the Lie algebra $\mathfrak{g}$ and it has $\mathfrak{g}_0$ as its subalgebra. Moreover, $\widehat{\mathfrak{g}}$ is the algebra of the left-invariant vector fields on G[14] and the vector space

$$\mathfrak{M} = \operatorname{span}\{\widehat{\mathfrak{l}}_1, \widehat{\mathfrak{l}}_3\} \subset \mathfrak{g} \tag{4.9}$$

is invariant under the adjoint action of G_0 on $\mathfrak{g}$ as $[\widehat{\mathfrak{l}}_2, \widehat{\mathfrak{l}}_1] = -\widehat{\mathfrak{l}}_3 \in \mathfrak{M}$ and $[\widehat{\mathfrak{l}}_2, \widehat{\mathfrak{l}}_3] = 0$. In fact, the vector space $\mathfrak{M}$ is a subalgebra of $\mathfrak{g}$, as $[\widehat{\mathfrak{l}}_1, \widehat{\mathfrak{l}}_3] = 0$, and the group G is a semidirect product of the isotropy group G_{x_0} and the additive subgroup $H = \{(a, 0, c) : a, c \in \mathbb{R}\} \subset G$ the Lie algebra of which is isomorphic to $\mathfrak{M}$.

In conclusion, the homogeneous space G/G_{x_0} of the lattice frame $l_1 = (1, 0), l_2 = (0, -x)$ is, as shown above, reductive via the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{M}$. The isotropy group G_{x_0} is isomorphic, via the isotropy linear representation, to the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} : \ b \in \mathbb{R} \right\} \subset GL(2, \mathbb{R}) \tag{4.10}$$

and the corresponding lattice canonical connection ω is both curvature and torsion free as $[\mathfrak{M}, \mathfrak{M}] = 0$; see Theorem 3. Thus, there exists a local coordinate system on $\mathbb{R}^2$ such that the corresponding Christoffel's symbols Γ^i_{jk}, $i, j, k = 1, 2$, vanish.

Example 3. Consider the continuous lattice on $\mathbb{R}^2$ given (in the standard coordinate system) by the frame

$$l_1 = (y, -x), \ l_2 = (\frac{1}{2}(1 + x^2 - y^2), xy). \tag{4.11}$$

As $[l_1, l_2] = \frac{1}{2}(2xy, 1 + y^2 - x^2) = l_3$ and as $[l_2, l_3] = l_1$ and $[l_3, l_1] = l_2$, the given lattice frame generates the three dimensional Lie algebra of vector fields $\mathfrak{l}$ which is isomorphic to the Lie algebra $\mathfrak{so}(3)$[15] of the special orthogonal group $SO(3)$. In turn, the algebra $\mathfrak{so}(3)$ is isomorphic to the Lie algebra $\mathfrak{su}(2)$ of the special unitary group $SU(2)$ which can be

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[14] The induced left action of the group G on its Lie algebra is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix}. \tag{4.8}$$

[15] One may select

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ R = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

as a basis of $\mathfrak{so}(3)$, [1].

spanned, for example, by the basis

$$E = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ F = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ H = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.12}$$

As the group $SU(2)$ is homomorphic to the group $SO(3)$ via the covering isomorphism $p : SU(2)/\{I, -I\} \to SO(3)$, rather than investigating the action of $SO(3)$ on $\mathbb{R}^2$ corresponding to our lattice algebra $\mathfrak{l}$, we shall consider the analogous action of $SU(2)$ on the complex space $\mathbb{C}$; viewed as $\mathbb{R}^2$. Namely, given

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1; a, b \in \mathbb{C} \right\} \tag{4.13}$$

consider the action $\Lambda : SU(2) \times \mathbb{C} \to \mathbb{C}$ such that

$$\Lambda\left(\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, z \right) = \frac{b + \bar{a}z}{a - \bar{b}z}. \tag{4.14}$$

As the action Λ is transitive, the isotropy groups at different points in $\mathbb{C}$ are conjugate to each other. Thus, to simplify our calculations let us consider $z_0 = 1$. It is then easy to show that the isotropy group of the action Λ at z_0 is

$$G_{z_0} = \left\{ \begin{pmatrix} \alpha & \beta i \\ \beta i & \alpha \end{pmatrix} : \alpha^2 + \beta^2 = 1; \alpha, \beta \in \mathbb{R} \right\} \tag{4.15}$$

and that its Lie algebra $\mathfrak{g}_0$ is spanned by $E = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. As $[E, F] = -H$ and $[E, H] = F$, one can see that the homogeneous space $SU(2)/G_{z_0}$ is reductive, that is, $\mathfrak{su}(3) = \mathfrak{g}_0 \oplus \mathfrak{M}$ where the vector space $\mathfrak{M} = \mathrm{span}\{H, F\}$ and $[\mathfrak{g}_0, \mathfrak{M}] \subset \mathfrak{M}$. Moreover, as $[H, F] = E$, that is, as $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{g}_0$, the lattice canonical connection ω, although torsion free, has non-vanishing curvature. In fact, as the isotropy group G_{z_0} is isomorphic to the special orthogonal group $SO(2)$:

$$\left\{ \begin{pmatrix} p & r \\ -r & p \end{pmatrix} :\ p^2 + r^2 = 1,\ p, r \in \mathbb{R} \right\} \tag{4.16}$$

the lattice canonical connection ω is a pull-back of the G_{z_0}-component of the Maurer-Cartan form of G to the manifold M isomorphic via the linear isotropy representation to the quotient $SO(3)/SO(2)$.

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