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Commutativity of the Strain Energy Density Expression for the Benefit of the FEM Implementation of Koiter's Initial Post-buckling Theory

Shuguang Li[1], Jiayi Yan[1], Guofan Zhang[2] and Shihui Duan[2]

[1]Faculty of Engineering, the University of Nottingham, Nottingham, NG7 2RD, UK

[2]AVIC Aircraft Strength Research Institute, Xi'an, 710065, PR China

Abstract

The concept of full commutativity of displacements in the expression for strain energy density for the geometrically nonlinear problem has been introduced for the first time and fully established in this paper. Its consequences for the FEM formulation have been demonstrated. As a result, the strain energy, equilibrium equation and incremental equilibrium equation for the geometrically nonlinear problem can all be presented in a unified manner involving various stiffness matrices which are all symmetric, unique and explicitly expressed. As an important application, the framework has been employed in the FEM implementation of Koiter's initial post-buckling theory, which has been handicapped by its mesh sensitivity in evaluating one of the initial post-buckling coefficients. This has largely prevented it from being incorporated in mainstream commercial FEM codes. Based on the outcomes of this paper, the mesh sensitivity problem has been completely resolved without the need to use any specially formulated element. As a result, Koiter's theory can be practically and straightforwardly implemented in any FEM code. The results have been verified against those found in the literature.

Keywords: Commutativity; Strain energy density; Geometrical nonlinearity; Finite element formulation; Koiter's theory; Initial post-buckling theory.

1 Introduction

The geometrically nonlinear problem has been perceived as well-established since analyses of this kind are routinely carried out by users of commercial FEM codes, e.g. [1-3], as a standard provision of these codes. However, there is one issue which has never been properly raised and investigated, namely, the lack of commutativity of displacements in the various stages of FEM formulation, all associated with the expression of strain energy density (SED) as will be explored in the paper. The issue has always been sidestepped when encountered without being identified as a problem but can be revealed straightforwardly if one attempts to derive the secant stiffness matrix. Using conventional notations which are provided in Appendix A to avoid any confusion, the strain energy in a finite element can be given as follows.

$$
\begin{aligned}
U^e &= \frac{1}{2}\int_{\Omega^e}\left(\boldsymbol{\varepsilon}_L^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}_L+\frac{1}{2}\boldsymbol{\varepsilon}_L^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}_N+\frac{1}{2}\boldsymbol{\varepsilon}_N^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}_L+\frac{1}{4}\boldsymbol{\varepsilon}_N^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}_N\right)d\Omega \\
&= \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left[\int_{\Omega^e}\left(\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_L+\frac{1}{2}\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_N(\mathbf{q})+\frac{1}{2}\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_L+\frac{1}{4}\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_N(\mathbf{q})\right)d\Omega\right]\mathbf{q}
\end{aligned}
\tag{1}
$$

where Ω^e is the domain of the finite element concerned and $\mathbf{q}$ the nodal displacement vector discretised from an otherwise continuous displacement field $\mathbf{u}$. In order to establish the equilibrium equation, variation of the strain energy (1) is required as given below, where advantage has been taken of the symmetry of the $\mathbf{C}$ matrix, the obvious commutativity between $\mathbf{q}$ and $\delta\mathbf{q}$ in $\mathbf{B}_N(\delta\mathbf{q})\mathbf{q}$ as explained in Appendix A and the fact that any term in the above expression, e.g. $\frac{1}{2}\mathbf{q}^{\mathrm{T}}\int_{\Omega^e}\frac{1}{2}\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\,\mathbf{B}_L d\Omega\,\mathbf{q}$, is a scalar and its transpose is equal to itself.

$$
\delta U^e = \delta\mathbf{q}^{\mathrm{T}}\left[\int_{\Omega^e}\left(\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_L+\frac{1}{2}\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_N(\mathbf{q})+\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_L+\frac{1}{2}\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_N(\mathbf{q})\right)d\Omega\right]\mathbf{q}. \tag{2}
$$

The integral inside the square brackets in (2) should give the secant stiffness matrix of the element but it has never been accepted because it lacks symmetry, as stated in the second edition of [4]. Whilst the first and fourth terms inside the integral are symmetric, the remaining part, i.e. the sum of the second and third terms, does not look symmetric superficially because

$$
\left(\frac{1}{2}\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\,\mathbf{B}_N(\mathbf{q})+\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_L\right)^{\mathrm{T}} = \frac{1}{2}\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_L+\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\,\mathbf{B}_N(\mathbf{q}) \neq \frac{1}{2}\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\,\mathbf{B}_N(\mathbf{q})+\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_L. \tag{3}
$$

The above derivation followed the so-called N-notation (a terminology introduced in [9] as being one approach to the FEM formulation, in contrast to the other commonly used approach referred to in [9] as the B-notation) but has always been avoided in textbooks, e.g. [4-8], because it was seen as a dead end. If one could swap the position of some of the displacements involved, e.g. the $\delta\mathbf{q}$ and the $\mathbf{q}$ at the end of the expression $\delta\mathbf{q}^{\mathrm{T}}\left(\int_{\Omega^e}\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\,\mathbf{B}_L d\Omega\right)\mathbf{q}$, to give $\mathbf{q}^{\mathrm{T}}\left(\int_{\Omega^e}\mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\,\mathbf{B}_L d\Omega\right)\delta\mathbf{q}$ without affecting the value of the expression, the inequality (3) would become an equality and the secant stiffness matrix could readily be made symmetric. Unfortunately, commutativity in matrix expressions is generally not applicable. Without a properly defined secant stiffness matrix, equilibrium conditions cannot be stated in explicit form. Common practice is to establish equilibrium indirectly by integrating stresses whilst jumping to the tangential stiffness matrix directly [5-8] in terms of the formulation of element stiffness matrices, employing the so-called B-notation [9]. There have been limited attempts [9-11] to unify expressions of secant and tangential stiffness matrices but the obtained expressions lacked uniqueness, as admitted in [10,11]. It will be shown later in this paper that the lack of symmetry and uniqueness is in fact a false impression. A more fundamental and generic issue surfaces when one tries to implement Koiter's initial post-buckling theory [12] through FEM as it places more demands on commutativity of displacements in energy expressions, referred to as *full commutativity*, as will be detailed later in this paper.

Existing accounts [13,14] of the FEM formulation of Koiter's initial post-buckling theory tended to shy away from having such commutativity fully established. There have not been many attempts to adopt the N-notation for the FEM formulation in general since the publication of [13-14], let alone its implementation for Koiter's theory. The difficulty has been circumvented by following the B-notation formulation of Koiter's theory [15]. The price paid for this convenience is, however, that the results become mesh sensitive as will be elaborated further in Section 6, noting that mesh sensitivity is undesirable in any FEM analysis.

The objectives of this paper are twofold. On the fundamental theoretical development side, it will establish the full commutativity of the SED as one of its intrinsic properties, and provide an explicitly and uniquely presented FEM formulation in N-notation based on the full commutativity established. The failure to appreciate such full commutativity is responsible for the fact that the secant stiffness matrix has never been explicitly and uniquely obtained. This long-standing issue will be resolved completely in this paper. On the practical side, and as a direct application of the obtained fully commutative SED, the FEM implementation of Koiter's theory using N-notation is herewith made possible thus freeing it from the problem of mesh sensitivity that occurs in available accounts. As will be introduced in more detail in Section 6 of this paper, Koiter's theory provides deep insight into the buckling problem of structures and hence has tremendous potential in practical engineering applications. However, due to the lack of an acceptable FEM implementation, this theory has been largely neglected, at least as far as mainstream commercial FEM codes are concerned, wasting the opportunity to exploit it as a potentially valuable approach to address the uncertainties associated with the buckling problem. The outcome of this paper will hopefully lead to the resurrection of Koiter's theory as a design analysis tool, providing structural designers and engineers with the means to evaluate the confidence on their obtained buckling strengths of structures. Lack of confidence in predicted results has been the main challenge to existing forms of buckling analysis, the reason being the fact that some structures are extremely sensitive to initial imperfections whilst others may be not. No assessment can currently be made of the sensitivity of the obtained buckling load to the initial imperfection from the conventional buckling analysis and Koiter's theory fills this gap precisely and neatly, as will be further discussed in this paper.

2 The concept of full commutativity and apparent symmetry

The strain energy density U exists for any geometrically nonlinear problem and can be expressed as a sum of homogeneous terms from the 1st to the 4th orders as follows

$$U = U_1 + U_2 + U_3 + U_4 \tag{4}$$

where the exact expressions of each term are provided in Appendix A.

Continuous displacement fields $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{u}'''$ are introduced to facilitate the following discussion, and they are assumed to be mutually independent. They are discretised into $\mathbf{q}$, $\mathbf{q}'$, $\mathbf{q}''$ and $\mathbf{q}'''$ for FEM applications, as already employed in expressing (1). In the context of Koiter's theory, these displacement fields can be the fundamental equilibrium path $\mathbf{u}_0$, various orders of perturbation displacements, $\mathbf{u}_1$, $\mathbf{u}_2$,…, etc., respectively, as well as displacement variation $\delta\mathbf{u}$. In order to illustrate the concept of commutativity, consider the strain energy at a specific deformation state which is described as the sum of independent displacement fields. Take the third order term of the SED as given in (A-5) in Appendix A for example

$$\begin{aligned}
&U_3(\mathbf{u}+\mathbf{u}'+\mathbf{u}'') = \frac{1}{2}(\mathbf{u}+\mathbf{u}'+\mathbf{u}'')^{\mathrm{T}}\,\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}+\mathbf{u}'+\mathbf{u}'')(\mathbf{u}+\mathbf{u}'+\mathbf{u}'') \\
&= U_{111}(\mathbf{u},\mathbf{u},\mathbf{u}) + U_{111}(\mathbf{u},\mathbf{u},\mathbf{u}') + U_{111}(\mathbf{u},\mathbf{u},\mathbf{u}'') + \cdots + U_{111}(\mathbf{u},\mathbf{u}'',\mathbf{u}) + U_{111}(\mathbf{u},\mathbf{u}'',\mathbf{u}') + U_{111}(\mathbf{u},\mathbf{u}'',\mathbf{u}'') \\
&+U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}) + U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}') + U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'') + \cdots + U_{111}(\mathbf{u}',\mathbf{u}'',\mathbf{u}) + U_{111}(\mathbf{u}',\mathbf{u}'',\mathbf{u}') + U_{111}(\mathbf{u}',\mathbf{u}'',\mathbf{u}'') \\
&+U_{111}(\mathbf{u}'',\mathbf{u},\mathbf{u}) + U_{111}(\mathbf{u}'',\mathbf{u},\mathbf{u}') + U_{111}(\mathbf{u}'',\mathbf{u},\mathbf{u}'') + \cdots + U_{111}(\mathbf{u}'',\mathbf{u}'',\mathbf{u}) + U_{111}(\mathbf{u}'',\mathbf{u}'',\mathbf{u}') + U_{111}(\mathbf{u}'',\mathbf{u}'',\mathbf{u}'')
\end{aligned} \tag{5}$$

where $U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}')\mathbf{u}''$ (6a)

$$U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'') = \frac{1}{2}(\mathbf{u}')^{\mathrm{T}}\,\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u})\mathbf{u}'',\ \ldots\ . \tag{6b}$$

Relationship (5) introduces a trilinear form and its typical expression has been given in (6a). Some of the displacements in (6a) can swap their positions without affecting the value of the expression, e.g. $\mathbf{u}'$ and $\mathbf{u}''$. In other words, $U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ is commutative as far as $\mathbf{u}'$ and $\mathbf{u}''$ are concerned. This commutativity is described as *partial commutativity* of the trilinear form $U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ in the present paper whilst $\mathbf{u}$ and $\mathbf{u}'$ cannot be swapped without affecting the value of $U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$, in general, because

$$U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}')\mathbf{u}'' \neq \frac{1}{2}(\mathbf{u}')^{\mathrm{T}}\,\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u})\mathbf{u}'' = U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}''). \tag{7}$$

In much the same way, a bilinear form $U_{11}(\mathbf{u},\mathbf{u}')$ and a quad-linear form $U_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''')$ can also be introduced from U_2 and U_4, respectively,

$$U_{11}(\mathbf{u},\mathbf{u}') = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}')\boldsymbol{\sigma}_I + \frac{1}{2}\mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_L\mathbf{u}' \tag{8}$$

$$U_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''') = \frac{1}{8}\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}')\mathbf{C}\,\nabla_N(\mathbf{u}'')\mathbf{u}'''\ . \tag{9}$$

A property of *full commutativity* for bilinear, trilinear and quad-linear forms can be introduced as one of the key subjects of the present paper if any pair of displacement fields involved in the expression concerned can swap their positions in an expression without affecting the value of the expression.

Along with full commutativity, another closely related property can be introduced as follows. To formulate element stiffness matrices in FEM, it is desirable to express energy and its first and second variations in a form of $\mathbf{u}^{\mathrm{T}}\mathbf{M}\mathbf{u}'$, with $\mathbf{M}$ being a matrix operator, as shown in (1) and (2), for instance. However, not every term is readily given in this form explicitly, e.g. $\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}\left(\mathbf{u}'\right)\boldsymbol{\sigma}_I$ as a part of $U_{11}\left(\mathbf{u},\mathbf{u}'\right)$ as given in (8). One of the tasks of the present paper is to make such terms into $\mathbf{u}^{\mathrm{T}}\mathbf{M}\mathbf{u}'$ form. Once a bilinear, trilinear or quad-linear form is so expressed, it will be referred to as *apparently symmetric* in the present paper if $\mathbf{M}$ is given as a symmetric matrix operator. The lack of symmetry in the secant stiffness matrix as shown in (2) resulted from the lack of such apparent symmetry in the third order term of the SED.

Some of the difficulties in the FEM formulation of problems involving geometrical nonlinearity resulting from the lack of full commutativity and apparent symmetry are listed as follows.

1) Whilst the B-notation [9] delivers a symmetric tangential stiffness matrix [5-8], the secant stiffness matrix cannot be obtained explicitly.
2) Following the N-notation [9], one can obtain the tangential and secant stiffness matrices directly and make them symmetric [9-11]. However, without achieving the full commutativity of SED, the obtained expressions of stiffness matrices lack uniqueness [10,11].

Just as the secant and tangential stiffness matrices can be considered as the outcomes of the first and second orders of variations of the strain energy, higher orders of variations are required in order to implement the Koiter theory [12]. The earliest versions of FEM implementation of Koiter's theory tended to follow the N-notation [13,14] for the expressions of the so-called initial post-buckling coefficients (IPBCs). However, the weaknesses of the N-notation as described above prevented it from being practically exploited. Instead, alternative expressions for these IPBCs were brought forward [15]. Unfortunately, they suffer from numerical 'locking' or mesh sensitivity [16,17]. The solution to the problem has so far been either to employ a much more refined mesh than that used for conventional buckling analysis in order to evaluate the IPBCs, or to employ specially formulated elements [18-23], to name but a few of the available unattractive options.

It should be pointed out that, in analytical applications of Koiter's theory, a displacement field, such as the buckling mode or any perturbation modes at high orders, is described by a single scalar giving the amplitude of an assumed pattern of the displacement field in the form of a scalar function. Scalars are always commutative and commutativity has therefore never been an issue there. In addition, the 'locking' problem is also absent in analytical approaches. However, the applicability of analytical approaches is of course restricted to simple structures under simple loading and boundary conditions, requiring FEM to be used for analysing structures of realistic complexity, giving rise to the difficulty of mesh sensitivity. The present paper aims to demonstrate that this difficulty can be removed by employing the N-notation after the establishment of the fully commutative SED.

3 Fully commutative SED

Due to the symmetries of stress, strain and stiffness tensors and the definition of the Green strain, the following relationships always hold.

$$\nabla_N(\mathbf{u})\mathbf{u}' = \nabla_N(\mathbf{u}')\mathbf{u} \tag{10a}$$

$$\mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\,\nabla_L\mathbf{u}' = (\mathbf{u}')^{\mathrm{T}}\,\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_L\mathbf{u} \tag{10b}$$

Given the above, it is clear that $U_{11}(\mathbf{u},\mathbf{u}')$ as given in (8) is already fully commutative, i.e.

$$U_{11}(\mathbf{u},\mathbf{u}') = U_{11}(\mathbf{u}',\mathbf{u}). \tag{11a}$$

However, relationships (10) deliver to $U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ and $U_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''')$ only partial commutativity between some pairs of $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{u}'''$ as follows.

$$U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') = U_{111}(\mathbf{u},\mathbf{u}'',\mathbf{u}') \tag{11b}$$

$$\begin{aligned}&U_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''') = U_{1111}(\mathbf{u}',\mathbf{u},\mathbf{u}'',\mathbf{u}''') = U_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}''',\mathbf{u}'') = U_{1111}(\mathbf{u}',\mathbf{u},\mathbf{u}''',\mathbf{u}'')\\&= U_{1111}(\mathbf{u}''',\mathbf{u}'',\mathbf{u}',\mathbf{u}) = U_{1111}(\mathbf{u}'',\mathbf{u}''',\mathbf{u}',\mathbf{u}) = U_{1111}(\mathbf{u}''',\mathbf{u}'',\mathbf{u},\mathbf{u}') = U_{1111}(\mathbf{u}'',\mathbf{u}''',\mathbf{u},\mathbf{u}').\end{aligned} \tag{11c}$$

In general, however, commutativity cannot be blindly assumed between other pairs of $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{u}'''$, as may be observed from (7).

It is desirable to find alternative but equivalent expressions to U_{111} and U_{1111}, denoted as Q_{111} and Q_{1111} which are fully commutative, i.e.

$$\begin{aligned}&Q_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') = Q_{111}(\mathbf{u},\mathbf{u}'',\mathbf{u}') = Q_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'')\\&= Q_{111}(\mathbf{u}',\mathbf{u}'',\mathbf{u}) = Q_{111}(\mathbf{u}'',\mathbf{u},\mathbf{u}') = Q_{111}(\mathbf{u}'',\mathbf{u}',\mathbf{u})\end{aligned} \tag{12a}$$

$$\begin{aligned}&Q_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''') = Q_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}''',\mathbf{u}'') = \cdots\cdots\\&= \text{ all together 24 permutations } = \cdots\cdots = Q_{1111}(\mathbf{u}''',\mathbf{u}'',\mathbf{u}',\mathbf{u}).\end{aligned} \tag{12b}$$

As U_{11} is fully commutative, Q_{11} is identical to U_{11}.

The trilinear form is considered first. The expansion given in (5) offers a hint to the construction of Q_{111} from U_{111}. If all possible permutations ($P_3^3 = 3! = 6$) of $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ in U_{111} as appearing in (5) have been included, one obtains

$$Q_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') = \begin{pmatrix} U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') + U_{111}(\mathbf{u},\mathbf{u}'',\mathbf{u}') + U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'') \\ +U_{111}(\mathbf{u}',\mathbf{u}'',\mathbf{u}) + U_{111}(\mathbf{u}'',\mathbf{u},\mathbf{u}') + U_{111}(\mathbf{u}'',\mathbf{u}',\mathbf{u}) \end{pmatrix}. \tag{13}$$

It is clear that swapping any pair of $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ only results in a reshuffle of the six terms inside the brackets, e.g. between pair $\mathbf{u}$ and $\mathbf{u}'$

$$Q_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'') = \begin{pmatrix} U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'') + U_{111}(\mathbf{u}',\mathbf{u}'',\mathbf{u}) + U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') \\ +U_{111}(\mathbf{u},\mathbf{u}'',\mathbf{u}') + U_{111}(\mathbf{u}'',\mathbf{u}',\mathbf{u}) + U_{111}(\mathbf{u}'',\mathbf{u},\mathbf{u}') \end{pmatrix} \equiv Q_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}''). \tag{14}$$

Given the partial commutativity of U_{111} as shown in (11b), $Q_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ given in (13) can be reduced to

$$\begin{aligned} &Q_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') = 2U_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') + 2U_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'') + 2U_{111}(\mathbf{u}'',\mathbf{u}',\mathbf{u}) \\ &= \mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}')\mathbf{u}'' + (\mathbf{u}')^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u})\mathbf{u}'' + (\mathbf{u}'')^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}')\mathbf{u} \end{aligned} \tag{15a}$$

The trilinear form $Q_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ given above is thus fully commutative and it is straightforward to verify that it satisfies (12a).

Similarly, a fully commutative fourth order expression $Q_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''')$ can be constructed from all $(P_4^4 = 4! = 24)$ possible permutations of $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{u}'''$ in U_{1111}. After making use of the partial commutativity (11c), it can be expressed as

$$\begin{aligned} &Q_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''') = 8U_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''') + 8U_{1111}(\mathbf{u},\mathbf{u}'',\mathbf{u}',\mathbf{u}''') + 8U_{1111}(\mathbf{u},\mathbf{u}''',\mathbf{u}',\mathbf{u}'') \\ &= \mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}')\mathbf{C}\nabla_N(\mathbf{u}'')\mathbf{u}''' + \mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}'')\mathbf{C}\nabla_N(\mathbf{u}')\mathbf{u}''' + \mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}''')\mathbf{C}\nabla_N(\mathbf{u}')\mathbf{u}''. \end{aligned} \tag{15b}$$

One can introduce functions $Q_3(\mathbf{u})$ and $Q_4(\mathbf{u})$ from Q_{111} and Q_{1111} as

$$Q_3(\mathbf{u}) = \frac{1}{6}Q_{111}(\mathbf{u},\mathbf{u},\mathbf{u}) = U_3(\mathbf{u}) = U_{111}(\mathbf{u},\mathbf{u},\mathbf{u}) \tag{16a}$$

$$Q_4(\mathbf{u}) = \frac{1}{24}Q_{1111}(\mathbf{u},\mathbf{u},\mathbf{u},\mathbf{u}) = U_4(\mathbf{u}) = U_{1111}(\mathbf{u},\mathbf{u},\mathbf{u},\mathbf{u}). \tag{16b}$$

It is clear that Q_3 and Q_4 are identical to U_3 and U_4 in their roles as parts of the SED, respectively, but Q_3 and Q_4 are associated with the fully commutative trilinear and quad-linear forms. If Q_3 and Q_4 are employed instead of U_3 and U_4 in the SED, they will not alter the value of SED in any way. However, any subsequent manipulations on the displacement in the relevant energy term, such as variations, will be indifferent to the position of the displacement within the expression, as a straightforward and important application of the full commutativity, i.e.

$$\delta Q_3(\mathbf{u}) = \frac{1}{6}Q_{111}(\delta\mathbf{u},\mathbf{u},\mathbf{u}) + \frac{1}{6}Q_{111}(\mathbf{u},\delta\mathbf{u},\mathbf{u})\frac{1}{6}Q_{111}(\mathbf{u},\mathbf{u},\delta\mathbf{u})$$
$$= \frac{1}{2}Q_{111}(\delta\mathbf{u},\mathbf{u},\mathbf{u}) = \frac{1}{2}Q_{111}(\mathbf{u},\delta\mathbf{u},\mathbf{u}) = \frac{1}{2}Q_{111}(\mathbf{u},\mathbf{u},\delta\mathbf{u}) = Q_{12}(\delta\mathbf{u},\mathbf{u}) \tag{17a}$$

and similarly

$$\delta Q_4 = \frac{1}{6}Q_{1111}(\delta\mathbf{u},\mathbf{u},\mathbf{u},\mathbf{u}) = Q_{13}(\delta\mathbf{u},\mathbf{u}) \tag{17b}$$

$$\delta^2 Q_3 = Q_{111}(\delta\mathbf{u},\mathbf{u},\delta\mathbf{u}) = 2Q_{21}(\delta\mathbf{u},\mathbf{u}) \quad \text{and} \quad \delta^2 Q_4 = \frac{1}{2}Q_{1111}(\delta\mathbf{u},\mathbf{u},\mathbf{u},\delta\mathbf{u}) = Q_{22}(\delta\mathbf{u},\mathbf{u}). \tag{18}$$

where the subscripts correspond to the order of the variable in the variable list in the same way as was originally introduced by Koiter in [12]. The SED expression obtained through Q_3 and Q_4 will be called the *fully commutative SED*. With this property, the variations of them resemble the derivatives of power functions. However, if δQ_3 and δQ_4 are directly derived from (16) with Q_3 and Q_4 with Q_{111} and Q_{1111} as expressed in (15), one might still find the lack of apparent symmetry at some points resembling the observation made to (2). It should be pointed out that the full commutativity was taken for granted in [12] by Koiter without proof or guidance how to obtain this property in general. Given the full commutativity as an intrinsic property of SED established mathematically above, it is now known with assurance that the matrix involved in (2) is symmetric, although this symmetry is not yet obvious. The lack of apparent symmetry in (2) is only due to the way in which the expression is presented. Apparent symmetry will emerge after various terms obtained above have been further streamlined by applying to them an identical transformation to be described in the next section.

4 The flip transformation

The second order term in the Green strain gives rise to a factor of $\nabla_N(\mathbf{u})\mathbf{u}$ appearing in various places as shown above. Whilst the displacements involved in it are commutative, as shown in (10a), many energy terms involving this do not have apparently symmetric appearances, i.e. not in a form of $\mathbf{u}^{\mathrm{T}}\mathbf{M}\mathbf{u}'$ with $\mathbf{M}$ being a symmetric matrix operator. For instance, $\mathbf{u}$ and $\mathbf{u}'$ in $\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}')\boldsymbol{\sigma}_I$ as a term in U_{11} are both located biasedly on the same side of $\boldsymbol{\sigma}_I$. They tend to cause difficulties in FEM formulation if not treated appropriately. In fact, these terms can be expressed differently in appearance but identically in value. One can easily prove the following relationship.

$$\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}')\boldsymbol{\sigma}_I \equiv (\mathbf{u}')^{\mathrm{T}}\mathbf{S}_I\mathbf{u} \equiv \mathbf{u}^{\mathrm{T}}\mathbf{S}_I\mathbf{u}' \tag{19a}$$

where $\mathbf{S}_I = \begin{bmatrix} \nabla_1 \\ \nabla_2 \\ \nabla_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{s}_I & 0 & 0 \\ 0 & \mathbf{s}_I & 0 \\ 0 & 0 & \mathbf{s}_I \end{bmatrix} \begin{bmatrix} \nabla_1 \\ \nabla_2 \\ \nabla_3 \end{bmatrix} = \sum_{i=1}^{3} \nabla_i^{\mathrm{T}} \mathbf{s}_I \nabla_i$, (19b)

$$\nabla_1 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & 0 \end{bmatrix}, \quad \nabla_2 = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \end{bmatrix} \quad \text{and} \quad \nabla_3 = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix}. \tag{19c}$$

$\boldsymbol{\sigma}_I$ and $\mathbf{s}_I$ represent the same stress but expressed in contracted and matrix forms, respectively, as introduced in Appendix A. (19a) is an identical transformation through which the asymmetric expression has been turned into an apparently symmetric one with $\mathbf{S}_I$ being a symmetric matrix operator. This transformation has often been employed, e.g. in [5,6], but the authors failed to trace it back to its source where it was originally introduced and it does not seem to have been named either. It is almost certain that it was first brought in when tackling the buckling problem using finite elements when the so-called geometric or initial stress stiffness matrix was introduced. For the sake of easy reference, the authors suggest naming it the ‘flip transformation’ because it flips one of the two displacements on one side of the stress to other side, leading to the two displacements being placed symmetrically on both sides of the symmetric matrix $\mathbf{S}_I$. The derivation can be found in [5,6] but is provided in Appendix B in the context of independent displacements to be consistent with the present context.

Similarly, the flip transformation also applies to higher order terms of stress

$$\mathbf{u}^{\mathrm{T}} \nabla_N^{\mathrm{T}}(\mathbf{u}') \mathbf{C} \nabla_L \mathbf{u}'' \equiv \mathbf{u}^{\mathrm{T}} \nabla_N^{\mathrm{T}}(\mathbf{u}') \mathbf{s}_L(\mathbf{u}'') \equiv \mathbf{u}^{\mathrm{T}} \mathbf{S}_L(\mathbf{u}'') \mathbf{u}' \tag{20a}$$

$$\mathbf{u}^{\mathrm{T}} \nabla_N^{\mathrm{T}}(\mathbf{u}') \mathbf{C} \nabla_N(\mathbf{u}'') \mathbf{u}'' \equiv \mathbf{u}^{\mathrm{T}} \nabla_N^{\mathrm{T}}(\mathbf{u}') \mathbf{s}_{11}(\mathbf{u}'', \mathbf{u}''') \equiv \mathbf{u}^{\mathrm{T}} \mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''') \mathbf{u}' \tag{20b}$$

where $\mathbf{S}_L$ is similar to $\mathbf{S}_I$ but with $\mathbf{s}_I$ replaced by $\mathbf{s}_L$, whilst $\mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''')$ is obtained by replacing $\mathbf{s}_I$ with $\mathbf{s}_{11}(\mathbf{u}'', \mathbf{u}''')$ whose contracted form can be obtained as

$$\boldsymbol{\sigma}_{11}(\mathbf{u}'', \mathbf{u}''') = \mathbf{C} \nabla_N(\mathbf{u}'') \mathbf{u}'''. \tag{20c}$$

One can obtain $\mathbf{S}_N(\mathbf{u}'')$ from $\mathbf{S}_{11}(\mathbf{u}'', \mathbf{u}''')$, as will be required later, if $\mathbf{u}''' = \mathbf{u}''$

$$\mathbf{S}_N(\mathbf{u}'') = \sum_{i=1}^{3} \nabla_i^{\mathrm{T}} \boldsymbol{\sigma}_N(\mathbf{u}'') \nabla_i = \frac{1}{2} \sum_{i=1}^{3} \nabla_i^{\mathrm{T}} \boldsymbol{\sigma}_{11}(\mathbf{u}'', \mathbf{u}'') \nabla_i. \tag{21}$$

To reflect the terminology of ‘apparent symmetry’, it would be wrong to call expressions, such as $\mathbf{u}^{\mathrm{T}} \nabla_N^{\mathrm{T}}(\mathbf{u}') \boldsymbol{\sigma}_I$, asymmetric, given its equivalence to $\mathbf{u}^{\mathrm{T}} \mathbf{S}_I \mathbf{u}'$. It only appears not to be symmetric, whilst the flip transformation makes the symmetry apparent without changing the content of the expression.

Although $Q_{11}(=U_{11})$, Q_{111} and Q_{1111} as given in (8) and (15a) and (15b) are fully commutative, they are not apparently symmetric. With the flip transformation they can be re-written into fully commutative and apparently symmetric forms as follows, denoted as ψ_{11}, ψ_{111} and ψ_{1111}, respectively.

$$\psi_{11}(\mathbf{u},\mathbf{u}') = \mathbf{u}^{\mathrm{T}}\mathbf{\Psi}_0\mathbf{u}' \tag{22a}$$

$$\psi_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'') = \mathbf{u}^{\mathrm{T}}\mathbf{\Psi}_1(\mathbf{u}'')\mathbf{u}' \tag{22b}$$

$$\psi_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''') = \mathbf{u}^{\mathrm{T}}\mathbf{\Psi}_{11}(\mathbf{u}'',\mathbf{u}''')\mathbf{u}' \tag{22c}$$

where $\mathbf{\Psi}_0 = \nabla_L^{\mathrm{T}}\mathbf{C}\,\nabla_L + \mathbf{S}_I$. (23a)

$$\mathbf{\Psi}_1(\mathbf{u}'') = \nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}'') + \nabla_N^{\mathrm{T}}(\mathbf{u}'')\mathbf{C}\nabla_L + \mathbf{S}_L(\mathbf{u}'') \tag{23b}$$

$$\mathbf{\Psi}_{11}(\mathbf{u}'',\mathbf{u}''') = \nabla_N^{\mathrm{T}}(\mathbf{u}'')\mathbf{C}\nabla_N(\mathbf{u}''') + \nabla_N^{\mathrm{T}}(\mathbf{u}''')\mathbf{C}\nabla_N(\mathbf{u}'') + \mathbf{S}_{11}(\mathbf{u}'',\mathbf{u}''') \tag{23c}$$

The first term in the expression of $\mathbf{\Psi}_0$ in (23a) is symmetric. The first two terms in the expressions of $\mathbf{\Psi}_1(\mathbf{u}'')$ in (23b) and $\mathbf{\Psi}_{11}(\mathbf{u}'',\mathbf{u}''')$ in (23c) forms a symmetric matrix operator. Since the remaining terms associated with $\mathbf{S}$ take a form similar to (19b), $\mathbf{\Psi}_0$, $\mathbf{\Psi}_1(\mathbf{u}'')$ and $\mathbf{\Psi}_{11}(\mathbf{u}'',\mathbf{u}''')$ are therefore all symmetric. $\psi_{11}(\mathbf{u},\mathbf{u}')$, $\psi_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ and $\psi_{1111}(\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}''')$ as given in (22) can therefore easily be shown to be fully commutative whilst being apparently symmetric. Taking $\psi_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ for example,

$$\begin{aligned}\psi_{111}(\mathbf{u}',\mathbf{u},\mathbf{u}'') &= (\mathbf{u}')^{\mathrm{T}}\left(\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}) + \nabla_N^{\mathrm{T}}(\mathbf{u})\mathbf{C}\nabla_L + \mathbf{S}_L(\mathbf{u})\right)\mathbf{u}'' \\ &= \mathbf{u}^{\mathrm{T}}\mathbf{S}_L(\mathbf{u}')\mathbf{u}'' + \mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}')\mathbf{C}\nabla_L\mathbf{u}'' + \mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}\mathbf{C}\nabla_N(\mathbf{u}')\mathbf{u}'' = \psi_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'').\end{aligned} \tag{24}$$

It can be seen that full commutativity may not be a property possessed by an individual term, e.g. $\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u}')\mathbf{C}\nabla_L\mathbf{u}''$, but when all three terms in $\psi_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ are put together, collectively they deliver the full commutativity for $\psi_{111}(\mathbf{u},\mathbf{u}',\mathbf{u}'')$ as a trilinear form which is closely associated with the third order term in the SED expression. The same applies to second and fourth order terms.

It should be pointed out that ψ_{11}, ψ_{111} and ψ_{1111} are identical to U_{11}, Q_{111} and Q_{1111}, respectively, since the flip transformation is meant to be an identical transformation. The difference between them is only the presentation. The second, third and fourth order terms of the SED can now be given in their fully commutative and apparently symmetric forms as

$$\Gamma_2 = \frac{1}{2}\psi_{11}(\mathbf{u},\mathbf{u}) = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{\Psi}_0\mathbf{u} \tag{25a}$$

$$\Gamma_3 = \frac{1}{6}\psi_{111}(\mathbf{u},\mathbf{u},\mathbf{u}) = \frac{1}{6}\mathbf{u}^{\mathrm{T}}\mathbf{\Psi}_1(\mathbf{u})\mathbf{u} \tag{25b}$$

$$\Gamma_4 = \frac{1}{24}\psi_{1111}\left(\mathbf{u},\mathbf{u},\mathbf{u},\mathbf{u}\right) = \frac{1}{12}\mathbf{u}^{\mathrm{T}}\boldsymbol{\Psi}_2\left(\mathbf{u}\right)\mathbf{u} \tag{25c}$$

where $\boldsymbol{\Psi}_2\left(\mathbf{u}\right) = \frac{1}{2}\boldsymbol{\Psi}_{11}\left(\mathbf{u},\mathbf{u}\right)$ (26)

with $\boldsymbol{\Psi}_0$ being introduced in (23a) whilst $\boldsymbol{\Psi}_1\left(\mathbf{u}\right)$ in (23b) having $\mathbf{u}''$ there replaced by $\mathbf{u}$. The SED and its variations can be obtained as follows.

$$U = U_1 + U_2 + U_3 + U_4 = U_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 = U_1 + \mathbf{u}^{\mathrm{T}}\left(\frac{1}{2}\boldsymbol{\Psi}_0 + \frac{1}{6}\boldsymbol{\Psi}_1 + \frac{1}{12}\boldsymbol{\Psi}_2\right)\mathbf{u} \tag{27a}$$

$$\delta U = \delta U_1 + \delta\Gamma_2 + \delta\Gamma_3 + \delta\Gamma_4 = \delta U_1 + \delta\mathbf{u}^{\mathrm{T}}\left(\boldsymbol{\Psi}_0 + \frac{1}{2}\boldsymbol{\Psi}_1 + \frac{1}{3}\boldsymbol{\Psi}_2\right)\mathbf{u} \tag{27b}$$

$$\delta^2 U = \delta^2\Gamma_2 + \delta^2\Gamma_3 + \delta^2\Gamma_4 = \delta\mathbf{u}^{\mathrm{T}}\left(\boldsymbol{\Psi}_0 + \boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2\right)\delta\mathbf{u} \tag{27c}$$

where the matrices of partial differential operators $\left(\frac{1}{2}\boldsymbol{\Psi}_0 + \frac{1}{6}\boldsymbol{\Psi}_1 + \frac{1}{12}\boldsymbol{\Psi}_2\right)$, $\left(\boldsymbol{\Psi}_0 + \frac{1}{2}\boldsymbol{\Psi}_1 + \frac{1}{3}\boldsymbol{\Psi}_2\right)$ and $\left(\boldsymbol{\Psi}_0 + \boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2\right)$ are all symmetric and expressed explicitly.

The expressions of the coefficient arrays for the second, third and fourth order terms of the SED can be claimed to be unique under the conditions of full commutativity of ψ_{11}, ψ_{111} and ψ_{1111}. Their uniqueness can be argued in exactly the same manner as coefficient matrix for the quadratic form [24] where the criterion is the symmetry of the coefficient matrix. A quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}$ can have an infinite number of different $\mathbf{M}$ matrices delivering the same function, e.g. $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x} = x_1^2 + 2x_1x_2 + x_2^2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \cdots$, but the commutativity of its corresponding bilinear form, i.e. $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}=\mathbf{y}^{\mathrm{T}}\mathbf{M}\mathbf{x}$ delivers a unique $\mathbf{M}$ which symmetric. The difference the third and fourth order terms is that the coefficient matrices are replaced by 3D and 4D arrays, respectively. Lack of appreciating the full commutativity prevented the reconciliation of different expressions for the same energy term, leading to the failure to deliver unique and symmetric stiffness matrices in the past.

The apparent symmetry as achieved through the use of flip transformation allows ψ_{11}, ψ_{111} and ψ_{1111} to be expressed explicitly as given in (22) and consequently the SED and its variations in the form of (27). They in turn deliver the secant and tangential stiffness matrices explicitly and systematically once applied to the FEM formulation as will be shown in the next section.

The full commutativity and apparent symmetry have thus been established as an intrinsic property of the SED which can be revealed only after the elaborations made above when it is presented uniquely and explicitly in the form of (27a).

5 Application in finite element formulation of the geometrically nonlinear problem

The stiffness matrices of a finite element for the geometrically nonlinear problem can be derived from the SED. Controversies in this field are rooted in the lack of full commutativity. To demonstrate the significance of the establishment of full commutativity, derivations are presented in this section, with the displacement field discretised as given in (A-6). With (25),

$$\Gamma_2 = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\Psi}_0\mathbf{A}\right)\mathbf{q} = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left(\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_L + \sum_{i=1}^{3}\mathbf{b}^{\mathrm{T}}\mathbf{s}_I\mathbf{b}\right)\mathbf{q} \tag{28a}$$

$$\Gamma_3 = \frac{1}{6}\mathbf{q}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\Psi}_1\mathbf{A}\right)\mathbf{q} = \frac{1}{6}\mathbf{q}^{\mathrm{T}}\left(\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_N\left(\mathbf{q}\right) + \mathbf{B}_N^{\mathrm{T}}\left(\mathbf{q}\right)\mathbf{C}\mathbf{B}_L + \sum_{i=1}^{3}\mathbf{b}^{\mathrm{T}}\mathbf{s}_L\mathbf{b}\right)\mathbf{q} \tag{28b}$$

$$\Gamma_4 = \frac{1}{12}\mathbf{q}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\Psi}_2\mathbf{A}\right)\mathbf{q} = \frac{1}{12}\mathbf{q}^{\mathrm{T}}\left(\mathbf{B}_N^{\mathrm{T}}\left(\mathbf{q}\right)\boldsymbol{C}\mathbf{B}_N\left(\mathbf{q}\right) + \sum_{i=1}^{3}\mathbf{b}^{\mathrm{T}}\mathbf{s}_N\mathbf{b}\right)\mathbf{q} \tag{28c}$$

The matrices involved in the above expressions can all be given explicitly, owing to the apparently symmetric presentation of the above energy terms, for a given type of element in which the displacement field is interpolated in the form of (A-6). In order to obtain the stiffness matrices for an element, one can substitute the fully commutative terms (28) into the SED and its variations (27) for the element.

$$U^e = \mathbf{q}^{\mathrm{T}}\mathbf{h}^e + \mathbf{q}^{\mathrm{T}}\mathbf{K}_e^e\mathbf{q} \tag{29a}$$

$$\delta U^e = \delta\mathbf{q}^{\mathrm{T}}\mathbf{h}^e + \left(\delta\mathbf{q}\right)^{\mathrm{T}}\mathbf{K}_s^e\,\mathbf{q} \tag{29b}$$

$$\delta^2 U^e = \delta\mathbf{q}^{\mathrm{T}}\mathbf{K}_t^e\,\delta\mathbf{q} \tag{29c}$$

where the superscript e indicates that the quantity is associated with the element concerned, and

$$\mathbf{h}^e = \int_{\Omega^e}\mathbf{B}_L^{\mathrm{T}}\boldsymbol{\sigma}_I d\Omega \tag{30a}$$

$$\mathbf{K}_e^e = \frac{1}{2}\mathbf{K}_0^e + \frac{1}{6}\mathbf{N}_1^e + \frac{1}{12}\mathbf{N}_2^e \tag{30b}$$

$$\mathbf{K}_s^e = \mathbf{K}_0^e + \frac{1}{2}\mathbf{N}_1^e + \frac{1}{3}\mathbf{N}_2^e \tag{30c}$$

$$\mathbf{K}_t^e = \mathbf{K}_0^e + \mathbf{N}_1^e + \mathbf{N}_2^e \tag{30d}$$

whilst $$\mathbf{K}_0^e = \int_{\Omega^e}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Psi}_0\,\mathbf{A}d\Omega = \mathbf{K}^e + \left(\mathbf{K}_0^{\sigma}\right)^e \tag{31a}$$

$$\mathbf{N}_1^e = \int_{\Omega^e}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Psi}_1\,\mathbf{A}d\Omega = \left(\mathbf{N}_1^q\right)^e + \left(\mathbf{K}_1^{\sigma}\right)^e \tag{31b}$$

$$\mathbf{N}_2^e = \int_{\Omega^e}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Psi}_2\,\mathbf{A}d\Omega = \left(\mathbf{N}_2^q\right)^e + \left(\mathbf{K}_2^{\sigma}\right)^e \tag{31c}$$

$$\mathbf{K}^e = \int_{\Omega^e}\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\,\mathbf{B}_L d\Omega \tag{32a}$$

$$\left(\mathbf{N}_1^q\right)^e = \int_{\Omega^e} \left(\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\,\mathbf{B}_N(\mathbf{q}) + \mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\mathbf{B}_L\right) d\Omega \tag{32b}$$

$$\left(\mathbf{N}_2^q\right)^e = \int_{\Omega^e} \mathbf{B}_N^{\mathrm{T}}(\mathbf{q})\mathbf{C}\,\mathbf{B}_N(\mathbf{q})\, d\Omega \tag{32c}$$

$$\left(\mathbf{K}_0^\sigma\right)^e = \sum_{i=1}^{3}\int_{\Omega^e} \mathbf{b}_i^{\mathrm{T}}\mathbf{s}_I\,\mathbf{b}_i\, d\Omega \qquad \text{referring to (A9b) for } \mathbf{b}_i \tag{32d}$$

$$\left(\mathbf{K}_1^\sigma\right)^e = \sum_{i=1}^{3}\int_{\Omega^e} \mathbf{b}_i^{\mathrm{T}}\mathbf{s}_L\,\mathbf{b}_i\, d\Omega \tag{32e}$$

$$\left(\mathbf{K}_2^\sigma\right)^e = \sum_{i=1}^{3}\int_{\Omega^e} \mathbf{b}_i^{\mathrm{T}}\mathbf{s}_N\,\mathbf{b}_i\, d\Omega. \tag{32f}$$

$\mathbf{K}_s^e$ and $\mathbf{K}_t^e$ are conventionally known as the secant and tangential stiffness matrices. For the ease of association with $\mathbf{K}_s^e$ and $\mathbf{K}_t^e$, it is suggested that $\mathbf{K}_e^e$ be called 'the energy stiffness matrix' since it is directly associated with the expression of strain energy. They have all been expressed explicitly and symmetrically above. $\mathbf{K}_s^e$ is related to $\mathbf{K}_e^e$ in a similar way as $\mathbf{K}_t^e$ is related to $\mathbf{K}_s^e$. This reveals the generic relationships amongst $\mathbf{K}_e^e$, $\mathbf{K}_s^e$ and $\mathbf{K}_t^e$. A different presentation of these stiffness matrices has also been obtained in [9-11] but they admitted the lack of uniqueness and left the problem behind unresolved. Given the full commutativity as an intrinsic property of the SED, the uniqueness of SED expression and stiffness matrices derived from it is guaranteed. It is straightforward to reveal that the secant stiffness matrix as shown in (2) can be identically transformed to $\mathbf{K}_s^e$ after the flip transformation has been applied to some of the terms as provided in Appendix B to bring a resolution to a longstanding issue associated with the secant stiffness matrix.

In (31a), $\mathbf{K}_0^e$ is the stiffness matrix in linear problems, which incorporates the contribution from the initial stress through the so-called initial stress stiffness matrix $\left(\mathbf{K}_0^\sigma\right)^e$. $\mathbf{h}^e$ can be considered to be an extra load resulting from initial stresses. For problems without initial stresses, $\left(\mathbf{K}_0^\sigma\right)^e$ and $\mathbf{h}^e$ can simply be dropped without affecting the other terms. $\mathbf{N}_1^e$ and $\mathbf{N}_2^e$ are the stiffness matrices due to the nonlinearity, being linear and quadratic functions of $\mathbf{q}$, respectively.

6 Koiter's initial post-buckling theory

The initial post-buckling theory was established by Koiter in the 1940s [12] although it did not become known to the rest of the world outside the Netherlands until some 20 years later [25]. It was the first ever attempt at offering an insight into various buckling phenomena through the concept of stability of equilibrium states, in particular, those in the initial post-buckling regime. Through this, the concept of imperfection sensitivity of a structure was quantitatively established and directly associated with the stability of the initial post-buckling equilibrium state. The stability of the initial post-buckling equilibrium state and the sensitivity of the buckling load to the initial imperfection could all be described through two non-dimensional parameters, *a* and *b*, often termed as initial post-buckling coefficients (IPBCs) [26]. Research activities on the theory peaked in the 1970s [26,27] when the development of commercial FEM codes was still in its infancy. Modern mainstream commercial FEM codes have never managed to incorporate Koiter's theory.

Buckling analysis is one of the basic modules in modern commercial FEM codes [1-3]. However, structural engineers and designers are very aware of the limitations of the predicted buckling load. Structures could collapse at a much lower load level than the predicted buckling load due to their imperfection sensitivity. As the buckling analysis alone is insufficient to provide any information in this regard, investigations have been taken to the post-buckling regime which could be traced as far back as to the 1940s [27] with intensified activities in the 1960s [28,29], to name but a few. The outcomes of such efforts, whilst having enhanced the understanding of the post-buckling deformation, proved to be a rather inefficient approach to understanding buckling behaviour. A modern equivalent of such efforts is to analyse structures with artificially introduced imperfections, often as an initial deflection taking the pattern of the buckling mode, taking advantage of the computing power nowadays and the options available in mainstream FEM codes. Such analyses are computationally expensive and time-consuming. A proper assessment will have to involve separate runs having imperfections of either sense one at a time and they will have to be run over a range of magnitudes of such artificially introduced imperfection in order to be representative. This is often missing from such attempts. The computational cost could be overwhelming, bearing in mind that each of such an analysis must be conducted as a nonlinear problem. In most of such endeavours, the ultimate objective is in fact some assurance on the predicted buckling load.

Much of the useful part of the outcomes of the above computationally expensive and time-consuming exercises could have been condensed into the two IPBCs, *a* and *b*, if Koiter's theory had been adopted. Their evaluations involve little more than a conventional buckling analysis, typically the extraction of the lowest order of eigenvalue, in terms of computational efforts. To be specific, the evaluation of *a* requires no more information than has been made available after the buckling analysis. That for *b* needs slightly more effort since one has to solve another specially formulated stiffness equation to obtain the 2nd order of displacement perturbation. This is a linear problem. Computationally, it is equivalent to a single iteration in a conventional incremental-iterative nonlinear analysis. Koiter's theory is called the initial post-buckling theory and it does produce a prediction of the post-buckling deformation. However, it is meant to be applicable to the initial phase of the post-buckling regime in an asymptotic sense. It is usually insufficient to offer meaningful understanding of post-buckling deformation in a broad sense, if such information is genuinely required. The true value of Koiter's theory is to provide an efficient tool for designers dealing with buckling as they will have much improved level of confidence on the buckling load predicted if it is accompanied by IPBCs *a* and *b*. In this regard, through the development of the theory in the 1970-80s, mostly by analytical means, a reasonable understanding has been achieved in terms of their imperfection sensitivity of simple structures, such as plates and cylindrical shells. However, modern structures tend to have intricate details, e.g. panels with hat-shaped stiffeners, often involving the use of new materials, such as composites, and their imperfection sensitivity is no longer obvious before an appropriate evaluation has been carried out.

In order to present this paper in self-contained form, the main procedure of Koiter's theory is provided briefly in its FEM presentation. With the FEM discretisation, after assembling all elements involved in the structure, the total potential energy for a conservative elastic system can be given as

$$\Pi(\mathbf{q}) = \Pi_1(\mathbf{q}) + \Pi_2(\mathbf{q}) + \Pi_3(\mathbf{q}) + \Pi_4(\mathbf{q}) \tag{33}$$

where the same $\mathbf{q}$ has been used for the nodal displacement vector for the whole structure without being confused with its counterpart for an element. Having employed the fully commutative and apparently symmetric SED to formulate element stiffness matrices as shown in the previous section, the full commutativity of $\mathbf{q}$ is present in Π_2, Π_3 and Π_4 for the structure. Equilibrium is defined by the first variation of Π

$$\delta\Pi(\mathbf{q}) = \Pi_1(\delta\mathbf{q}) + \Pi_{11}(\mathbf{q}, \delta\mathbf{q}) + \Pi_{21}(\mathbf{q}, \delta\mathbf{q}) + \Pi_{31}(\mathbf{q}, \delta\mathbf{q}) = 0 \tag{34}$$

where the subscript rules for Π remain the same as for Q in previous sections as illustrated in (17) and (18) in line with Koiter's original notation [12].

As conventionally assumed, there exists a fundamental equilibrium path given as

$$\mathbf{q}_0 = \mathbf{q}_0(\lambda) \quad \text{satisfying} \quad \mathbf{q}_0(0) = \mathbf{0}. \tag{35}$$

where λ the load parameter and λ^c the critical load. The critical condition for buckling is met as λ increases from 0 to λ^c, when the second variation as follows seizes to be positive definite, i.e.

$$\delta^2\Pi(\mathbf{q}) = \Pi_{11}(\delta\mathbf{q}, \delta\mathbf{q}) + \Pi_{111}(\mathbf{q}_0, \delta\mathbf{q}, \delta\mathbf{q}) + \Pi_{211}(\mathbf{q}_0, \delta\mathbf{q}, \delta\mathbf{q}) \geq 0 \tag{36}$$

whilst there exists a nontrivial displacement variation $\delta\mathbf{q} = \mathbf{q}_1$ that results in a zero value for the second variation above, i.e.

$$\Pi_{11}(\mathbf{q}_1, \delta\mathbf{q}) + \Pi_{111}(\mathbf{q}_0, \mathbf{q}_1, \delta\mathbf{q}) + \Pi_{211}(\mathbf{q}_0, \mathbf{q}_1, \delta\mathbf{q}) = 0 \tag{37}$$

which gives the governing equation for the eigenvalue problem where the lowest eigenvalue determines the magnitude of $\mathbf{q}_0$ at buckling, i.e. λ^c, and eigenvector $\mathbf{q}_1$ provides the buckling mode.

The classical buckling problem finishes at this point. However, structures tend to exhibit a range of dramatically different behaviours. Some can support further loads after buckling whilst others collapse long before the classical buckling load is reached. Koiter established [12,25] that the different responses were directly related to the state of stability of the post-buckling equilibrium path at the bifurcation point. If it were stable, the structure would be insensitive to the initial imperfection and it could sustain higher levels of load without catastrophic collapse. Otherwise, the structure would be sensitive to the initial imperfection and it would tend to collapse at a load significantly lower than the predicted buckling load, depending on the magnitude of the initial imperfection. In order to obtain the initial post-buckling path and to assess the stability of the equilibrium state on the initial post-buckling path, an asymptotic expression of it can be assumed as an implicit function of $\mathbf{q} = \mathbf{q}(\lambda)$ in its parametric form with parameter ξ as follows.

$$\mathbf{q} = \mathbf{q}_0 + \xi\mathbf{q}_1 + \xi^2\mathbf{q}_2 + \xi^3\mathbf{q}_3 + \cdots\cdots \tag{38a}$$

$$\lambda = \lambda^c\left(1 + a\xi + b\xi^2 + c\xi^3 + \cdots\cdots\right) \tag{38b}$$

where ξ is also called the perturbation parameter in the context of asymptotic analysis which signifies the amplitude of the buckling mode as a measure of the magnitude of the initial post-buckling deformation, *a*, *b*, *c*,… are the so-called IPBCs, $\mathbf{q}_1$ is the buckling mode assumed to have already being obtained from (37), $\mathbf{q}_2$ is the secondary perturbation mode for the determination of *b* and $\mathbf{q}_3 \ldots$ are higher order perturbation modes. For initial post-buckling analysis, the fundamental path $\mathbf{q}_0$ is assumed to be known and it can be expanded as a Taylor series around the bifurcation point as

$$\mathbf{q}_0 = \mathbf{q}_0^c + \left(\lambda - \lambda^c\right)\dot{\mathbf{q}}_0^c + \cdots\cdots. \tag{38c}$$

For cases where Koiter's theory applies, buckling is characterised by bifurcation from the fundamental equilibrium path. In most of such cases, the pre-buckling phase of the fundamental path involves small deformation and can be determined from a linear analysis as a reasonable approximation which is usually incorporated as a part of the buckling analysis. Koiter's theory does not apply to the other type of buckling at a maximum load where snap-though takes place, in which pre-buckling deformation is typically highly nonlinear.

When the asymptotic series expressions (38a-c) are substituted into the total potential energy expression, its first derivative with respect to ξ gives the equilibrium condition for the post-buckling path and can be considered as an implicit function $\lambda = \lambda(\xi)$. If a and b in (38b) are considered as the 1st and 2nd order of derivatives of λ with respective to ξ at $\xi = 0$ and hence $\lambda = \lambda^c$, their expressions can be obtained as follows.

$$a = -\frac{3}{2\lambda^c} \frac{\Pi_3(\mathbf{q}_1) + \Pi_{13}(\mathbf{q}_0^c, \mathbf{q}_1)}{\Pi_{12}(\dot{\mathbf{q}}_0^c, \mathbf{q}_1) + \Pi_{112}(\mathbf{q}_0^c, \dot{\mathbf{q}}_0^c, \mathbf{q}_1)} \tag{39}$$

$$b = -\frac{2}{\lambda^c} \frac{\Pi_4(\mathbf{q}_1) - \left(\Pi_2(\mathbf{q}_2) + \Pi_{12}(\mathbf{q}_0^c, \mathbf{q}_2) + \Pi_{22}(\mathbf{q}_0^c, \mathbf{q}_2)\right)}{\Pi_{12}(\dot{\mathbf{q}}_0^c, \mathbf{q}_1) + \Pi_{112}(\mathbf{q}_0^c, \dot{\mathbf{q}}_0^c, \mathbf{q}_1)} \qquad (a \neq 0) \tag{40}$$

In most practical analyses, determination of a and b is sufficient. As established in Koiter's theory, the most important characteristics of the initial post-buckling behaviour can be determined using only these constants, provided that they do not vanish simultaneously.

If $a \neq 0$, the initial post-buckling path given in (38b) can be truncated at the first order of the asymptotic series as an approximation and one does not even need to evaluate b as it contributes only an asymptotically higher order term. In this case, an asymmetric bifurcation is expected which predicts an unstable initial post-buckling behaviour in general and leads to imperfection sensitivity. Given the asymmetric nature of the bifurcation, if the structure is sensitive to an imperfection of one sense, e.g. positive, it is insensitive to the same imperfection of the other sense, i.e. negative. This is why if one adopts the nonlinear post-buckling analysis approach, imperfections of both senses must be assessed separately to form a complete view.

When a=0, one has to proceed to the second order perturbation. If $b \neq 0$, function (38b) can be truncated at the second order term for initial post-buckling behaviour leading to a symmetric bifurcation. In order to evaluate b, the second order displacement perturbation $\mathbf{q}_2$ is required. The governing equation can be obtained from the second order terms of ξ in the asymptotically expanded total potential energy as [12,25]

$$\Pi_{11}(\delta\mathbf{q}, \mathbf{q}_2) + \Pi_{111}(\delta\mathbf{q}, \mathbf{q}_0, \mathbf{q}_2) + \Pi_{121}(\delta\mathbf{q}, \mathbf{q}_0, \mathbf{q}_2) = -\Pi_{12}(\delta\mathbf{q}, \mathbf{q}_1) - \Pi_{112}(\delta\mathbf{q}, \mathbf{q}_0, \mathbf{q}_1) \tag{41}$$

subject to an orthogonality condition between $\mathbf{q}_1$ and $\mathbf{q}_2$ as

$$\Pi_{11}(\dot{\mathbf{q}}_0^c, \mathbf{q}_1, \mathbf{q}_2) + \Pi_{1111}(\mathbf{q}_0^c, \dot{\mathbf{q}}_0^c, \mathbf{q}_1, \mathbf{q}_2) = 0\,. \tag{42}$$

Higher orders of perturbation could be made in theory in the case that b vanishes but this situation rarely arise in practice.

Koiter's theory has been summarised as above. However, it has not been implemented in any mainstream commercial FEM code. In fact, there has been no lack of desire to achieve this, early attempts could be traced back to the 1970s [13, 14] and numerous more recent efforts could be found including those in [16-23]. The development of the FEM implementation of Koiter's theory has suffered from a number of key obstacles. One of them was referred to as the 'locking' phenomenon as originally defined in [31]. It will be resolved as one of the main outcomes of the present paper. The remaining obstacles have also been resolved by the present authors who plan to present them soon in subsequent publications.

Making use of the SED and FEM formulation as presented in previous section, key governing equations as presented above can be expressed in the FEM context as follow.

$$\Pi = \mathbf{q}^{\mathrm{T}}\left(\frac{1}{2}\mathbf{K}_0 + \frac{1}{6}\mathbf{N}_1 + \frac{1}{12}\mathbf{N}_2\right)\mathbf{q} - \mathbf{q}^{\mathrm{T}}\mathbf{F} \qquad \textit{cf.}\ (33) \tag{43}$$

$$\left(\mathbf{K}_0 + \frac{1}{2}\mathbf{N}_1 + \frac{1}{3}\mathbf{N}_2\right)\mathbf{q} - \mathbf{F} = 0 \qquad \textit{cf.}\ (34) \tag{44}$$

$$\left(\mathbf{K}_0 + \mathbf{N}_1 + \mathbf{N}_2\right)\mathbf{q}_1 = \mathbf{0} \qquad \textit{cf.}\ (37) \tag{45}$$

$$a = -\frac{1}{2\lambda^c}\frac{\mathbf{q}_1^{\mathrm{T}}\mathbf{N}_1\left(\mathbf{q}_1\right)\mathbf{q}_1 + 2\mathbf{q}_1^{\mathrm{T}}\mathbf{N}_2\left(\mathbf{q}_1\right)\mathbf{q}_0^c}{\mathbf{q}_1^{\mathrm{T}}\mathbf{N}_1\left(\mathbf{q}_1\right)\dot{\mathbf{q}}_0^c + 2\mathbf{q}_1^{\mathrm{T}}\mathbf{N}_2\left(\mathbf{q}_0^c\right)\dot{\mathbf{q}}_0^c} \qquad \textit{cf.}\ (39) \tag{46}$$

$$b = -\frac{1}{3\lambda^c}\frac{\mathbf{q}_1^{\mathrm{T}}\mathbf{N}_2\left(\mathbf{q}_1\right)\mathbf{q}_1 - 6\mathbf{q}_2^{\mathrm{T}}\left(\mathbf{K}_0 + \mathbf{N}_1\left(\mathbf{q}_0^c\right) + \mathbf{N}_2\left(\mathbf{q}_0^c\right)\right)\mathbf{q}_2}{\mathbf{q}_1^{\mathrm{T}}\mathbf{N}_1\left(\mathbf{q}_1\right)\dot{\mathbf{q}}_0^c + 2\mathbf{q}_1^{\mathrm{T}}\mathbf{N}_2\left(\mathbf{q}_0^c\right)\dot{\mathbf{q}}_0^c} \qquad \textit{cf.}\ (40) \tag{47}$$

$$\left(\mathbf{K}_0 + \mathbf{N}_1 + \mathbf{N}_2\right)\mathbf{q}_2 = -\frac{1}{2}\mathbf{N}_1\left(\mathbf{q}_1\right)\mathbf{q}_1 - \mathbf{N}_2\left(\mathbf{q}_1\right)\mathbf{q}_0^c \qquad \textit{cf.}\ (41) \tag{48}$$

$$\mathbf{q}_1^{T}\left(-\mathbf{N}_1\left(\dot{\mathbf{q}}_0^c\right) - \mathbf{N}_{11}\left(\dot{\mathbf{q}}_0^c, \mathbf{q}_0^c\right)\right)\mathbf{q}_2 = 0 \qquad \textit{cf.}\ (42) \tag{49}$$

where $\mathbf{F}$ is the external loading vector whilst various symbols associated with the FEM formulation are inherited from the previous section in a straightforward manner except that they have been assembled from all elements involved in the structure.

The expressions of *a* and *b* above were obtained as early as in 1971 [13]. However, they have remained unimplemented because the nonlinear stiffness matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ have been a 'symbolism' as referred to in the first edition of [4]. In obtaining them in [13], the full commutativity of various displacements within each of the terms appearing in the expressions had been taken for granted, which did not stand scrutiny if one wished to adopt it in a commercial code.

Based on the full commutativity and the apparent symmetry as established earlier in this paper, all expressions now are not only well-defined and explicit but also proven to be practically effective in applications as will be demonstrated in the next section.

In the literature, most attempts of evaluating the IPBCs *a* and *b* through FEM have employed the expressions as provided in Appendix C as an alternative. They are mathematically equivalent to (39) and (40), respectively. However, the equivalence for *b* was greatly compromised numerically when implemented through FEM, as the two terms in the numerator of (C-2) are of opposite senses whilst having similar magnitude. The sum is often one or more orders of magnitude smaller, resulting in the so-called 'locking' phenomenon [31]. The fact that it has never appeared in analytical solutions [15] suggests that it is a numerical problem associated with accuracy. The underlying reason was in fact the generic shortcoming of FEM in which stresses obtained are inherently found to a lower order of accuracy. As expression (C-2) involves stresses explicitly, the accuracy of the results would have to rely on mesh refinement. Such mesh sensitivity for this particular problem has been well-observed and documented in the literature, e.g. [16-23]. The consequence is that a much more refined mesh would be required for initial post-buckling analysis than that which would usually be sufficient to deliver satisfactory buckling analysis. This would undermine the practical usefulness of an FEM code. To improve the accuracy in order to avoid the mesh sensitivity, specially formulated finite elements have been adopted using unconventional shape functions, e.g. those having higher orders of continuity [16]. Neither enduring the mesh sensitivity nor employing specially formulated elements is an attractive property to FEM code developers. It was undoubtedly one of the major reasons why the theory has never been incorporated in any mainstream FEM code.

Since (C-2) on longer relies on stresses, one can expect (47) to offer improved performance. This will be demonstrated in the next section through example applications. It will be shown that it has in fact removed the mesh sensitivity completely without having to use a specially formulated element.

7 Numerical examples

To demonstrate the advantages of the formulation as presented in Section 6, it is applied here to the plates presented in [16]. The formulation can be simplified accordingly for plates and shells where all higher order terms of in-plane displacements can be neglected, as has been conventionally performed in [6-8]. The plate is considered to take a range of aspect ratios and is assumed to be simply supported (a), (b-1) and (c-1) or clamped (b-2) and (c-2) with in-plane constraints as specified in Figure 1. It is subjected to a number of loading conditions, *viz*. (a) uniaxial compression, (b) pure in-plane bending and (c) pure shear, as illustrated in Figure 1. The Young's modulus of the material of the plate is $E = 210\,\text{GPa}$ and its Poisson's ratio $\nu = 0.3$. The thickness of the plate is $t = 1mm$.

Given the absence of Koiter's theory from any mainstream commercial FEM code, the present analyses were conducted using an in-house developed FEM code employing the 3D degenerated 8-noded shell element with reduced integration [4,5]. Prior to the applications shown in this paper, the code had been extensively verified against various sources, including analytical solutions wherever available and the results obtained from Abaqus. For the present application, the obtained results have been listed in Table 1 and compared with available results in [16]. As an illustration of the reliability of the in-house code, the results for the buckling loads obtained using Abaqus have also been included, where agreement is seen to be excellent.

The IPBCs as given in (46) and (47) have been evaluated. As the problem is for a flat plate, IPBC *a* should vanish identically. This was also verified in the numerical results obtained through the in-house code, which offered further assurance of the accuracy of the in-house code from another perspective in the post-buckling regime, although there was no need to show this trivial result in Table 1.

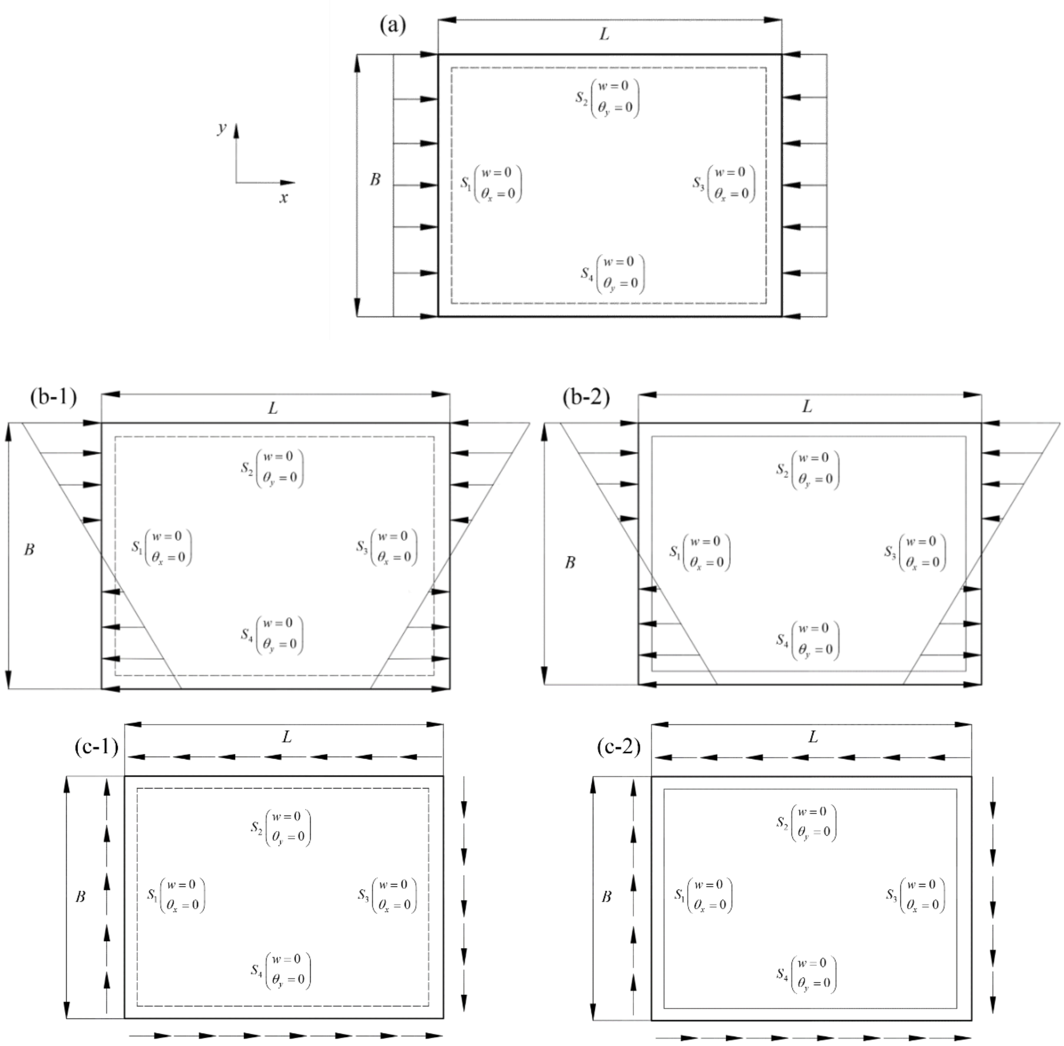

Figure 1. Rectangular plate under different loading and boundary conditions (dashed inner frame indicating simply supported and solid clamped boundary conditions against bending) [16]

The predicted IPBC *b* using the in-house FEM code has been shown in Table 1. As Abaqus does not incorporate Koiter's theory, no results can be obtained from it. Results published in [16] have been resorted to in order to verify the present predictions where excellent agreement is obvious. Over the range of mesh densities considered, the mesh convergence occurs in a consistent way between the buckling load and IPBC b, i.e. a mesh good enough for buckling analysis is also good enough for initial post-buckling analysis.

Table 1. Initial post-buckling coefficient of simply supported rectangular plate

Case	Mesh	Normalised buckling load $12\left(1-\nu^2\right)B^2\lambda^c/\pi^2Et^3$			IPBC b		Partitions of b such that $b=b_1-b_2$	
		Ref.[16]	Abaqus	Present	Ref.[16]	Present	b_1	b_2
(a) L/B=3	21×7	4.033	3.969	3.968	0.2203	0.2108	1.7716	1.5608
	33×11	4.014	3.974	3.974	0.2213	0.2190	1.9215	1.7025
	49×15	4.007	3.974	3.974	0.2217	0.2149	1.8838	1.6689
(b-1) L/B=1	9×9	26.18	25.49	25.31	0.2236	0.2160	2.0664	1.8504
	15×15	25.76	25.32	25.25	0.2252	0.2156	2.0441	1.8285
	25×25	25.61	25.27	25.25	0.2194	0.2097	2.0129	1.8032
	33×33	25.58	25.26	25.25	0.2193	0.2104	2.0257	1.8153
(b-2) L/B=1	9×9	53.49	48.90	48.44	**0.3400**	**0.2838**	1.3404	1.0566
	15×15	49.74	47.71	47.55	0.2978	0.2926	1.3913	1.0987
	25×25	48.46	47.55	47.50	0.2978	0.2806	1.3352	1.0546
	33×33	48.16	47.53	47.50	0.2924	0.2854	1.3569	1.0715
(c-1) L/B=2	15×7	6.777	6.537	6.537	0.07297	0.07305	1.2558	1.1828
	25×11	6.639	6.522	6.523	0.07174	0.07273	1.2635	1.1908
	33×17	6.587	6.521	6.521	0.07164	0.07226	1.2758	1.2036
(c-2) L/B=2	15×7	11.41	10.37	10.37	**0.1566**	**0.1357**	1.3057	1.1700
	25×11	10.67	10.23	10.23	0.1383	0.1330	1.2982	1.1652
	33×17	10.42	10.22	10.22	0.1340	0.1326	1.2924	1.1598

The agreements with [16] and Abaqus verify the correctness of the formulation and the implementation. A much more important aspect is that the present analysis employs a conventional type of finite element, *viz*. 3D degenerated 8-noded shell with reduced integration, available commonly in commercial FEM codes, whilst that in [16] was based on specially formulated finite elements with a higher order of continuity which is unavailable in any mainstream commercial code.

It should be pointed out that an analytical solution for b is available in [27] for a problem corresponding to Case (a), given as

$$b=\frac{3}{8}\left(1-\nu^2\right)=0.34125 \tag{50}$$

which is significantly different from what was obtained in [16] as well as the present prediction. It is most disturbing to enquire why the disparity has never been flagged up and explained appropriately. There appeared to be reluctance in the literature, e.g. [20], to refer to Case (a). It would be extremely negative for any serious developer of FEM code considering adaption of Koiter's theory, if such the discrepancy were left unaddressed. The authors of this paper have identified the source of the problem and resolved it. Since it is not simply a numerical error and takes a considerable elaboration on both numerical and analytical sides in order to clarify the position, it will have to be presented in a subsequent publication. For readers' assurance, the result as presented in Table 1 is correct for the problem as defined in Figure 1(a), which is however subtly different from that solved analytically in [27].

As mesh sensitivity has been one of the key issues regarding the FEM implementation of Koiter's theory, it has been investigated by plotting the predicted b for Case (b-1) shown in Figure 1 against the degrees of freedom (dofs) involved in the mesh with the template found in [20], as shown in Figure 2. Over the range of the number of dofs as shown, no mesh sensitivity can be observed using (47). It should be noted that the full commutativity and apparent symmetry come without any extra computational effort or cost. If anything, it has made the FEM formulation clearer and more streamlined. In contrast, the curve as was found in [20] with conventional elements using the popular expression (C-2) showed strong mesh sensitivity over the same range of number of dofs. Other results shown in Figure 2 including those from [16] show improvements in mesh sensitivity but at the price of employing specially formulated finite elements. In fact, the present formulation demonstrated better convergence in general with a number of more pronounced cases shown bold in Table 1.

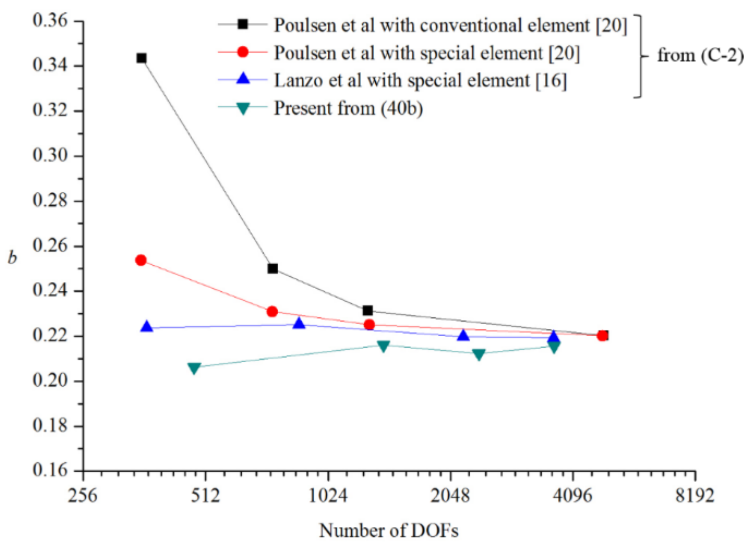

Figure 2 Mesh convergence for Case (b-1) as sketched in Figure 1

The reason for the improvement in the numerical accuracy when using (47) instead of (C-2) is believed to have resulted from the much reduced direct involvement of stresses in the expression of b and therefore any undue numerical errors are avoided. Also included in Table 1 are the values of the two terms of b if expression (47) is split according to its numerator into two separate terms, b_1 and b_2, so that $b=b_1-b_2$. It can be observed that b is indeed an order of magnitude smaller than either b_1 or b_2 and hence 'locking' is likely [16,17], especially if the terms involved could not be evaluated sufficiently accurately. However, having reduced the direct involvement of stresses in these expressions, numerical errors due to FEM should have been reduced in comparison with that evaluated using (C-2). The freedom from mesh sensitivity in the present results is therefore by no accident.

Use of (47) brings forward a satisfactory solution and locking does not appear whilst using a conventional type of C_0 element. This has been achieved based on the N-notation which was not possible practically previously. With the fully commutative and apparently symmetric SED, the N-notation is now fully established. This should open the gate for Koiter's theory to be incorporated in mainstream commercial FEM codes.

8 Conclusions

In this paper, the concept of full commutativity of displacements in the expressions for the strain energy density within the geometrically nonlinear problem has been introduced for the first time, and established firmly such that full commutativity can now be identified as an intrinsic property of the SED.

When this fully commutative SED is employed in the formulation of the FEM, as a direct application, all relevant stiffness matrices in the so-called N-notation can be obtained uniquely. In order for them to be expressed explicitly, the SED also needs to be put into an apparently symmetric form by applying the mathematically identical flip transformation to some of the terms of the SED, as has also been established in this paper. The stiffness matrices thus obtained can then be used to construct the strain energy, equilibrium equation and the incremental equilibrium equation in a consistent manner. The N-notation FEM formulation has thus been fully and rigorously established. Prior to this, the N-notation formulation has remained as a symbolism, particularly within the FEM implementation of Koiter's theory. Without a functional N-notation formulation, the option for the FEM implementation of Koiter's theory is through the B-notation instead, which has suffered from the well-known mesh sensitivity issue in the evaluation of the IPBC b unless some unconventional types of elements have been employed. With the expression of b in N-notation, the mesh sensitivity issue has been resolved naturally whilst using a conventional type of element. There will be no need either to employ specially formulated elements or to endure the inconvenience of mesh sensitivity, as has been demonstrated in the present paper. The unavailability of Koiter's theory from commercial FEM codes is a missed opportunity for structural designer and engineers to improve dramatically the confidence in the predicted load carrying capacity of structures when designing against buckling. Using any existing commercial FEM code, it is easy to predict the buckling load but difficult to place confidence on it; Koiter's theory would overcome this problem.

The most important contribution of this paper is to pave a way for the wide adoption of Koiter's theory in mainstream FEM codes.

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Appendix A Basic equations and notations

Basic notations involved in this paper are defined as follows. For a linearly elastic continuum as dealt with in this paper, the strain energy density U exists and it can be expressed as

$$U(\mathbf{u}) = \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}_I + \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} \tag{A-1}$$

where $\boldsymbol{\varepsilon}$ is the Green strain, $\boldsymbol{\sigma}_I$ the initial Cauchy stress and $\boldsymbol{\sigma}$ the second Piola-Kirchhoff stress [8].

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_N, \quad \boldsymbol{\varepsilon}_L = \nabla_L \mathbf{u}, \quad \boldsymbol{\varepsilon}_N = \frac{1}{2}\nabla_N(\mathbf{u})\mathbf{u} \tag{A-2}$$

where $\mathbf{u} = \{u \quad v \quad w\}^{\mathrm{T}}$ is the displacement field, and

$$\nabla_L = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \qquad \nabla_N(\mathbf{u}) = \begin{bmatrix} \frac{\partial u}{\partial x}\frac{\partial}{\partial x} & \frac{\partial v}{\partial x}\frac{\partial}{\partial x} & \frac{\partial w}{\partial x}\frac{\partial}{\partial x} \\ \frac{\partial u}{\partial y}\frac{\partial}{\partial y} & \frac{\partial v}{\partial y}\frac{\partial}{\partial y} & \frac{\partial w}{\partial y}\frac{\partial}{\partial y} \\ \frac{\partial u}{\partial z}\frac{\partial}{\partial z} & \frac{\partial v}{\partial z}\frac{\partial}{\partial z} & \frac{\partial w}{\partial z}\frac{\partial}{\partial z} \\ \frac{\partial u}{\partial y}\frac{\partial}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial}{\partial y} & \frac{\partial v}{\partial y}\frac{\partial}{\partial z}+\frac{\partial v}{\partial z}\frac{\partial}{\partial y} & \frac{\partial w}{\partial y}\frac{\partial}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial}{\partial y} \\ \frac{\partial u}{\partial z}\frac{\partial}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial}{\partial z} & \frac{\partial v}{\partial z}\frac{\partial}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial}{\partial z} & \frac{\partial w}{\partial z}\frac{\partial}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial}{\partial x} & \frac{\partial v}{\partial x}\frac{\partial}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial}{\partial x} & \frac{\partial w}{\partial x}\frac{\partial}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial}{\partial x} \end{bmatrix} \tag{A-3}$$

where operator ∇_L and $\nabla_N(\mathbf{u})$ are both defined according to the Green strain with the latter containing homogeneous linear expressions of the derivatives of $\mathbf{u}$. These operators apply to the displacement field on its right, e.g. $\nabla_L \mathbf{u}$ and $\nabla_N(\mathbf{u})\mathbf{u}$. Transposing these expressions reverses the sequence of these operators, e.g. $\mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}$ and $\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}(\mathbf{u})$. Given the expression of $\nabla_N(\mathbf{u})$, $\mathbf{u}$ and $\delta\mathbf{u}$ in $\nabla_N(\delta\mathbf{u})\mathbf{u}$ are commutative, i.e. $\nabla_N(\delta\mathbf{u})\mathbf{u} = \nabla_N(\mathbf{u})\delta\mathbf{u}$.

Under the assumption of linear elasticity, the constitutive relationship can be given as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_I + \mathbf{C}\,\boldsymbol{\varepsilon} = \boldsymbol{\sigma}_I + \mathbf{C}\,\boldsymbol{\varepsilon}_L + \mathbf{C}\,\boldsymbol{\varepsilon}_N = \boldsymbol{\sigma}_I + \boldsymbol{\sigma}_L + \boldsymbol{\sigma}_N \tag{A-4}$$

where $\mathbf{C}$ is the material stiffness, $\boldsymbol{\sigma}_L$ and $\boldsymbol{\sigma}_N$ are linear and quadratic parts of the second Piola-Kirchhoff stress in their contracted form, with $\boldsymbol{\sigma}_N$ representing the effects of geometrical nonlinearity. In some parts of the derivation involved in the paper, the matrix form these stresses are required and they will be denoted as $\mathbf{s}$, $\mathbf{s}_I$, $\mathbf{s}_L$ and $\mathbf{s}_N$, respectively.

Having introduced the notations above, (A-1) can be partitioned according to the orders of displacements as follows.

$$U = U_1 + U_2 + U_3 + U_4 \tag{A-5a}$$

where $U_1 = \boldsymbol{\varepsilon}_L^{\mathrm{T}}\,\boldsymbol{\sigma}_I = \mathbf{u}^{\mathrm{T}}\nabla_L^{\mathrm{T}}\,\boldsymbol{\sigma}_I$ (A-5b)

$$U_2 = \boldsymbol{\varepsilon}_N^{\mathrm{T}} \boldsymbol{\sigma}_I + \frac{1}{2} \boldsymbol{\varepsilon}_L^{\mathrm{T}} \boldsymbol{C}\, \boldsymbol{\varepsilon}_L = \frac{1}{2} \mathbf{u}^{\mathrm{T}} \nabla_N^{\mathrm{T}} (\mathbf{u}) \boldsymbol{\sigma}_I + \frac{1}{2} \mathbf{u}^{\mathrm{T}} \nabla_L^{\mathrm{T}} \mathbf{C} \nabla_L \mathbf{u} \tag{A-5c}$$

$$U_3 = \frac{1}{2} \left(\boldsymbol{\varepsilon}_L^{\mathrm{T}} \mathbf{C}\, \boldsymbol{\varepsilon}_N + \boldsymbol{\varepsilon}_N^{\mathrm{T}} \mathbf{C} \boldsymbol{\varepsilon}_L \right) = \boldsymbol{\varepsilon}_L^{\mathrm{T}} \mathbf{C}\, \boldsymbol{\varepsilon}_N = \frac{1}{2} \mathbf{u}^{\mathrm{T}} \nabla_L^{\mathrm{T}} \mathbf{C}\, \nabla_N (\mathbf{u}) \mathbf{u} \tag{A-5d}$$

$$U_4 = \frac{1}{2} \boldsymbol{\varepsilon}_N^{\mathrm{T}} \mathbf{C}\, \boldsymbol{\varepsilon}_N = \frac{1}{8} \mathbf{u}^{\mathrm{T}} \nabla_N^{\mathrm{T}} (\mathbf{u}) \mathbf{C}\, \nabla_N (\mathbf{u}) \mathbf{u}\,. \tag{A-5e}$$

The subscripts to various terms of U above indicate the orders of power in displacement. Each of U_1 to U_4 is a homogeneous function of **u** and hence can be expressed as a linear, quadratic, cubic or quartic form of **u** as in (A-5b) to (A5-e), respectively. Geometrical nonlinearity is brought into the problem through U_3 and U_4 which are often referred to as higher order terms whilst U_1 and U_2 pertain to the conventional linear problem.

To derive the stiffness matrices of a finite element for the geometrically nonlinear problem the displacement field in a finite element can be discretised as follows.

$$\mathbf{u} = \mathbf{A}\mathbf{q} \tag{A-6}$$

where $\mathbf{q}$ is the nodal displacement vector for the element corresponding to the displacement field **u** and **A** is the displacement interpolation matrix which is formed from shape functions.

$$\boldsymbol{\varepsilon}_L = \mathbf{B}_L \mathbf{q}, \qquad \boldsymbol{\varepsilon}_N = \frac{1}{2} \mathbf{B}_N (\mathbf{q}) \mathbf{q} \tag{A-7}$$

where $\mathbf{B}_L = \nabla_L \mathbf{A}$ (A-8a)

$$\mathbf{B}_N (\mathbf{q}) = \nabla_N (\mathbf{u}) \mathbf{A} = \nabla_N (\mathbf{q}) \mathbf{A} \qquad \text{with } \nabla_N (\mathbf{A}\mathbf{q}) \text{ denoted later as } \nabla_N (\mathbf{q}) \tag{A-8b}$$

$\mathbf{B}_L$ is independent of $\mathbf{q}$, whilst all the elements of $\mathbf{B}_N$ are homogeneous linear expressions of $\mathbf{q}$, which represents the effects of geometrical nonlinearity in the kinematic equation. Matrices $\mathbf{B}_L$ and $\mathbf{B}_N$ may vary with the type of element, including truss, membrane and brick continuum elements and beam, plate and shell structural elements. Following an argument that was made earlier, $\mathbf{q}$ and $\delta\mathbf{q}$ involved in $\mathbf{B}_N(\delta\mathbf{q})\mathbf{q}$ are commutative, i.e. $\mathbf{B}_N(\delta\mathbf{q})\mathbf{q} = \mathbf{B}_N(\mathbf{q})\delta\mathbf{q}$.

Discretising the term in ψ_{11} associated with $\mathbf{S}_I$ as in (22a) leads to

$$\mathbf{u}^{\mathrm{T}} \mathbf{S}_I \mathbf{u}' = \mathbf{q} \left(\sum_{i=1}^{3} \mathbf{b}_i^{\mathrm{T}} \mathbf{s}_I \mathbf{b}_i \right) \mathbf{q}' \tag{A9a}$$

where $\mathbf{b}_i = \nabla_i \mathbf{A}$. (A9b)

Similarly, those in ψ_{111} and ψ_{1111} associated with $\mathbf{S}_L$ and $\mathbf{S}_{11}$ as in (22b-c) are discretised to

$$\mathbf{u}^{\mathrm{T}} \mathbf{S}_L (\mathbf{q}'') \mathbf{u}' = \mathbf{q} \left(\sum_{i=1}^{3} \mathbf{b}_i^{\mathrm{T}} \mathbf{s}_L (\mathbf{u}'') \mathbf{b}_i \right) \mathbf{q}' \qquad \text{with } \mathbf{s}_L (\mathbf{q}'') = \mathbf{s}_L (\mathbf{A}\mathbf{q}'') = \mathbf{s}_L (\mathbf{u}'') \tag{A10}$$

$$\mathbf{u}^{\mathrm{T}}\mathbf{S}_{11}\left(\mathbf{u}'',\mathbf{u}'''\right)\mathbf{u}' = \mathbf{q}\left(\sum_{i=1}^{3}\mathbf{b}_i^{\mathrm{T}}\mathbf{s}_{11}\left(\mathbf{u}'',\mathbf{u}'''\right)\mathbf{b}_i\right)\mathbf{q}' \qquad \text{with } \mathbf{s}_{11}\left(\mathbf{q}'',\mathbf{q}'''\right) = \mathbf{s}_{11}\left(\mathbf{u}'',\mathbf{u}'''\right) \tag{A11}$$

whilst

$$\mathbf{u}^{\mathrm{T}}\mathbf{S}_{N}\left(\mathbf{u}\right)\mathbf{u} = \frac{1}{2}\mathbf{q}\left(\sum_{i=1}^{3}\mathbf{b}_i^{\mathrm{T}}\mathbf{s}_{11}\left(\mathbf{u},\mathbf{u}\right)\mathbf{b}_i\right)\mathbf{q} \qquad \text{with } \mathbf{s}_{N}\left(\mathbf{q}\right) = \mathbf{s}_{N}\left(\mathbf{u}\right) \tag{A12}$$

as a part of (25c) after discretisation.

Appendix B Derivation of the flip transformation and an example of its application

$$\begin{aligned}
\mathbf{u}^{\mathrm{T}}\nabla_N^{\mathrm{T}}\left(\mathbf{u}'\right)\boldsymbol{\sigma}_I &= \left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x}+\cdots\right)\sigma_{Ix} + \left(\frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}+\cdots\right)\sigma_{Iy} + \left(\frac{\partial u}{\partial z}\frac{\partial u'}{\partial z}+\cdots\right)\sigma_{Iz} \\
&\quad + \left(\frac{\partial u}{\partial y}\frac{\partial u'}{\partial z}+\frac{\partial u'}{\partial y}\frac{\partial u}{\partial z}+\cdots\right)\tau_{Iyz} + \left(\frac{\partial u}{\partial z}\frac{\partial u'}{\partial x}+\frac{\partial u'}{\partial z}\frac{\partial u}{\partial x}+\cdots\right)\tau_{Izx} + \left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial y}+\frac{\partial u'}{\partial x}\frac{\partial u}{\partial y}+\cdots\right)\tau_{Ixy}
\end{aligned}$$

$$\begin{aligned}
&= \begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial z}\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}\sigma_{Ix} & \tau_{Ixy} & \tau_{Ixz}\\ \tau_{Ixy} & \sigma_{Iy} & \tau_{Iyz}\\ \tau_{Ixz} & \tau_{Iyz} & \sigma_{Iz}\end{bmatrix} \begin{Bmatrix}\frac{\partial u'}{\partial x}\\ \frac{\partial u'}{\partial y}\\ \frac{\partial u'}{\partial z}\end{Bmatrix} + \begin{Bmatrix}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial z}\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}\sigma_{Ix} & \tau_{Ixy} & \tau_{Ixz}\\ \tau_{Ixy} & \sigma_{Iy} & \tau_{Iyz}\\ \tau_{Ixz} & \tau_{Iyz} & \sigma_{Iz}\end{bmatrix} \begin{Bmatrix}\frac{\partial v'}{\partial x}\\ \frac{\partial v'}{\partial y}\\ \frac{\partial v'}{\partial z}\end{Bmatrix} + \begin{Bmatrix}\frac{\partial w}{\partial x}\\ \frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial z}\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}\sigma_{Ix} & \tau_{Ixy} & \tau_{Ixz}\\ \tau_{Ixy} & \sigma_{Iy} & \tau_{Iyz}\\ \tau_{Ixz} & \tau_{Iyz} & \sigma_{Iz}\end{bmatrix} \begin{Bmatrix}\frac{\partial w'}{\partial x}\\ \frac{\partial w'}{\partial y}\\ \frac{\partial w'}{\partial z}\end{Bmatrix} \\
&= u\nabla^{\mathrm{T}}\mathbf{s}_I\nabla u' + v\nabla^{\mathrm{T}}\mathbf{s}_I\nabla v' + w\nabla^{\mathrm{T}}\mathbf{s}_I\nabla w' \\
&= \mathbf{u}^{\mathrm{T}}\nabla_1^{\mathrm{T}}\,\mathbf{s}_I\,\nabla_1\mathbf{u}' + \mathbf{u}^{\mathrm{T}}\nabla_2^{\mathrm{T}}\,\mathbf{s}_I\,\nabla_2\mathbf{u}' + \mathbf{u}^{\mathrm{T}}\nabla_3^{\mathrm{T}}\,\mathbf{s}_I\,\nabla_3\mathbf{u}' \\
&= \mathbf{u}^{\mathrm{T}}\mathbf{S}_I\mathbf{u}'
\end{aligned} \tag{B1}$$

where $\mathbf{s}_I$ and ∇_i have been given in (19b) and (19c). Applying the flip transformation to discretised energy terms, e.g. those involved in (2), one can readily recover the symmetry of the secant stiffness matrix as in (2).

$$\begin{aligned}
\delta U^e &= \delta\mathbf{q}^{\mathrm{T}}\left[\int_{\Omega^e}\left(\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_L + \frac{1}{2}\mathbf{B}_L^{\mathrm{T}}\mathbf{C}\mathbf{B}_N\left(\mathbf{q}\right) + \mathbf{B}_N^{\mathrm{T}}\left(\mathbf{q}\right)\mathbf{C}\mathbf{B}_L + \frac{1}{2}\mathbf{B}_N^{\mathrm{T}}\left(\mathbf{q}\right)\mathbf{C}\mathbf{B}_N\left(\mathbf{q}\right)\right)d\Omega\right]\mathbf{q} \\
&= \delta\mathbf{q}^{\mathrm{T}}\left[\mathbf{K}_0^e + \frac{1}{2}\left(\mathbf{N}_1^q\right)^e + \frac{1}{3}\left(\mathbf{N}_2^q\right)^e + \int_{\Omega^e}\left(\frac{1}{2}\mathbf{B}_N^{\mathrm{T}}\left(\mathbf{q}\right)\mathbf{C}\mathbf{B}_L + \frac{1}{6}\mathbf{B}_N^{\mathrm{T}}\left(\mathbf{q}\right)\mathbf{C}\mathbf{B}_N\left(\mathbf{q}\right)\right)d\Omega\right]\mathbf{q} \\
&= \delta\mathbf{q}^{\mathrm{T}}\left[\mathbf{K}_0^e + \frac{1}{2}\left(\mathbf{N}_1^q\right)^e + \frac{1}{3}\left(\mathbf{N}_2^q\right)^e + \frac{1}{2}\sum_{i=1}^{3}\int_{\Omega^e}\mathbf{b}_i^{\mathrm{T}}\mathbf{s}_L\,\mathbf{b}_i d\Omega + \frac{1}{3}\sum_{i=1}^{3}\int_{\Omega^e}\mathbf{b}_i^{\mathrm{T}}\mathbf{s}_N\,\mathbf{b}_i d\Omega\right]\mathbf{q} \\
&= \delta\mathbf{q}^{\mathrm{T}}\left[\mathbf{K}_0^e + \frac{1}{2}\left(\mathbf{N}_1^q\right)^e + \frac{1}{3}\left(\mathbf{N}_2^q\right)^e + \frac{1}{2}\left(\mathbf{K}_1^\sigma\right)^e + \frac{1}{3}\left(\mathbf{K}_2^\sigma\right)^e\right]\mathbf{q} \\
&= \delta\mathbf{q}^{\mathrm{T}}\left[\mathbf{K}_0^e + \frac{1}{2}\mathbf{N}_1^e + \frac{1}{3}\mathbf{N}_2^e\right]\mathbf{q} = \delta\mathbf{q}^{\mathrm{T}}\mathbf{K}_s^e\mathbf{q}
\end{aligned} \tag{B2}$$

where various stiffness matrices have been given in (30) to (32). Without making the relevant expression apparently symmetric through the flip transformation, the secant stiffness matrix cannot be obtained explicitly.

Appendix C Alternative expressions for the IPBCs and their drawback

Alternative expressions of the IPBCs a and b were obtained in [15]. Presented in the notations of this paper, they were expressed as follows.

$$a = -\frac{3}{2\lambda^c}\frac{\int_\Omega \boldsymbol{\sigma}_1^{\mathrm{T}} \nabla_N\left(\mathbf{q}_1\right)\mathbf{q}_1 d\Omega}{\int_\Omega \left(\boldsymbol{\sigma}_0^c\right)^{\mathrm{T}} \nabla_N\left(\mathbf{q}_1\right)\mathbf{q}_1 d\Omega} \tag{C-1}$$

$$b = -\frac{1}{\lambda^c}\frac{\int_\Omega \left[\boldsymbol{\sigma}_2^{\mathrm{T}} \nabla_N\left(\mathbf{q}_1\right)\mathbf{q}_1 + 2\boldsymbol{\sigma}_1^{\mathrm{T}} \nabla_N\left(\mathbf{q}_1\right)\mathbf{q}_2\right] d\Omega}{\int_\Omega \left(\boldsymbol{\sigma}_0^c\right)^{\mathrm{T}} \nabla_N\left(\mathbf{q}_1\right)\mathbf{q}_1 d\Omega} \tag{C-2}$$

where $\nabla_N\left(\mathbf{q}_1\right)$ is defined in the same way as (A-8b) but with $\mathbf{q}$ replaced by $\mathbf{q}_1$, $\boldsymbol{\sigma}_0^c$ the stress state on the fundamental equilibrium path at the point of buckling, and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the first and second order perturbation stresses defined as follows (to be distinguished from $\boldsymbol{\sigma}_L$ and $\boldsymbol{\sigma}_N$).

$$\boldsymbol{\sigma}_0^c = \mathbf{C}\left(\nabla_L + \frac{1}{2}\nabla_N\left(\mathbf{q}_0^c\right)\right)\mathbf{q}_0^c \tag{C-3}$$

$$\boldsymbol{\sigma}_1 = \mathbf{C}\left(\nabla_L + \nabla_N\left(\mathbf{q}_0^c\right)\right)\mathbf{q}_1 \tag{C-4}$$

$$\boldsymbol{\sigma}_2 = \mathbf{C}\left(\left(\nabla_L + \nabla_N\left(\mathbf{q}_0^c\right)\right)\mathbf{q}_2 + \frac{1}{2}\nabla_N\left(\mathbf{q}_1\right)\mathbf{q}_1\right). \tag{C-5}$$