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The infinite Fibonacci groups and relative asphericity

Martin Edjvet and Arye Juhász

Dedicated to David L. Johnson

Abstract

We prove that the generalised Fibonacci group $F(r, n)$ is infinite for $(r, n) \in \{(7 + 5k, 5), (8 + 5k, 5) : k \geqslant 0\}$. This together with previously known results yields a complete classification of the finite $F(r, n)$, a problem that has its origins in a question by J. H. Conway in 1965. The method is to show that a related relative presentation is aspherical from which it can be deduced that the groups are infinite.

1. *Introduction*

The generalised Fibonacci group $F(r, n)$ is the group defined by the *cyclic presentation*

$$\langle x_1, \dots, x_n | x_1x_2 \dots x_r x_{r+1}^{-1}, x_2x_3 \dots x_{r+1}x_{r+2}^{-1}, \dots, x_{n-1}x_nx_1 \dots x_{r-2}x_{r-1}^{-1}, x_nx_1x_2 \dots x_{r-1}x_r^{-1}\rangle,$$

where $r > 1$, $n > 1$. Thus there are n generators and n relators each of length $r + 1$ and each relator is obtained from the first relator by cyclically permuting the subscripts and reducing modulo n [**10**, Section 7.3]. There has been a great deal of interest in the study of these groups since the question in [**5**] by Conway about the order of $F(2, 5)$. Up to now the order of $F(r, n)$ was known except for the two infinite families $F\{7, 5\}$ and $F\{8, 5\}$, where $F\{r, n\} := \{F(r + kn, n) : k \geqslant 0\}$. The reader is referred to [**15**], and the references therein together with [**3**, **14**] for further details. In this paper we will show that each group in $F\{7, 5\}$ or $F\{8, 5\}$ is infinite. This together with previous results yields the following theorem.

Theorem 1.1. *The generalised Fibonacci group $F(r, n)$ is finite if and only if one of the following conditions is satisfied:*

- (i) $r = 2$ *and* $n \in \{2, 3, 4, 5, 7\}$: *indeed* $F(2, 2)$ *is trivial;* $F(2, 3) \cong Q_8$, *the quaternion group of order* 8; $F(2, 4) \cong \mathbb{Z}_5$; $F(2, 5) \cong \mathbb{Z}_{11}$; *and* $F(2, 7) \cong \mathbb{Z}_{29}$;
- (ii) $r = 3$ *and* $n \in \{2, 3, 5, 6\}$: *indeed* $F(3, 2) \cong Q_8$; $F(3, 3) \cong \mathbb{Z}_2$; $F(3, 5) \cong \mathbb{Z}_{22}$; *and* $F(3, 6)$ *is non-metacyclic, soluble of order* 1512;
- (iii) $r \geqslant 4$ *and* $r \equiv 0 \pmod n$, *in which case* $F(r, n) \cong \mathbb{Z}_{r-1}$;
- (iv) $r \geqslant 4$ *and* $r \equiv 1 \pmod n$, *in which case* $F(r, n)$ *is metacyclic of order* $r^n - 1$;
- (v) $r \geqslant 4$, $n = 4$ *and* $r \equiv 2 \pmod n$, *in which case* $F(r, n) = F(4k + 2, 4)$ $(k \geqslant 1)$ *is metacyclic of order* $(4k + 1)(2(4)^{2k} + 2(-4)^k + 1)$.

A *relative group presentation* is a presentation of the form $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r}\rangle$ where G is a group, $\boldsymbol{x}$ a set disjoint from G and $\boldsymbol{r}$ a set of cyclically reduced words in the free product $G * \langle \boldsymbol{x}\rangle$ where $\langle \boldsymbol{x}\rangle$ denotes the free group on $\boldsymbol{x}$ [**2**]. If $G(\mathcal{P})$ denotes the group defined by $\mathcal{P}$ then $G(\mathcal{P})$ is the quotient group $G * \langle \boldsymbol{x}\rangle/N$, where N denotes the normal closure in $G * \langle \boldsymbol{x}\rangle$ of $\boldsymbol{r}$. A

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relative group presentation is defined in [2] to be *aspherical* if every spherical picture over it contains a dipole, that is, fails to be reduced. There is interest in when a relative presentation is aspherical, see, for example, [1, 2, 6, 8, 9, 13]. In this paper we consider the situation when $G = \langle t | t^5 \rangle$, $\boldsymbol{x} = \{u\}$ and $\boldsymbol{r} = \{t^2 utu^{-n}\}$ and prove the following theorem.

THEOREM 1.2. *The relative presentation* $\mathcal{P}_n = \langle t, u | t^5, t^2 utu^{-n} \rangle$ *is aspherical for* $n \geqslant 7$.

Applying, for example, statement (0.4) in the introduction of [2] and the fact that the group defined by $\mathcal{P}_n$ is neither trivial nor cyclic of order 5 we immediately obtain

COROLLARY 1.3. *If* $G(\mathcal{P}_n)$ *is the group defined by* $\mathcal{P}_n$ *then* $G(\mathcal{P}_n)$ *is infinite for* $n \geqslant 7$, *indeed* u *has infinite order in* $G(\mathcal{P}_n)$ *for* $n \geqslant 7$.

We will show in Section 2 that Corollary 1.3 implies that each group in $F\{7,5\}, F\{8,5\}$ is infinite. The remaining Sections 3–11 of the paper will be devoted entirely to proving Theorem 1.2.

2. *Fibonacci groups*

Consider the generalised Fibonacci group $F(r,n)$ of the introduction. If $r = 2$ or $2 \leqslant n \leqslant 4$ or $(r,n) \in \{(3,5),(3,6)\}$ or n divides r or $r \equiv 1 \pmod{n}$ then Theorem 1.1 applies and these cases are discussed fully with relevant references in [15]. Assume then that none of these conditions holds. In particular $r \geqslant 3$ and $n \geqslant 5$. In [14] it is shown that if n does not divide any of $r \pm 1$, $r+2$, $2r$, $2r+1$ or $3r$ then $F(r,n)$ is infinite. If n divides $r+1$ then $F(r,n)$ is infinite for $r \geqslant 3$ [11] so assume otherwise. We are left therefore to consider the families $F\{r, r+2\}$; $F\{r, 2r\}$; $F\{r, 2r+1\}$ and $F\{r, 3r\}$. In [3] it is shown that if $r \geqslant 4$ then each group in $F\{r, r+2\}$ and $F\{r, 2r\}$ is infinite; and if $r \geqslant 3$ then each group in $F\{r, 2r+1\}$ is infinite. This leaves $F\{8,5\}$, $F\{9,6\}$, $F\{7,5\}$ and $F\{r,3r\}$. In [14] it is also shown that if n does not divide any of $r \pm 1$, $r \pm 2$, $r+3$, $2r$, $2r+1$ then $F(r,n)$ is infinite. If n divides $3r$ and $r+2$ we obtain the family $F\{4,6\}$ which is $F\{r, r+2\}$ for $r = 4$; if n divides $3r$ and $r-2$ we obtain $F\{8,6\}$ and each group in this family is infinite [4]; and if n divides $3r$ and $r+3$ we obtain $F\{6,9\}$. By our assumptions n does not divide $3r$ together with any of $r \pm 1$, $2r$ or $2r+1$. It is also shown in [3] that each group in $F\{9,6\}$ or $F\{6,9\}$ is infinite, all of which leaves $F\{7,5\}$ and $F\{8,5\}$. These families are

$$\{F(7+5k,5)\colon k \geqslant 0\} \quad \text{and} \quad \{F(8+5k,5)\colon k \geqslant 0\},$$

where $F(7+5k,5)$ and $F(8+5k,5)$ are defined, respectively, by the presentations

$$\langle x_1,x_2,x_3,x_4,x_5 \mid (x_1x_2x_3x_4x_5)^{k+1}x_1x_2x_3^{-1},\ldots,(x_5x_1x_2x_3x_4)^{k+1}x_5x_1x_2^{-1}\rangle,$$

$$\langle x_1,x_2,x_3,x_4,x_5 \mid (x_1x_2x_3x_4x_5)^{k+1}x_1x_2x_3x_4^{-1},\ldots,(x_5x_1x_2x_3x_4)^{k+1}x_5x_1x_2x_3^{-1}\rangle.$$

We show how Corollary 1.3 can be used to prove Theorem 1.1. Since cyclically permuting the generators induces an automorphism we can form a semi-direct product with the cyclic group of order 5 in the way described, for example, in [10, Section 10.2] and this yields the groups $E(7+5k,5)$ and $E(8+5k,5)$ defined, respectively, by the presentations

$$\langle x,t \mid t^5,(xt^{-1})^{7+5k}x^{-1}t^2\rangle,$$

$$\langle x,t \mid t^5,(xt^{-1})^{8+5k}x^{-1}t^3\rangle.$$

Now

$$\begin{aligned}\langle x,t \mid t^5,(xt^{-1})^{7+5k}x^{-1}t^2\rangle &= \langle x,t,y \mid t^5,(xt^{-1})^{7+5k}x^{-1}t^2, y^{-1}xt^{-1}\rangle \\ &= \langle y,t \mid t^5, y^{7+5k}t^{-1}y^{-1}t^2\rangle\end{aligned}$$

$$\begin{aligned}
&= \langle y, t \mid t^5, y^{7+5k} t y^{-1} t^3 \rangle \quad (\text{replacing } t \text{ by } t^{-1}) \\
&= \langle y, t, s \mid t^5, y^{7+5k} t y^{-1} t^3, s t^{-3} \rangle \quad (s = t^3) \\
&= \langle y, s \mid s^5, y^{7+5k} s^2 y^{-1} s \rangle \quad (s^2 = t^6 = t) \\
&= \langle u, t \mid t^5, t^2 u t u^{-(7+5k)} \rangle \quad (s \leftrightarrow t,\ y = u^{-1}) \quad (\text{cyclic conjugate})
\end{aligned}$$

and

$$\begin{aligned}
\langle x, t \mid t^5, (xt^{-1})^{8+5k} x^{-1} t^3 \rangle &= \langle x, t, y \mid t^5, (xt^{-1})^{8+5k} x^{-1} t^3, y^{-1} x t^{-1} \rangle \\
&= \langle t, y \mid t^5, y^{8+5k} t^{-1} y^{-1} t^3 \rangle \\
&= \langle u, t \mid t^5, t^2 u t u^{-(8+5k)} \rangle \quad (\text{inverse, } t^{-3} = t^2, y = u).
\end{aligned}$$

Therefore Corollary 1.3 implies that each group in $\{E(7+5k, 5) \text{ and } E(8+5k, 5) \colon k \geqslant 0\}$ is infinite and, given this, Theorem 1.1 now follows.

3. *The amended picture and curvature*

The reader is referred to [**2**, **12**] for definitions of many of the basic terms used in this and subsequent sections.

Suppose by way of contradiction that the relative presentation

$$\mathcal{P}_n = \langle t, u \mid t^5, t^2 u t u^{-n} \rangle \quad (n \geqslant 7)$$

is not aspherical, that is, there exists a *reduced spherical picture* $\boldsymbol{P}$ over $\mathcal{P}_n$. Then each *arc* of $\boldsymbol{P}$ is equipped with a normal orientation and labelled by an element of $\{u, u^{-1}\}$; each *corner* of $\boldsymbol{P}$ is labelled by an element of $\{t^i \colon -2 \leqslant i \leqslant 2\}$; reading the labels clockwise on the corners and arcs at a given vertex yields $t^2 u t u^{-n}$ (up to cyclic permutation and inversion); and, since t has order 5 in $G(\mathcal{P}_n)$, the product of the sequence of corner labels encountered in an anti-clockwise traversal of any given *region* of $\boldsymbol{P}$ yields the identity in $G = \langle t \mid t^5 \rangle$.

Now let $\boldsymbol{D}$ be the dual of the picture $\boldsymbol{P}$ with the labelling of $\boldsymbol{D}$ inherited from that of $\boldsymbol{P}$. Then $\boldsymbol{D}$ is a (spherical) diagram such that: each corner label of $\boldsymbol{D}$ is t^i, where $-2 \leqslant i \leqslant 2$; and each edge is oriented and labelled u or u^{-1}. For convenience we adopt the notation

$$b u a u^{-1} \lambda_{n-1}^{-1} u^{-1} \lambda_{n-2}^{-1} \dots u^{-1} \lambda_1^{-1} u^{-1}$$

for $t^2 u t u^{-n}$. Thus $a = t^1$; $b = t^2$; and $\lambda_i = t^0$ $(1 \leqslant i \leqslant n-1)$. Each (oriented) region Δ of $\boldsymbol{D}$ is given (up to cyclic permutation and inversion) by Figure 1(i); and an example of how the regions are oriented is illustrated by Figure 1(iii). (In subsequent figures we will not show the orientation of the regions or edges nor the edge labels u, u^{-1}.) Note that the sum of the powers of t read around any given vertex of $\boldsymbol{D}$ is congruent to 0 modulo 5.

For ease of presentation and to simplify matters further we will use λ to denote λ_i and μ to denote λ_j^{-1} $(1 \leqslant i, j \leqslant n-1)$ throughout what follows. This way the *star graph* Γ for $\boldsymbol{D}$ is given by Figure 1(ii) with the understanding that the edges labelled λ and μ in Γ are traversed only in the direction indicated. Thus the edge labelled λ represents the $n-1$ edges, labelled λ_i; and μ represents the $n-1$ inverse edges. Recall that the vertex labels in $\boldsymbol{D}$ yield closed admissible paths in Γ.

We can make the following assumptions without any loss of generality.

A1. $\boldsymbol{D}$ is minimal with respect to number of regions and so, in particular, is *reduced*.
A2. Subject to **A1**, $\boldsymbol{D}$ is maximal with respect to number of vertices of degree 2.

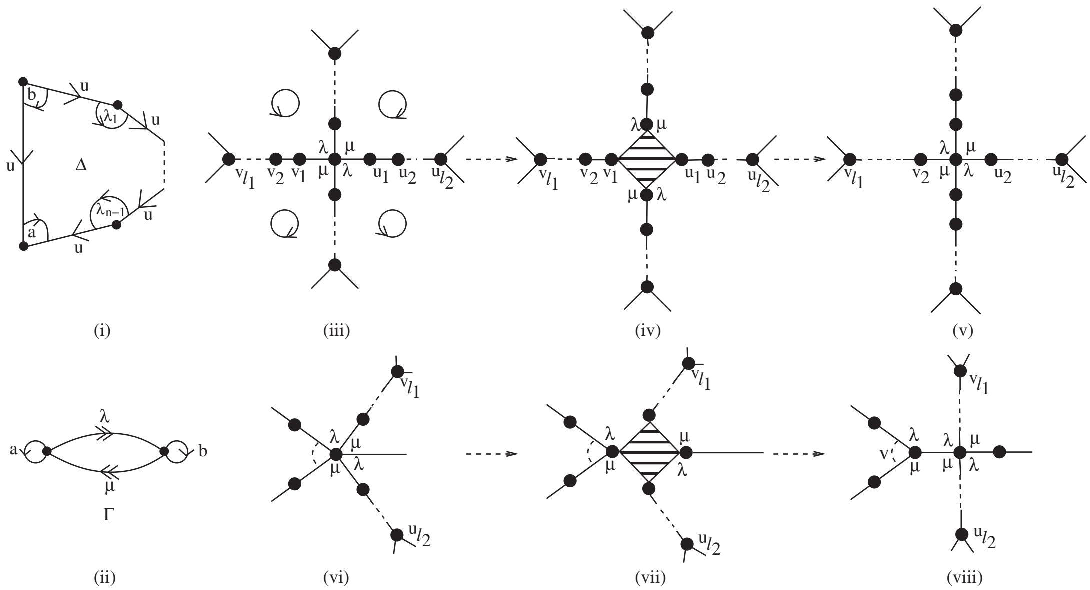

FIGURE 1. *Region Δ, bridge moves and star graph.*

We introduce some further notation. If v is a vertex of $\boldsymbol{D}$ then $l(v)$, the *label* of v, is the cyclic word obtained from the corner labels of v in a *clockwise* direction; and $d(v)$ denotes the *degree* of v. A (v_1, v_2)*-edge* is an edge with endpoints v_1 and v_2; and an edge is a (θ_1, θ_2)*-edge relative to the region* Δ if its corner labels in Δ are, in no particular order, θ_1 and θ_2. (Sometimes we will simply talk of a (θ_1, θ_2)-edge with the understanding that the corner labels are either θ_1 and θ_2 or θ_1^{-1} and θ_2^{-1}.)

LEMMA 3.1. *If v is a vertex of $\boldsymbol{D}$ then $l(v) \neq (\lambda\mu)^{\pm k}$ for $k \geqslant 2$.*

Proof. The proof is by induction on k. Consider the vertex of Figure 1(iii) having label $(\lambda\mu)^2$. Apply $m = \min\{l_1, l_2\}$ bridge moves of the type shown in Figure 1(iv) and (v). Then each of the first $m-1$ bridge moves will create and destroy two vertices of degree 2, leaving the total number unchanged. The mth bridge move however will create two vertices of degree 2 but destroy at most 1. Since bridge moves leave the total number of regions unchanged, we obtain a contradiction to assumption **A2**. Now consider the vertex of Figure 1(vi) having label $(\lambda\mu)^k$, where $k \geqslant 3$. Again apply $m = \min\{l_1, l_2\}$ bridge moves of the type shown in Figure 1(vii) and (viii). The first such bridge move may decrease the total number of vertices of degree 2 by 1, each subsequent bridge move creates two and destroys two until the mth bridge which increases the number by at least 1. This produces a new diagram with at most the same number of vertices of degree 2 as $\boldsymbol{D}$. But applying an induction argument to the vertex v of Figure 1(viii) where $l(v) = (\lambda\mu)^{k-1}$ will yield a contradiction to **A2** as before. □

LEMMA 3.2. *Let $v \in \boldsymbol{D}$.* (i) *If $d(v) = 2$ then $l(v) = (\lambda\mu)^{\pm 1}$ and* (ii) *if $d(v) > 2$ then $l(v)$ contains at least three occurrences of $a^{\pm 1}, b^{\pm 1}$.*

Proof. Both statements follow from the fact that the sum of the corner labels is congruent to $0 \bmod 5$ together with Lemma 3.1 for (ii) and the fact that, since $\boldsymbol{D}$ is reduced, no adjacent corner labels are inverse to each other (that is, no sublabels of the form $aa^{-1}, bb^{-1}, \lambda_i\lambda_i^{-1}$). □

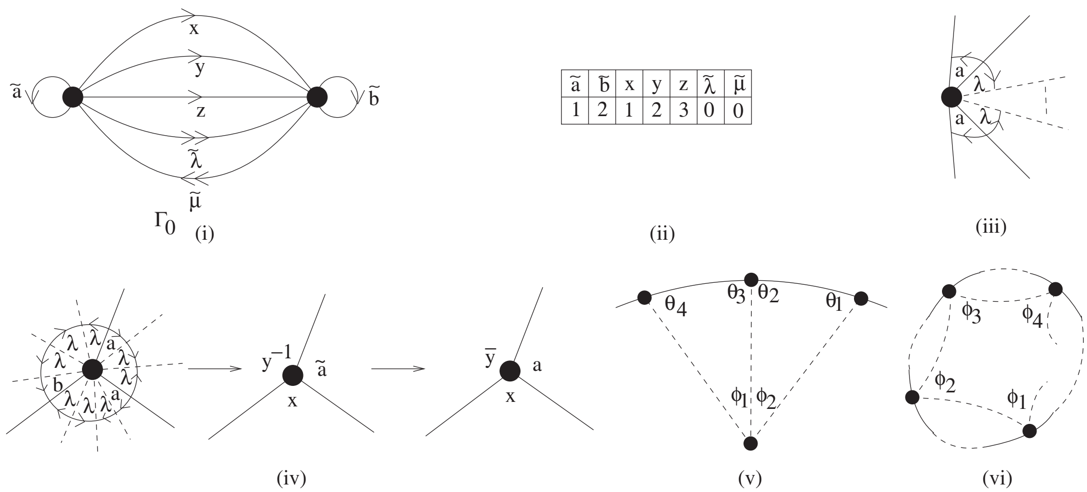

FIGURE 2. *New star graph and vertex labels.*

We amend $\boldsymbol{D}$ as follows. Delete all vertices of degree 2 and remove all edges that are not (b,a)-edges relative to any region, in so doing merging the adjacent regions. This results in the diagram $\boldsymbol{K}$.

LEMMA 3.3. *If $v \in \boldsymbol{K}$ then $d(v) \geqslant 3$.*

Proof. We see from Lemma 3.2 that $l(v)$ contains at least three occurrences of $a^{\pm 1}, b^{\pm 1}$ and each such occurrence contributes uniquely to $d(v)$. □

We claim that $\boldsymbol{K}$ contains a subdiagram $\boldsymbol{K}_0$ with the following properties: (1) $\boldsymbol{K}_0$ has connected interior and is simply connected; and (2) every connected component of $\boldsymbol{K}_0 \setminus \boldsymbol{K}_0^{(1)}$ is homeomorphic to an open disc. If $\boldsymbol{K}$ satisfies these two properties then take $\boldsymbol{K} = \boldsymbol{K}_0$. If not then $\boldsymbol{K} \backslash \boldsymbol{K}^{(1)}$ has a connected component $\boldsymbol{L}_1$ satisfying (1) which fails to satisfy (2). (The merging of regions may produce open annuli.) Consider $\boldsymbol{K} \backslash \boldsymbol{L}_1$. It is the disjoint union of subdiagrams at least one of which $\boldsymbol{L}_2$, say, satisfies (1). If $\boldsymbol{L}_2$ satisfies (2) then put $\boldsymbol{L}_2 = \boldsymbol{K}_0$; if not then repeat the argument with $\boldsymbol{L}_2 \backslash \boldsymbol{L}_2^{(1)}$ instead of $\boldsymbol{K} \backslash \boldsymbol{K}^{(1)}$ and so on. This procedure will terminate in a finite number of steps with a subdiagram $\boldsymbol{K}_0$ satisfying conditions (1) and (2). Observe that if Δ is a region of $\boldsymbol{K}_0$ then it follows from Lemma 3.1 that any 2-segment in Δ when regarded as a region of $\boldsymbol{D}$ will have its endpoints on $\partial\Delta$. (Recall that a 2-*segment* is a segment where endpoints have degree greater than 2 and whose intermediate vertices each have degree 2.)

The corner labels of $\boldsymbol{K}_0$ are obtained by taking the product of the corner labels of $\boldsymbol{D}$ used in forming each corner of $\boldsymbol{K}_0$. (An example is shown in Figure 2(iv).) Since each corner of $\boldsymbol{K}_0$ is between two (b,a)-edges it follows that the corner labels of $\boldsymbol{K}_0$ are

$$\begin{aligned}
\tilde{a} &= a(\lambda\mu)^{k_1} && \text{(odd length)}\\
\tilde{b} &= (\mu\lambda)^{k_2} b && \text{(odd length)}\\
\tilde{\lambda} &= (\lambda\mu)^{k_3}\lambda && \text{(odd length)}\\
x &= \tilde{a}\lambda && \text{(even length)}\\
y &= \lambda\tilde{b} && \text{(even length)}\\
z &= \tilde{a}\lambda\tilde{b} && \text{(odd length)},
\end{aligned} \tag{3.1}$$

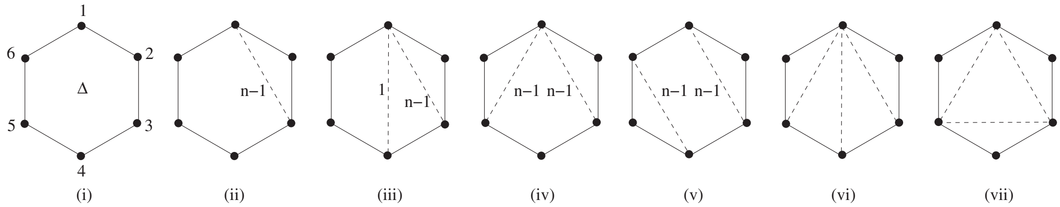

FIGURE 3. *Possible regions of degree* 6.

where $k_i \geqslant 0$ $(1 \leqslant i \leqslant 3)$. The star graph Γ_0 for $\boldsymbol{K}_0$ is given by Figure 2(i), and the table in Figure 2(ii) gives the power of t each corner label represents. Observe that $(\lambda\mu)^k$ for $k \geqslant 1$ cannot be a corner label in $\boldsymbol{K}_0$ for otherwise $\boldsymbol{D}$ would contain a subdiagram of the form shown in Figure 2(iii) and this contradicts **A1** since after *bridge moves* and cancellation it would be possible to reduce the number of regions of $\boldsymbol{D}$ by at least 2.

LEMMA 3.4. *Let $v \in \boldsymbol{K}_0$. If $d(v) < 6$ then $l(v)$ is one of the following:*

(i) $d(v) = 3:$ $\tilde{a}xy^{-1}$ $\tilde{b}\tilde{\mu}z;$
(ii) $d(v) = 4:$ $\tilde{a}\tilde{a}z\tilde{\mu}$ $\tilde{b}\tilde{b}x^{-1}y;$
(iii) $d(v) = 5:$ $\tilde{a}\tilde{a}\tilde{a}\tilde{a}\tilde{a}$ $\tilde{b}\tilde{b}\tilde{b}\tilde{b}\tilde{b}$ $\tilde{a}zx^{-1}y\tilde{\mu}$ $\tilde{b}x^{-1}\tilde{\lambda}z^{-1}y.$

Proof. This follows from checking all reduced closed paths in Γ_0 whose exponent sum is 0 modulo 5 together with the fact that equations (3.1) can be used to show that the following paths of length 2 together with their inverses do not occur as sublabels: $\tilde{a}\tilde{\lambda}$; $\tilde{a}y$; $\tilde{a}^{-1}x$; $\tilde{a}^{-1}z$; $\tilde{b}y^{-1}$; $\tilde{b}z^{-1}$; $\tilde{b}^{-1}x^{-1}$; $\tilde{b}^{-1}\tilde{\mu}$; $\tilde{\lambda}x^{-1}$; $\tilde{\lambda}\mu$; $\tilde{\mu}y$; $\tilde{\mu}\tilde{\lambda}$; $x^{-1}z$; yz^{-1}. For example, $\tilde{a}\tilde{\lambda} = a(\lambda\mu)^{k_1}(\lambda\mu)^{k_2}\lambda = a(\lambda\mu)^{k_1+k_2}\lambda = \tilde{a}\lambda = x$ after rewriting using equations (3.1). Indeed if $\tilde{a}$ and $\tilde{\lambda}$ are adjacent corner labels then this would imply the existence of a non(b,a)-edge in $\boldsymbol{K}_0$. □

Convention: We will usually write a, b, λ, μ for $\tilde{a}, \tilde{b}, \tilde{\lambda}, \tilde{\mu}$ simply for ease of presentation. For example, if $v \in \boldsymbol{D}$ has label $l(v) = a\lambda\mu a\lambda\mu\lambda b^{-1}\mu\lambda\mu$ then in $\boldsymbol{K}_0$ this transforms uniquely to $(a\lambda\mu)(a\lambda\mu\lambda)(b^{-1}\mu\lambda\mu) = \tilde{a}xy^{-1}$ which we write as axy^{-1} or as $ax\bar{y}$ in the figures. This is illustrated in Figure 2(iv). Moreover, when drawing diagrams we use $\bar{\theta}$ for θ^{-1}, where $\theta \in \{a, b, x, y, z\}$.

We turn now to the regions of $\boldsymbol{K}_0$. The edges or 2-segments deleted in forming $\boldsymbol{K}$ from $\boldsymbol{D}$ will be referred to as *shadow edges* and will usually be denoted by dotted edges in our figures. The number of edges in a 2-segment will be called its *length*. Much use will be of the fact that *the number of edges in a region of* $\boldsymbol{D}$ *is* $n+1$. By *length contradiction* we mean either a contradiction to this fact or to the fact that $n \geqslant 7$.

We will also use the fact that no region of $\boldsymbol{K}_0$ can contain the configuration of edges and shadow edges shown in Figure 2(v) and (vi). To see this observe in Figure 2(v) that $\{\phi_1, \phi_2\} \subseteq \{\lambda, \mu\}$ forcing each $\theta_i \in \{a^{\pm 1}, b^{\pm 1}\}$ and any attempt at labelling forces $\theta_2\theta_3 = aa^{-1}$ or bb^{-1}, a contradiction to $\boldsymbol{D}$ being reduced. In Figure 2(vi) each $\phi_i \in \{\lambda, \mu\}$ and this produces a region in $\boldsymbol{D}$ without corner label $a^{\pm 1}$ or $b^{\pm 1}$. We refer to the existence of each of these situations as a *basic labelling contradiction*.

If Δ is a region of $\boldsymbol{K}_0$ then $d(\Delta)$ denotes the degree of Δ, that is, the number of sides Δ has. For example, suppose $\Delta \in \boldsymbol{K}_0$ and $d(\Delta) = 6$. If Δ contains no shadow edges as in Figure 3(i) then we obtain the length contradiction $n+1 = 6$. Let (pq) denote the shadow edge with endpoints p and q. If Δ contains exactly one shadow edge e then, working modulo cyclic permutation and inversion, $e \in \{(13), (14)\}$. But $e = (13)$ yields the length contradiction $n+1 = n+3$ as shown in Figure 3(ii) since the length of (13) must be $n-1$. If Δ contains exactly two shadow edges e_1 and e_2 then $(e_1, e_2) \in \{((13),(14)), ((13),(15)), ((13),(46))\}$. But

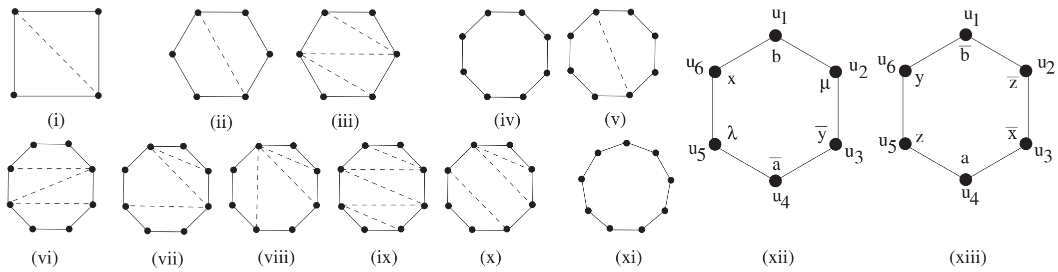

FIGURE 4. *Permitted regions of degree at most* 9.

TABLE 1. *Degree* 4 *curvature formulae.*

$c(3,3,3,3) = \frac{2\pi}{3}$	$c(3,3,5,5) = \frac{2\pi}{15}$	$c(3,4,5,5) = -\frac{\pi}{30}$
$c(3,3,3,4) = \frac{\pi}{2}$	$c(3,3,5,6) = \frac{\pi}{15}$	$c(3,4,5,6) = -\frac{\pi}{10}$
$c(3,3,3,5) = \frac{2\pi}{5}$	$c(3,3,5,7) = \frac{2\pi}{105}$	$c(3,4,5,7) = -\frac{31\pi}{210}$
$c(3,3,3,6) = \frac{\pi}{3}$	$c(3,3,6,6) = 0$	$c(3,4,6,6) = -\frac{\pi}{6}$
$c(3,3,4,4) = \frac{\pi}{3}$	$c(3,4,4,5) = \frac{\pi}{15}$	$c(3,5,5,5) = -\frac{2\pi}{15}$
$c(3,3,4,5) = \frac{7\pi}{30}$	$c(3,4,4,6) = 0$	$c(4,4,4,6) = -\frac{\pi}{6}$
$c(3,3,4,6) = \frac{\pi}{6}$	$c(3,4,4,7) = -\frac{\pi}{21}$	$c(4,4,6,6) = -\frac{\pi}{3}$

$(e_1, e_2) = ((13),(14))$ yields the length contradiction $n+1=4$; and $(e_1,e_2) = ((13),(15))$ or $((13),(46))$ implies $n+1=2n$ (see Figure 3(iii)–(v)). Finally if Δ contains three shadow edges e_1, e_2 and e_3 then $(e_1,e_2,e_3) = ((13),(14),(15))$ or $((13),(15),(35))$ yielding a basic labelling contradiction (see Figure 3(vi) and (vii)); or $(e_1,e_2,e_3) = ((13),(14),(46))$.

Similar elementary but somewhat lengthy arguments can be used to prove the following. (Full details can be found at http://arxiv.org/abs/1708.01194.)

LEMMA 3.5. *Let Δ be a region of $\boldsymbol{K}_0$. If $d(\Delta) \leqslant 9$ then $d(\Delta) \in \{4,6,8,9\}$ and Δ is given by Figure* 4(i)–(xi)*.*

For example, it follows from Lemma 3.5 that if $d(\Delta) = 6$ then up to cyclic permutation and inversion Δ is given by Figure 4(xii) and (xiii). In particular, if Δ contains an (a,b)-edge or (x,y)-edge then $d(\Delta) \geqslant 8$.

We will use similar curvature arguments to those used in [7]. Briefly, each corner at a vertex of degree d is given the angle $\frac{2\pi}{d}$ and so the curvature of each vertex is 0. Thus, if Δ is an m-gon of $\boldsymbol{K}_0$ and the degrees of the vertices of Δ are d_i $(1 \leqslant i \leqslant m)$, then the *curvature* of Δ is given by

$$c(\Delta) = c(d_1, \dots, d_m) = (2-m)\pi + 2\pi \sum_{i=1}^{m} \frac{1}{d_i}. \tag{3.2}$$

(Observe that if ρ is any permutation of $\{1,\dots,m\}$ then $c(\Delta) = c(d_{\rho(1)}, \dots, d_{\rho(m)})$. This fact will be used throughout without explicit reference.) A list of some of the $c(d_1,\dots,d_m)$ used in the paper is given in the tables below for the reader's benefit.

TABLE 2. *Degree 6 curvature formulae.*

$c(3,3,3,3,3,4) = -\frac{\pi}{6}$	$c(3,3,3,4,4,4) = -\frac{\pi}{2}$
$c(3,3,3,3,4,4) = -\frac{\pi}{3}$	$c(3,3,3,4,4,5) = -\frac{3\pi}{5}$
$c(3,3,3,3,4,5) = -\frac{13\pi}{30}$	$c(3,3,3,4,4,6) = -\frac{2\pi}{3}$
$c(3,3,3,3,4,6) = -\frac{\pi}{2}$	$c(3,3,4,4,4,4) = -\frac{2\pi}{3}$

4. *Proof of Theorem* 1.2

It was assumed by way of contradiction that there is a reduced spherical picture $\boldsymbol{P}$ over $\mathcal{P}_n$. As described in Section 3, the dual $\boldsymbol{D}$ of $\boldsymbol{P}$ was amended to produce the spherical diagram $\boldsymbol{K}$ and then the subdiagram $\boldsymbol{K}_0$.

Suppose first $\boldsymbol{K}_0 = \boldsymbol{K}$. By Lemma 3.5, $\boldsymbol{K}_0$ has no regions of degree 5 or 7 and, since the curvature of the vertices are 0, the total curvature $c(\boldsymbol{K}_0)$ is given by

$$c(\boldsymbol{K}_0) = \sum_{d(\Delta)=4} c(\Delta) + \sum_{d(\Delta)=6} c(\Delta) + \sum_{d(\Delta)\geqslant 8} c(\Delta).$$

Now suppose $\boldsymbol{K}_0 \neq \boldsymbol{K}$. In this case delete all vertices and edges in $\boldsymbol{K}\backslash\boldsymbol{K}_0$ to produce a spherical diagram $\boldsymbol{K}_1$ consisting of the union of $\boldsymbol{K}_0$ and a single region Δ_0 (which has essentially been obtained by merging all the regions of $\boldsymbol{K}$ not in $\boldsymbol{K}_0$). Note that Lemma 3.5 holds for $\boldsymbol{K}_1$ and so

$$c(\boldsymbol{K}_1) = \sum_{\substack{d(\Delta)=4\\ \Delta\neq\Delta_0}} c(\Delta) + \sum_{\substack{d(\Delta)=6\\ \Delta\neq\Delta_0}} c(\Delta) + \sum_{\substack{d(\Delta)\geqslant 8\\ \Delta\neq\Delta_0}} c(\Delta) + c(\Delta_0).$$

An elementary argument using Euler's formula for the sphere shows $c(\boldsymbol{K}_0) = c(\boldsymbol{K}_1) = 4\pi$ and it is this fact we seek to contradict.

The first step, given in detail in Section 5, is to define a *positive curvature distribution scheme* for regions of degree 4. That is, regions $\Delta \neq \Delta_0$ are located for which $c(\Delta) > 0$, and so $d(\Delta) = 4$, and $c(\Delta)$ is distributed to near regions $\hat{\Delta}$ of Δ. (*Remark*. Throughout the paper Δ or Δ_i will usually be used to denote regions from which positive curvature is initially transferred, and $\hat{\Delta}, \hat{\Delta}_j$ regions that receive, and possibly distribute further, positive curvature.)

For the region $\hat{\Delta}$ define $c^*(\hat{\Delta})$ to equal $c(\hat{\Delta})$ plus all the positive curvature $\hat{\Delta}$ receives minus all the positive curvature $\hat{\Delta}$ distributes as a result of the positive curvature distribution scheme that has been defined.

After completion of the first step, the following is proved in Section 6.

PROPOSITION 4.1. *If* $\boldsymbol{K}_0 = \boldsymbol{K}$ *then* $c(\boldsymbol{K}_0) \leqslant \sum_{d(\hat{\Delta})\geqslant 6} c^*(\hat{\Delta})$; *or if* $\boldsymbol{K}_0 \neq \boldsymbol{K}$ *then* $c(\boldsymbol{K}_1) \leqslant \sum_{\substack{d(\hat{\Delta})\geqslant 6\\ \hat{\Delta}\neq\Delta_0}} c^*(\hat{\Delta}) + c^*(\Delta_0)$.

The second step of the proof, given in detail in Section 7, is to define a positive curvature distribution scheme for regions $\hat{\Delta}$ of degree 6. That is, regions $\hat{\Delta} \neq \Delta_0$ of degree 6 are located for which $c^*(\hat{\Delta}) > 0$ and $c^*(\hat{\Delta})$ is distributed to near regions of $\hat{\Delta}$.

After completion of the second step, the following is proved in Section 8.

PROPOSITION 4.2. *If* $\boldsymbol{K}_0 = \boldsymbol{K}$ *then* $c(\boldsymbol{K}_0) \leqslant \sum_{d(\hat{\Delta})\geqslant 8} c^*(\hat{\Delta})$; *or if* $\boldsymbol{K}_0 \neq \boldsymbol{K}$ *then* $c(\boldsymbol{K}_1) \leqslant \sum_{\substack{d(\hat{\Delta})\geqslant 8\\ \hat{\Delta}\neq\Delta_0}} c^*(\hat{\Delta}) + c^*(\Delta_0)$.

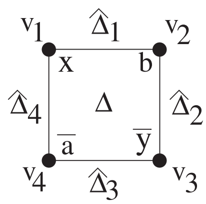

FIGURE 5. $d(\Delta) = 4$.

After completion of the first two steps, for the third and final step the following is proved in Sections 10 and 11.

PROPOSITION 4.3. *If* $d(\hat{\Delta}) \geqslant 8$ *and* $\hat{\Delta} \neq \Delta_0$ *then* $c^*(\hat{\Delta}) \leqslant 0$.

If $\boldsymbol{K}_0 = \boldsymbol{K}$ then Theorem 1.2 follows immediately since, combining the above results, we obtain the contradiction $c(\boldsymbol{K}_0) \leqslant 0$.

If $\boldsymbol{K}_0 \neq \boldsymbol{K}$ then noting that the positive curvature distribution schemes are exactly the same with the proviso that if at any stage positive curvature is transferred to Δ_0 then it remains with Δ_0, it follows that $c(\boldsymbol{K}_1) \leqslant c^*(\Delta_0)$. Let $d(\Delta_0) = k$. It follows by inspection of steps one and two above (in Sections 5–7) that the maximum amount of curvature any region receives across an edge is $\frac{\pi}{3}$. Therefore it can be seen from equation (3.2) that $c^*(\Delta_0) \leqslant (2-k)\pi + k(\frac{2\pi}{3}) + k(\frac{\pi}{3}) = 2\pi$. This final contradiction completes the proof of Theorem 1.2.

5. *Distribution of positive curvature from 4-gons*

In this section we will describe the distribution of positive curvature from regions $\Delta \neq \Delta_0$ of the diagram $\boldsymbol{K}_0$ such that $c(\Delta) > 0$. It follows from Lemma 3.5 that $d(\Delta) = 4$ and Δ is given by Figure 5 with neighbouring regions $\hat{\Delta}_i$ $(1 \leqslant i \leqslant 4)$ and vertices v_i $(1 \leqslant i \leqslant 4)$ which we fix for the remainder of this section. There are fifteen cases to consider according to which vertices of Δ have degree 3. Our approach will be to consider neighbouring regions of Δ, the valency and labels of their vertices and, if necessary, the neighbours of these also.

There will be exactly fourteen exceptions to the distribution of positive curvature rules given for the fifteen cases. These are contained within six exceptional Configurations A–F and will be fully described later in this section.

For the benefit of the reader let us indicate briefly that a general rule for distribution is to try whenever possible to add curvature from Δ to neighbouring regions of degree greater than 4. Given this, we try to keep the number of times $\frac{\pi}{6}$ is exceeded to a minimum; and, given this, to keep the number of times $\frac{\pi}{5}$ is exceeded to a minimum, see, for example, Figure 7(iii). When avoidance of neighbouring regions of degree 4 is not possible we usually introduce distribution paths from Δ to nearby regions of degree greater than 4. For example, in Figure 17(iv) there is a distribution path of length 2 from Δ to $\hat{\Delta}_6$; or in Figure 19(iii) there is a distribution path of length 3 from Δ to $\hat{\Delta}_{10}$. This approach turns out to be sufficient in almost all cases in terms of compensating positive curvature by negatively curved regions. However, a more complicated distribution scheme was required for some exceptional case and these are Configurations A–F mentioned immediately above and treated in detail in what follows.

Notes. (1) In the figures, the upper bound of the amount of curvature transferred will generally be indicated.

(2) It should be emphasised that whenever we identify regions, we do so modulo cyclic permutation and inversion. For example, in what follows we will identify Δ of Figure 20(vi) with Δ_1 of Figure 31(i).

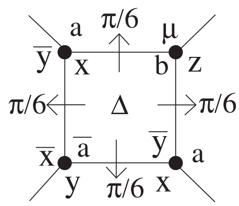

FIGURE 6. $d(v_i) = 3$ $(1 \leqslant i \leqslant 4)$.

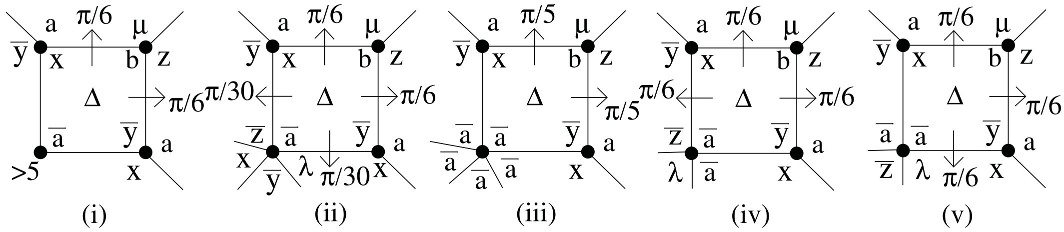

FIGURE 7. $d(v_i) = 3$ $(1 \leqslant i \leqslant 3)$.

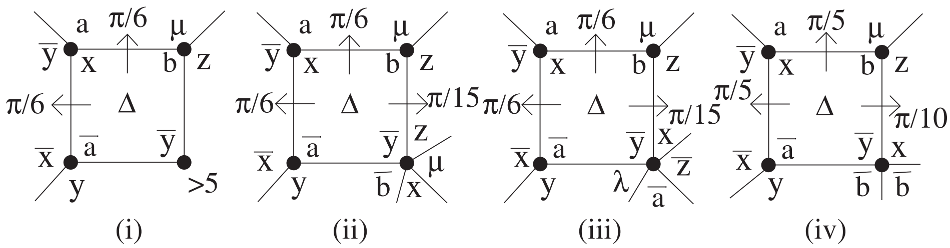

FIGURE 8. $d(v_1) = d(v_2) = d(v_4) = 3$.

(3) Here and in what follows we use Lemma 3.4 to classify possible labelling of vertices.

(4) If $\boldsymbol{K}_0 \neq \boldsymbol{K}$ then it is assumed $\Delta \neq \Delta_0$ and if at any point positive curvature is transferred to Δ_0 then it remains with Δ_0.

A complete description of the distribution of positive curvature from regions of degree 4 is given by Figures 6–32. We give below an explanation for each case.

$\boldsymbol{d(v_i) = 3}$ **$(1 \leqslant i \leqslant 4)$**. *Figure* 6. $c(\Delta) = \frac{2\pi}{3}$ is distributed as shown.

$\boldsymbol{d(v_i) = 3}$ **$(1 \leqslant i \leqslant 3)$**. *Figure* 7. Either $d(v_4) > 5$ and $c(\Delta) \leqslant \frac{\pi}{3}$; or $d(v_4) = 5$ and $c(\Delta) = \frac{2\pi}{5}$; or $d(v_4) = 4$ and $c(\Delta) = \frac{\pi}{2}$ (and distribute $c(\Delta)$ accordingly, as shown).

$\boldsymbol{d(v_1) = d(v_2) = d(v_4) = 3}$. *Figure* 8. Either $d(v_3) > 5$ or $d(v_3) = 5$ or $d(v_3) = 4$.

$\boldsymbol{d(v_1) = d(v_3) = d(v_4) = 3}$. *Figure* 9. Either $d(v_2) > 4$ or $d(v_2) = 4$ and distribution of $c(\Delta)$ is given by Figure 9(i)–(iii). There are two exceptions to these rules. There is an exception to Figure 9(ii) when $\Delta = \Delta_1$ of Configuration F in Figure 32(vi) and which, for convenience, we have reproduced in Figure 9(iv) with a rotation of $\pi/2$ so that Figures 9(ii) and (iv) match up. Thus the exceptional rule (which is again described later for Configuration F) is that in Figure 9(iv) $\frac{\pi}{5}, \frac{2\pi}{15}$ is added from Δ to $\hat{\Delta}_3, \hat{\Delta}_4$ instead of $\frac{\pi}{6}, \frac{\pi}{6}$ (respectively) as in Figure 9(ii), and the dotted lines in Figure 9(iv) indicate the changes made. The second exception is to Figure 9(iii) and is when $\Delta = \Delta_1$ of Configuration E in Figure 32(iv). Again, for convenience, this has been reproduced (after inverting and rotating) in Figure 9(v). Thus the exceptional rule (which is again described later for Configuration E) is that in Figure 9(v) $\frac{2\pi}{15}, \frac{\pi}{5}$ is added

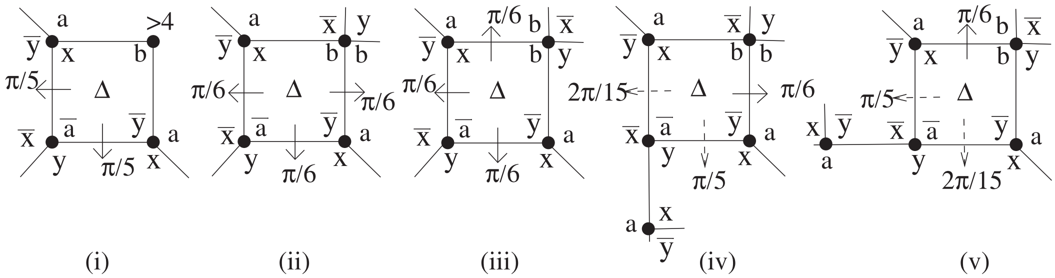

FIGURE 9. $d(v_1) = d(v_3) = d(v_4) = 3$.

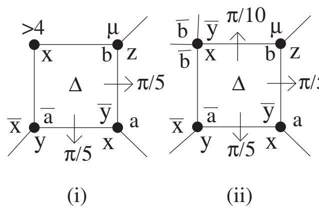

FIGURE 10. $d(v_i) = 3$ $(2 \leqslant i \leqslant 4)$.

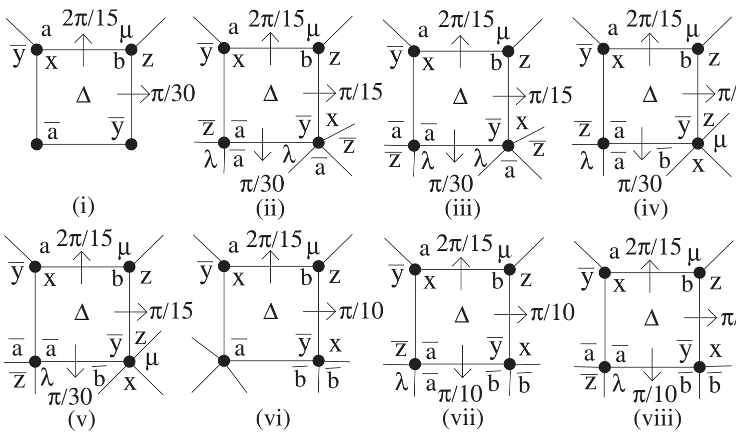

FIGURE 11. $d(v_1) = d(v_2) = 3$.

from Δ to $\hat{\Delta}_3, \hat{\Delta}_4$ instead of $\frac{\pi}{6}, \frac{\pi}{6}$ (respectively) as in Figure 9(iii), and once more the dotted lines in Figure 9(v) indicate the changes made.

$\boldsymbol{d(v_i) = 3}$ $\boldsymbol{(2 \leqslant i \leqslant 4)}$. *Figure* 10. Either $d(v_1) > 4$ or $d(v_1) = 4$.

$\boldsymbol{d(v_1) = d(v_2) = 3}$. *Figure* 11. If $d(v_3) = 4$ and $d(v_4) \geqslant 6$ or $d(v_3) \geqslant 5$ and $d(v_4) \geqslant 5$ or $d(v_3) \geqslant 6$ and $d(v_4) = 4$ then $c(\Delta) \leqslant c(3,3,4,6) = \frac{\pi}{6}$ is distributed as in Figure 11(i); otherwise the distribution is described by Figure 11(ii)–(viii).

$\boldsymbol{d(v_2) = d(v_3) = 3}$. *Figure* 12. If $d(v_1) = 4$ and $d(v_4) \geqslant 6$ or $d(v_1) \geqslant 5$ and $d(v_4) \geqslant 5$ or $d(v_1) \geqslant 6$ and $d(v_4) = 4$ then $c(\Delta) \leqslant \frac{\pi}{6}$ is distributed as in Figure 12(i); otherwise the distribution is described by Figure 12(ii)–(viii).

$\boldsymbol{d(v_3) = d(v_4) = 3}$. *Figure* 13. Distribution of $c(\Delta)$ is given by Figure 13(i). There are five exceptions to this rule. When $\Delta = \Delta_3$ of Configuration A in Figure 31(ii)–(iv) and, for convenience, $\Delta = \Delta_3$ has been reproduced (after inverting and rotating) in Figure 13(ii); when $\Delta = \Delta$ of Configuration C in Figure 32(i) and Δ has been reproduced in Figure 13(iii);

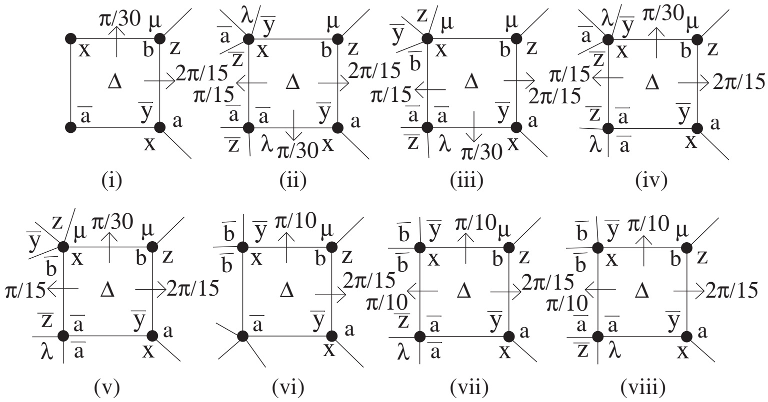

FIGURE 12. $d(v_2) = d(v_3) = 3$.

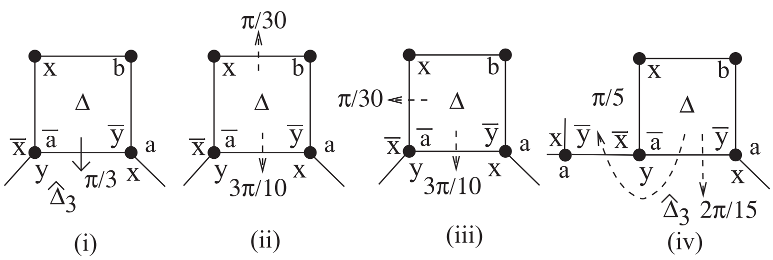

FIGURE 13. $d(v_3) = d(v_4) = 3$.

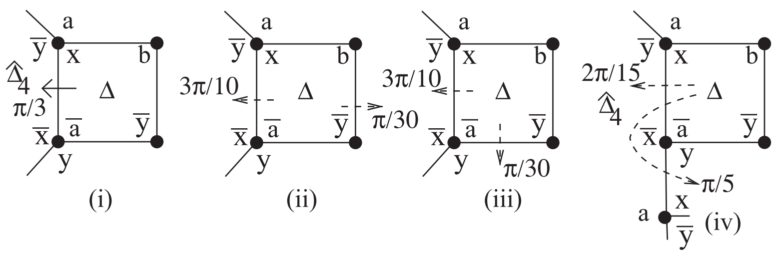

FIGURE 14. $d(v_1) = d(v_4) = 3$.

and when $\Delta = \Delta_1$ of Configuration E in Figure 32(iii) and $\Delta = \Delta_1$ has been reproduced (after inverting and rotating) in Figure 13(iv). Dotted lines in Figure 13(ii)–(iv) indicate the exceptional rules, so, for example, Δ adds $\frac{\pi}{30}, \frac{3\pi}{10}$ to $\hat{\Delta}_1, \hat{\Delta}_3$ (respectively) in Figure 13(ii) instead of simply adding $\frac{\pi}{3}$ to $\hat{\Delta}_3$ as in Figure 13(i).

$\boldsymbol{d(v_4) = d(v_1) = 3}$. *Figure* 14. Distribution of $c(\Delta)$ is given by Figure 14(i). There are five exceptions to this rule. When $\Delta = \Delta_3$ of Configuration B in Figure 31(vi)–(viii) and, for convenience, $\Delta = \Delta_3$ has been reproduced (after inverting and rotating) in Figure 14(ii); when $\Delta = \Delta$ of Configuration D and Δ has been reproduced (after rotating) in Figure 14(iii); and when $\Delta = \Delta_1$ of Configuration F in Figure 32(v) and $\Delta = \Delta_1$ has been reproduced (after rotating) in Figure 14(iv). Dotted lines in Figure 14(ii)–(iv) indicate the exceptional rules. so, for example, Δ adds $\frac{\pi}{30}, \frac{3\pi}{10}$ to $\hat{\Delta}_2, \hat{\Delta}_4$ (respectively) in Figure 14(ii) instead of simply adding $\frac{\pi}{3}$ to $\hat{\Delta}_4$ as in Figure 13(i).

$\boldsymbol{d(v_2) = d(v_4) = 4}$. *Figures* 15–19. There are four subcases.

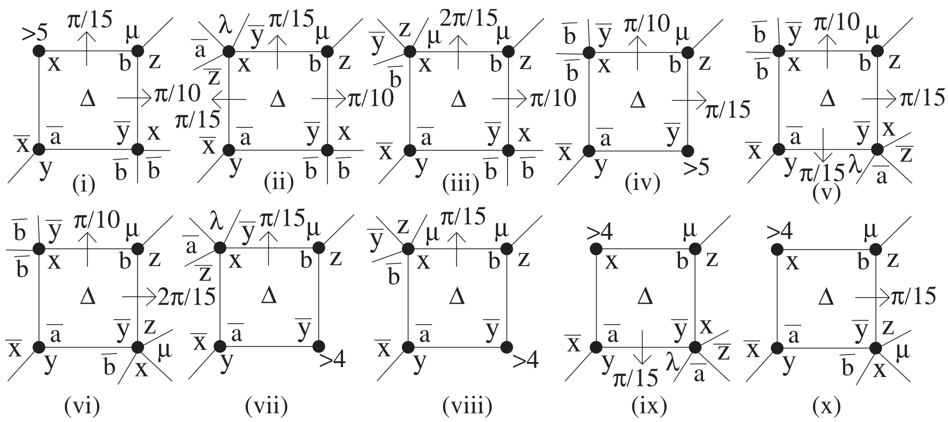

FIGURE 15. $d(v_2) = d(v_4) = 3(subcase1)$.

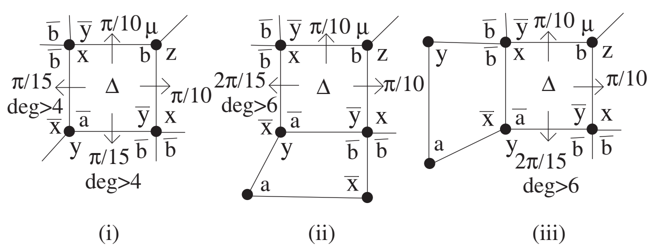

FIGURE 16. $d(v_2) = d(v_4) = 3(subcase2)$.

(1)$\boldsymbol{(d(v_1), d(v_3)) \neq (4,4)}$. *Figure* 15. Either $(d(v_1), d(v_3)) \in \{(>5, 4), (5, 4), (4, >5), (4, 5)\}$ and distribution of $c(\Delta)$ is given by Figure 15(i)–(vi) or $d(v_1) \geqslant 5$ and $d(v_3) \geqslant 5$, $c(\Delta) \leqslant \frac{2\pi}{15}$ and distribution is given by Figure 15(vii)–(x).

Suppose now $d(v_1) = d(v_3) = 4$.

(2)$\boldsymbol{(d(\hat{\Delta}_3), d(\hat{\Delta}_4)) \notin \{(4,6),(4,4),(6,4)\}}$. *Figure* 16. $c(\Delta)$ is distributed as shown.

(3)$\boldsymbol{(d(\hat{\Delta}_3), d(\hat{\Delta}_4)) \in \{(4,6),(6,4)\}}$. *Figure* 17. In each case add $\frac{\pi}{10}$ from $c(\Delta)$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$; and add $\frac{\pi}{15}$ from $c(\Delta)$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$. Let $d(\hat{\Delta}_3) = 4$ and $d(\hat{\Delta}_4) = 6$. This is shown in Figure 17(i) in which $d(u_1) \geqslant 3$ and $d(u_2) \geqslant 4$. It remains to describe the further transfer (if any) of positive curvature from $c(\hat{\Delta}_3)$.

If $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{15}$ then the $\frac{\pi}{15}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_3)$ as in Figure 17(i); and if $-\frac{\pi}{15} < c(\hat{\Delta}_3) \leqslant 0$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) \leqslant \frac{\pi}{15}$ is added to $c(\hat{\Delta}_4)$ as in Figure 17(ii). Assume $c(\hat{\Delta}_3) > 0$. We now proceed according to the values of $d(u_1)$ and $d(u_2)$. If $d(u_1) = 4$ and $d(u_2) = 5$ then $(c(\hat{\Delta}_3) = c(3,4,4,5) = \frac{\pi}{15}$ and) $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ so add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_4)$ and $c(\hat{\Delta}_6)$ as in Figure 17(iii); if $d(u_1) = 4 = d(u_2)$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{7\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ and $\frac{\pi}{6}$ to $c(\hat{\Delta}_6)$ as in (iv); if $d(u_1) = 5$ and $d(u_2) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_4)$ as in (v); if $d(u_1) = 3$ and $d(u_2) \geqslant 6$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) \leqslant \frac{7\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ and $\frac{\pi}{6}$ to $c(\hat{\Delta}_5)$ as in (vi); if $d(u_1) = 3$ and $d(u_2) = 5$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{9\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ and $c(\hat{\Delta}_6)$, and add $\frac{\pi}{6}$ to $c(\hat{\Delta}_5)$ as in (vii); and if $d(u_1) = 3$ and $d(u_2) = 4$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{6\pi}{15}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_4)$ and $\frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ as in (viii).

Now let $d(\hat{\Delta}_3) = 6$ and $d(\hat{\Delta}_4) = 4$. This is shown in Figure 17(ix) where $d(u_3) \geqslant 3$ and $d(u_2) \geqslant 4$. It remains to describe the further transfer (if any) of positive curvature from $c(\hat{\Delta}_4)$. If $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{15}$ then the $\frac{\pi}{15}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_4)$ as in Figure 17(ix); and if $-\frac{\pi}{15} <$

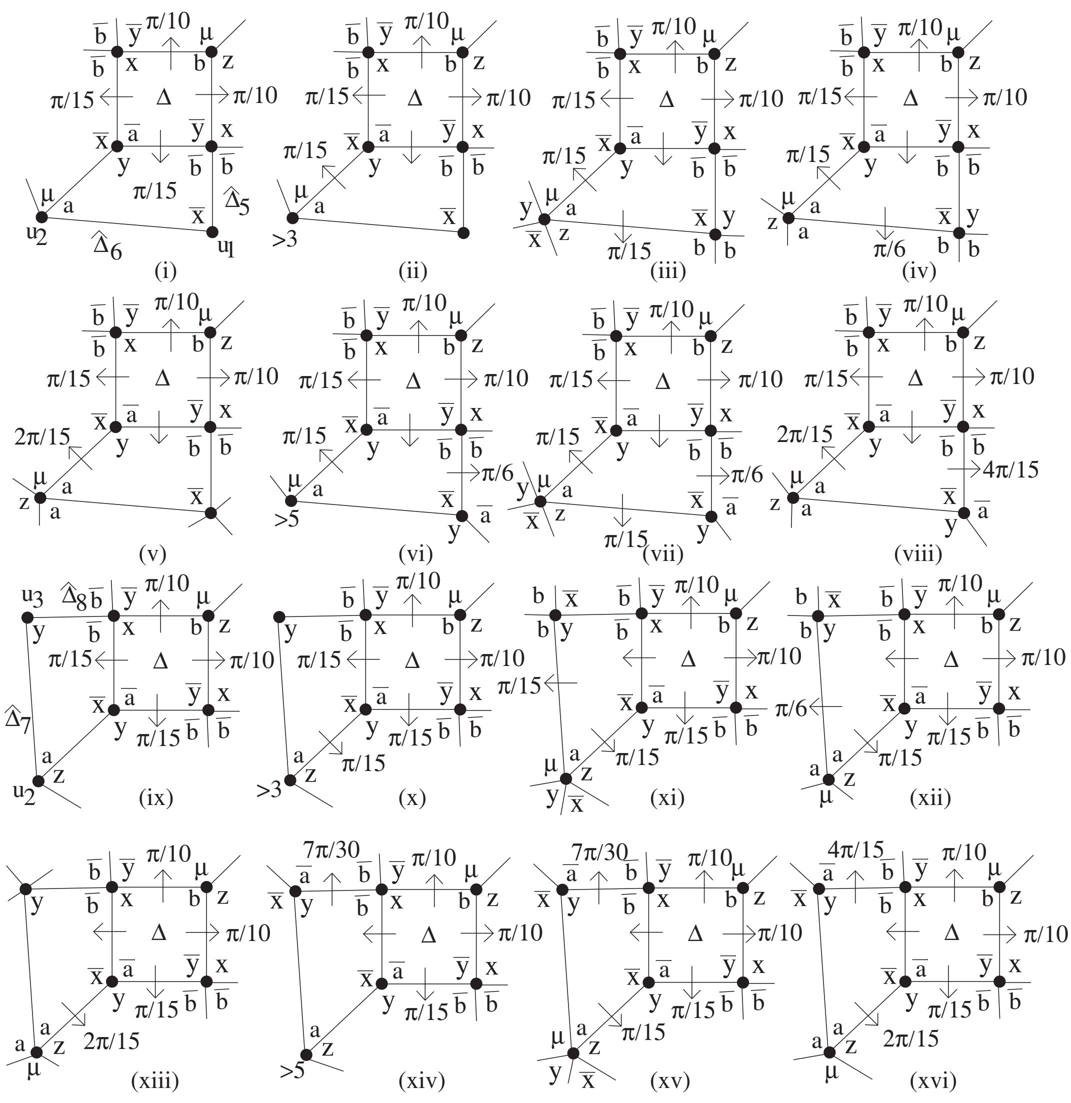

FIGURE 17. $d(v_2) = d(v_4) = 3(subcase3)$.

$c(\hat{\Delta}_4) \leqslant 0$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) \leqslant \frac{\pi}{15}$ is added to $c(\hat{\Delta}_3)$ as in Figure 17(x). Assume $c(\hat{\Delta}_4) > 0$. We proceed according to the values of $d(u_2)$ and $d(u_3)$. If $d(u_3) = 4$ and $d(u_2) = 5$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ so add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_7)$ as in Figure 17(xi); if $d(u_3) = 4 = d(u_2)$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{7\pi}{30}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ and $\frac{\pi}{6}$ to $c(\hat{\Delta}_7)$ as in (xii); if $d(u_3) = 5$ and $d(u_2) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_3)$ as in (xiii); if $d(u_3) = 3$ and $d(u_2) \geqslant 6$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leqslant \frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (xiv); if $d(u_3) = 3$ and $d(u_2) = 5$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{3\pi}{10}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ and $\frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (xv); and if $d(u_3) = 3$ and $d(u_2) = 4$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{6\pi}{15}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_3)$ and $\frac{4\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (xvi).

(4)$\boldsymbol{d(\hat{\Delta}_3) = d(\hat{\Delta}_4) = 4}$. *Figures 18 and 19*. This subcase is shown in Figure 18(i) in which $d(u_1) \geqslant 3$, $d(u_2) \geqslant 4$ and $d(u_3) \geqslant 3$ and $\frac{\pi}{10}$ is distributed from $c(\Delta)$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$ with $\frac{2\pi}{15}$ remaining with $c(\Delta)$. If $d(u_1) = d(u_3) = 4$, $d(u_2) \geqslant 6$ and $d(\hat{\Delta}_6) > 4$ then distribute this remaining $\frac{2\pi}{15}$ from $c(\Delta)$ to $c(\hat{\Delta}_6)$ as shown in Figure 18(ii), noting that $c(\hat{\Delta}_3) \leqslant 0$; or

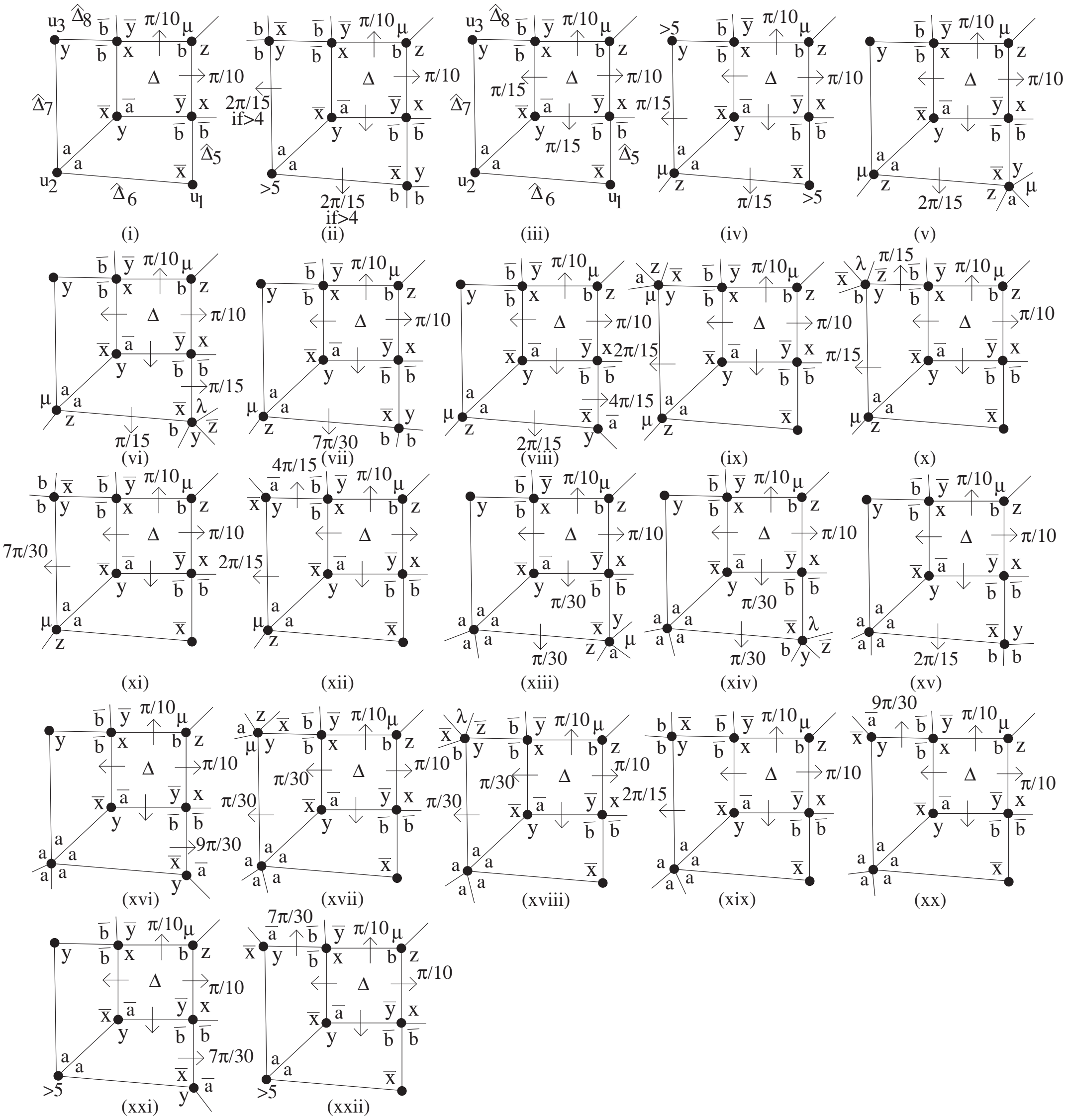

FIGURE 18. $d(v_2) = d(v_4) = 3(subcase 4)$.

if $d(u_1) = d(u_3) = 4$, $d(u_2) \geqslant 6$ and $d(\hat{\Delta}_7) > 4$ then distribute the $\frac{2\pi}{15}$ from $c(\Delta)$ to $c(\hat{\Delta}_7)$ as shown in Figure 18(ii), noting that $c(\hat{\Delta}_4) \leqslant 0$. Assume from now on that neither of these sets of conditions occur. Then $\frac{\pi}{15}$ is distributed from $c(\Delta)$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ as shown in Figure 18(iii). If $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{15}$ then the $\frac{\pi}{15}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_3)$ and similarly for $c(\hat{\Delta}_4)$, so assume from now on that $c(\hat{\Delta}_3) > -\frac{\pi}{15}$ and $c(\hat{\Delta}_4) > -\frac{\pi}{15}$. It remains to describe further transfer of positive curvature from $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$ (and possibly $c(\hat{\Delta}_6)$ when $d(\hat{\Delta}_6) = 4$ and $c(\hat{\Delta}_7)$ when $d(\hat{\Delta}_7) = 4$).

Let $d(u_2) = 4$. If $d(u_1) \geqslant 6$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_6)$ as in Figure 18(iv); if $d(u_1) = 5$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ if $l(u_1)$ is given by (v), or add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_5)$ and $c(\hat{\Delta}_6)$ if $l(u_1)$ is given by (vi); if $d(u_1) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_6)$ as in (vii); if $d(u_1) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{6\pi}{15}$ so add $\frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (viii); if $d(u_3) \geqslant 6$

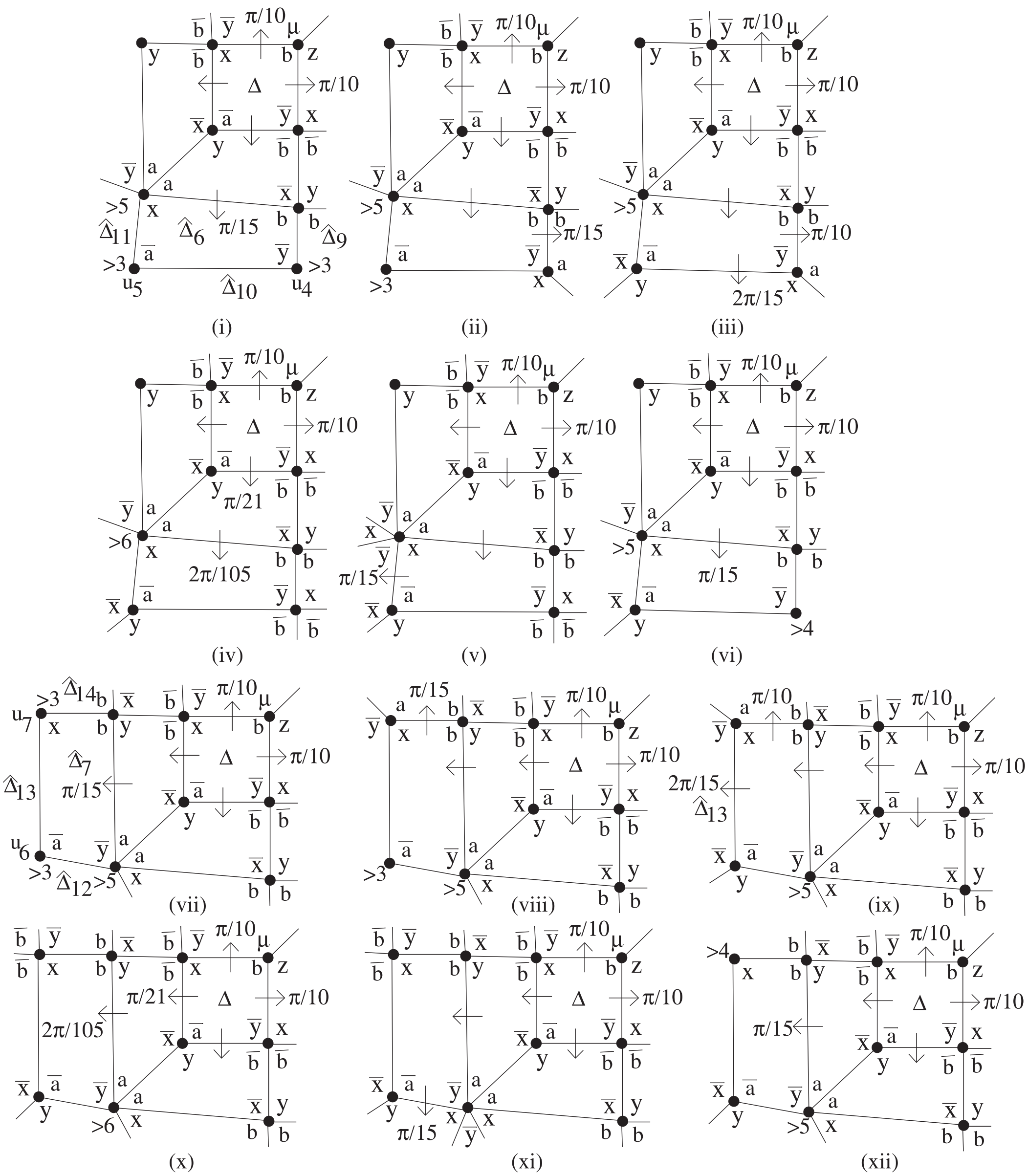

FIGURE 19. $d(v_2) = d(v_4) = 3(subcase 4)$.

then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ as in (iv); if $d(u_3) = 5$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_7)$ if $l(u_3)$ is given by (ix), or add $\frac{\pi}{15}$ to each of $c(\hat{\Delta}_7)$ and $c(\hat{\Delta}_8)$ if $l(u_3)$ is given by (x); if $d(u_3) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_7)$ as in (xi); and if $d(u_3) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{6\pi}{15}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_7)$ and $\frac{4\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (xii).

Let $d(u_2) = 5$ and so $l(u_2) = a^5$. If $d(u_1) = 5$ then add $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_3) = c(3,4,5,5) = -\frac{\pi}{30}$ and $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_6)$ as in Figure 18(xiii) and (xiv); if $d(u_1) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (xv); if $d(u_1) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) = \frac{9\pi}{30}$ to $c(\hat{\Delta}_5)$ as in (xvi); if $d(u_3) = 5$ then add $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_4)$ and $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_7)$ as in (xvii) and

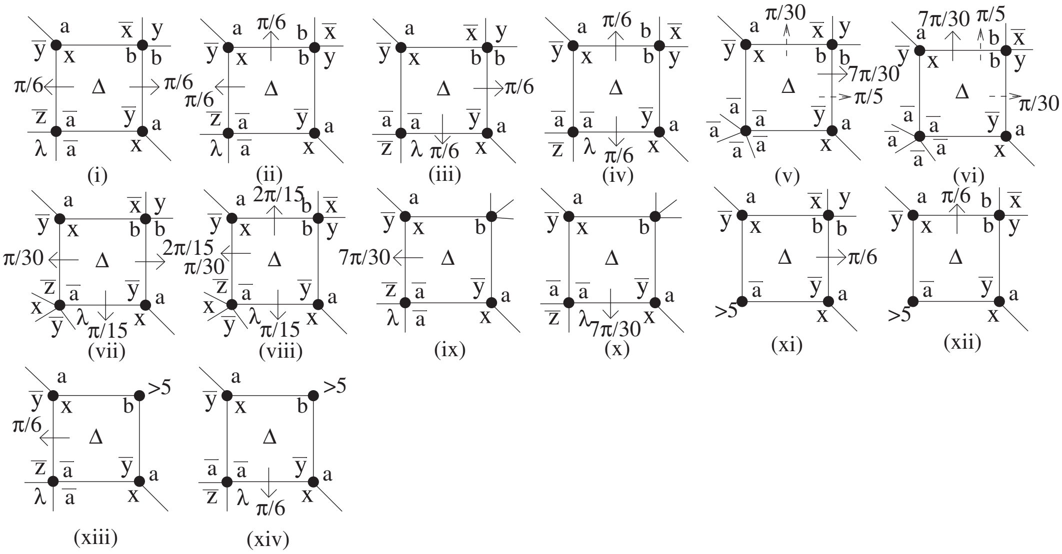

FIGURE 20. $d(v_1) = d(v_3) = 3(subcase1)$.

(xviii); if $d(u_3) = 4$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_7)$ as in (xix); and if $d(u_3) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) = \frac{3\pi}{10}$ to $c(\hat{\Delta}_8)$ as in (xx).

Let $d(u_2) \geqslant 6$ so that by assumption $3 \leqslant d(u_1),\ d(u_3) \leqslant 4$ (since $c(3,4,5,6) = -\frac{\pi}{10}$). If $d(u_1) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leqslant \frac{7\pi}{30}$ to $c(\hat{\Delta}_5)$ as in Figure 18(xxi); and if $d(u_3) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leqslant \frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (xxii).

This leaves $d(u_1) = d(u_3) = d(\hat{\Delta}_6) = d(\hat{\Delta}_7) = 4$. First consider $\hat{\Delta}_6$ as shown in Figure 19(i). If $d(u_4) > 3$ and $d(u_5) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_6) \leqslant c(4,4,4,6) = -\frac{\pi}{6}$ as in 19(i); if $d(u_4) = 3$ and $d(u_5) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_9)$ as in (ii); if $d(u_4) = 3 = d(u_5)$ then $\frac{\pi}{15} + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) \leqslant \frac{7\pi}{30}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_{10})$ as in (iii); if $d(u_4) = 4$, $d(u_5) = 3$ and $d(u_2) > 6$ then $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{21}$ so add $\frac{\pi}{21}$ from $c(\Delta)$ to $c(\hat{\Delta}_3)$ and the remaining $\frac{\pi}{15} - \frac{\pi}{21} = \frac{2\pi}{105}$ to $c(\hat{\Delta}_6) \leqslant -\frac{\pi}{21}$ as in (iv); if $d(u_4) = 4$, $d(u_5) = 3$ and $d(u_2) = 6$ then (checking the star graph Γ_0 for possible labels shows that) u_2 is given by (v) in which case add $\frac{\pi}{15} + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) = \frac{\pi}{15}$ to $c(\hat{\Delta}_{11})$ as in (v); and if $d(u_4) > 4$ and $d(u_5) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_3) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_6) \leqslant -\frac{\pi}{10}$ as in (vi).

Now consider $\hat{\Delta}_7$ as in Figure 19(vii). If $d(u_6) > 3$ and $d(u_7) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_7) \leqslant -\frac{\pi}{6}$ as in 19(vii); if $d(u_7) = 3$ and $d(u_6) > 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_6) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_{14})$ as in (viii); if $d(u_7) = 3 = d(u_6)$ then $\frac{\pi}{15} + c(\hat{\Delta}_4) + c(\hat{\Delta}_7) = \frac{7\pi}{30}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_{13})$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{14})$ as in (ix); if $d(u_7) = 4$, $d(u_6) = 3$ and $d(u_2) > 6$ then $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{21}$ so add $\frac{\pi}{21}$ from $c(\Delta)$ to $c(\hat{\Delta}_4)$ and the remaining $\frac{\pi}{15} - \frac{\pi}{21} = \frac{2\pi}{105}$ to $c(\hat{\Delta}_7) \leqslant -\frac{\pi}{21}$ as in (x); if $d(u_7) = 4$, $d(u_6) = 3$ and $d(u_2) = 6$ then u_2 is given by (xi) in which case add $\frac{\pi}{15} + c(\hat{\Delta}_4) + c(\hat{\Delta}_7) = \frac{\pi}{15}$ to $c(\hat{\Delta}_{12})$ as in (xi); and if $d(u_7) > 4$ and $d(u_6) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_4) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_7) \leqslant -\frac{\pi}{10}$ as in (xii).

$\boldsymbol{d(v_1) = d(v_3) = 3}$. *Figures* 20–23. There are three subcases.

(1) $\boldsymbol{((d(v_2), d(v_4)) \notin \{(\geqslant 6, 5), (5, \geqslant 5)\}}$. *Figure* 20. Distribution of $c(\Delta)$ is described in Figure 20 according to possible $d(v_2)$ and $d(v_4)$. There are two exceptions to these rules. When Δ is given by Figure 20(v), $\frac{7\pi}{30}$ is added to $\hat{\Delta}_2$ as indicated, except when $\Delta = \Delta_1$ of Configuration B in Figure 31(v,) in which case the dotted lines in Figure 20(v) indicate the new rule, that is, $\frac{\pi}{30}, \frac{\pi}{5}$ is added to $\hat{\Delta}_1, \hat{\Delta}_2$, respectively; and when Δ is given by Figure 20(v),

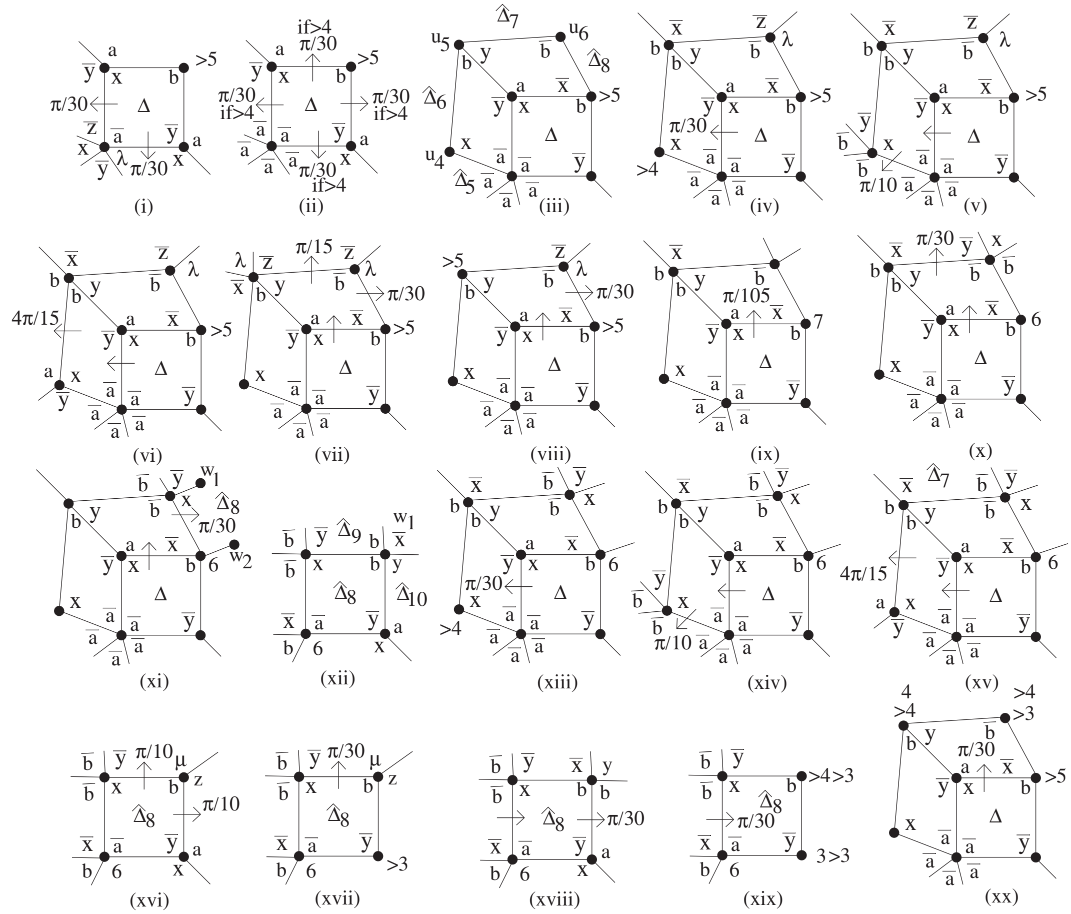

FIGURE 21. $d(v_1) = d(v_3) = 3(subcase2)$.

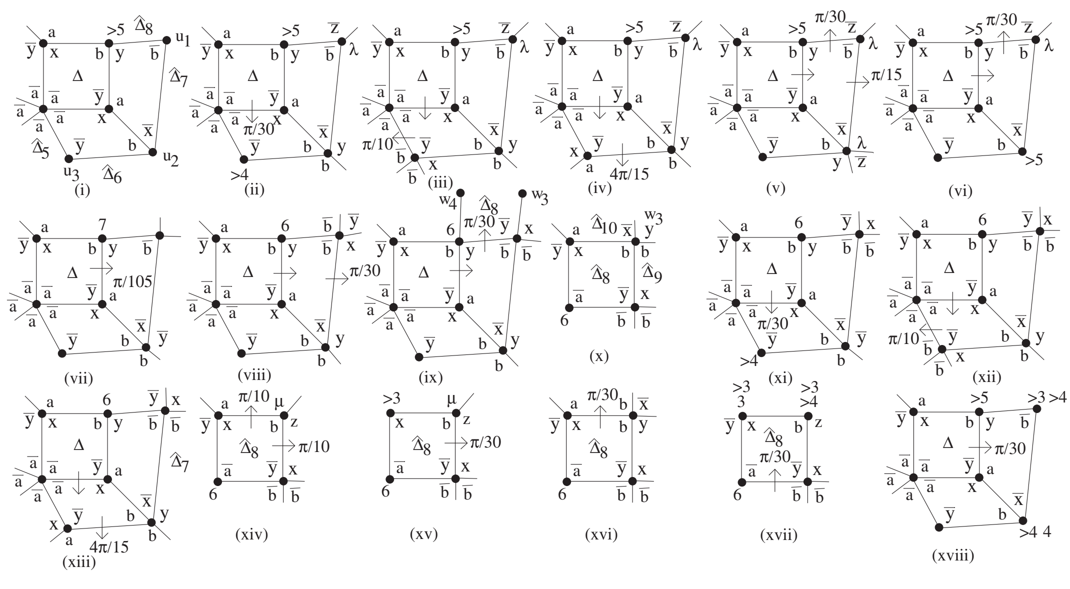

FIGURE 22. $d(v_1) = d(v_3) = 3(subcase2)$.

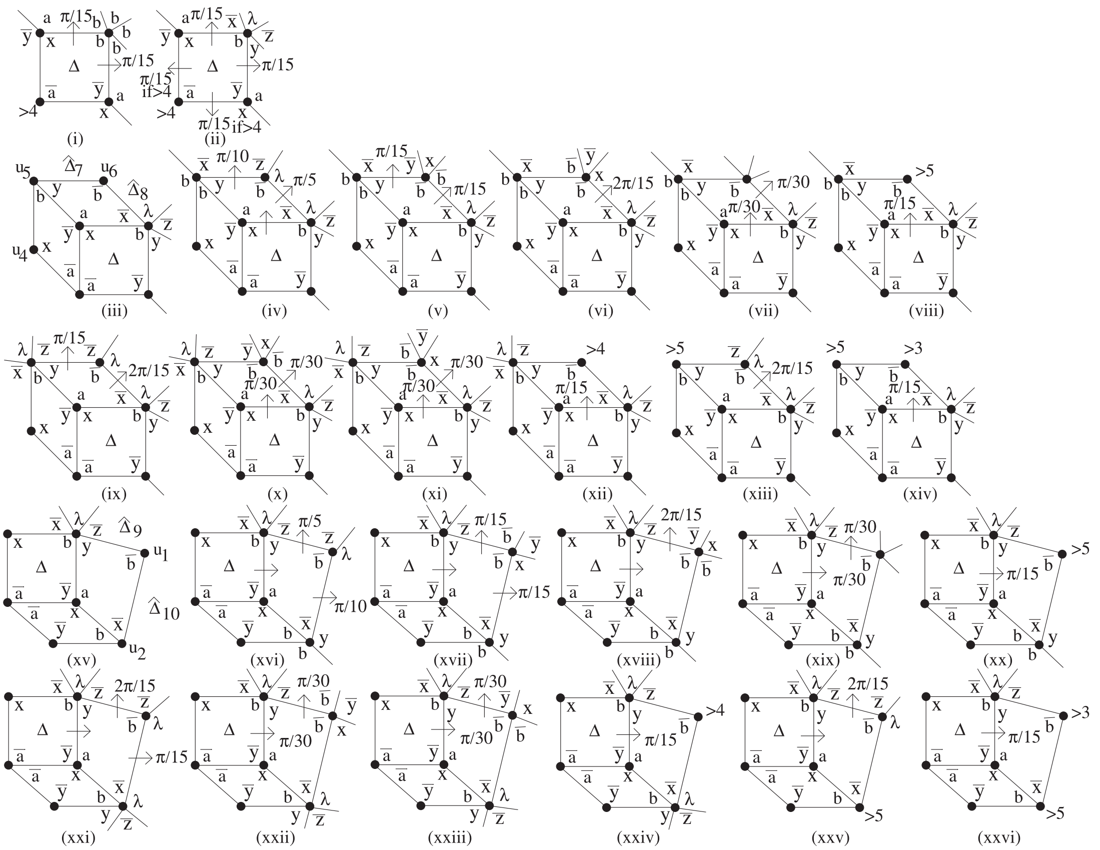

FIGURE 23. $d(v_1) = d(v_3) = 3(subcase3)$.

$\frac{7\pi}{30}$ is added to $\hat{\Delta}_1$ as indicated, except when $\Delta = \Delta_1$ of Configuration A in Figure 31(i), in which case the dotted lines in Figure 20(vi) indicate the new rule, that is, $\frac{\pi}{5}, \frac{\pi}{30}$ is added to $\hat{\Delta}_1, \hat{\Delta}_2$ respectively.

(2) $\boldsymbol{d(v_2) \geqslant 6}$ *and* $\boldsymbol{d(v_4) = 5}$. *Figures* 21 *and* 22. If Δ is given by Figure 21(i) then $c(\Delta)$ is distributed as shown. Otherwise $l(v_4) = a^5$ and this subcase is now considered using Figures 21(ii)–(xx) and 22.

Let $d(v_2) \geqslant 6$ and $l(v_4) = a^5$. Then $c(\Delta) \leqslant \frac{\pi}{15}$, half of which ($\leqslant \frac{\pi}{30}$) is distributed to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ whilst the other half is distributed to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ (and this will be described in the next paragraph). The distribution of $\frac{1}{2}c(\Delta)$ to $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_4)$ is as follows. If $d(\hat{\Delta}_1) > 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_1)$ as in Figure 21(ii), or if $d(\hat{\Delta}_1) = 4$ and $d(\hat{\Delta}_4) > 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ again as in (ii). It can be assumed therefore that Δ, $\hat{\Delta}_1$ and $\hat{\Delta}_j$ ($4 \leqslant j \leqslant 8$) are given by Figure 21(iii). We proceed according to $d(u_4) \geqslant 3$, $d(u_5) \geqslant 4$, $d(u_6) \geqslant 3$ of Figure 21(iii). If $d(u_6) = 3$, $d(u_5) = 4$ and $d(u_4) \geqslant 5$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{30}$ as in Figure 21(iv); if $d(u_6) = 3$, $d(u_5) = 4$ and $d(u_4) = 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leqslant \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (v); if $d(u_6) = 3$, $d(u_5) = 4$ and $d(u_4) = 3$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leqslant \frac{4\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (vi); if $d(u_6) = 3$ and $d(u_5) = 5$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_1) \leqslant \frac{\pi}{15}$ and then add $\frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ and add $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vii); if $d(u_6) = 3$ and $d(u_5) \geqslant 6$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_1) \leqslant 0$ and then add $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (viii). This completes $d(u_6) = 3$. If $d(u_6) = 4$, $d(u_5) = 4$ and $d(v_2) = 7$ (note that $c(3, 3, 5, 8) < 0$) then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{105}$ to $c(\hat{\Delta}_1) \leqslant -\frac{\pi}{21}$ as in (ix). Let $d(u_6) = 4$, $d(u_5) = 4$ and $d(v_2) = 6$ so, in particular, $c(\hat{\Delta}_1) = 0$. If u_6 is given by Figure 21(x) then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_7)$, so from now on suppose that u_6 is given by Figure 21(xi). If $d(\hat{\Delta}_8) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as shown in Figure 21(xi), so suppose from now on $d(\hat{\Delta}_8) = 4$. *Suppose that* $\hat{\Delta}_8$ *is given by Figure* 21(xii). If $d(u_4) \geqslant 5$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{30}$ as in Figure 21(xiii); if $d(u_4) = 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leqslant \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (xiv); and if $d(u_4) = 3$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_4)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_4) \leqslant \frac{4\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (xv). *Suppose now that* $\hat{\Delta}_8$ *is not given by Figure* 21(xii). Then again add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as in Figure 21(xi). We proceed according to the degrees of the vertices w_1 and w_2 of Figure 21(xi). If $d(w_1) = d(w_2) = 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{5}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{10})$ as shown in Figure 21(xvi); if $d(w_1) = 3$ and $d(w_2) > 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30}$ so add $\frac{\pi}{30}$ to $c(\hat{\Delta}_9)$ as shown in (xvii); if $d(w_1) = 4$ and $d(w_2) = 3$ then by assumption $\hat{\Delta}_8$ is given by (xviii) and $c(\hat{\Delta}_8) = 0$ so add $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30}$ to $c(\hat{\Delta}_{10})$ as shown; and if either $d(w_1) \geqslant 5$ and $d(w_2) = 3$ or $d(w_1) \geqslant 4$ and $d(w_2) \geqslant 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_8) \leqslant -\frac{\pi}{10}$ as shown in (xix). This completes $d(u_6) = d(u_5) = 4$. Finally if $d(u_6) \geqslant 4$ and $d(u_5) \geqslant 5$ or $d(u_6) \geqslant 5$ and $d(u_5) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_1) \leqslant c(3,4,5,6) = -\frac{\pi}{10}$ as shown in Figure 21(xx).

The remaining $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ is distributed to $c(\hat{\Delta}_2)$ and $c(\hat{\Delta}_3)$ as follows. If $d(\hat{\Delta}_2) > 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_2)$ as in Figure 21(ii), or if $d(\hat{\Delta}_2) = 4$ and $d(\hat{\Delta}_3) > 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ again as in (ii). It can be assumed therefore that Δ, $\hat{\Delta}_2$, $\hat{\Delta}_3$ and $\hat{\Delta}_j$ $(5 \leqslant j \leqslant 8)$ are now given by Figure 22(i). We proceed according to $d(u_1) \geqslant 3$, $d(u_2) \geqslant 4$, $d(u_3) \geqslant 3$ of Figure 22(i). If $d(u_1) = 3$, $d(u_2) = 4$ and $d(u_3) \geqslant 5$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{30}$ as in Figure 22(ii); if $d(u_1) = 3$, $d(u_2) = 4$ and $d(u_3) = 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leqslant \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (iii); if $d(u_1) = 3$, $d(u_2) = 4$ and $d(u_3) = 3$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leqslant \frac{4\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (iv); if $d(u_1) = 3$ and $d(u_2) = 5$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_2) \leqslant \frac{\pi}{15}$ and add $\frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ and $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (v); if $d(u_1) = 3$ and $d(u_2) \geqslant 6$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_2) \leqslant 0$ and then add $\frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vi). This completes $d(u_1) = 3$. If $d(u_1) = 4$, $d(u_2) = 4$ and $d(v_2) = 7$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{105}$ to $c(\hat{\Delta}_2) \leqslant -\frac{\pi}{21}$ as in (vii). Let $d(u_1) = 4$, $d(u_2) = 4$ and $d(v_2) = 6$ so, in particular, $c(\hat{\Delta}_2) = 0$. If u_2 is given by Figure 22(viii) then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_7)$, so from now on suppose that u_2 is given by Figure 22(ix). If $d(\hat{\Delta}_8) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as shown in Figure 22(ix), so suppose from now on $d(\hat{\Delta}_8) = 4$. *Suppose that* $\hat{\Delta}_8$ *is given by Figure* 22(x). If $d(u_3) \geqslant 5$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{30}$ as in Figure 22(xi); if $d(u_3) = 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leqslant \frac{\pi}{10}$ to $c(\hat{\Delta}_5)$ as in (xii); and if $d(u_3) = 3$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_3)$ and then add $\frac{\pi}{30} + c(\hat{\Delta}_3) \leqslant \frac{4\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (xiii). *Suppose now that* $\hat{\Delta}_8$ *is not given by Figure* 22(x). Then again add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $\hat{\Delta}_8$ as in Figure 22(ix). We proceed according to the degrees of the vertices w_3 and w_4 of Figure 22(ix). If $d(w_3) = d(w_4) = 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30} + \frac{\pi}{6} = \frac{\pi}{5}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{10})$ as shown in Figure 22(xiv); if $d(w_3) = 3$ and $d(w_4) > 3$ then $\frac{1}{2}c(\Delta) + c(\hat{\Delta}_8) = \frac{\pi}{30}$ so add $\frac{\pi}{30}$ to $c(\hat{\Delta}_9)$ as shown in (xv); if $d(w_3) = 4$ and $d(w_4) = 3$ then by assumption $\hat{\Delta}_8$ is given by (xvi) and $c(\hat{\Delta}_8) = 0$ so add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_{10})$ as shown; and if either $d(w_3) \geqslant 5$ and $d(w_4) = 3$ or $d(w_3) \geqslant 4$ and $d(w_4) \geqslant 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_8) \leqslant -\frac{\pi}{10}$ as shown in (xvii). This completes $d(u_1) = d(u_2) = 4$. Finally if $d(u_1) \geqslant 4$ and $d(u_2) \geqslant 5$ or $d(u_1) \geqslant 5$ and $d(u_2) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to $c(\hat{\Delta}_2) \leqslant c(3,4,5,6) = -\frac{\pi}{10}$ as shown in Figure 22(xviii).

(3) $\boldsymbol{d(v_2) = 5}$ *and* $\boldsymbol{d(v_4) \geqslant 5}$. *Figure* 23. If Δ is given by Figure 23(i) then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to each of $c(\hat{\Delta}_1)$ and $c(\hat{\Delta}_2)$. Otherwise $l(v_2) = bx^{-1}\lambda z^{-1}y$ and Δ is given by Figure 23(ii).

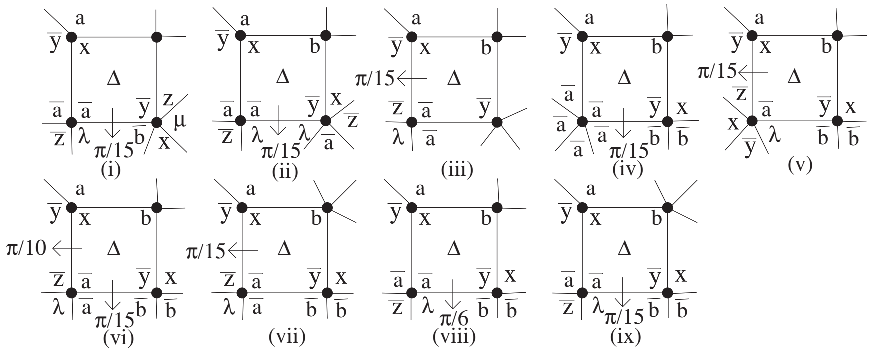

FIGURE 24. $d(v_1) = 3$.

Here add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ if $d(\hat{\Delta}_4) > 4$, otherwise add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_1)$; and add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ if $d(\hat{\Delta}_3) > 4$, otherwise add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_2)$. If $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ is added to $c(\hat{\Delta}_1)$ and $d(\hat{\Delta}_1) > 4$ there is no further distribution of curvature from $\hat{\Delta}_1$ and the same statement holds for $\hat{\Delta}_2$. This leaves the subcases $d(\hat{\Delta}_1) = d(\hat{\Delta}_4) = 4$ and $d(\hat{\Delta}_2) = d(\hat{\Delta}_3) = 4$.

Assume first $d(\hat{\Delta}_1) = d(\hat{\Delta}_4) = 4$ in Figure 23(ii). Then Δ is given by Figure 23(iii). We proceed according to $d(u_5) \geqslant 4$ and $d(u_6) \geqslant 3$. If $d(u_5) = 4$ and $d(u_6) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_1) \leqslant \frac{3\pi}{10}$ so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_7)$ and $\frac{\pi}{5}$ to $c(\hat{\Delta}_8)$ as in Figure 23(iv); if $d(u_5) = 4$ and $d(u_6) = 4$ then add $\frac{1}{2}(\frac{\pi}{15} + c(\hat{\Delta}_1)) \leqslant \frac{\pi}{15}$ to each of $c(\hat{\Delta}_7)$ and $c(\hat{\Delta}_8)$ if u_6 is given by (v), or add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leqslant \frac{2\pi}{15}$ to $c(\hat{\Delta}_8)$ if u_6 is given by (vi); if $d(u_5) = 4$ and $d(u_6) = 5$ then $c(\hat{\Delta}_1) = -\frac{\pi}{30}$ so add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as in (vii); if $d(u_5) = 4$ and $d(u_6) \geqslant 6$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_1) \leqslant -\frac{\pi}{10}$ as in (viii); if $d(u_5) = 5$ and $d(u_6) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_1) \leqslant \frac{\pi}{5}$ so add $\frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (ix); if $d(u_5) = 5$ and $d(u_6) = 4$ then $c(\hat{\Delta}_1) \leqslant -\frac{\pi}{30}$ so add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leqslant \frac{\pi}{30}$ to $c(\hat{\Delta}_8)$ as shown in the two possibilities for u_6, namely (x) and (xi); if $d(u_5) = 5$ and $d(u_6) \geqslant 5$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_1) \leqslant -\frac{2\pi}{15}$ as in (xii); if $d(u_5) > 5$ and $d(u_6) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_1) \leqslant \frac{2\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (xiii); and if $d(u_5) > 5$ and $d(u_6) > 3$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_1) \leqslant -\frac{\pi}{10}$ as in (xiv).

Now assume $d(\hat{\Delta}_2) = d(\hat{\Delta}_3) = 4$ in Figure 23(ii). Then Δ is given by Figure 23(xv). We proceed according to $d(u_1) \geqslant 3$ and $d(u_2) \geqslant 4$. If $d(u_2) = 4$ and $d(u_1) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_2) \leqslant \frac{3\pi}{10}$ so add $\frac{\pi}{5}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{10}$ to $c(\hat{\Delta}_{10})$ as in Figure 23(xvi); if $d(u_2) = 4$ and $d(u_1) = 4$ then add $\frac{1}{2}(\frac{\pi}{15} + c(\hat{\Delta})) \leqslant \frac{\pi}{15}$ to each of $c(\hat{\Delta}_9)$ and $c(\hat{\Delta}_{10})$ if u_1 is given by (xvii), or $\frac{\pi}{15} + c(\hat{\Delta}_2) \leqslant \frac{2\pi}{15}$ to $c(\hat{\Delta}_9)$ if u_1 is given by (xviii); if $d(u_2) = 4$ and $d(u_1) = 5$ then $c(\hat{\Delta}_2) = -\frac{\pi}{30}$ so $\frac{\pi}{15} + c(\hat{\Delta}_2) \leqslant \frac{\pi}{30}$ is added to $c(\hat{\Delta}_9)$ as shown in (xix); if $d(u_2) = 4$ and $d(u_1) \geqslant 6$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_2) \leqslant -\frac{\pi}{10}$ as in (xx); if $d(u_2) = 5$ and $d(u_1) = 3$ then $\frac{\pi}{15} + c(\hat{\Delta}_2) \leqslant \frac{\pi}{5}$ so add $\frac{2\pi}{15}$ to $c(\hat{\Delta}_9)$ and $\frac{\pi}{15}$ to $c(\hat{\Delta}_{10})$ as in (xxi); if $d(u_2) = 5$ and $d(u_1) = 4$ then $c(\hat{\Delta}_2) = -\frac{\pi}{30}$ so add $\frac{\pi}{15} + c(\hat{\Delta}_2) = \frac{\pi}{30}$ to $c(\hat{\Delta}_9)$ as shown in the two possibilities (xxii) and (xxiii); if $d(u_2) = 5$ and $d(u_1) > 4$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_2) \leqslant -\frac{2\pi}{15}$ as in (xxiv); if $d(u_2) > 5$ and $d(u_1) = 3$ then add $\frac{\pi}{15} + c(\hat{\Delta}_2) \leqslant \frac{2\pi}{15}$ to $c(\hat{\Delta}_9)$ as in (xxv); and if $d(u_2) > 5$ and $d(u_1) > 3$ then add $\frac{1}{2}c(\Delta) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_2) \leqslant -\frac{\pi}{10}$ as in (xxvi).

$\boldsymbol{d(v_1) = 3}$. *Figure* 24. Either $d(v_3) = 5$ or $d(v_4) = 5$ or $d(v_3) = d(v_4) = 4$.
$\boldsymbol{d(v_2) = 3}$. *Figure* 25. Either $d(v_1) = 5$ or $d(v_3) = 5$ or $d(v_1) = d(v_3) = 4$.
$\boldsymbol{d(v_3) = 3}$. *Figure* 26. Either $d(v_1) = 5$ or $d(v_4) = 5$ or $d(v_1) = d(v_4) = 4$.
$\boldsymbol{d(v_4) = 3}$. *Figures* 27–30. There are four subcases.

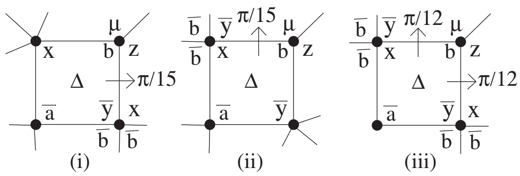

FIGURE 25. $d(v_2) = 3$.

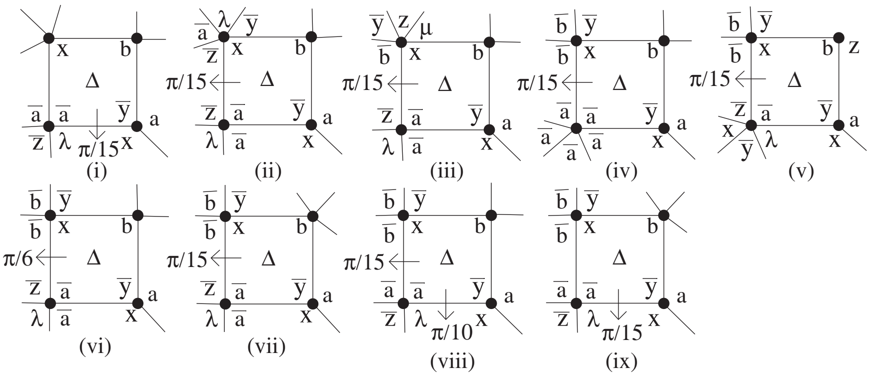

FIGURE 26. $d(v_3) = 3$.

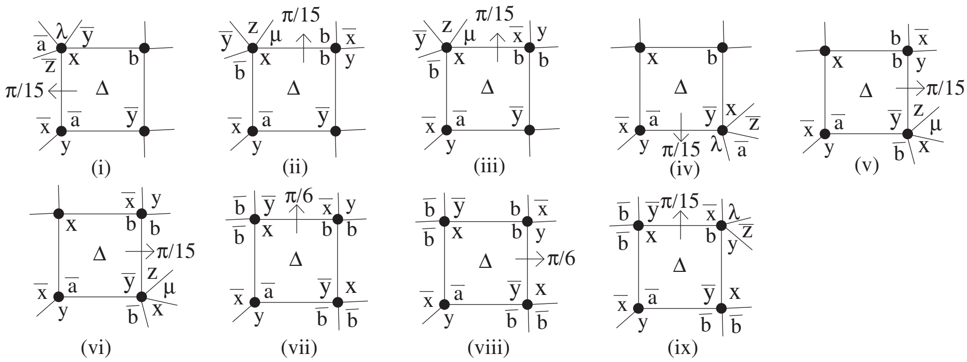

FIGURE 27. $d(v_4) = 3(subcase1)$.

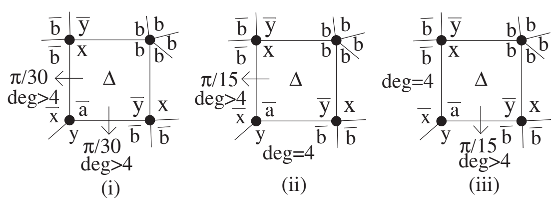

FIGURE 28. $d(v_4) = 3(subcase2)$.

(1) $\boldsymbol{(d(v_1), d(v_3), l(v_2)) \neq (4, 4, b^5)}$. *Figure* 27. Either $d(v_1) = 5$ or $d(v_3) = 5$ or $d(v_1) = d(v_3) = 4$ but $l(v_2) \neq b^5$ and the distribution of curvature is as shown.

Assume from now on $d(v_1) = d(v_3) = 4$ and $l(v_2) = b^5$.

(2) $\boldsymbol{(d(\hat{\Delta}_3), d(\hat{\Delta}_4)) \neq (4, 4)}$. *Figure* 28. $c(\Delta) = \frac{\pi}{15}$ is distributed as shown.

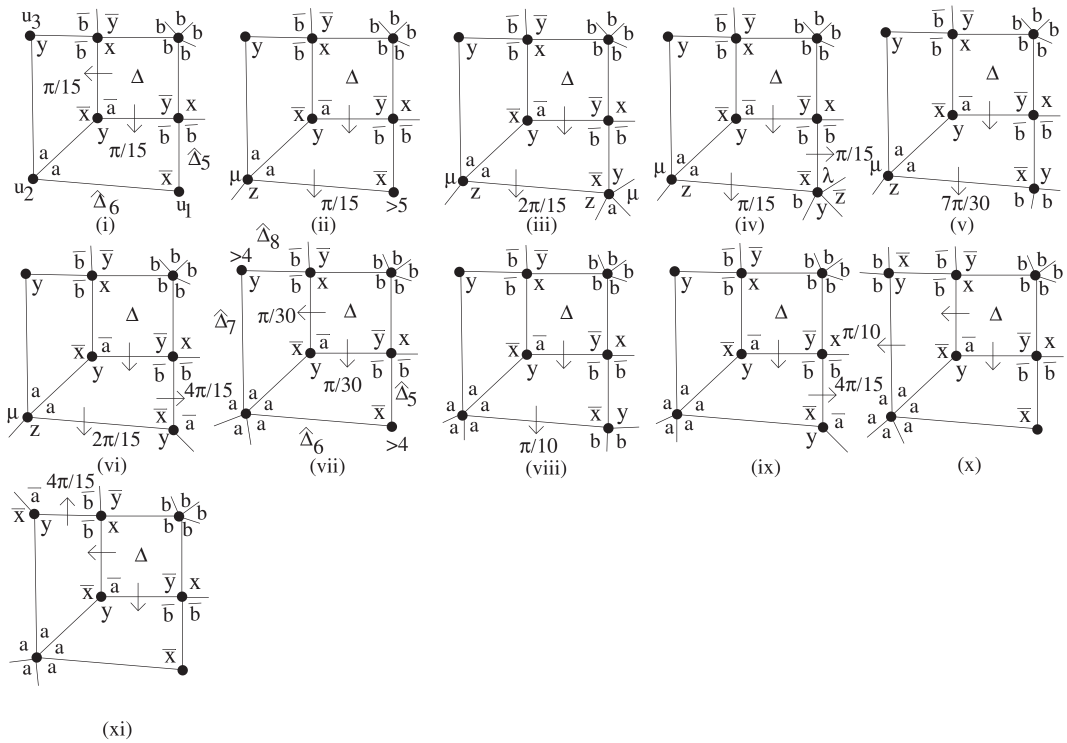

FIGURE 29. $d(v_4) = 3(subcase 3)$.

Now assume $d(\hat{\Delta}_3) = d(\hat{\Delta}_4) = 4$ as shown in Figure 29(i). If $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{15}$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3)$ or if $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{15}$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4)$ as shown in Figure 29(i). Assume $c(\hat{\Delta}_3) > -\frac{\pi}{15}$ and $c(\hat{\Delta}_4) > -\frac{\pi}{15}$. There are two subcases according to $d(u_2) \geqslant 4$.

(3) $\boldsymbol{4 \leqslant d(u_2) \leqslant 5}$. *Figure* 29.

Let $d(u_2) = 4$. If $d(u_1) \geqslant 6$ then add $c(\Delta) + c(\hat{\Delta}_3) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_6)$ as in Figure 29(ii); if $d(u_1) = 5$ then add $c(\Delta) + c(\hat{\Delta}_3) = \frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ if $\hat{\Delta}_3$ is given by (iii), or add $\frac{1}{2}(c(\Delta) + c(\hat{\Delta}_3)) = \frac{\pi}{15}$ to each of $c(\hat{\Delta}_5)$ and $c(\hat{\Delta}_6)$ if $\hat{\Delta}_3$ is given by (iv); if $d(u_1) = 4$ then add $c(\Delta) + c(\hat{\Delta}_3) = \frac{7\pi}{30}$ to $c(\hat{\Delta}_6)$ as in (v); and if $d(u_1) = 3$ then $c(\Delta) + c(\hat{\Delta}_3) = \frac{6\pi}{15}$ so add $\frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ and $\frac{2\pi}{15}$ to $c(\hat{\Delta}_6)$ as in (vi).

Let $d(u_2) = 5$ in which case $l(u_2) = a^5$. In this case add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to each of $c(\hat{\Delta}_3)$ and $c(\hat{\Delta}_4)$. If $d(u_1) \geqslant 5$ then $c(\hat{\Delta}_3) \leqslant c(3,4,5,5) = -\frac{\pi}{30}$ and the $\frac{\pi}{30}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_3)$ as shown in Figure 29(vii); if $d(u_1) = 4$ then add $\frac{\pi}{30} + c(\hat{\Delta}_3) = \frac{\pi}{10}$ to $c(\hat{\Delta}_6)$ as in (viii); and if $d(u_1) = 3$ then add $\frac{\pi}{30} + c(\hat{\Delta}_3) = \frac{4\pi}{15}$ to $c(\hat{\Delta}_5)$ as in (ix). If $d(u_3) \geqslant 5$ then $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{30}$ and the $\frac{\pi}{30}$ from $c(\Delta)$ remains with $c(\hat{\Delta}_4)$ as shown in Figure 29(vii); if $d(u_3) = 4$ then add $\frac{\pi}{30} + c(\hat{\Delta}_4) = \frac{\pi}{10}$ to $c(\hat{\Delta}_7)$ as in (x); and if $d(u_3) = 3$ then add $\frac{\pi}{30} + c(\hat{\Delta}_4) = \frac{4\pi}{15}$ to $c(\hat{\Delta}_8)$ as in (xi).

(4) $\boldsymbol{d(u_2) \geqslant 6}$. *Figure* 30. If $d(u_1) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{10}$; or if $d(u_3) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{10}$ as in Figure 30(i). If $d(u_1) = 3$ then add $c(\Delta) + c(\hat{\Delta}_3) \leqslant \frac{\pi}{15} + \frac{\pi}{6} = \frac{7\pi}{30}$ to $c(\hat{\Delta}_5)$ as shown in Figure 30(ii); or if $d(u_3) = 3$ then add $c(\Delta) + c(\hat{\Delta}_4) \leqslant \frac{7\pi}{30}$ to $c(\hat{\Delta}_8)$ as shown in Figure 30(ii). This leaves $d(u_1) = d(u_3) = 4$. If $d(u_2) \geqslant 7$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{30}$ to each of $c(\hat{\Delta}_3) \leqslant -\frac{\pi}{21}$ and $c(\hat{\Delta}_4) \leqslant -\frac{\pi}{21}$ as in Figure 30(iii), so assume $d(u_2) = 6$. If $d(\hat{\Delta}_6) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_6)$, or if $d(\hat{\Delta}_7) > 4$ then add $c(\Delta) = \frac{\pi}{15}$ to $c(\hat{\Delta}_7)$ as

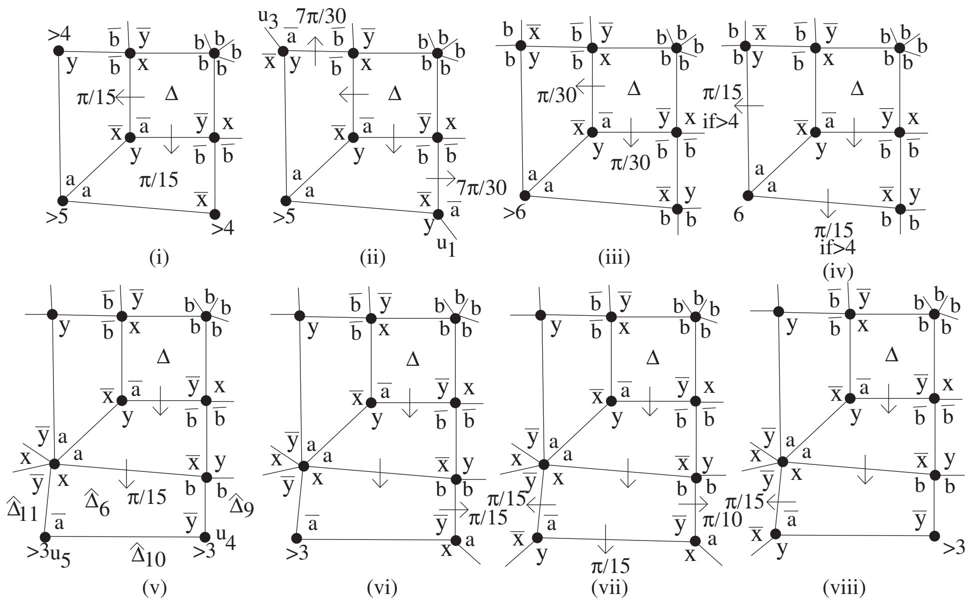

FIGURE 30. $d(v_4) = 3(subcase 4)$.

in Figure 30(iv). It can be assumed $d(\hat{\Delta}_6) = d(\hat{\Delta}_7) = 4$ which forces $l(u_2) = aaxy^{-1}xy^{-1}$ as shown in Figure 30(v). If $d(u_4) > 3$ and $d(u_5) > 3$ in Figure 30(v) then add $c(\Delta) + c(\hat{\Delta}_3) = \frac{\pi}{15}$ to $c(\hat{\Delta}_6) \leqslant -\frac{\pi}{6}$ as shown; if $d(u_4) = 3$ and $d(u_5) > 3$ then add $c(\Delta) + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) \leqslant \frac{\pi}{15}$ to $c(\hat{\Delta}_9)$ as in (vi); if $d(u_4) = d(u_5) = 3$ then $c(\Delta) + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) = \frac{7\pi}{30}$ and so add $\frac{\pi}{10}$ to $c(\hat{\Delta}_9)$, $\frac{\pi}{15}$ to $c(\hat{\Delta}_{10})$ and $\frac{\pi}{15}$ to $c(\hat{\Delta}_{11})$ as in (vii); and if $d(u_4) > 3$ and $d(u_5) = 3$ then add $c(\Delta) + c(\hat{\Delta}_3) + c(\hat{\Delta}_6) = \frac{\pi}{15}$ to $c(\hat{\Delta}_{11})$ as in (viii).

This completes the description of distribution of curvature from Δ when $d(\Delta) = 4$ except for six exceptional configurations which we now describe and for which there is an amendment to the rules given above. (Indeed the amendments, as they relate to Figures 9, 13, 14 and 20 have already been described. In what follows we detail the amendments as they relate to Figures 31 and 32.)

Configuration A. This is shown in Figure 31(i) in which $c(\Delta_1) = \frac{7\pi}{30}$ and $c(\Delta_3) = \frac{\pi}{3}$. The region Δ_1 in Figure 31(i) corresponds to the region Δ in Figure 20(vi); and the region Δ_3 in Figure 31(i)–(iv) corresponds to the region Δ in Figure 13(ii). The new rule is: add $\frac{\pi}{5}$ from $c(\Delta_1)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ from $c(\Delta_1)$ to $c(\hat{\Delta}_1)$ as shown by dotted lines in Figure 31(i) *except* when the neighbouring regions of Δ_3 are given by Figure 31(ii)–(iv). There it is assumed that $\hat{\Delta}_2$ receives $\frac{\pi}{5}$ from Δ_4; and so the region Δ_4 of Figure 31(ii) and (iii) corresponds to the region Δ of Figure 7(iii), and the region Δ_4 of Figure 31(iv) corresponds to the region Δ of Figure 10. When Δ_3 is given by 31(ii)–(iv) add *all* of $c(\Delta_1) = \frac{7\pi}{30}$ to $c(\hat{\Delta})$ (as shown in Figure 31(i)) as usual and the new rule is as follows: add $\frac{3\pi}{10}$ from $c(\Delta_3)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ from $c(\Delta_3)$ to $c(\hat{\Delta}_3)$ as shown by dotted lines. Moreover, if $d(\hat{\Delta}_3) = 4$ then add $\frac{\pi}{30} + c(\hat{\Delta}_3) = \frac{\pi}{10}$ to $c(\hat{\Delta}_4)$ as shown by dotted line in Figure 31(ii). Note that it is being assumed $d(\hat{\Delta}_3) \neq 4$ in Figure 31(iii) and (iv), in which case $\hat{\Delta}_3$ is not given by Figure 4(ii) or (iii) and so $d(\hat{\Delta}_3) \geqslant 8$.

Configuration B. This is shown in Figure 31(v) in which $c(\Delta_1) = \frac{7\pi}{30}$ and $c(\Delta_3) = \frac{\pi}{3}$. The region Δ_1 in Figure 31(v) corresponds to the region Δ in Figure 20(v); and the region Δ_3 in Figure 31(v)–(viii) corresponds to the region Δ in Figure 14(ii). The new rule is: add $\frac{\pi}{5}$ from

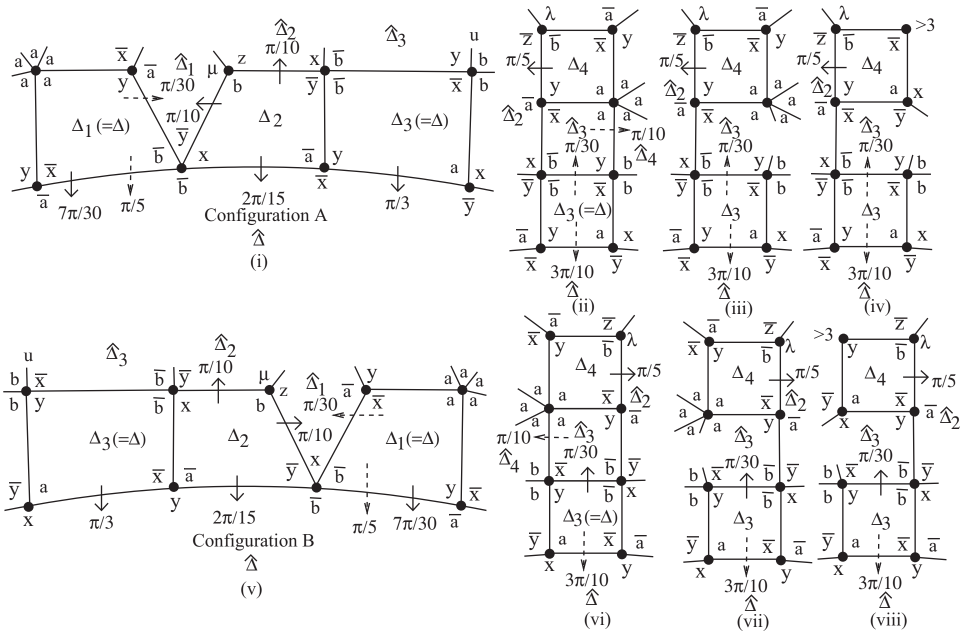

FIGURE 31. *Configurations A and B.*

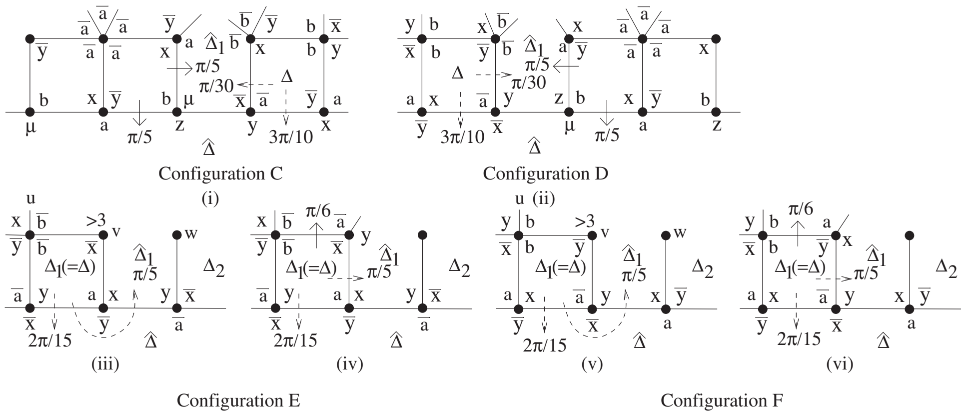

FIGURE 32. *Configurations C to F.*

$c(\Delta_1)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ from $c(\Delta_1)$ to $c(\hat{\Delta}_1)$ as shown by dotted lines in Figure 31(v) *except* when the neighbouring regions of Δ_3 are given by Figure 31(vi)–(viii). There it is assumed that $\hat{\Delta}_2$ receives $\frac{\pi}{5}$ from Δ_4; and so the region Δ_4 in Figure 31(vi) and (vii) corresponds to the region Δ in Figure 7(iii), and the region Δ_4 in Figure 31(viii) corresponds to the region Δ in Figure 8(iv). When Δ_3 is given by Figure 31(vi)–(viii) add *all* of $c(\Delta_1) = \frac{7\pi}{30}$ to $c(\hat{\Delta})$ (as shown in Figure 31(v)) as usual and the new rule is as follows: add $\frac{3\pi}{10}$ from $c(\Delta_3)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ as shown by dotted lines. Moreover, if $d(\hat{\Delta}_3) = 4$ then add $\frac{\pi}{30} + c(\hat{\Delta}_3) = \frac{\pi}{10}$ to $c(\hat{\Delta}_4)$ as shown by dotted line in Figure 31(vi). Note that it is being assumed $d(\hat{\Delta}_3) \neq 4$ in Figure 31(vii) and (viii), in which case $\hat{\Delta}_3$ is not given by Figure 4(ii) or (iii) and so $d(\hat{\Delta}_3) \geqslant 8$.

Configurations C and D. These are shown in Figure 32(i) and (ii). The region Δ of Figure 32(i) corresponds to the region Δ of Figure 13(iii); and the region Δ of Figure 32(ii) corresponds to the region Δ of Figure 14(iii). In both cases the new rule (given by dotted lines) is: add $\frac{3\pi}{10}$ from $c(\Delta)$ to $c(\hat{\Delta})$ and add $\frac{\pi}{30}$ from $c(\Delta)$ to $c(\hat{\Delta}_1)$.

Configurations E and F. This is shown in Figure 32. There are two cases, namely when $d(v) \geqslant 4$ and when $d(v) = 3$ for the vertex v indicated. If $d(v) \geqslant 4$ then the region Δ_1 in Figure 32(iii) corresponds to the region Δ of Figure 13(iv); and the region Δ_1 of Figure 32(v) corresponds to the region Δ in Figure 14(iv). If $d(v) = 3$ then the region Δ_1 in Figure 32(iv) corresponds to the region Δ in Figure 9(v); and the region Δ_1 in Figure 32(vi) corresponds to the region Δ in Figure 9(iv). For Figure 32(iii) and (v) the new rule is as follows: instead of adding $c(\Delta_1) \leqslant \frac{\pi}{3}$ to $c(\hat{\Delta})$, as in Figures 13(i) and 14(i), add $\min\{c(\Delta_1), \frac{\pi}{5}\}$ from $c(\Delta_1)$ to $c(\hat{\Delta}_1)$ via $\hat{\Delta}$ across the edge shown; and add (at most) $\frac{2\pi}{15}$ from $c(\Delta_1)$ to $c(\hat{\Delta})$. For Figure 32(iv) and (vi) the new rule is as follows: instead of adding $\frac{\pi}{6}$ from $c(\Delta_1)$ to each of $c(\hat{\Delta})$ and $c(\hat{\Delta}_1)$, as in Figure 9(ii) and (iii), add $\frac{\pi}{5}$ from $c(\Delta_1)$ to $c(\hat{\Delta}_1)$ and add $\frac{2\pi}{15}$ from $c(\Delta_1)$ to $c(\hat{\Delta})$ as shown by the dotted lines. Note that $d(\hat{\Delta}_1) \geqslant 8$ in Figure 32(iii)–(vi).

6. *Proof of Proposition* 4.1

Let $\hat{\Delta}$ ($\neq \Delta_0$) be a region that receives positive curvature in Figures 6–32. Then inspection of these figures shows $d(\hat{\Delta}) \geqslant 6$ in Figures 6– 12; 13(i), (iii), (iv); 14(i), (iii), (iv); 15 and 16; 20; 24–28 and 32.

Lemma 6.1. *Let $\hat{\Delta}$ be a region of degree 4 that receives positive curvature across at least one edge in Figures 6–32. Then one of the following holds.*

(i) *$\hat{\Delta}$ occurs in Figure 17, 18 or 19, in which case we say that $\hat{\Delta}$ is a T24 region.*
(ii) *$\hat{\Delta}$ occurs in Figure 21, 22 or 23, in which case we say that $\hat{\Delta}$ is a T13 region.*
(iii) *$\hat{\Delta}$ occurs in Figure 29 or 30, in which case we say that $\hat{\Delta}$ is a T4 region.*
(iv) *$\hat{\Delta} = \hat{\Delta}_3$ of Figure 31(ii) $= \hat{\Delta}$ of Figure 13(ii).*
(v) *$\hat{\Delta} = \hat{\Delta}_3$ of Figure 31(vi) $= \hat{\Delta}$ of Figure 14(ii).*

Proof. The result for Figures 6–30 and 32 follows immediately from the statement preceding the lemma. To complete the proof observe that all regions in Figure 31 other than $\hat{\Delta}_3$ of (ii) and (vi) that receive positive curvature have degree greater than 4. □

We remark here that if $\hat{\Delta}$ is a T24 region then an inspection of Figures 17–19 shows that there are essentially six cases for $\hat{\Delta}$, namely $\hat{\Delta} = \hat{\Delta}_3$ of Figure 17(i) and this is again shown in Figure 33(i); $\hat{\Delta} = \hat{\Delta}_4$ of Figure 17(ix), see Figure 33(ii); $\hat{\Delta} = \hat{\Delta}_3$ or $\hat{\Delta}_4$ of Figure 18(iii), see Figure 33(iii); $\hat{\Delta} = \hat{\Delta}_6$ of Figure 19(i) for which it is no longer assumed $d(u_4) > 3$ or $d(u_5) > 3$, see Figure 33(iv); or $\hat{\Delta} = \hat{\Delta}_7$ of Figure 19(vii) for which it is no longer assumed $d(u_6) > 3$ or that $d(u_7) > 3$, see Figure 33(v). If $\hat{\Delta}$ is a T13 region then an inspection of Figures 21– 23 shows that there are six cases for $\hat{\Delta}$, namely $\hat{\Delta} = \hat{\Delta}_1$ of Figure 21(iii) but with $d(v_2) \geqslant 5$ to take Figure 23(iii)–(xiv) into account, see Figure 33(vi); $\hat{\Delta} = \hat{\Delta}_4$ of Figure 21(iii), see Figure 33(vi); $\hat{\Delta} = \hat{\Delta}_8$ of Figure 21(xi) *under the assumption* that $d(\hat{\Delta}_8) = 4$ and that $\hat{\Delta}_8$ is *not* given by Figure 21(xii), see Figure 33(vii); $\hat{\Delta} = \hat{\Delta}_2$ of Figure 22(i) but with $d(v_2) \geqslant 5$ to take Figure 23(xv)–(xxvi) into account, see Figure 33(viii); $\hat{\Delta} = \hat{\Delta}_3$ of Figure 22(i), see Figure 33(viii); or $\hat{\Delta} = \hat{\Delta}_8$ of Figure 22(ix) *under the assumption* that $d(\hat{\Delta}_8) = 4$ and that $\hat{\Delta}_8$ is *not* given by Figure 22(x), see Figure 33(ix). If $\hat{\Delta}$ is a T4 region then inspecting Figures 29 and 30 shows that there are three cases for $\hat{\Delta}$, namely $\hat{\Delta} = \hat{\Delta}_3$ or $\hat{\Delta}_4$ of Figure 29(i), see Figure 33(x); or $\hat{\Delta} = \hat{\Delta}_6$ of Figure 30(v) where it is no longer assumed $d(u_4) > 3$ or $d(u_5) > 3$, see Figure 33(xi). The two remaining possibilities for $\hat{\Delta}$, namely $\hat{\Delta} = \hat{\Delta}_3$ of Figure 31(ii) and (vi) are given by Figure 33(xii) and (xiii).

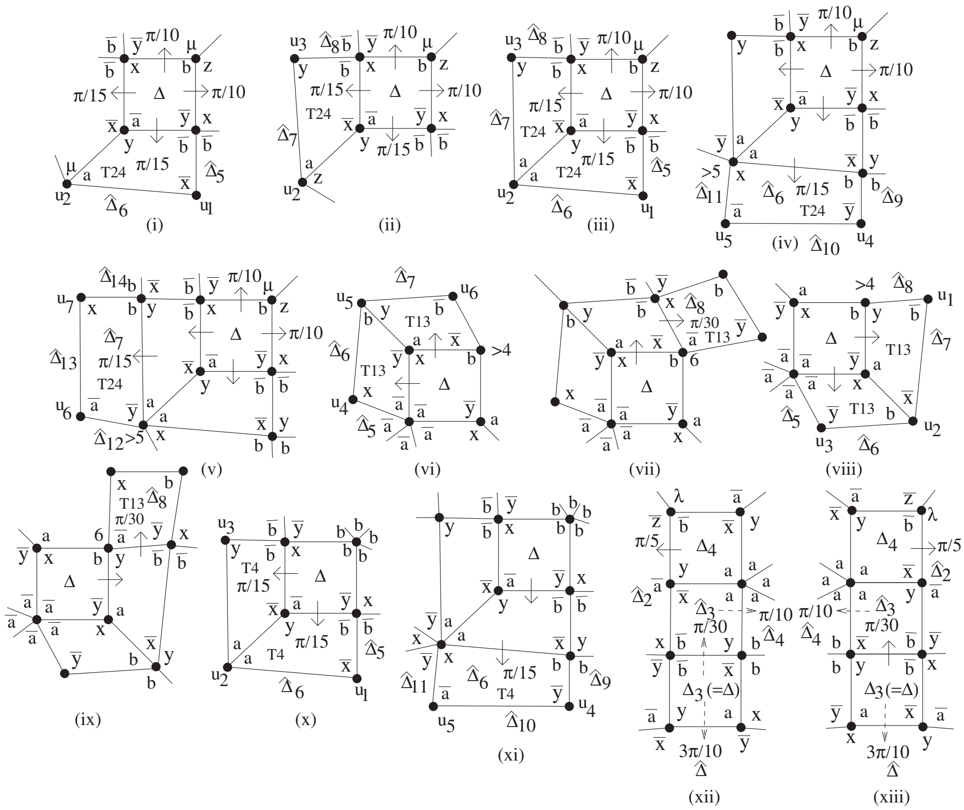

FIGURE 33. *Curvature to degree* 4 *regions.*

LEMMA 6.2. *Let* $\hat{\Delta}$ *be a region of degree* 4 *that receives positive curvature across at least one edge. Then one of the following occurs:*

(i) $c^*(\hat{\Delta}) \leqslant 0$;
(ii) $c^*(\hat{\Delta}) > 0$ *is distributed to a region of degree greater than* 4;
(iii) $c^*(\hat{\Delta}) > 0$ *is distributed to a region* Δ' *of degree* 4 *and either* $c^*(\Delta') \leqslant 0$ *or* $c^*(\Delta') > 0$ *is distributed to a region of degree greater than* 4.

Proof. Let $d(\hat{\Delta}) = 4$. By Lemma 6.1, $\hat{\Delta}$ is a T24, T13 or T4 region or $\hat{\Delta} = \hat{\Delta}_3$ of Figure 31(ii) and (vi). We divide the proof of the lemma into two parts. The first deals with the cases when $\hat{\Delta}$ receives positive curvature across exactly one edge and the second part deals with the cases in which $\hat{\Delta}$ receives positive curvature across at least two edges.

If $\hat{\Delta}$ receives positive curvature across exactly one edge then we see by inspection of Figures 17–19, 21–23, 29 and 30, 31(ii) and (vi) that in all cases either $c^*(\hat{\Delta}) \leqslant 0$ or $c^*(\hat{\Delta})$ is distributed from $\hat{\Delta}$ to a neighbouring region of degree greater than 4 except when $\hat{\Delta}$ is given by Figures 19, 21(xi), 22(ix) or 30(v)–(viii) where $c^*(\hat{\Delta})$ may initially be distributed further to a region Δ' of degree 4. But in each of these cases either $c^*(\Delta') \leqslant 0$ (under the assumption that Δ' receives positive curvature across exactly one edge – the case when Δ' may receive across more than one edge is considered below) or $c^*(\Delta')$ is again distributed to a region of degree greater than 4.

Now suppose that $\hat{\Delta}$ receives positive curvature across at least two edges. An inspection of the labelling and degrees of the vertices in each of these 17 possibilities for $\hat{\Delta}$ shown in Figure 33 immediately rules out the following combinations: a T24 region with a T24; a

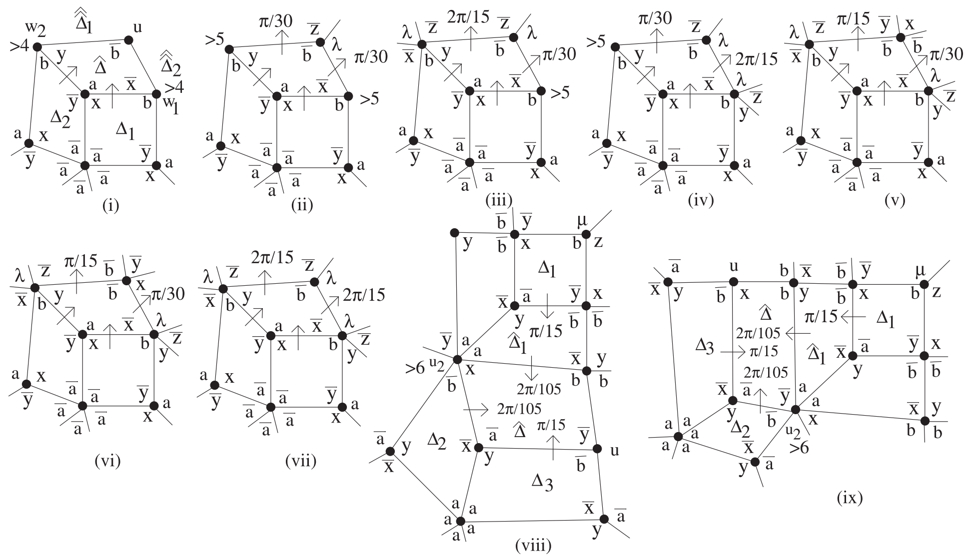

FIGURE 34. *Curvature across more than one edge.*

T24 with a T4; a T4 with a T4; and either Figure 33(xii) or 33(xiii) with any of the other sixteen possibilities. For example, Δ of Figure 33(viii) cannot coincide with the inverse of Δ_4 of (xii) as the degrees of the b-corner vertices differ. This leaves the possibility that at least two T13 regions coincide or a T13 coincides with a T24 or a T13 coincides with a T4.

Suppose that at least two T13 regions coincide. An inspection of the six T13 regions of Figure 33(vi)–(ix) shows that all combinations are immediately ruled out by the labelling and degree of vertices except for three cases. The first case is $\hat{\Delta} = \hat{\Delta}_4$ of Figure 33(vi) $= \hat{\Delta}_3$ of Figure 33(vii). This, for example, forces $l(u_5) = ybx^{-1}w$ in Figure 33(vi), in particular, $d(u_5) > 4$. But observe that if $\hat{\Delta}_4$ has degree 4 and receives positive curvature from Δ in Figures 21–23 then $d(u_5) = 4$, a contradiction. The second case is $\hat{\Delta} = \hat{\Delta}_8$ of Figure 33(vii) $= \hat{\Delta}_8$ of Figure 33(ix). But this forces $l(v_2) = bx^{-1}a^{-1}ybw$ in Figure 33(vii), and the fact that $d(v_2) = 6$ then forces $l(v_2) = bx^{-1}a^{-1}ybb$, a label whose t-exponent sum is equal to 6, a contradiction. The third case is $\hat{\Delta} = \hat{\Delta}_1$ of Figure 33(vi) $= \hat{\Delta}_2$ of Figure 33(viii). This case can occur and is shown in Figure 34(i). It follows that a combination of more than two T13 regions cannot occur.

Consider Figure 34(i) in which $\hat{\Delta}$ receives positive curvature from the regions Δ_1 and Δ_2 each contributing at most $\frac{\pi}{15}$ to $c(\hat{\Delta})$. (Note that we use Δ_1, Δ_2 and not Δ as before to denote regions from which positive curvature is distributed.) Let $d(w_1) > 5$ and $d(w_2) > 5$. If $d(u) > 3$ then $c^*(\hat{\Delta}) \leqslant c(3,4,6,6) + 2(\frac{\pi}{30}) < 0$; and if $d(u) = 3$ then $c(\hat{\Delta}) + 2(\frac{\pi}{30}) \leqslant c(3,3,6,6) + 2(\frac{\pi}{30}) = \frac{\pi}{15}$ so add $\frac{\pi}{30}$ to each of $c(\hat{\hat{\Delta}}_1), c(\hat{\hat{\Delta}}_2)$ as shown in Figure 34(ii). Let $d(w_1) > 5$ and $d(w_2) = 5$. If $d(u) > 3$ then $c^*(\hat{\Delta}) \leqslant c(3,4,5,6) + \frac{\pi}{30} + \frac{\pi}{15} = 0$; and if $d(u) = 3$ then $c(\hat{\Delta}) + \frac{\pi}{30} + \frac{\pi}{15} \leqslant c(3,3,5,6) + \frac{\pi}{30} + \frac{\pi}{15} = \frac{\pi}{6}$ so add $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{\pi}{30}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figure 34(iii). Let $d(w_1) = 5$ and $d(w_2) > 5$. If $d(u) > 3$ then $c^*(\hat{\Delta}) \leqslant c(3,4,5,6) + \frac{\pi}{15} + \frac{\pi}{30} = 0$; and if $d(u) = 3$ then $c(\hat{\Delta}) + \frac{\pi}{15} + \frac{\pi}{30} \leqslant c(3,3,5,6) + \frac{\pi}{15} + \frac{\pi}{30} = \frac{\pi}{6}$ so add $\frac{\pi}{30}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figure 34(iv). This leaves $d(w_1) = d(w_2) = 5$. If $d(u) > 4$ then $c^*(\hat{\Delta}) \leqslant c(3,5,5,5) + 2(\frac{\pi}{15}) = 0$; if $d(u) = 4$ then $c(\hat{\Delta}) + 2(\frac{\pi}{15}) = c(3,4,5,5) + 2(\frac{\pi}{15}) = \frac{\pi}{10}$ so add $\frac{\pi}{15}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{\pi}{30}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figure 34(v) and (vi); and if $d(u) = 3$ then $c(\hat{\Delta}) + 2(\frac{\pi}{15}) \leqslant c(3,3,5,5) + \frac{2\pi}{15} = \frac{4\pi}{15}$

so add $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_1)$ and $\frac{2\pi}{15}$ to $c(\hat{\hat{\Delta}}_2)$ as shown in Figure 34(vii). Observe that $d(\hat{\hat{\Delta}}_1) \geqslant 6$ and $d(\hat{\hat{\Delta}}_2) \geqslant 6$ in Figure 34(ii)–(vii).

Now suppose that a T4 region and a T13 region coincide. Again an inspection of Figure 33 of the labelling and degrees of the vertices involved immediately rules out all combinations except for three cases. The first case is $\hat{\Delta}_3$ of Figure 33(x) with $\hat{\Delta}_8$ of Figure 33(vii), but this forces $\hat{\Delta}_8$ to be given by Figure 21(xii), a contradiction; and the second case is $\hat{\Delta}_4$ of Figure 33(x) with $\hat{\Delta}_8$ of Figure 33(ix), but this forces $\hat{\Delta}_8$ to be given by Figure 22(x), a contradiction. The third case is when $\hat{\Delta} = \hat{\Delta}_6$ of Figure 33(xi) and $\hat{\Delta} = \hat{\Delta}_8$ of Figure 33(ix). But then $c^*(\hat{\Delta}) \leqslant c(4,4,6,6) + \frac{\pi}{15} + \frac{\pi}{30} < 0$. Note that we have also shown that a T4 region coincides with at most one T13 region.

Finally suppose that a T24 region and a T13 region coincide. An inspection of the 36 possible combinations immediately rules out all but the following 10 cases. If $\hat{\Delta} = \hat{\Delta}_3$ of Figure 33(i) or 33(iii) and $\hat{\Delta} = \hat{\Delta}_8$ of Figure 33(vii) then this forces $\hat{\Delta}_8$ to be given by Figure 21(xii), a contradiction; or if $\hat{\Delta} = \hat{\Delta}_4$ of Figure 33(ii) or 33(iii) and $\hat{\Delta} = \hat{\Delta}_8$ of Figure 33(ix) then this forces $\hat{\Delta}_8$ to be given by Figure 22(x), a contradiction. If $\hat{\Delta} = \hat{\Delta}_6$ of Figure 33(iv) and $\hat{\Delta} = \hat{\Delta}_8$ of Figure 33(ix), or if $\hat{\Delta} = \hat{\Delta}_7$ of Figure 33(v) and $\hat{\Delta} = \hat{\Delta}_8$ of Figure 33(vii) then $c^*(\hat{\Delta}) \leqslant c(4,4,6,6) + \frac{\pi}{15} + \frac{\pi}{30} < 0$.

This leaves $\hat{\Delta} = \hat{\Delta}_6$ of Figure 33(iv) and either $\hat{\Delta} = \hat{\Delta}_1$ of 33(vi) or or $\hat{\Delta} = \hat{\Delta}_2$ of 33(viii); or $\hat{\Delta} = \hat{\Delta}_7$ of Figure 33(v) and again either $\hat{\Delta} = \hat{\Delta}_1$ of 33(vi) or $\hat{\Delta} = \hat{\Delta}_2$ of 33(viii). Observe that if $\hat{\Delta}_6$ of Figure 33(iv) $= \hat{\Delta}_1$ of 33(vi) or if $\hat{\Delta}_6$ of Figure 33(iv)$= \hat{\Delta}_2$ of 33(viii) then $l(u_2) = y^{-1}a^2xb^{-1}w$ in Figure 33(iv) and so $d(u_2) \geqslant 7$, and this is shown in Figure 34(viii). Moreover if $\hat{\Delta}_7$ of Figure 33(v) $= \hat{\Delta}_1$ of 33(vi) or if $\hat{\Delta}_7$ of Figure 33(v)$= \hat{\Delta}_2$ of 33(viii) then $l(u_2) = b^{-1}y^{-1}a^2xw$ in Figure 33(v) and again $d(u_2) \geqslant 7$, and this is shown in Figure 34(ix). It follows in both Figure 34(viii) and (ix) that $c(\hat{\Delta}_1) \leqslant c(3,4,4,7) = -\frac{\pi}{21}$ and $c(\Delta_2) \leqslant c(3,3,5,7) = \frac{2\pi}{105}$. In both configurations $\frac{\pi}{21}$ is added from $c(\Delta_1) = \frac{\pi}{15}$ to $c(\hat{\Delta}_1)$ and the remaining $\frac{\pi}{15} - \frac{\pi}{21} = \frac{2\pi}{105}$ to $\hat{\Delta}$ as shown. If $\hat{\Delta}$ does not receive positive curvature from Δ_3 then $c^*(\hat{\Delta}) \leqslant c(3,4,4,7) + 2(\frac{2\pi}{105}) < 0$ so it can be assumed without any loss that $\hat{\Delta}$ receives from Δ_1 (via $\hat{\Delta}_1$), Δ_2 and Δ_3. But then $\hat{\Delta} = \hat{\Delta}_2$ of Figure 33(viii) forces $d(u) \geqslant 5$ in Figure 34(viii), and $\hat{\Delta} = \hat{\Delta}_1$ of Figure 33(vi) forces $d(u) \geqslant 5$ in Figure 34(ix); therefore in each case $c^*(\hat{\Delta} \leqslant c(3,4,5,7) + \frac{1}{2}c(3,3,5,5) + 2(\frac{2\pi}{105}) < 0$. □

Proposition 4.1 follows immediately from Lemma 6.2 together with the fact that all possibilities for distribution of curvature from a region of degree 4 have been covered by Figures 6–32.

We end this section with a summary that will be helpful in subsequent sections.

Note. In Figure 35(i) the maximum amount of curvature, denoted by $c(u,v)$, distributed across an edge e_i with endpoints u, v according to the description of curvature given in Figures 6–32 and 34 is shown for each choice of corner labels. The list excludes (b,a)-edges and excludes the (x,y)-edges of Figures 13 and 14. In Figure 35(ii) and (iii) $c(u,v)$ is shown when at least one of $d(u), d(v)$ is greater than 4 apart from the two exceptional cases shown in (iv) and (v) (see Figure 23(xvi) and (iv)). The integers shown are multiples of $\frac{\pi}{30}$ with 7 or 5, 4 or 2 meaning that if $c(u,v) < \frac{7\pi}{30}, \frac{2\pi}{15}$ then $c(u,v) = \frac{\pi}{6}, \frac{\pi}{15}$, respectively. This will be used throughout what follows often without explicit reference.

7. *Distribution of positive curvature from 6-gons*

We turn now to step 2 of the proof of Theorem1.2 as described in Section 4. Let $d(\hat{\Delta}) = 6$ and so $\hat{\Delta}$ is given by Figure 4(xii) and (xiii). In Figure 36 we fix the labelling of the six neighbours $\hat{\Delta}_i$ $(1 \leqslant i \leqslant 6)$ of $\hat{\Delta}$ as shown. We consider regions $\hat{\Delta}$ $(\neq \Delta_0)$ of degree 6 that have received positive curvature in step 1 of Sections 5 and 6.

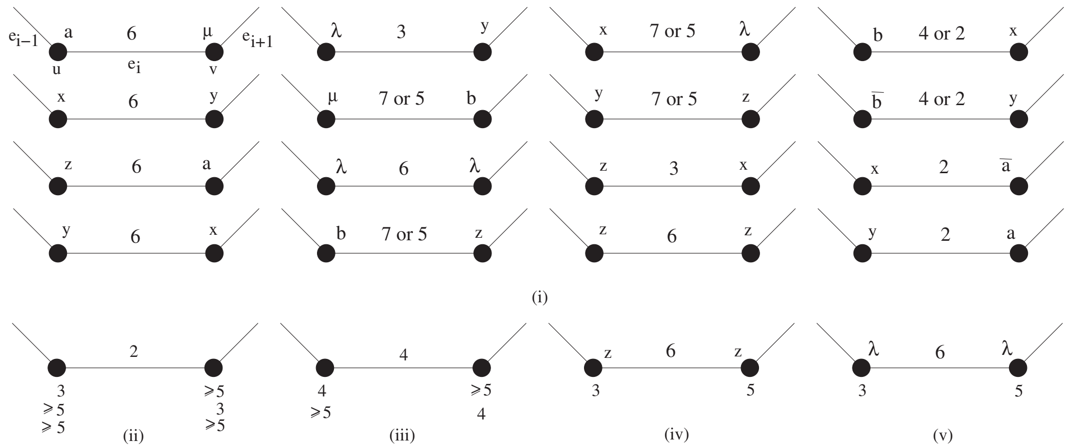

FIGURE 35. *Curvature distribution maxima.*

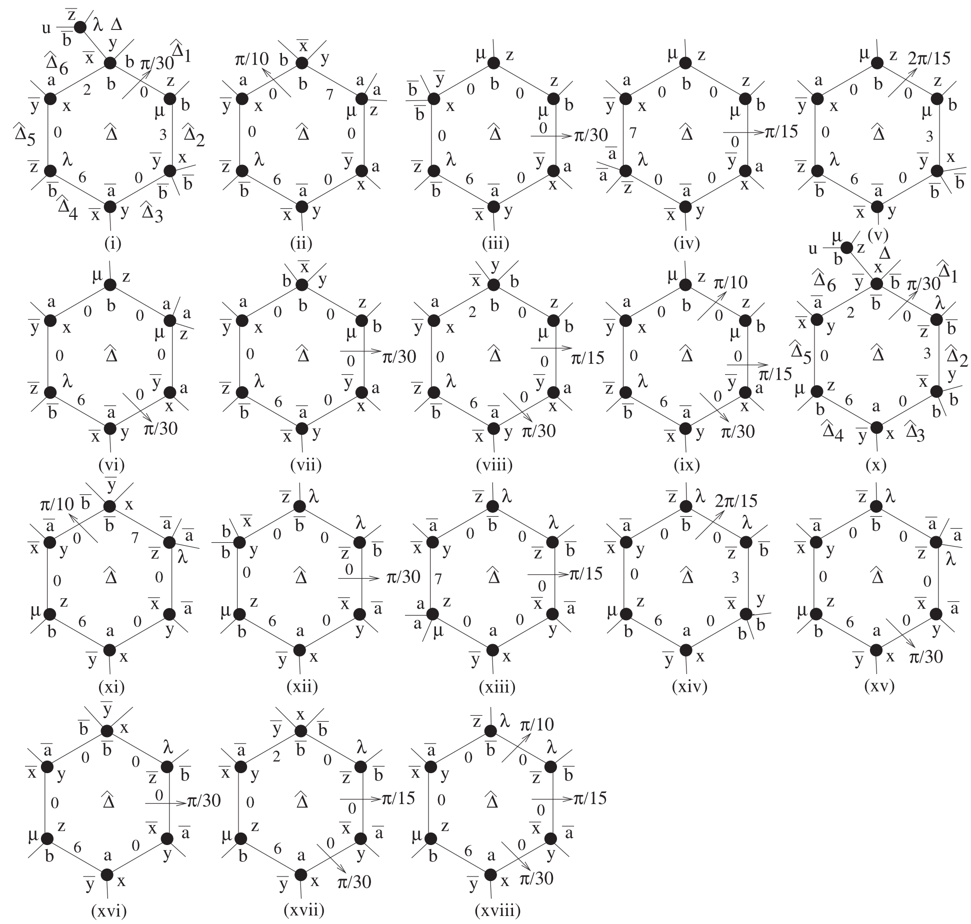

FIGURE 36. *Curvature from degree* 6 *regions.*

Again for the benefit of the reader let us indicate briefly that a general rule for distribution is to try whenever possible to add the positive curvature from $\hat{\Delta}$ to neighbouring regions of degree greater than 6. It turns out that this is not possible for exactly four cases, namely, $\hat{\Delta}_1$ of Figure 36(i) and (x); and $\hat{\Delta}_2$ of Figures 37(iv) and 38(iv). These four exceptions are dealt

TABLE 3. $c(u_i, u_{i+1})$ *for regions of degree* 6.

$d(u_i)$	$d(u_{i+1})$	$c(u_1, u_2)$	$c(u_2, u_3)$	$c(u_3, u_4)$	$c(u_4, u_5)$	$c(u_5, u_6)$	$c(u_6, u_1)$
3	3	0	0	0	6	0	0
3	4	0	3	0	0	0	2
4	3	0	0	0	0	7	0
3	5	0	2	2	0	0	0
5	3	0	2	2	2	2	0
3	6^+	0	2	2	2	0	0
6^+	3	0	2	2	2	2	0
4	4	7	0	0	0	0	0
4	5	2	0	0	2	0	0
5	4	2	2	0	0	4	0
4	6^+	4	0	1	0	2	0
6^+	4	2	0	0	0	1	4
5^+	5^+	1	0	0	1	1	0

with in greater detail in the next section. In all other cases in Figures 36 and 38 the curvature is added to a region of degree at least 8.

First assume that $\hat{\Delta}$ is *not* $\hat{\Delta}_1$ of Figure 31(i) (Configuration A) or Figure 31((v) (Configuration B). Then checking the distribution of curvature described in Figures 6–32 and 34 yields Table 3 in which vertex subscripts are modulo 6; the entries under $c(u_i, u_{i+1})$ are multiples of $\frac{\pi}{30}$ and denote the maximum amount of curvature that $\hat{\Delta}$ can receive across the edge with endpoints u_i, u_{i+1} according to Figure 35; and 5^+, 6^+ means $\geqslant 5$, $\geqslant 6$. Moreover Table 3 applies to $\hat{\Delta}$ both of Figure 4(xii) and (xiii).

Notes. (1) (See Figures 4 and 36.) $d(u_1) = 3$ $(\Rightarrow d(\hat{\Delta}_1) > 4,\ d(\hat{\Delta}_2) > 4) \Rightarrow c(u_1, u_2) = c(u_6, u_1) = 0$; $d(u_2) = 3 \Rightarrow c(u_1, u_2) = 0$; $d(u_2) = 4 \Rightarrow c(u_2, u_3) = 0$; $d(u_5) = 3 \Rightarrow c(u_5, u_6) = 0$; and $d(u_5) = 4 \Rightarrow c(u_4, u_5) = 0$.

(2) $c(u_1, u_2) > 0$ and $c(u_2, u_3) > 0 \Rightarrow$ (Table 3) $c(u_1, u_2) + c(u_2, u_3) \leqslant \frac{2\pi}{15} + \frac{\pi}{15}$ and since $c(u_1, u_2) \leqslant \frac{7\pi}{30}$, $c(u_2, u_3) \leqslant \frac{\pi}{10}$ we have $c(u_1, u_2) + c(u_2, u_3) \leqslant \frac{7\pi}{30}$.

(3) $c(u_4, u_5) > 0$ and $c(u_5, u_6) > 0 \Rightarrow c(u_4, u_5) + c(u_5, u_6) \leqslant \frac{\pi}{15} + \frac{2\pi}{15}$ and since $c(u_4, u_5) \leqslant \frac{\pi}{5}$, $c(u_5, u_6) \leqslant \frac{7\pi}{30}$ we have $c(u_4, u_5) + c(u_5, u_6) \leqslant \frac{7\pi}{30}$.

(4) Let $d(u_5) = 5$, $d(u_6) = 4$. If $c(u_5, u_6) = \frac{2\pi}{15}$ then checking $l(u_5), l(u_6)$ shows $c(u_4, u_5) = \frac{2\pi}{15}$ (see Figure 23(vi) and (xviii)); moreover (see Figure 35(ii) and (iii)) if $c(u_5, u_6) \neq \frac{2\pi}{15}$ then $c(u_5, u_6) = \frac{\pi}{15}$.

In what follows much use will be made of Lemma 3.4 when determining the vertex labels and Table 3 when determining $c(u, v)$.

LEMMA 7.1. *If $\hat{\Delta}$ is given by Figure* 4(xii)–(xiii) (*with the assumption that $\hat{\Delta}$ is not $\hat{\Delta}_1$ of Figure* 31) *and $\hat{\Delta}$ receives positive curvature across at least one edge then $c^*(\hat{\Delta}) \leqslant \frac{2\pi}{15}$ and if $c^*(\hat{\Delta}) > 0$ then $\hat{\Delta}$ is given by one of the regions of Figure* 36.

Proof. It follows from Table 3 and Notes 1–4 above that $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + (c(u_1, u_2) + c(u_2, u_3)) + c(u_3, u_4) + (c(u_4, u_5) + c(u_5, u_6)) + c(u_6, u_1) \leqslant c(\hat{\Delta}) + \frac{7\pi}{30} + \frac{\pi}{15} + \frac{7\pi}{30} + \frac{2\pi}{15} = c(\hat{\Delta}) + \frac{2\pi}{3}$. Therefore if $\hat{\Delta}$ has at most two vertices of degree 3 then $c^*(\hat{\Delta}) \leqslant c(3, 3, 4, 4, 4, 4) + \frac{2\pi}{3} = 0$.

Let $\hat{\Delta}$ have exactly three vertices of degree 3 so that $c(\hat{\Delta}) \leqslant -\frac{\pi}{2}$. If $d(u_1) = 3$ then $c(u_1, u_2) = c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{\pi}{10} + \frac{\pi}{15} + \frac{7\pi}{30} < 0$, so assume $d(u_1) \geqslant 4$. If $d(u_2) = 3$ then $c(u_1, u_2) = 0$ so if $d(u_6) \geqslant 6$ then $c^*(\hat{\Delta}) \leqslant c(3,3,3,4,4,6) + \frac{\pi}{10} + \frac{\pi}{15} + \frac{7\pi}{30} + \frac{2\pi}{15} < 0$; otherwise $c(u_6, u_1) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{\pi}{10} + \frac{\pi}{15} + \frac{7\pi}{30} + \frac{\pi}{15} < 0$; so assume $d(u_2) \geqslant 4$. This leaves four subcases. First let $d(u_3) = d(u_4) = d(u_5) = 3$. Then $c(u_3, u_4) = c(u_5, u_6) = 0$. Moreover if $d(u_6) < 6$ then $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{\pi}{5} = 0$; and if $d(u_6) \geqslant 6$ then $c^*(\hat{\Delta}) \leqslant c(3,3,3,4,4,6) + \frac{7\pi}{30} + \frac{\pi}{5} + \frac{2\pi}{15} < 0$. Let $d(u_3) = d(u_4) = d(u_6) = 3$. Then $c(u_2, u_3) = \frac{\pi}{15}$, $c(u_3, u_4) = 0$, $c(u_4, u_5) + c(u_5, u_6) \leqslant \frac{7\pi}{30}$ and $c(u_6, u_1) = \frac{\pi}{15}$. If either $d(u_1) > 4$ or $d(u_2) > 4$ then $c(u_1, u_2) \leqslant \frac{2\pi}{15}$ and $c^*(\hat{\Delta}) \leqslant c(3,3,3,4,4,5) + \frac{\pi}{2} < 0$; otherwise $d(u_1) = d(u_2) = 4$ which implies $c(u_2, u_3) = 0$ and the labelling (of u_1, u_2 and u_6) either forces $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{3\pi}{10} < 0$ or forces $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{7\pi}{30} < 0$. Let $d(u_3) = d(u_5) = d(u_6) = 3$. Then $c(u_4, u_5) = \frac{\pi}{15}$, $c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. Therefore $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{\pi}{15} + \frac{\pi}{15} + \frac{\pi}{15} < 0$. Finally let $d(u_4) = d(u_5) = d(u_6) = 3$. Then $c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. If $d(u_1) > 4$ or $d(u_2) > 4$ then $c(u_1, u_2) = \frac{2\pi}{15}$ and $c^*(\hat{\Delta}) \leqslant -\frac{3\pi}{5} + \frac{2\pi}{15} + \frac{\pi}{15} + \frac{\pi}{15} + \frac{\pi}{5} + \frac{\pi}{15} < 0$; otherwise $d(u_1) = d(u_2) = 4$ so $c(u_2, u_3) = 0$ and the labelling either forces $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{\pi}{3} < 0$ or forces $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{7\pi}{30} + \frac{\pi}{15} + \frac{\pi}{5} = 0$.

Now let $\hat{\Delta}$ have exactly four vertices of degree 3 so that $c(\hat{\Delta}) \leqslant -\frac{\pi}{3}$. There are fifteen cases to consider. In fact if $(d(u_1), d(u_2), d(u_3), d(u_4), d(u_5), d(u_6)) \in \{(3,3,3,3,*,*), (3,3,3,*,3,*), (3,3,3,*,*,3), (3,3,*,*,3,3), (3,*,3,3,3,*), (3,*,3,3,*,3), (3,*,3,*,3,3), (3,*,*,3,3,3), (*,3,3,3,3,*), (*,3,3,3,*,3), (*,3,3,*,3,3)\}$ then a straightforward check using Table 3 and Notes 1–4 shows $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + \frac{\pi}{3} = 0$. Let $d(u_1) = d(u_2) = d(u_4) = d(u_5) = 3$. Then $c(u_1, u_2) = c(u_5, u_6) = c(u_6, u_1) = 0$. If $d(u_3) > 4$ then $c^*(\hat{\Delta}) \leqslant -\frac{13\pi}{30} + \frac{11\pi}{30} < 0$; otherwise $d(u_3) = 4$ forcing $c(u_3, u_4) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + \frac{3\pi}{10} < 0$. Let $d(u_1) = d(u_2) = d(u_4) = d(u_6) = 3$. Then $c(u_1, u_2) = c(u_6, u_1) = 0$. If $d(u_3) > 4$ then $c^*(\hat{\Delta}) \leqslant -\frac{13\pi}{30} + \frac{2\pi}{5} < 0$; otherwise $d(u_3) = 4$ forcing $c(u_3, u_4) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + \frac{\pi}{3} = 0$. Let $d(u_2) = d(u_4) = d(u_5) = d(u_6) = 3$. Then $c(u_1, u_2) = c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. If $d(u_1) > 4$ or $d(u_3) > 4$ then $c^*(\hat{\Delta}) \leqslant -\frac{13\pi}{30} + \frac{13\pi}{30} = 0$, so assume $d(u_1) = d(u_3) = 4$. Then $c(u_3, u_4) = 0$ and $l(u_1)$ either forces $c(u_6, u_1) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + \frac{3\pi}{10} < 0$ or $\hat{\Delta}$ is given by Figure 36(i) or (x)) in which the numbers assigned to each edge is the value of $c(u_i, u_{i+1})$ in multiples of $\frac{\pi}{30}$ and so $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + \frac{\pi}{10} + \frac{\pi}{5} + \frac{\pi}{15} = \frac{\pi}{30}$. (Note that if $c^*(\hat{\Delta}) > 0$ then $\hat{\Delta}$ must receive $\frac{\pi}{15}$ from $\hat{\Delta}_6$ and, since $d(\hat{\Delta}_5) > 4$, this forces $\hat{\Delta}_6 = \Delta$ where Δ is given by Figure 16(i) which in turn forces $l(u) = b^{-1}z^{-1}\lambda$ in Figure 36(i), and $l(u) = b\mu z$ in Figure 36(x); and $\hat{\Delta}$ must receive $\frac{\pi}{5}$ from $\hat{\Delta}_4$.) This leaves the case $d(u_j) = 3$ $(3 \leqslant j \leqslant 6)$. Then $c(u_3, u_4) = c(u_5, u_6) = 0$ and $c(u_6, u_1) = \frac{\pi}{15}$. If $d(u_1) \geqslant 5$ and $d(u_2) \geqslant 5$ then $c^*(\hat{\Delta}) \leqslant -\frac{8\pi}{15} + \frac{\pi}{2} < 0$. If $d(u_1) = 4$ and $d(u_2) = 5$ or $d(u_1) \geqslant 5$ and $d(u_2) = 4$ then $c(u_1, u_2) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leqslant c(3,3,3,3,4,5) + \frac{\pi}{15} + \frac{\pi}{10} + \frac{\pi}{5} + \frac{\pi}{15} = 0$; and if $d(u_1) = 4$ and $d(u_2) \geqslant 6$ then $c^*(\hat{\Delta}) \leqslant c(3,3,3,3,4,6) + \frac{2\pi}{15} + \frac{\pi}{10} + \frac{\pi}{5} + \frac{\pi}{15} = 0$. Let $d(u_1) = d(u_2) = 4$ so $c(u_2, u_3) = 0$. Then $l(u_1)$ either forces $c(u_1, u_2) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + \frac{\pi}{5} + \frac{\pi}{15} < 0$ or $\hat{\Delta}$ is given by Figure 36(ii) or (xi) where $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + \frac{7\pi}{30} + \frac{\pi}{5} = \frac{\pi}{10}$.

Now suppose that $\hat{\Delta}$ has exactly five vertices of degree 3 so that $c(\hat{\Delta}) \leqslant -\frac{\pi}{6}$. If $d(u_6) > 3$ then $c(u_1, u_2) = c(u_2, u_3) = c(u_3, u_4) = c(u_5, u_6) = c(u_6, u_1) = 0$, $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6} + \frac{\pi}{5} = \frac{\pi}{30}$ and $\hat{\Delta}$ is given by Figure 36(iii) or (xii). If $d(u_5) > 3$ then $c(u_i, u_{i+1}) = 0$ except for $c(u_4, u_5)$ and $c(u_5, u_6)$. If $d(u_5) \geqslant 5$ then $c^*(\hat{\Delta}) \leqslant -\frac{4\pi}{15} + \frac{7\pi}{30} < 0$; and if $d(u_5) = 4$ then $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6} + \frac{7\pi}{30} = \frac{\pi}{15}$ and $\hat{\Delta}$ is given by Figure 36(iv) or (xiii). If $d(u_4) > 3$ then $c(u_i, u_{i+1}) = 0$ except for $c(u_3, u_4) = c(u_4, u_5) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6} + \frac{2\pi}{15} < 0$. Let $d(u_3) > 3$. Then $c(u_1, u_2) = c(u_5, u_6) = c(u_6, u_1) = 0$. If $d(u_3) \geqslant 6$ then $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3} + 2(\frac{\pi}{15}) + \frac{\pi}{5} = 0$; if $d(u_3) = 5$ then

$l(u_3)$ forces either $c(u_2,u_3)=0$ or $c(u_3,u_4)=0$ so $c^*(\hat{\Delta}) \leqslant -\frac{4\pi}{15}+\frac{\pi}{15}+\frac{\pi}{5}=0$; and if $d(u_3)=4$ then $c(u_3,u_4)=0$, $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{\pi}{10}+\frac{\pi}{5}=\frac{2\pi}{15}$ and $\hat{\Delta}$ is given by Figure 36(v) or (xiv). If $d(u_2)>3$ then $c(u_1,u_2)=c(u_3,u_4)=c(u_5,u_6)=c(u_6,u_1)=0$. If $d(u_2)\geqslant 5$ then $c^*(\hat{\Delta}) \leqslant -\frac{4\pi}{15}+\frac{\pi}{15}+\frac{\pi}{5}=0$; and if $d(u_2)=4$ then $c(u_2,u_3)=0$, $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{\pi}{5}=\frac{\pi}{30}$ and $\hat{\Delta}$ is given by Figure 36(vi) or (xv). Finally if $d(u_1)>3$ then $c(u_i,u_{i+1})=0$ except for $c(u_4,u_5)=\frac{\pi}{5}$ and $c(u_6,u_1)=\frac{\pi}{15}$. So if $d(u_1)\geqslant 5$ then $c^*(\hat{\Delta}) \leqslant -\frac{4\pi}{15}+\frac{4\pi}{15}=0$; and if $d(u_1)=4$ then either $c(u_6,u_1)=0$ or $c(u_6,u_1)=\frac{\pi}{15}$, so either $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{\pi}{5}=\frac{\pi}{30}$ or $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{\pi}{5}+\frac{\pi}{15}=\frac{\pi}{10}$ and the two cases for $\hat{\Delta}$ are shown in Figure 36(vii) and (viii) or Figure 36(xvi) and (xvii).

This leaves the case $d(u_i)=3$ $(1\leqslant i\leqslant 6)$. Then $c(u_i,u_{i+1})=0$ except for $c(u_4,u_5)=\frac{\pi}{5}$, $c^*(\hat{\Delta}) \leqslant 0+\frac{\pi}{5}=\frac{\pi}{5}$ and $\hat{\Delta}$ is given by Figure 36(ix) or (xviii). □

We now describe the distribution of curvature from each of the eighteen regions $\hat{\Delta}$ of Figure 36.

Figure 36(i) *and* (x): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3}+\frac{11\pi}{30}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_1$ in each case.

Figure 36(ii) *and* (xi): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{3}+\frac{13\pi}{30}$; distribute $\frac{\pi}{10}$ from $\hat{\Delta}$ to $\hat{\Delta}_6$ in each case.

Figure 36(iii) *and* (xii): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{\pi}{5}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ in each case.

Figure 36(iv) *and* (xiii): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{7\pi}{30}$; distribute $\frac{\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ in each case.

Figure 36(v) *and* (xiv): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{3\pi}{10}$; distribute $\frac{2\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_1$ in each case.

Figure 36(vi) *and* (xv): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{\pi}{5}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_3$ in each case.

Figure 36(vii) *and* (xvi): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{\pi}{5}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ in each case.

Figure 36(viii) *and* (xvii): $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{6}+\frac{4\pi}{15}$; distribute $\frac{\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ and $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_3$ in each case.

Figure 36(ix) *and* (xviii): $c^*(\hat{\Delta}) \leqslant 0+\frac{\pi}{5}$; distribute $\frac{\pi}{10}$ from $\hat{\Delta}$ to $\hat{\Delta}_1$, $\frac{\pi}{15}$ from $\hat{\Delta}$ to $\hat{\Delta}_2$ and $\frac{\pi}{30}$ from $\hat{\Delta}$ to $\hat{\Delta}_3$ in each case.

Note: in all of the above cases $d(\hat{\Delta}_i)>6$ for each region $\hat{\Delta}_i$ that receives positive curvature from $\hat{\Delta}$ except possibly for $\hat{\Delta}_1$ in Figure 36(i) and (x).

Now assume that $\hat{\Delta}$ *is* $\hat{\Delta}_1$ of Figure 31(i) or (v). Then $\hat{\Delta}_1$ is given by Figures 37(i) and 38(i). (Recall that for now we are only considering distribution of curvature from Sections 5 and 6.)

First assume $d(u_3)\geqslant 5$. Then $c(w_1,u_3)=\frac{\pi}{15}$ and $c(u_3,u_2)=\frac{2\pi}{15}$ by Figure 35(ii)–(v). Since $c(u_1,u_2)=\frac{2\pi}{15}$ it follows that $c^*(\hat{\Delta}_1) \leqslant c(\hat{\Delta})+\frac{7\pi}{15}$. If $d(u_1)>3$ then $c(\hat{\Delta}_1) \leqslant c(3,3,3,4,4,5)=-\frac{9\pi}{15}$; on the other hand if $d(u_1)=3$ then $c(u_1,u_2)=0$ and $c^*(\hat{\Delta}_1) \leqslant c(3,3,3,3,4,5)+\frac{\pi}{3}<0$. Now let $d(u_3)=4$. Then $c(u_1,u_2)=\frac{2\pi}{15}$, $c(u_2,u_3)=\frac{7\pi}{30}$ and $c(u_3,w_1)=0$ so $c^*(\hat{\Delta}_1) \leqslant c(\hat{\Delta}_1)+\frac{\pi}{2}$. If $d(u_1)>3$ or $d(u_2)>3$ then $c(\hat{\Delta}_1) \leqslant -\frac{\pi}{2}$; on the other hand if $d(u_1)=d(u_2)=3$ then $c^*(\hat{\Delta}_1) \leqslant c(3,3,3,3,4,4)+\frac{11\pi}{30}=\frac{\pi}{30}$ as shown in Figures 37(ii) and 38(ii). Finally let $d(u_3)=3$. Then $c(u_3,w_1)=\frac{\pi}{5}$, $c(u_2,u_1)=\frac{2\pi}{15}$ and $c(u_2,u_3)=0$ so $c^*(\hat{\Delta}_1) \leqslant c(\hat{\Delta}_1)+\frac{7\pi}{15}$. If $d(u_1)=3$ then $c(u_1,u_2)=0$ and so $d(u_2)\geqslant 4$ would imply $c^*(\hat{\Delta}_1) \leqslant c(3,3,3,3,4,4)+\frac{\pi}{3}=0$, whereas if also $d(u_2)=3$ then $c^*(\hat{\Delta}_1) \leqslant \frac{\pi}{6}$ as shown in Figures 37(iii) and 38(iii). Let $d(u_1)=4$. If $d(u_2)\geqslant 4$ then $c^*(\hat{\Delta}_1) \leqslant c(3,3,3,4,4,4)+\frac{7\pi}{15}<0$ so assume $d(u_2)=3$. Reading clockwise from the $\hat{\Delta}_1$ corner label if $l(u_1)=bbx^{-1}y, bx^{-1}yb$ in Figures 37(i) and 38(i), respectively, then $c(u_1,u_2)=0$ and $c^*(\hat{\Delta}_2) \leqslant -\frac{\pi}{3}+\frac{\pi}{3}=0$; otherwise $c(u_1,u_2)=\frac{\pi}{15}$ and $\hat{\Delta}_1$ is given by Figures 37(iv) and 38(iv) and $c^*(\hat{\Delta}_1) \leqslant -\frac{\pi}{3}+\frac{6\pi}{15}=\frac{\pi}{15}$ as shown. This leaves $d(u_1)\geqslant 5$ in which case $c(u_1,u_2)=\frac{\pi}{15}$ and $c^*(\hat{\Delta}_1) \leqslant c(3,3,3,3,4,5)+\frac{6\pi}{15}<0$.

The distribution of curvature in Figures 37 and 38 is as follows.

Figure 37(ii) *and* 38(ii): $c^*(\hat{\Delta}_1) \leqslant -\frac{\pi}{3}+\frac{11\pi}{30}$; distribute $\frac{\pi}{30}$ from $\hat{\Delta}_1$ to $\hat{\Delta}_2$ in each case.

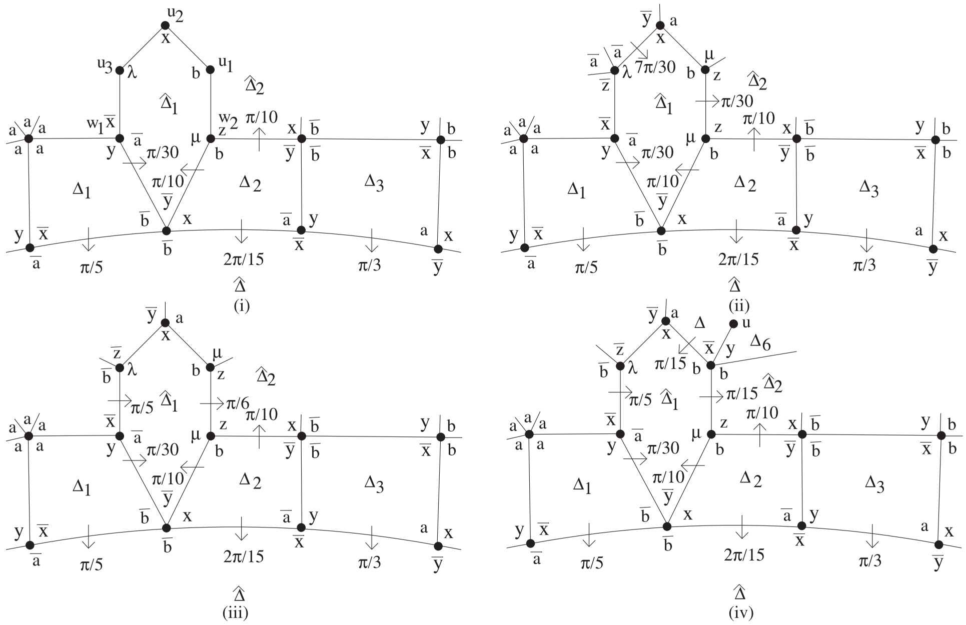

FIGURE 37. *Configuration A.*

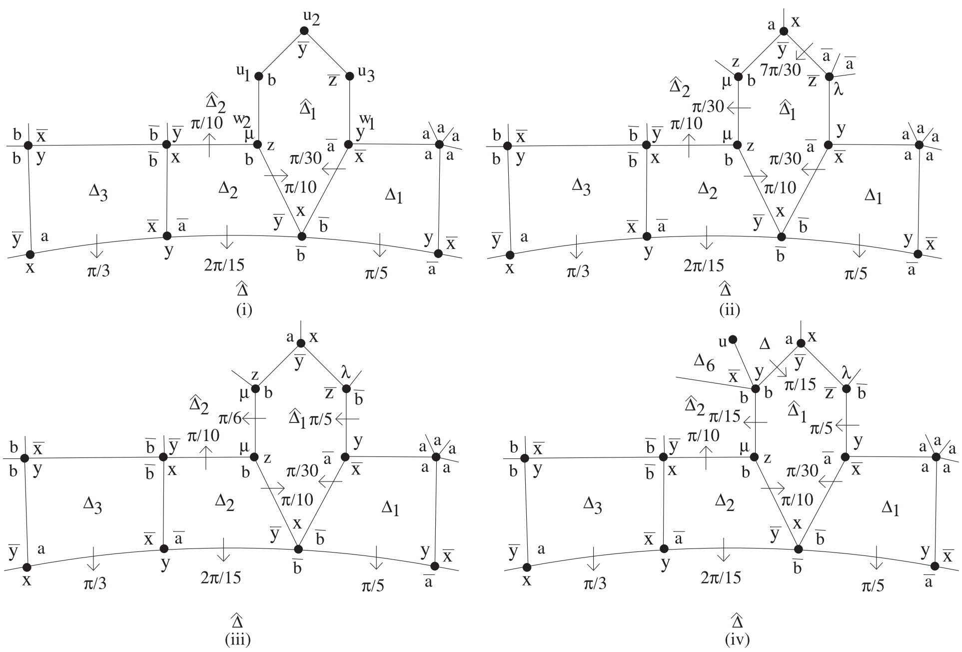

FIGURE 38. *Configuration B.*

Figure 37(iiii) *and* 38(ii): $c^*(\hat{\Delta}_1) \leqslant -\frac{\pi}{6} + \frac{\pi}{3}$; distribute $\frac{\pi}{6}$ from $\hat{\Delta}_1$ to $\hat{\Delta}_2$ in each case.
Figure 37(iv) *and* 38(iv): $c^*(\hat{\Delta}_1) \leqslant -\frac{\pi}{3} + \frac{6\pi}{15}$; distribute $\frac{\pi}{15}$ from $\hat{\Delta}_1$ to $\hat{\Delta}_2$ in each case.

LEMMA 7.2. *According to the distribution of curvature so far, that is, in Figures 6–32, 34 and 36–38, $\hat{\Delta}_1$ of Figures 37*(i) *and 38*(i) *does not receive positive curvature from $\hat{\Delta}_2$, that is, $c(w_2, u_1) = 0$.*

Proof. Consider $\hat{\Delta}_1$ of Figure 37(i). If $c(w_2, u_1) > 0$ then $\hat{\Delta}_2$ of Figure 37(i) is (the inverse of) $\hat{\Delta}$ of Figure 36(x) or $\hat{\Delta}_2$ is $\hat{\Delta}_1$ of Figure 38(iv). Suppose that $\hat{\Delta}_2$ is $\hat{\Delta}$ of Figure 36(x) Then since $\hat{\Delta}$ of Figure 36(x) must receive $\frac{\pi}{5}$ across its (v_4, v_5)-edge, the region $\hat{\Delta}_4$ of Figure 36(x) is given by Δ of Figure 7(iii); and this in turn forces $\hat{\Delta}_2$ of Figure 37(i) to be given by Figure 31(ii)–(iv) and not Configuration A of Figure 31(i), a contradiction. Moreover the region $\hat{\Delta}_2$ of Figure 37(i) cannot coincide with the region $\hat{\Delta}_1$ of Figure 38(iv) since, for example, the distribution of curvature from the region Δ_2 of Figure 37(i) is not the same as the distribution of curvature from the corresponding region Δ_2 of Figure 38(iv).

Consider $\hat{\Delta}_1$ of Figure 38(i). If $c(w_2, u_1) > 0$ then $\hat{\Delta}_2$ of Figure 38(i) is $\hat{\Delta}$ of Figure 36(i) or $\hat{\Delta}_2$ is $\hat{\Delta}_1$ of Figure 37(iv). If $\hat{\Delta}_2$ is $\hat{\Delta}$ of Figure 36(i) then a similar argument to the one above using Figures 36(i), 7(iii) and 31(vi)–(viii) applies to yield a contradiction; and $\hat{\Delta}_2$ of Figure 38(i) cannot coincide with the region $\hat{\Delta}_1$ of Figure 37(iv) since as above the distribution of curvature from the corresponding Δ_2 differs. □

Note. The upper bounds $c(u, v)$ of Figure 35 remain unchanged as a result of the distribution of curvature described in this section.

8. *Proof of Proposition* 4.2

An inspection of all distribution of curvature described so far yields the following. If positive curvature is distributed across an (x, a^{-1})-edge e into a region of degree greater than 4 then e is given by Figure 21(ii) (two cases), Figure 23(ii) (two cases), Figure 21(xi) and Figure 31(v). In particular if the x-corner vertex has degree 4 and the a^{-1}-corner vertex has degree 3 then e is given by Figure 31(v) (Configuration B). If positive curvature is distributed across an (a^{-1}, y^{-1})-edge e into a region of degree greater than 4 then e is given by Figure 21(ii) (two cases), Figure 23(ii) (two cases), Figure 22(ix) and Figure 31(i). In particular if the a^{-1}-corner has degree 3 and the y^{-1}-corner has degree 4 then e is given by Figure 31(i) (Configuration A).

LEMMA 8.1. *Let $\hat{\Delta}$ be a region of degree* 6 *that receives positive curvature across at least one edge. Then one of the following occurs.*

(i) $c^*(\hat{\Delta}) \leqslant 0$;
(ii) $c^*(\hat{\Delta}) > 0$ *is distributed to a region of degree greater than* 6;
(iii) $c^*(\hat{\Delta}) \in \left\{\frac{\pi}{30}, \frac{\pi}{15}\right\}$ *is distributed to a region Δ' of degree* 6 *and $c^*(\Delta') \leqslant 0$.*

Proof. It is clear from Figures 36–38 that if (i) and (ii) do not hold then $c^*(\hat{\Delta}) \in \left\{\frac{\pi}{30}, \frac{\pi}{15}\right)$ is distributed to $\hat{\Delta}_1$ of Figure 36(i) and (x) or $\hat{\Delta}_2$ of Figures 37(iv) and 38(iv). It follows that a region of degree 6 receives positive curvature from at most one region of degree 6. We treat each pair of cases in turn.

Consider $\hat{\Delta}_1$ of Figure 36(i) and (x). Then $\hat{\Delta}_1$ is given by Figure 39(i) and (ii) in which $d(w_4) > 3$. Observe that $d(\hat{\hat{\Delta}}) > 4$ and it follows that $\hat{\Delta}_1$ does not receive any positive curvature from $\hat{\hat{\Delta}}$ in Figure 39(i) and (ii). Note also from Figure 35 that $c(w_3, w_4) = \frac{2\pi}{15}$ or $\frac{\pi}{15}$; $c(u_2, w_4) = \frac{\pi}{10}$; and, from Note 3 following Table 3 at the start of Section 7, $c(w_1, w_2) + c(w_2, w_3) = \frac{7\pi}{30}$. Therefore $c^*(\hat{\Delta}_1) \leqslant c(\hat{\Delta}_1) + \frac{\pi}{2}$. If however $c(w_3, w_4) = \frac{2\pi}{15}$ then from Figure 35(iii) it follows that $c(\hat{\Delta}_1) \leqslant c(3, 3, 3, 4, 4, 5) = -\frac{3\pi}{5}$ and so $c^*(\hat{\Delta}_1) \leqslant 0$; so assume $c(w_3, w_4) = \frac{\pi}{15}$, $c^*(\hat{\Delta}_1) \leqslant$

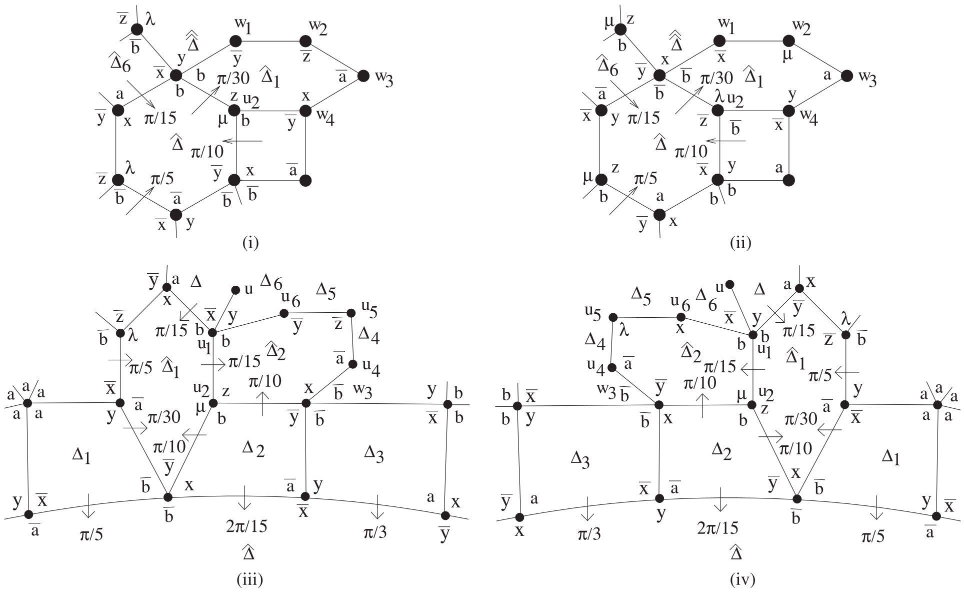

FIGURE 39. *Curvature to degree* 6 *regions.*

$c(\hat{\Delta}_1) + \frac{13\pi}{30}$. If $\hat{\Delta}_1$ has at least three vertices of degree greater than 3 then $c(\hat{\Delta}_1) \leqslant -\frac{\pi}{2}$; and if $d(w_4) \geqslant 5$ then $c(\hat{\Delta}_1) \leqslant c(3,3,3,3,4,5) = -\frac{13\pi}{30}$; this leaves $d(w_i) = 3$ $(1 \leqslant i \leqslant 3)$ and $d(w_4) = 4$ in which case $c(w_1, w_2) = 0$ and $c(w_2, w_3) = \frac{\pi}{5}$. If $c(w_3, w_4) = 0$ then $c^*(\hat{\Delta}_1) \leqslant -\frac{\pi}{3} + \frac{\pi}{3} = 0$. On the other hand if $c(w_3, w_4) > 0$ then it follows from the remark preceding the statement of the lemma that $\hat{\Delta}_1$ of Figure 39(i) and (ii) must coincide with $\hat{\Delta}_1$ of Figure 38(i) (Configuration B), Figure 37(i) (Configuration A). But the fact that $\hat{\Delta}_1$ receives $\frac{\pi}{30}$ from $\hat{\Delta}$ in Figure 39(i) and (ii) contradicts Lemma 7.2

Now consider $\hat{\Delta}_2$ of Figures 37(iv) and 38(iv) and assume $d(\hat{\Delta}_2) = 6$. Then $\hat{\Delta}_2$ is given by Figure 39(iii) and (iv) in which (see Figure 35) the following hold: $c(u_2, w_3) = \frac{\pi}{10}$; $c(u_5, u_6) = \frac{7\pi}{30}$ if $d(u_6) < 6$; and $c(u_5, u_6) = \frac{2\pi}{15}$ if $d(u_6) \geqslant 6$. Note that if $\hat{\Delta}_1$ of Figure 39(iii) and (iv) does not receive $\frac{\pi}{30}$ from Δ_1 then we are back in the previous case, so assume otherwise. In particular, according to Configurations A and B of Figure 31, this implies $c(u_4, u_5) \neq \frac{\pi}{5}$ and so $c(u_4, u_5) = \frac{\pi}{6}$; and note that if $d(u_4) = 6$ then $c(u_4, u_5) = \frac{2\pi}{15}$. Applying the statement at the beginning of this section, it follows by inspection of Figures 21(ii) and (xi), 22(ix), 23(ii) and 31(i) and (v) that if $c(w_3, u_4) > 0$ then $\hat{\Delta}_2$ of Figure 39(iii) and (iv) coincides with region $\hat{\Delta}_8$ of Figures 21(xi) and 22(xi) in which case $c(w_3, u_4) = \frac{\pi}{30}$ and $d(u_4) = 6$. Finally if $c(u_1, u_2) = \frac{\pi}{15}$ then $\hat{\Delta}_1$ must receive $\frac{\pi}{15}$ from Δ which implies $d(u) = 3$ and $d(\Delta_6) > 4$ and so $c(u_1, u_6) = 0$. On the other hand if $c(u_1, u_2) = \frac{\pi}{30}$ then (see Figure 35) either $c(u_1, u_6) = \frac{2\pi}{15}$ in which case $\hat{\Delta}_2$ is given by $\hat{\Delta}_3$ or $\hat{\Delta}_4$ of Figure 18(ii), in particular $d(u_6) \geqslant 6$; or $d(u_6) < 6$ and $c(u_1, u_6) = \frac{\pi}{15}$.

It follows that if $d(u_6) < 6$ then $c^*(\hat{\Delta}_2) = c(\hat{\Delta}_2) + c(u_2, w_3) + c(w_3, u_4) + c(u_4, u_5) + c(u_5, u_6) + (c(u_1, u_2) + c(u_1, u_6)) \leqslant c(\hat{\Delta}_2) + \frac{\pi}{10} + \frac{\pi}{30} + \frac{\pi}{6} + \frac{7\pi}{30} + \frac{\pi}{10} = c(\hat{\Delta}_2) + \frac{19\pi}{30}$; or if $d(u_6) \geqslant 6$ then $c^*(\hat{\Delta}_2) \leqslant c(\hat{\Delta}_2) + \frac{\pi}{10} + \frac{\pi}{30} + \frac{\pi}{6} + \frac{2\pi}{15} + \frac{\pi}{6} = c(\hat{\Delta}_2) + \frac{18\pi}{30}$.

Let $d(u_4) \geqslant 4$. If $d(u_6) \geqslant 4$ or $d(u_5) \geqslant 4$ then $c^*(\hat{\Delta}_2) \leqslant -\frac{2\pi}{3} + \frac{19\pi}{30} < 0$; on the other hand if $d(u_6) = d(u_5) = 3$ then $c(u_5, u_6) = 0$ and $c^*(\hat{\Delta}_2) \leqslant c(3,3,3,4,4,4) + (\frac{19\pi}{30} - \frac{7\pi}{30}) < 0$.

Let $d(u_4) = 3$ so, in particular, $c(w_3, u_4) = 0$. If $d(u_6) \geqslant 4$ and $d(u_5) \geqslant 4$ or if $d(u_6) = 3$ and $d(u_5) \geqslant 5$ or if $d(u_6) \geqslant 5$ and $d(u_5) = 3$ then $c(\hat{\Delta}_2) \leqslant -\frac{3\pi}{5}$ and it follows that $c^*(\hat{\Delta}_2) \leqslant 0$. If $d(u_6) = 4$ and $d(u_5) = 3$ then $d(\Delta_5) > 4$, $c(u_5, u_6) = 0$ and $c^*(\hat{\Delta}_2) \leqslant -\frac{\pi}{2} + \frac{\pi}{10} + 0 + \frac{\pi}{6} +$

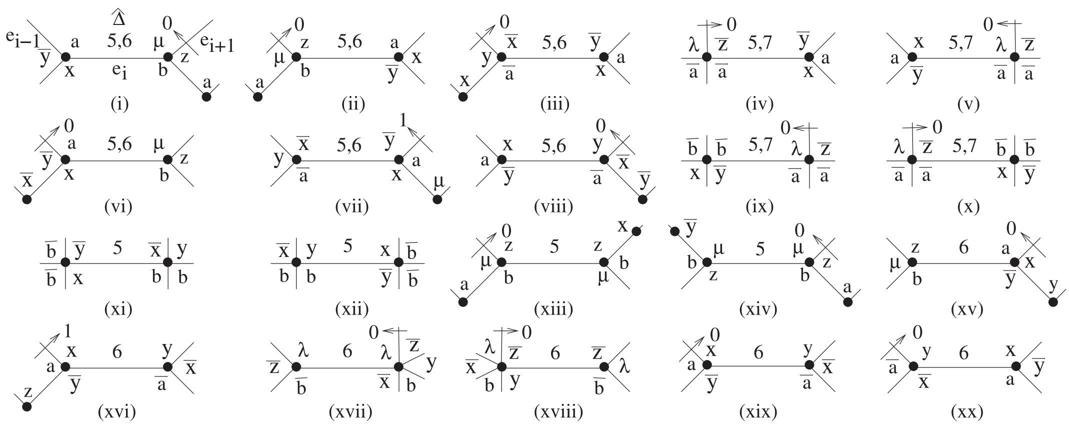

FIGURE 40. $c(u,v) > 2\pi/15$.

$0 + \frac{\pi}{10} < 0$; and if $d(u_6) = 3$ and $d(u_5) = 4$ then $d(\Delta_4) > 4$, $c(u_4, u_5) = 0$ and $c^*(\hat{\Delta}) \leqslant -\frac{\pi}{2} + \frac{\pi}{10} + 0 + 0 + \frac{7\pi}{30} + \frac{\pi}{10} < 0$. This leaves $d(u_5) = d(u_6) = 3$ in which case $c(u_5, u_6) = 0$. Moreover $d(\Delta_5) > 4$ also means that if $c(u_1, u_6) = \frac{\pi}{15}$ then Δ_6 is given by Δ of Figure 16(i) forcing the region Δ of Figure 39(iii) and (iv) to have degree greater than 4, a contradiction, so $c(u_1, u_6) = \frac{\pi}{30}$. Since, as noted above, $c(u_1, u_2) = \frac{\pi}{15}$ implies $c(u_1, u_6) = 0$ it follows that $c(u_1, u_2) + c(u_1, u_6) = \frac{\pi}{15}$ and $c^*(\hat{\Delta}) \leqslant c(3,3,3,3,4,4) + \frac{\pi}{10} + 0 + \frac{\pi}{6} + 0 + \frac{\pi}{15} = 0$. □

Proposition 4.2 follows immediately from Lemma 8.1.

9. *Two lemmas*

The first two steps of the proof have now been completed. Given this, only step three remains, that is, it remains to consider regions $\hat{\Delta}$ of degree at least 8. To do this we partition such $\hat{\Delta} \neq \Delta_0$ into regions of type $\mathcal{A}$ or type $\mathcal{B}$.

We say that $\hat{\Delta}$ is a *region of type* $\mathcal{B}$ if $\hat{\Delta}$ receives positive curvature from a region Δ of degree 4 shown in Figure 5 such that Δ has not received any positive curvature from any other region of degree 4 and such that either $d(v_3) = d(v_4) = 3$ only or $d(v_4) = d(v_1) = 3$ only. Thus $\hat{\Delta}$ is given by $\hat{\Delta}_3$ of Figure 13(i) or $\hat{\Delta}_4$ of Figure 14(i) or $\hat{\Delta}$ of Figure 31 or $\hat{\Delta}$ of Figure 32(i), (ii), (iii) or (v). Otherwise we will say that $\hat{\Delta}$ is a *region of type* $\mathcal{A}$.

There will be no further distribution of curvature in what follows and so we collect together in this section results that will be useful in Sections 10 and 11. The statements in the following lemma can be verified by inspecting Figures 6–39. Further details will appear in the proof of Lemma 10.1.

LEMMA 9.1. *Let e_i be an edge with endpoint u, v such that e_i is neither a (b,a)-edge nor is the edge of a region Δ across which positive curvature is transferred to a type $\mathcal{B}$ region.*

(i) *If $c(e_i) := c(u,v) > \frac{2\pi}{15}$ then $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}\}$.*
(ii) *If $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}\}$ then e_i is given by Figure 40 (in which possible $c(e_i)$ is given by multiples of $\frac{\pi}{30}$).*
(iii) *If $c(e_i) > \frac{2\pi}{15}$ then either $c(e_{i-1}) = 0$ or $c(e_{i+1}) = 0$ except for e_i of Figure 40*(vii), (xi), (xii) *and* (xvi).

Now assume that e_i be a (b,a)-edge and that transfer of curvature to a type $\mathcal{B}$ region is allowed.

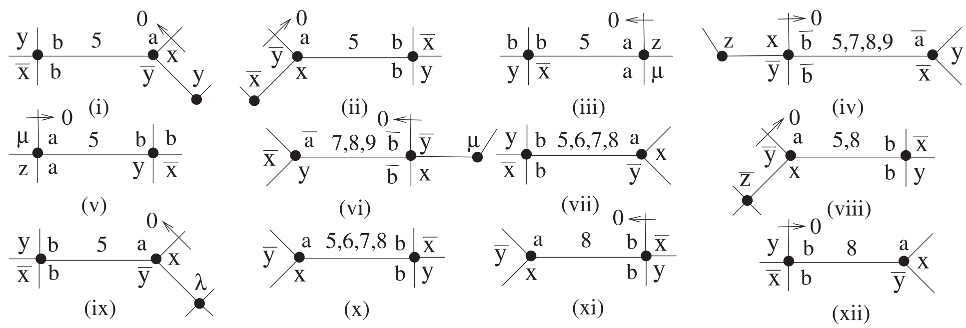

FIGURE 41. $c(u, v) > 2\pi/15$.

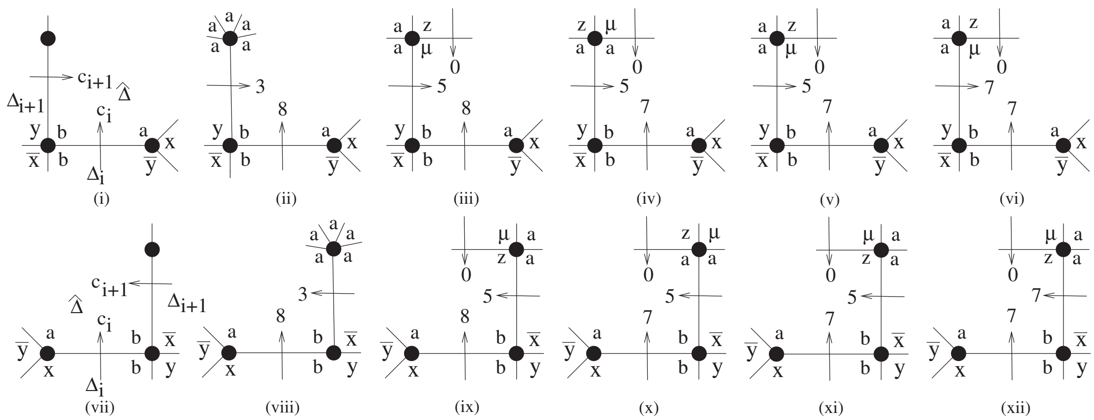

FIGURE 42. *Curvature across adjacent edges.*

(iv) *If* $c(e_i) > \frac{2\pi}{15}$ *then* $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}, \frac{4\pi}{15}, \frac{3\pi}{10}\}$.
(v) *If* $c(e_i) \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}, \frac{4\pi}{15}, \frac{3\pi}{10}\}$ *then* c_i *is given by Figure* 41.
(vi) *If* $c(e_i) > \frac{2\pi}{15}$ *then either* $c(e_{i-1}) = 0$ *or* $c(e_{i+1}) = 0$ *except for* e_i *of Figure* 41(vii) *and* (x).

REMARKS. (1) In verifying statement (iii) note that $\hat{\Delta}$ of Figure 40(xix) and (xx) corresponds to $\hat{\Delta}_1$ of Figure 32(iii) and (v), respectively.

(2) In Figure 40(vii) if $c(u, v) = \frac{\pi}{6}$ then $\hat{\Delta} = \hat{\Delta}_4$ of Figure 8(i)–(iii); if $c(u, v) = \frac{\pi}{5}$ then $\hat{\Delta} = \hat{\Delta}_4$ of Figure 8(iv); moreover the $\frac{\pi}{30}$ distributed across the e_{i+1} edge is given by Figure 36(viii) and (ix). In Figure 40(xi), $\hat{\Delta} = \hat{\Delta}_1$ of Figure 27(vii). In Figure 40(xii), $\hat{\Delta} = \hat{\Delta}_2$ of Figure 27(viii). In Figure 40(xiii), $\hat{\Delta} = \hat{\Delta}_2$ of Figure 37(iii). In Figure 40(xiv), $\hat{\Delta} = \hat{\Delta}_2$ of Figure 38(iii). In Figure 40(xvi), $\hat{\Delta} = \hat{\Delta}_3$ of Figure 10(i) and (ii); moreover the $\frac{\pi}{30}$ distributed across the e_{i-1} edge is given by Figure 36(xvii) and (xviii).

(3) In Figure 41(vii) if $c(u, v) = \frac{\pi}{5}$ then $\hat{\Delta}$ is given by Figure 31(v); and in Figure 41(x) if $c(u, v) = \frac{\pi}{5}$ then $\hat{\Delta}$ is given by Figure 31(i), in particular, $\hat{\Delta}$ in both cases is a type $\mathcal{B}$ region.

LEMMA 9.2. *Let the regions* $\hat{\Delta}$, Δ_i *and* Δ_{i+1} *be given by Figure* 42(i) *or* (vii).

(i) *If* $c_i = \frac{9\pi}{30}$ *then* $c_{i+1} = 0$.
(ii) *If* $c_i = \frac{8\pi}{30}$ *then* $c_{i+1} \leqslant \frac{5\pi}{30}$.

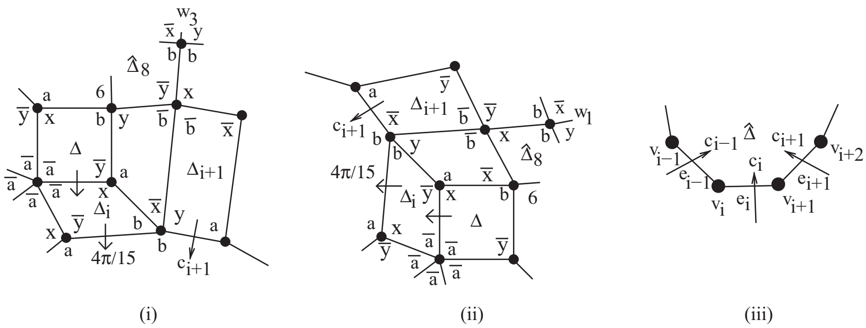

FIGURE 43. *Curvature from more than one region.*

(iii) *If* $c_i = \frac{8\pi}{30}$ *and* $c_{i+1} = \frac{3\pi}{30}$ *then* $\hat{\Delta}$ *of Figure* 42(i) *is given by Figure* 42(ii) *in which* $\Delta_{i+1} = \hat{\Delta}_3$ *of Figure* 22(xii) *or* $\Delta_{i+1} = \hat{\Delta}_4$ *of Figure* 29(x); *and* $\hat{\Delta}$ *of Figure* 42(vii) *is given by* $\hat{\Delta}$ *of Figure* 42(viii) *in which* $\Delta_{i+1} = \hat{\Delta}_4$ *of Figure* 21(xiv) *or* $\Delta_{i+1} = \hat{\Delta}_3$ *of Figure* 29(viii).

(iv) *If* $c_i = \frac{8\pi}{30}$ *then* $c_{i+1} \neq \frac{4\pi}{30}$.

(v) *If* $c_i = \frac{8\pi}{30}$ *and* $c_{i+1} = \frac{5\pi}{30}$ *then* $\hat{\Delta}$ *of Figure* 42(i) *is given by Figure* 42(iii) *in which* $\Delta_{i+1} = \Delta$ *of Figure* 24(viii); *and* $\hat{\Delta}$ *of Figure* 42(vii) *is given by Figure* 42(ix) *in which* $\Delta_{i+1} = \Delta$ *of Figure* 26(vi).

(vi) *If* $c_i = \frac{7\pi}{30}$ *then* $c_{i+1} \leqslant \frac{7\pi}{30}$.

(vii) *If* $c_i = \frac{7\pi}{30}$ *and* $c_{i+1} = \frac{5\pi}{30}$ *then* $\hat{\Delta}$ *of Figure* 42(i) *is given by Figure* 42(iv) *and* (v) *in which* $\Delta_{i+1} = \hat{\Delta}_4$ *of Figure* 17(xii) *and* $\Delta_{i+1} = \Delta$ *of Figure* 24(viii), *respectively; and* $\hat{\Delta}$ *of Figure* 42(vii) *is given by Figure* 42(x) *and* (xi) *in which* $\Delta_{i+1} = \hat{\Delta}_2$ *of Figure* 17(iv) *and* $\Delta_{i+1} = \Delta$ *of Figure* 26(vi), *respectively.*

(viii) *If* $c_i = \frac{7\pi}{30}$ *then* $c_{i+1} \neq \frac{6\pi}{30}$.

(ix) *If* $c_i = c_{i+1} = \frac{7\pi}{30}$ *then* $\hat{\Delta}$ *of Figure* 42(i) *is given by Figure* 42(vi) *in which* $\Delta_{i+1} = \hat{\Delta}_4$ *of Figure* 18(xi); *and* $\hat{\Delta}$ *of Figure* 42(vii) *is given by Figure* 42(xii) *in which* $\Delta_{i+1} = \hat{\Delta}_2$ *of Figure* 18(vii) *or of Figure* 29(v).

Proof. Statements (i), (vi) and (viii) follow from an inspection of Figures 40 and 41.

Moreover if $\hat{\Delta}$ is given by Figure 42(i) and $c_i = \frac{8\pi}{30}$ then it can be assumed without any loss that either $\Delta_i = \hat{\Delta}_2$ of Figure 22(iv) or (xiii) or $\Delta_i = \hat{\Delta}_4$ of Figure 29(xi); and if $\hat{\Delta}$ is given by Figure 42(vii) and $c_i = \frac{8\pi}{30}$ then it can be assumed without any loss that either $\Delta_i = \hat{\Delta}_4$ of Figure 21(vi) or (xv) or $\Delta_i = \hat{\Delta}_2$ of Figure 29(ix).

(ii) Let $\hat{\Delta}$ be given by Figure 42(i). If $c_{i+1} > \frac{5\pi}{30}$ then the only possibility is given by Figure 40(ix) in which case $c_{i+1} = \frac{7\pi}{30}$ and $\Delta_{i+1} = \hat{\Delta}_2$ of Figure 18(xi) where we note that (in Δ) $d(v_1) = 4$ and $d(v_2) = 3$. However if $\Delta_i = \hat{\Delta}_2$ of Figure 22(iv) then the vertex corresponding to v_1 is u_1 (see Figure 22(i)) which has degree 3; or if $\Delta_i = \hat{\Delta}_4$ of Figure 29(xi) then the vertex corresponding to v_1 is v_2 of Δ which has degree 5, in each case a contradiction. This leaves $\Delta_i = \hat{\Delta}_2$ of Figure 22(xiii), where $\Delta_{i+1} = \hat{\Delta}_7$ and this is shown in Figure 43(i) (recall that $\hat{\Delta}_8$ of Figure 22(xiii) is given by $\hat{\Delta}_8$ of Figure 22(x), hence w_3 of Figure 43(i)). But observe that w_3 is the vertex of Figure 43(i) that corresponds to v_2 of Figure 18(xi) and since w_3 has degree 4 again there is a contradiction.

Now let $\hat{\Delta}$ be given by Figure 42(vii). If $c_{i+1} > \frac{5\pi}{30}$ then the only possibility is given by Figure 40(x) in which case $c_{i+1} = \frac{7\pi}{30}$ and either $\Delta_{i+1} = \hat{\Delta}_2$ of Figure 18(vii) where in Δ $d(v_2) = 3$ and $d(v_3) = 4$ or $\Delta_{i+1} = \hat{\Delta}_2$ of Figure 29(v) where in $\Delta d(v_2) = 5$ and $d(v_3) = 4$. However if $\Delta_i = \hat{\Delta}_4$ of Figure 21(vi) the vertex corresponding to v_3 (both cases) is u_6 (see

Figure 21(iii)) which has degree 3; or if $\Delta_i = \hat{\Delta}_2$ of Figure 29(ix) the vertex corresponding to v_3 (both cases) is v_2 which has degree 5, in all cases a contradiction. This leaves $\Delta_i = \hat{\Delta}_4$ of Figure 21(xv) where $\Delta_{i+1} = \hat{\Delta}_7$ and this is shown in Figure 43(ii) (and here recall that $\hat{\Delta}_8$ of Figure 21(xv) is given by $\hat{\Delta}_8$ of Figure 21(xii), hence w_1 of Figure 43(ii)). But observe that the vertex of Figure 43(ii) corresponding to v_2 of Figures 18(vii), 29(v) is w_1 which has degree 4, again a contradiction.

(iii) Checking Figures 6–39 shows if $c_{i+1} = \frac{3\pi}{30}$ in Figure 42(i) then Δ_{i+1} must be one of Figures 11(vii) and (viii), 22(iii) and (xii), 29(x) or 31(ii). Given that $\Delta_i = \hat{\Delta}_2$ of Figure 22(iv) or (xiii) or $\Delta_i = \hat{\Delta}_4$ of Figure 29(xi) there is a vertex (degree or labelling) contradiction in each possible combination except when Δ_{i+1} is given by Figure 22(xii) or Figure 29(x) and these each yield Figure 42(ii). If $c_{i+1} = \frac{3\pi}{30}$ in Figure 43(ii) then Δ_{i+1} must be one of Figures 12(vii) and (viii), 21(v) and (xiv), 29(viii) or 31(vi). Given that $\Delta_i = \hat{\Delta}_4$ of Figure 21(vi) or (xv) or $\Delta_i = \hat{\Delta}_2$ of Figure 29(ix) again there is a vertex contradiction in each case except when Δ_{i+1} is given by Figure 21(xiv) or Figure 29(viii) and these yield Figure 42(viii).

(iv) If $c_{i+1} = \frac{4\pi}{30}$ in Figure 42(i) then Δ_{i+1} must be one of the Figures 16(iii), 18(ii) and (xix) or 31(v), but in each case there is a vertex contradiction when compared with Figure 22(iv) and (xiii) or 29(xi). (When comparing Figures 18(xix) and 22(xiii) we use Figure 43(i) for 22(xiii) as in case (ii) above.) If $c_{i+1} = \frac{4\pi}{30}$ in Figure 42(vii) then Δ_{i+1} must be one of Figures 16(ii), 18(ii), 18(xv) or 31(i), and again in each case there is a vertex contradiction when compared Figure 21(vi) and (xv) or 29(ix). (When comparing Figures 18(xv) and 21(xv) we use Figure 43(ii) for 21(xv) again as in case (ii) above.)

(v) The possibilities for Δ_{i+1} of Figure 42(i) are (see Figures 40(ix) and 41(v)) $\hat{\Delta}_4$ of Figure 17(xii) which yields a vertex contradiction when compared with Figure 22(iv) and (xiii) or 29(xi) and Δ of Figure 24(viii) which is given by Figure 42(iii); and for Δ_{i+1} of Figure 42(vii) are (see Figures 40(x) and 41(iii)) $\hat{\Delta}_2$ of Figure 17(iv) which yields a vertex contradiction when compared with Figure 21(vi) and 21(xv) or Figure 29(ix) and Δ of Figure 26(vi) which is given by Figure 42(ix).

Finally statement (vii) appears in the proof of (v); and statement (ix) appears in the proof of (ii). □

10. *Type $\mathcal{A}$ regions*

Throughout this section many assertions will be based on previous lemmas. Moreover *checking* means checking Figures 6–34 and 36–39. The reader is also referred to Figures 35, 40, 41 and 42.

The *surplus* s_i of an edge e_i is defined by $s_i = c_i - \frac{2\pi}{15}$ $(1 \leqslant i \leqslant k)$ where c_i is the maximum amount of curvature that is transferred across e_i. If we add s_i to c_{i+1}, c_{i-1} we will say that e_{i+1}, e_{i-1} (respectively) *absorbs* s_i from c_i. Checking Figures 40 and 41 shows, for example, that if $d(u_i) = d(u_{i+1}) = 3$ in Figure 43(iii) then $s_i \leqslant \frac{\pi}{15}$. The *deficit* δ_i of a vertex u_i of degree d_i is defined by $\delta_i = 2\pi(\frac{1}{d_i} - \frac{1}{3})$ and so if $d_i \geqslant 4$ then $\delta_i \leqslant -\frac{\pi}{6}$. If we add s_{i-1}, s_i (respectively) to δ_i we will say that u_i *absorbs* s_{i-1}, s_i from e_{i-1}, e_i (respectively).

LEMMA 10.1. *Let $\hat{\Delta}$ be a type $\mathcal{A}$ region of degree k. Then the following statement holds.* $c^*(\hat{\Delta}) \leqslant (2-k) + k.\frac{2\pi}{3} + k.\frac{2\pi}{15}$.

Proof. Denote the vertices of $\hat{\Delta}$ by v_i $(1 \leqslant i \leqslant k)$, the edges by e_i $(1 \leqslant i \leqslant k)$ and the degrees of the v_i by d_i $(1 \leqslant i \leqslant k)$. Let c_i denote the amount of curvature $\hat{\Delta}$ receives across the edge e_i $(1 \leqslant i \leqslant k)$. Consider the edge e_i of $\hat{\Delta}$ as shown in Figure 43(iii). If $c_i \leqslant \frac{2\pi}{15}$ there is nothing to consider, so let $c_i > \frac{2\pi}{15}$. Then by Lemma 9.1, $c_i \in \{\frac{\pi}{6}, \frac{\pi}{5}, \frac{7\pi}{30}, \frac{4\pi}{15}, \frac{3\pi}{10}\}$ and $\hat{\Delta}$ is given by Figures 40 and 41. First assume that e_i is not given by Figure 32(iii) or (v).

Let $\hat{\Delta}$ be given by Figure 40. If $\hat{\Delta}$ is given by Figure 40(i), (vii), (viii), (xiv) or (xv) then the edge e_{i+1} absorbs $s_i \leqslant \frac{\pi}{15}$ (from c_i). Note that in these cases $(d_{i+1}, c_{i+1}) \in \{(3,0),(3,\frac{\pi}{30})\}$. If $\hat{\Delta}$ is given by Figure 40(ii), (iii), (vi), (xiii), (xvi), (xix) or (xx) then e_{i-1} absorbs $s_i \leqslant \frac{\pi}{15}$. Note that $(d_i, c_{i-1}) \in \{(3,0),(3,\frac{\pi}{30})\}$. If $\hat{\Delta}$ is given by Figure 40(iv), (x) or (xviii) then the vertex v_i absorbs $s_i \leqslant \frac{\pi}{10}$. Note that $(d_i, c_{i-1}) \in \{(4,0),(5,0)\}$. If $\hat{\Delta}$ is given by Figure 40(v), (ix) or (xvii) then v_{i+1} absorbs $s_i \leqslant \frac{\pi}{10}$. Note that $(d_{i+1}, c_{i+1}) \in \{(4,0),(5,0)\}$. This leaves Figure 40(xi) and (xii) to be considered. If $\hat{\Delta}$ is given by Figure 40(xi) or (xii) then v_i absorbs $s_i = \frac{\pi}{30}$. Note that $d_i = 4$.

Now let $\hat{\Delta}$ be given by Figure 41. If $\hat{\Delta}$ is given by Figure 41(i) or (ix) then the edge e_{i+1} absorbs $s_i = \frac{\pi}{30}$. Note that $(d_{i+1}, c_{i+1}) = (3,0)$. If $\hat{\Delta}$ is given by Figure 41(ii) or (viii) restricted to the case $c_i = \frac{5\pi}{30}$ then e_{i-1} absorbs $s_i = \frac{\pi}{30}$. Note that $(d_i, c_{i-1}) = (3,0)$. If $\hat{\Delta}$ is given by Figure 41(iii), (vi) or (xi) then v_{i+1} absorbs $s_i \leqslant \frac{\pi}{6}$. Note that $(d_{i+1}, c_{i+1}) = (4,0)$. If $\hat{\Delta}$ is given by Figure 41(iv), (v) or (xii) then v_i absorbs $s_i \leqslant \frac{\pi}{6}$. Note that $(d_i, c_{i-1}) = (4,0)$. This leaves the cases Figure 41(vii) and (viii) with $c_i = \frac{8\pi}{30}$ and (x). If $\hat{\Delta}$ is given by Figure 41(vii) then v_i absorbs $s_i \leqslant \frac{2\pi}{15}$. Note that $d_i = 4$. If $\hat{\Delta}$ is given by Figure 41(viii) or (x) then v_{i+1} absorbs $s_i \leqslant \frac{2\pi}{15}$. Note that $d_{i+1} = 4$.

This completes absorption by edges or vertices when e_i is not given by Figure 32(iii) or (v) (and these correspond to cases of Figure 40(xix) and (xx)). Observe that if an edge e_j absorbs positive curvature a_j, say, then $a_j \leqslant \frac{\pi}{15}$ and either $c_j = 0$ or $c_j = \frac{\pi}{30}$; moreover e_j *always absorbs across a vertex of degree* 3. If $c_j = 0$ then $c_j + a_j \leqslant \frac{2\pi}{15}$ so let $c_j = \frac{\pi}{30}$. We claim that in this case we also have $c_j + a_j \leqslant \frac{2\pi}{15}$. The only possible way this fails is if $s_{j-1} = s_{j+1} = \frac{\pi}{15}$, that is, $c_{j-1} = c_{j+1} = \frac{\pi}{5}$. Thus $e_j = e_{i+1}$ of Figure 40(vii) and $\hat{\Delta} = \hat{\Delta}_4$ of Figure 8(iv); and also $e_j = e_{i-1}$ of Figure 40(xvi) and $\hat{\Delta} = \hat{\Delta}_3$ of Figure 10(i) and (ii). But any attempt at labelling shows that this is impossible and so our claim follows. Observe further that any pair of vertices each absorbing more than $\frac{\pi}{30}$ cannot coincide. This follows immediately from the fact that either $c_{i-1} = 0$ or $c_{i+1} = 0$ or the vertex is given by v_i of Figure 41(vii) or v_{i+1} of Figure 41(x) and clearly these cannot coincide. Also observe that if a vertex v_i say absorbs more than $\frac{2\pi}{15}$ from e_i or e_{i-1} (respectively) then it absorbs 0 from e_{i-1} or e_i (respectively). Therefore any given vertex can absorb at most $\frac{\pi}{6} + 0 = \frac{\pi}{6}$ as in Figure 41(iv) and (vi), or at most $\frac{2\pi}{15} + \frac{\pi}{30} = \frac{\pi}{6}$. But since any vertex that absorbs curvature *has degree at least* 4 and so a deficit of at most $-\frac{\pi}{6}$, the statement of the lemma holds for these cases.

Finally let e_i be given by Figure 32(iii) or (v). Since $d(v) \geqslant 4$ in both figures it follows that e_{i-1} does not absorb any surplus from e_{i-2}. If $s_{i+1} > \frac{\pi}{15}$ then according to the above it must be absorbed by $v_{i+2} = w$ (of Figure 32(iii) and (v)) and in this case e_{i-1} absorbs $s_i \leqslant \frac{\pi}{15}$; or if $s_{i+1} \leqslant \frac{\pi}{15}$ then let e_{i-1} absorb $s_i + s_{i+1} \leqslant \frac{2\pi}{15}$. Again the statement follows. □

PROPOSITION 10.2. *If $\hat{\Delta}$ is a type $\mathcal{A}$ region of degree k and $k \geqslant 10$ then $c^*(\hat{\Delta}) \leqslant 0$.*

Proof. This follows from Lemma 10.1 and the fact that $(2-k) + k.\frac{2\pi}{3} + k.\frac{2\pi}{15} \leqslant 0$ if and only if $k \geqslant 10$. □

It follows from Proposition 10.2 that we need only consider type $\mathcal{A}$ regions of degree at most 9. The following lemma applies to all regions $\hat{\Delta}$.

LEMMA 10.3. *If $7 \leqslant d(\hat{\Delta}) \leqslant 9$ then (up to cycle-permutation and corner labelling) either $d(\hat{\Delta}) = 8$ and $\hat{\Delta}$ is given by Figure 44(i)–(xi) or $d(\hat{\Delta}) = 9$ and $\hat{\Delta}$ is given by Figure 44(xii).*

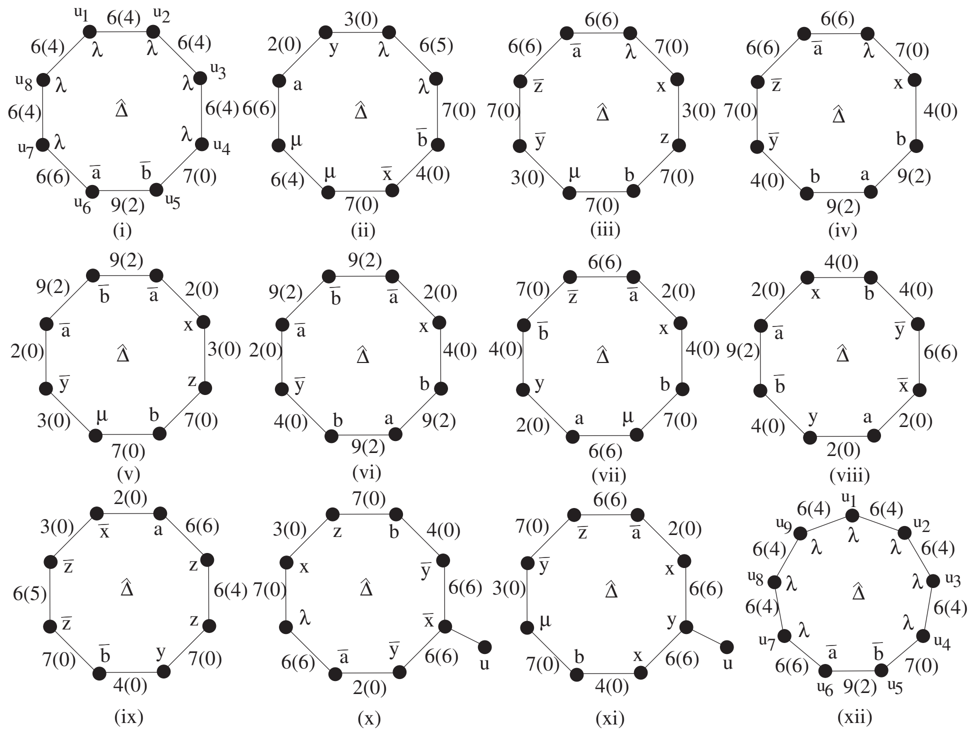

FIGURE 44. *Degree* 8 *regions.*

Proof. If $7 \leqslant d(\hat{\Delta}) \leqslant 9$ then $\hat{\Delta}$ is given by Figure 4(iv)–(xi). It turns out that there is (up to cyclic permutation and inversion) exactly one way to label $\hat{\Delta}$ of Figure 4(iv), (v), (ix) and (xi); four ways to label $\hat{\Delta}$ of (vi); six ways to label $\hat{\Delta}$ of (vii); and two ways to label $\hat{\Delta}$ of (viii) and (x). The resulting set of seventeen labelled regions contains some repeats with respect to corner labelling and deleting these leaves the twelve $\hat{\Delta}$ of Figure 44(i)–(xii). □

NOTATION. Let $d(\hat{\Delta}) = k$ and suppose that the vertices of $\hat{\Delta}$ are u_i $(1 \leqslant i \leqslant k)$. We write $cv(\hat{\Delta}) = (a_1, \ldots, a_k)$, where each a_i is a non-negative integer, to denote the fact that the total amount of curvature $\hat{\Delta}$ receives is bounded above by $(a_1 + \cdots + a_k)\frac{\pi}{30}$ with the understanding that $a_i \frac{\pi}{30}$ is transferred to $\hat{\Delta}$ across the (u_i, u_{i+1})-edge (subscripts mod k).

NOTATION. In the proof of Proposition 10.4 we will use non-negative integers a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, h_1, h_2, where

$$a_1 + a_2 = 7; b_1 + b_2 = 8; c_1 + c_2 = 9; d_1 + d_2 = 10; e_1 + e_2 = 11; h_1 + h_2 = 14.$$

Let $c(\Delta) = c(d_1, \ldots, d_m)$. Suppose $m = m_1 + m_2 + m_3 = 8 + k$ where $k \geqslant 0$ and suppose further that Δ contains m_1, m_2, m_3 vertices of degrees 3, 4, 5 (respectively). Then we will use the following formula (here and in the next section)

$$c(\Delta) = c(3, \ldots, 3, 4, \ldots, 4, 5, \ldots, 5) = -\frac{(20 + 10k + 5m_2 + 8m_3)\pi}{30}.$$

REMARK. Much use will be made here and in Section 11 of the fact that the region $\hat{\Delta}_1$ of Figure 36(i) and (x) receives no curvature from the region Δ shown. If $\hat{\Delta}_2$ of Figure 37(iv), 38(iv) receives $\frac{\pi}{15}$ from $\hat{\Delta}_1$ then $\hat{\Delta}_2$ receives no curvature from the region Δ_6 shown; however if $\hat{\Delta}_2$ receives $\frac{\pi}{30}$ from $\hat{\Delta}_1$ then $\hat{\Delta}_2$ may receive curvature from Δ_6 although note that $\hat{\Delta}_2$ has at least two vertices of degree greater than 3.

PROPOSITION 10.4. *If $\hat{\Delta}$ is a type $\mathcal{A}$ region and $7 \leqslant d(\hat{\Delta}) \leqslant 9$ then $c^*(\hat{\Delta}) \leqslant 0$.*

Proof. It follows from Lemma 10.3 that we need only consider $\hat{\Delta}$ of Figure 44 in which the label $\alpha(\beta)$ at the edge with endpoints u, v indicates $c(u,v) = \frac{\alpha\pi}{30}$ or $c(u,v) = \frac{\beta\pi}{30}$ when $d(u) = d(v) = 3$. We treat each of the twelve cases of Figure 44 in turn. (We will make extensive use of *checking* and Figures 35, 40, 41 and 42 often without explicit reference although for the reader's benefit full details will be given in Cases 1 and 4.)

Case 1. Let $\hat{\Delta}$ be given by Figure 44(i). Note that $\hat{\Delta}$ cannot be $\hat{\Delta}_2$ of Figure 37 or 38. If $c(u_1,u_2) > \frac{2\pi}{15}$ then, noting that Figure 40(xiv) does not apply to $\hat{\Delta}$, $d(u_1) > 3$ and $c(u_8,u_1) = 0$ (see Figure 40(xvii)); and if $\frac{\pi}{15} < c(u_1,u_2) < \frac{\pi}{5}$ then $c(u_1,u_2) \in \{\frac{2\pi}{15}, \frac{\pi}{10}\}$ and either $c(u_8,u_1) = 0$ or $c(u_2,u_3) = 0$ (see Figures 15(iii), 18(ix), 23(ix) and (xiii), 34(iv) and (vii) and 36(xiv) and (xviii)). Similar statements hold for each of (u_2,u_3), (u_3,u_4), (u_7,u_8) and (u_8,u_1). In particular it follows that $c(u_7,u_8)+c(u_8,u_1)+c(u_1,u_2)+c(u_2,u_3)+c(u_3,u_4) \leqslant \frac{2\pi}{3}$. Indeed the maximum is given by $(6+4+0+6+4)\frac{\pi}{30}$. If $c(u_4,u_5) > \frac{2\pi}{15}$ then (see Figure 40(ix)) $d(u_4) = d(u_5) = 4$ and $c(u_3,u_4) = 0$; if $c(u_6,u_7) > \frac{2\pi}{15}$ then (see Figure 40(i) and (vi)) $d(u_6) = d(u_7) = 3$ and either $c(u_5,u_6) = 0$ or $c(u_7,u_8) = 0$; and by Lemma 9.2 (see Figure 42(vi)), $c(u_4,u_5)+c(u_5,u_6) \leqslant \frac{7\pi}{15}$. Therefore if $c(u_4,u_5) > \frac{2\pi}{15}$ then $cv(\hat{\Delta}) = (0,6,0,h_1,h_2,6,0,6)$; and if $c(u_4,u_5) \leqslant \frac{2\pi}{15}$ then $c(u_4,u_5)+c(u_5,u_6) \leqslant \frac{11\pi}{30}$ (see Figure 42) and $cv(\hat{\Delta}) = (4,0,6,e_1,e_2,6,0,6)$. So if $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{35\pi}{30} + \frac{33\pi}{30} < 0$. If $d(u_1) = d(u_2) = 3$ and $c(u_1,u_2) > 0$ then $\hat{\Delta}$ is given by $\hat{\Delta}_1$ of Figure 36(xiv) or (xviii) therefore $c(u_1,u_2) = \frac{2\pi}{15}$ and $c(u_2,u_3) = 0$. Again similar statements hold for (u_2,u_3), (u_3,u_4), (u_7,u_8) and (u_8,u_1). Suppose that $\hat{\Delta}$ has no vertices of degree greater than 3. In particular $l(u_4) = \lambda b^{-1}z^{-1}$ and $l(u_5) = b^{-1}z^{-1}\lambda$. Then $c(u_4,u_5) = 0$ and $c(u_5,u_6) = \frac{\pi}{15}$ (see Figure 36(i)–(ix)) so it follows that $cv(\hat{\Delta}) = (4,0,4,0,2,4,4,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{3\pi}{5} < 0$. Suppose that $\hat{\Delta}$ has exactly one vertex of degree greater than 3. If $d(u_5) = 3$ then $c(u_4,u_5) = 0$ and $c(u_5,u_6) = \frac{\pi}{15}$ (see Figure 36) and it follows that $cv(\hat{\Delta}) = (0,6,4,0,2,6,0,4)$; and if $d(u_5) > 3$ then $d(u_4) = 3$ and $c(u_4,u_5) = \frac{\pi}{30}$ (see Figure 36(x)–(xviii)) so $cv(\hat{\Delta}) = (4,0,4,c_1,c_2,4,4,0)$. Therefore $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{5\pi}{6} = 0$. Finally suppose that $\hat{\Delta}$ has exactly two vertices u_i, u_j of degree greater than 3. If $d(u_5) = 3$ then $c^*(\hat{\Delta}) < 0$ so it can be assumed without any loss that $i = 5$. If $j = 1$ then (since $l(u_8) = \lambda b^{-1}z^{-1}$) $c(u_8,u_1) = 0$ and if $c(u_4,u_5) > 0$ then $c(u_4,u_5) = \frac{\pi}{30}$ and $c(u_5,u_6) = 0$ (see Figure 36(x)) so $cv(\hat{\Delta}) = (6,0,4,c_1,c_2,4,4,0)$; if $j = 2$ then $c(u_1,u_2) = 0$ and $cv(\hat{\Delta}) = (0,6,4,c_1,c_2,6,0,4)$; if $j = 3$ then $c(u_2,u_3) = 0$ and $cv(\hat{\Delta}) = (4,0,6,c_1,c_2,4,4,0)$; if $j = 4$ then $c(u_3,u_4) = 0$ and $cv(\hat{\Delta}) = (0,4,0,h_1,h_2,6,0,4)$; if $j = 6$ then $c(u_4,u_5) = 0$, $c(u_5,u_6) = \frac{\pi}{6}$ and $cv(\hat{\Delta}) = (4,0,4,0,5,4,4,0)$; if $j = 7$ then $cv(\hat{\Delta}) = (4,0,4,c_1,c_2,4,4,0)$; and if $j = 8$ then $c(u_7,u_8) = 0$ and $cv(\hat{\Delta}) = (4,0,4,c_1,c_2,6,0,6)$. It follows that $c^*(\hat{\Delta}) \leqslant -\pi + \frac{29\pi}{30} < 0$.

Case 2. Let $\hat{\Delta}$ be given by Figure 44(ii). If $c(u_3,u_4) > \frac{2\pi}{15}$ then, see Figure 40(ix), $(d(u_3), d(u_4)) = (4,4)$ and $c(u_2,u_3) = 0$; and if $c(u_5,u_6) > \frac{2\pi}{15}$ then, see Figure 40(v), $(d(u_5), d(u_6)) = (3,4)$ and $c(u_6,u_7) = 0$. It follows that if at least three of u_i have degree at least 4 then $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{7\pi}{6} = 0$, so assume otherwise. Note that if $d(u_2) = d(u_3) = 3$ and $c(u_2,u_3) > \frac{2\pi}{15}$ then $\hat{\Delta}$ is given by $\hat{\Delta}_2$ of Figure 38(iii); in particular, $c(u_2,u_3) = \frac{\pi}{6}$ and $d(u_1) = 4$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see (from Figure 44(ii) and Figure 38(iii)) that $cv(\hat{\Delta}) = (0,4,0,0,0,4,6,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{7\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. Then the following holds.

If $i=1$ then $cv(\hat{\Delta})=(3,5,0,0,0,4,6,2)$; if $i=2$ then $cv(\hat{\Delta})=(3,6,0,0,0,4,6,0)$; if $i=3$ then $cv(\hat{\Delta})=(0,6,4,0,0,4,6,0)$; if $i=4$ then $cv(\hat{\Delta})=(0,4,4,4,0,4,6,0)$; if $i=5$ then $cv(\hat{\Delta})=(0,4,0,4,4,4,6,0)$; if $i=6$ then $cv(\hat{\Delta})=(0,4,0,0,d_1,d_2,6,0)$; if $i=7$ then $cv(\hat{\Delta})=(0,4,0,0,0,6,6,0)$; and if $i=8$ then $cv(\hat{\Delta})=(0,4,0,0,0,4,6,2)$. It follows that $c^*(\hat{\Delta})\leqslant -\frac{5\pi}{6}+\frac{11\pi}{15}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_3)=d(u_4)=d(u_5)=3$ or $d(u_4)=d(u_5)=d(u_6)=3$ then $c^*(\hat{\Delta})\leqslant -\pi+\pi=0$. This leaves 14 of 28 cases to be considered. If $(i,j)=(1,4)$ then $cv(\hat{\Delta})=(3,5,4,4,0,4,6,2)$; if $(1,5)$ then $(3,5,0,4,4,4,6,2)$; if $(2,4)$ then $(3,6,4,4,0,4,6,0)$; if $(2,5)$ then $(3,6,0,4,4,4,6,0)$; if $(3,4)$ then $(0,d_1,d_2,4,0,4,6,0)$; if $(3,5)$ then $(0,6,4,4,4,4,6,0)$; if $(3,6)$ then $(0,6,4,0,d_1,d_2,6,0)$; if $(4,5)$ then $(0,4,4,4,4,4,6,0)$; if $(4,6)$ then $(0,4,4,4,d_1,d_2,6,0)$; if $(4,7)$ then $(0,4,4,4,0,6,6,0)$; if $(4,8)$ then $(0,4,4,4,0,4,6,2)$; if $(5,6)$ then $(0,4,0,4,4,6,6,0)$; if $(5,7)$ then $(0,4,0,4,4,6,6,0)$; and if $(5,8)$ then $(0,4,0,4,4,4,6,2)$. It follows that $c^*(\hat{\Delta})\leqslant -\pi+\frac{14\pi}{15}<0$.

Case 3. Let $\hat{\Delta}$ be given by Figure 44(iii). If $c(u_2,u_3)>\frac{2\pi}{15}$ then $(d(u_2),d(u_3))=(4,3)$ and $c(u_1,u_2)=0$; if $c(u_4,u_5)>\frac{2\pi}{15}$ then, see Figure 40(x), $(d(u_4),d(u_5))=(4,4)$ and $c(u_3,u_4)=0$; if $c(u_5,u_6)>\frac{2\pi}{15}$ then $(d(u_5),d(u_6))=(4,4)$ and $c(u_6,u_7)=0$; if $c(u_7,u_8)>\frac{2\pi}{15}$ then, see Figure 40(iv), $(d(u_7),d(u_8))=(3,4)$ and $c(u_8,u_1)=0$. Moreover if $c(u_1,u_2)>\frac{2\pi}{15}$ then $d(u_2)=3$ and $c(u_2,u_3)=0$; and if $c(u_8,u_1)>\frac{2\pi}{15}$ then $d(u_8)=3$ and $c(u_7,u_8)=0$. It follows that $c(u_1,u_2)+c(u_2,u_3)\leqslant\frac{4\pi}{15}$; $c(u_3,u_4)+c(u_4,u_5)\leqslant\frac{7\pi}{30}$; $c(u_5,u_6)+c(u_6,u_7)\leqslant\frac{7\pi}{30}$; and $c(u_7,u_8)+c(u_8,u_1)\leqslant\frac{4\pi}{15}$. Therefore if $\hat{\Delta}$ has at least two vertices of degree greater than 3 then $c^*(\hat{\Delta})\leqslant -\pi+\pi=0$. Suppose that $\hat{\Delta}$ contains no vertices of degree greater than 3. Then we see (from Figure 44(iii)) that $cv(\hat{\Delta})=(6,0,0,0,0,0,0,6)$ and $c^*(\hat{\Delta})\leqslant -\frac{2\pi}{3}+\frac{2\pi}{5}<0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. Then the following holds: if $i=1$ then $cv(\hat{\Delta})=(6,0,0,0,0,0,0,6)$; if $i=2$ then $cv(\hat{\Delta})=(b_1,b_2,0,0,0,0,0,6)$; if $i=3$ then $cv(\hat{\Delta})=(6,0,3,0,0,0,0,6)$; if $i=4$ then $cv(\hat{\Delta})=(6,0,3,4,0,0,0,6)$; if $i=5$ then $cv(\hat{\Delta})=(6,0,0,4,4,0,0,6)$; if $i=6$ then $cv(\hat{\Delta})=(6,0,0,0,4,3,0,6)$; if $i=7$ then $cv(\hat{\Delta})=(6,0,0,0,0,3,0,6)$; and if $i=8$ then $cv(\hat{\Delta})=(6,0,0,0,0,0,b_1,b_2)$. Therefore $c^*(\hat{\Delta})\leqslant -\frac{5\pi}{6}+\frac{2\pi}{3}<0$.

Case 4. Let $\hat{\Delta}$ be given by Figure 44(iv). If $c(u_1,u_2)>\frac{2\pi}{15}$ then $c(u_2,u_3)=0$; if $c(u_2,u_3)>\frac{2\pi}{15}$ then $c(u_1,u_2)=0$; if $c(u_8,u_1)>\frac{2\pi}{15}$ then $c(u_7,u_8)=0$; if $c(u_7,u_8)>\frac{2\pi}{15}$ then $c(u_8,u_1)=0$; if $c(u_4,u_5)=\frac{3\pi}{10}$ then, see Figure 41(iv), $c(u_3,u_4)=0$; if $c(u_4,u_5)=\frac{4\pi}{15}$ then $c(u_3,u_4)=\frac{\pi}{15}$ (see Figure 42); if $c(u_5,u_6)=\frac{3\pi}{10}$ then, see Figure 41(vi), $c(u_6,u_7)=0$; and if $c(u_5,u_6)=\frac{4\pi}{15}$ then $c(u_6,u_7)=\frac{\pi}{15}$ (see Figure 42). It follows that $c(u_3,u_4)+c(u_4,u_5)+c(u_5,u_6)+c(u_6,u_7)=\frac{11\pi}{15}$. Therefore $c^*(\hat{\Delta})\leqslant c(\Delta)+\frac{19\pi}{15}$ so if $\hat{\Delta}$ has at least four vertices of degree greater than 3 then $c^*(\hat{\Delta})<0$. Let $\hat{\Delta}$ have no vertices of degree greater than 3. Then $cv(\hat{\Delta})=(6,0,0,2,2,0,0,6)$ and $c^*(\hat{\Delta})\leqslant -\frac{2\pi}{3}+\frac{8\pi}{15}<0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. If $d(u_4)=3$ then $c(u_3,u_4)=0$ and $c(u_4,u_5)=\frac{\pi}{15}$; and if $d(u_6)=3$ then $c(u_5,u_6)=\frac{\pi}{15}$ and $c(u_6,u_7)=0$. Thus if $d(u_4)=d(u_6)=3$ then $c^*(\hat{\Delta})\leqslant -\frac{5\pi}{6}+\frac{2\pi}{3}<0$; if $d(u_4)>3$ then $cv(\hat{\Delta})=(6,0,e_1,e_2,2,0,0,6)$; and if $d(u_6)>3$ then $cv(\hat{\Delta})=(6,0,0,2,e_1,e_2,0,6)$. Therefore $c^*(\hat{\Delta})\leqslant -\frac{5\pi}{6}+\frac{5\pi}{6}=0$. Let $\hat{\Delta}$ have exactly two vertices of degree greater than 3. If $d(u_4)=3$ or $d(u_6)=3$ then $c^*(\hat{\Delta})\leqslant -\pi+\frac{29\pi}{30}<0$ so it can be assumed $d(u_4)>3$ and $d(u_6)>3$. Then $d(u_2)=3$ implies $d(\Delta_2)>4$ and $c(u_2,u_3)=0$; and $d(u_8)=3$ implies $d(\Delta_7)>4$ and $c(u_7,u_8)=0$. This then prevents $c(u_3,u_4)=\frac{2\pi}{15}$ or $c(u_6,u_7)=\frac{2\pi}{15}$ (see Figure 16(ii) and (iii)) so $c(u_3,u_4)=c(u_7,u_8)=\frac{\pi}{15}$. Since $c(u_1,u_2)=c(u_8,u_1)=\frac{\pi}{5}$ it follows that if $c(u_4,u_5)\neq\frac{4\pi}{15}$ and $c(u_5,u_6)\neq\frac{4\pi}{15}$ then $cv(\hat{\Delta})=(6,0,c_1,c_2,c_1,c_2,0,6)$ and $c^*(\hat{\Delta})\leqslant -\pi+\pi=0$. Suppose $c(u_4,u_5)=\frac{4\pi}{15}$, say. If $c(u_3,u_4)>0$ then $\hat{\Delta}=\hat{\Delta}_5$ of Figure 29(vi) or (ix) where $l(v_2)=b^5$ and so $c(u_3,u_4)>0$ implies $\Delta_3=\Delta$ of Figure 28. But $d(\Delta_2)>4$ forces $\Delta_3=\Delta$ of Figure 28(i) and $c(u_3,u_4)=\frac{\pi}{30}$. Similarly if $c(u_5,u_6)=\frac{4\pi}{15}$ and $c(u_6,u_7)>0$ then we see from Figure 29(xi) and Figure 28(i) and (iii) that $c(u_6,u_7)=\frac{\pi}{30}$. It follows that if $c(u_4,u_5)=\frac{4\pi}{15}$ or $c(u_5,u_6)=\frac{4\pi}{15}$ then $c^*(\hat{\Delta})\leqslant -\pi+\pi=0$. Finally let $\hat{\Delta}$ have exactly three vertices u_i,u_j,j_k of degree greater than 3. Then $c(\hat{\Delta})\leqslant -\frac{7\pi}{6}$. If $d(u_2)=d(u_8)=3$ then $c(u_2,u_3)=c(u_7,u_8)=0$ and $cv(\hat{\Delta})=(6,0,e_1,e_2,e_1,e_2,0,6)$; if $d(u_4)=3$ then $cv(\hat{\Delta})=(b_1,b_2,0,2,e_1,e_2,b_1,b_2)$; and if $d(u_6)=3$ then $cv(\hat{\Delta})=(b_1,b_2,e_1,e_2,2,0,b_1,b_2)$. So it can be assumed $(i,j,k)=(2,4,6)$ or $(4,6,8)$ and in both cases $c^*(\hat{\Delta})\leqslant c(\hat{\Delta})+\frac{36\pi}{30}$. If $d(u_i)$ or $d(u_j)$ or $d(u_k)$ is greater than 4 then $c^*(\hat{\Delta})<0$ so assume otherwise. But now $d(u_2)=4$ implies $l(u_2)=\lambda z^{-1}a^{-2}$ and $c(u_1,u_2)=0$; and $d(u_8)=4$ forces $c(u_8,u_1)=0$. It follows that $cv(\hat{\Delta})=(0,7,e_1,e_2,e_1,e_2,0,6)$ or $(6,0,e_1,e_2,e_1,e_2,7,0)$ so $c^*(\hat{\Delta})\leqslant -\frac{7\pi}{6}+\frac{7\pi}{6}=0$.

Case 5. Let $\hat{\Delta}$ be given by Figure 44(v). If $c(u_1,u_2)=\frac{9\pi}{30}$ then $c(u_8,u_1)=0$; if $c(u_1,u_2)=\frac{4\pi}{15}$ then $c(u_8,u_1)=\frac{\pi}{10}$ (see Figure 42); if $c(u_8,u_1)=\frac{3\pi}{10}$ then $c(u_1,u_2)=0$; if $c(u_8,u_1)=\frac{4\pi}{15}$ then $c(u_1,u_2)=\frac{\pi}{10}$; if $c(u_4,u_5)>\frac{2\pi}{15}$ then $c(u_3,u_4)=0$; and if $c(u_5,u_6)>\frac{2\pi}{15}$ then $c(u_6,u_7)=0$. It follows that $c(u_8,u_1)+c(u_1,u_2)=\frac{7\pi}{15}$; $c(u_3,u_4)+c(u_4,u_5)=\frac{7\pi}{30}$; and $c(u_5,u_6)+c(u_6,u_7)=\frac{7\pi}{30}$ so $c^*(\hat{\Delta})\leqslant c(\hat{\Delta})+\frac{16\pi}{15}$. Therefore if $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta})<0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see (from Figure 44(v)) that $c^*(\hat{\Delta})\leqslant\frac{2\pi}{3}+\frac{2\pi}{15}<0$. Observe that if $d(u_1)=3$ then $c(u_8,u_1)+c(u_1,u_2)=\frac{2\pi}{15}$; and if $d(u_5)=3$ then $c(u_4,u_5)=c(u_5,u_6)=0$. It follows that if $d(u_1)=3$ or $d(u_5)=3$ then $c^*(\hat{\Delta})\leqslant c(\hat{\Delta})+\frac{11\pi}{15}<0$ and so if $\hat{\Delta}$ has exactly one vertex of degree greater than 3 then $c^*(\hat{\Delta})<0$. If $\hat{\Delta}$ has exactly two vertices u_i,u_j of degree greater than 3 it can be assumed $(i,j)=(1,5)$ in which case $cv(\hat{\Delta})=(h_1,0,0,7,7,0,0,h_2)$. Therefore $c^*(\hat{\Delta})\leqslant -\pi+\frac{14\pi}{15}<0$.

Case 6. Let $\hat{\Delta}$ be given by Figure 44(vi). If $c(u_4,u_5)=\frac{3\pi}{10}$ then $c(u_3,u_4)=0$; if $c(u_4,u_5)=\frac{4\pi}{15}$ then $c(u_3,u_4)=\frac{\pi}{15}$ (see Figure 42); if $c(u_5,u_6)=\frac{3\pi}{10}$ then $c(u_6,u_7)=0$; if $c(u_5,u_6)=\frac{4\pi}{15}$ then $c(u_6,u_7)=\frac{\pi}{15}$; and as in Case 5, $c(u_8,u_1)+c(u_1,u_2)=\frac{7\pi}{15}$. It follows that $c^*(\hat{\Delta})\leqslant c(\hat{\Delta})+\frac{4\pi}{3}$ so if $\hat{\Delta}$ has at least four vertices of degree greater than 3 then $c^*(\hat{\Delta})\leqslant 0$. Let $\hat{\Delta}$ have no vertices of degree greater than 3. Then $cv(\hat{\Delta})=(2,0,0,2,2,0,0,2)$ and $c^*(\hat{\Delta})\leqslant -\frac{2\pi}{3}+\frac{4\pi}{15}<0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. Note that if $d(u_1)=3$ then $c(u_8,u_1)=c(u_1,u_2)=\frac{\pi}{15}$; if $d(u_4)=3$ then $c(u_3,u_4)=0$ and $c(u_4,u_5)=\frac{\pi}{15}$; and if $d(u_6)=3$ then $c(u_5,u_6)=\frac{\pi}{15}$ and $c(u_6,u_7)=0$. If $i=1$ then $cv(\hat{\Delta})=(h_1,0,0,2,2,0,0,h_2)$; if $i=2$ then $cv(\hat{\Delta})=(2,2,0,2,2,0,0,2)$; if $i=3$ then $cv(\hat{\Delta})=(2,2,4,2,2,0,0,2)$; if $i=4$ then $cv(\hat{\Delta})=(2,0,e_1,e_2,2,0,0,2)$; if $i=5$ then $cv(\hat{\Delta})=(2,0,0,9,9,0,0,2)$; if $i=6$ then $cv(\hat{\Delta})=(2,0,0,2,e_1,e_2,0,2)$; if $i=7$ then $cv(\hat{\Delta})=(2,0,0,2,2,4,2,2)$; and if $i=8$ then $cv(\hat{\Delta})=(2,0,0,2,2,0,2,2)$. Therefore $c^*(\hat{\Delta})\leqslant -\frac{5\pi}{6}+\frac{11\pi}{15}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. Then $c(\hat{\Delta})\leqslant -\pi$. If $d(u_1)=3$ then $cv(\hat{\Delta})=(2,2,e_1,e_2,e_1,e_2,2,2)$ and $c^*(\hat{\Delta})\leqslant 0$ so it can be assumed $i=1$. If $j=2$ then $cv(\hat{\Delta})=(h_1,2,0,2,2,0,0,h_2)$; if $j=3$ then $cv(\hat{\Delta})=(h_1,2,4,2,2,0,0,h_2)$; if $j=4$ then $cv(\hat{\Delta})=(h_1,0,e_1,e_2,2,0,0,h_2)$; if $j=5$ then $cv(\hat{\Delta})=(h_1,0,0,2,2,0,0,h_2)$; if $j=6$ then $cv(\hat{\Delta})=(h_1,0,0,2,e_1,e_2,0,h_2)$; if $j=7$ then $cv(\hat{\Delta})=(h_1,0,0,2,2,4,2,h_2)$; and if $j=8$ then $cv(\hat{\Delta})=(h_1,0,0,2,2,0,2,h_2)$. Therefore $c^*(\hat{\Delta})\leqslant -\pi+\frac{9\pi}{10}<0$. Let $\hat{\Delta}$ have exactly three vertices of degree greater than 3 so that $c(\hat{\Delta})\leqslant -\frac{7\pi}{6}$. If $d(u_1)=3$ then $c^*(\hat{\Delta})\leqslant -\frac{7\pi}{6}+\pi$; if $d(u_4)=3$ then $c^*(\hat{\Delta})\leqslant -\frac{7\pi}{6}+\frac{31\pi}{30}$; and if $d(u_6)=3$ then $c^*(\hat{\Delta})\leqslant\frac{7\pi}{6}+\frac{31\pi}{30}$. So it can be assumed $d(u_1)>3$, $d(u_4)>3$ and $d(u_6)>3$ in which case $cv(\hat{\Delta})=(h_1,0,e_1,e_2,e_1,e_2,0,h_2)$. If $d(u_1)>4$ then $c^*(\hat{\Delta})\leqslant -\frac{19}{15}\pi+\frac{18}{15}\pi<0$, whereas if $d(u_1)=4$ then the fact that $d(u_2)=d(u_8)=3$ means that $l(u_1)=bbx^{-1}y$ forces either $c(u_1,u_2)=0$ or $c(u_8,u_1)=0$ and $c^*(\hat{\Delta})\leqslant -\frac{7\pi}{6}+\frac{31\pi}{30}<0$.

Case 7. Let $\hat{\Delta}$ be given by Figure 44(vii) and note that $\hat{\Delta}$ cannot be $\hat{\Delta}_2$ of Figure 37(iv) or 38(iv). If $c(u_1,u_2)>\frac{2\pi}{15}$ then $d(u_1)=3$ and $c(u_8,u_1)=\frac{\pi}{30}$; if $c(u_4,u_5)>\frac{2\pi}{15}$ then $c(u_5,u_6)=0$;

if $c(u_5,u_6) > \frac{2\pi}{15}$ then $d(u_5)=3$ and $c(u_4,u_5)=\frac{\pi}{30}$; and if $c(u_8,u_1) > \frac{2\pi}{15}$ then $c(u_1,u_2)=0$. It follows that $c(u_8,u_1)+c(u_1,u_2) \leqslant \frac{4\pi}{15}$ and $c(u_4,u_5)+c(u_5,u_6) \leqslant \frac{4\pi}{15}$. If $\hat{\Delta}$ has at least two vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\pi + \frac{14\pi}{15} < 0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see (from Figure 44(vii)) that $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{8\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly one vertex of degree greater than 3. If $d(u_3)=d(u_4)=3$ then $c(u_3,u_4)=0$ and $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{4\pi}{5} < 0$; if $d(u_3)>3$ then $cv(\hat{\Delta})=(6,2,4,0,6,0,0,0)$; if $d(u_4)>3$ then $cv(\hat{\Delta})=(6,0,4,b_1,b_2,0,0,0)$; and it follows that $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{18\pi}{30} < 0$.

Case 8. Let $\hat{\Delta}$ be given by Figure 44(viii). Then $cv(\hat{\Delta})=(4,4,6,2,2,4,9,2)$ so if $\hat{\Delta}$ has at least three vertices of degree 2 then $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$. Note that if $d(u_2)=3$ then $c(u_1,u_2)=c(u_2,u_3)=0$ and if $d(u_7)=3$ then $c(u_6,u_7)=0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then $cv(\hat{\Delta})=(0,0,6,0,0,0,2,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{4\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. If $i=1$ then $cv(\hat{\Delta})=(0,0,6,0,0,0,2,2)$; if $i=2$ then $cv(\hat{\Delta})=(4,4,6,0,0,0,2,0)$; if $i=3$ then $cv(\hat{\Delta})=(0,0,6,0,0,0,2,0)$; if $i=4$ then $cv(\hat{\Delta})=(0,0,6,2,0,0,2,0)$; if $i=5$ then $cv(\hat{\Delta})=(0,0,6,2,2,0,2,0)$; if $i=6$ then $cv(\hat{\Delta})=(0,0,6,0,2,0,2,0)$; if $i=7$ then $cv(\hat{\Delta})=(0,0,0,6,0,0,4,9,0)$; and if $i=8$ then $cv(\hat{\Delta})=(0,0,6,0,0,0,9,2)$. Therefore $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{19\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices of degree greater than 3. If $d(u_2)=3$ or $d(u_7)=3$ then $c^*(\hat{\Delta}) \leqslant -\pi + \frac{29\pi}{30} < 0$ so assume that $d(u_2)>3$ and $d(u_7)>3$. Then $cv(\hat{\Delta})=(4,4,6,0,0,4,9,0)$ and $c^*(\hat{\Delta}) \leqslant -\pi + \frac{9\pi}{10} < 0$.

Case 9. Let $\hat{\Delta}$ be given by Figure 44(ix). If $c(u_4,u_5) > \frac{2\pi}{15}$ then $d(u_4)=4$ and $c(u_3,u_4)=0$; and if $c(u_6,u_7) > \frac{2\pi}{15}$ then $(d(u_6),d(u_7))=(4,4)$ and $c(u_7,u_8)=0$. It follows that if at least three of the u_i have degree at least 4 then $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{7\pi}{6} = 0$, so assume otherwise. If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see (from Figure 44(ix) and the fact that $\hat{\Delta}$ cannot then be $\hat{\Delta}_2$ of Figure 37(iii)) that $cv(\hat{\Delta})=(0,6,4,0,0,0,4,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{7\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. Then the following holds. If $i=1$ then $cv(\hat{\Delta})=(2,6,4,0,0,0,5,3)$; if $i=2$ then $cv(\hat{\Delta})=(2,6,4,0,0,0,4,0)$; if $i=3$ then $cv(\hat{\Delta})=(0,6,6,0,0,0,4,0)$; if $i=4$ then $cv(\hat{\Delta})=(0,6,d_1,d_2,0,0,4,0)$; if $i=5$ then $cv(\hat{\Delta})=(0,6,4,4,4,0,4,0)$; if $i=6$ then $cv(\hat{\Delta})=(0,6,4,0,4,4,4,0)$; if $i=7$ then $cv(\hat{\Delta})=(0,6,4,0,0,4,6,0)$; and if $i=8$ then $cv(\hat{\Delta})=(0,6,4,0,0,0,6,3)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{11\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_4)=d(u_5)=d(u_6)=3$ or $d(u_5)=d(u_6)=d(u_7)=3$ then $c^*(\hat{\Delta}) \leqslant -\pi+\pi=0$. This leaves 14 of 28 cases to be considered. If $(i,j)=(1,5)$ then $cv(\hat{\Delta})=(2,6,4,4,4,0,5,3)$; if $(i,j)=(1,6)$ then $cv(\hat{\Delta})=(2,6,4,0,4,4,5,3)$; if $(i,j)=(2,5)$ then $cv(\hat{\Delta})=(2,6,4,4,4,0,4,0)$; if $(i,j)=(2,6)$ then $cv(\hat{\Delta})=(2,6,4,0,4,4,4,0)$; if $(i,j)=(3,5)$ then $cv(\hat{\Delta})=(0,6,6,4,4,0,4,0)$; if $(i,j)=(3,6)$ then $cv(\hat{\Delta})=(0,6,6,0,4,4,4,0)$; if $(i,j)=(4,5)$ then $cv(\hat{\Delta})=(0,6,d_1,d_2,4,0,4,0)$; if $(i,j)=(4,6)$ then $cv(\hat{\Delta})=(0,6,d_1,d_2,4,4,4,0)$; if $(i,j)=(4,7)$ then $cv(\hat{\Delta})=(0,6,d_1,d_2,0,4,4,0)$; if $(i,j)=(5,6)$ then $cv(\hat{\Delta})=(0,6,4,4,4,4,4,0)$; if $(i,j)=(5,7)$ then $cv(\hat{\Delta})=(0,6,4,4,4,4,4,0)$; if $(i,j)=(5,8)$ then $cv(\hat{\Delta})=(0,6,4,4,4,0,4,3)$; if $(i,j)=(6,7)$ then $cv(\hat{\Delta})=(0,6,4,0,4,d_1,d_2,0)$; and if $(i,j)=(6,8)$ then $cv(\hat{\Delta})=(0,6,4,0,4,4,4,3)$. It follows that $c^*(\hat{\Delta}) \leqslant= -\pi + \frac{14\pi}{15} < 0$.

Case 10. Let $\hat{\Delta}$ be given by Figure 44(x). If $c(u_1,u_2) > \frac{2\pi}{15}$ then $(d(u_1),d(u_2))=(4,4)$ and $c(u_8,u_1)=0$; if $c(u_7,u_8) > \frac{2\pi}{15}$ then $(d(u_7),d(u_8))=(4,3)$ and $c(u_6,u_7)=0$; and if $c(u_6,u_7) > \frac{2\pi}{15}$ then $d(u_7)=3$ forcing $c(u_7,u_8)=0$. It follows that $c(u_8,u_1)+c(u_1,u_2) \leqslant \frac{7\pi}{30}$; and $c(u_6,u_7)+c(u_7,u_8) \leqslant \frac{4\pi}{15}$. If $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see (from Figure 44(x)) that $cv(\hat{\Delta})=(0,0,6,6,0,6,0,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{18\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. Then the following holds. If $i=1$ then $d(u_2)=3$ and so

$cv(\hat{\Delta}) = (0,0,6,6,0,6,0,3)$; if $i = 2$ then $d(u_1) = 3$ and so $cv(\hat{\Delta}) = (x_1, y_1, 6, 6, 0, 6, 0, 0)$ where, by the remark preceding this lemma, $x_1 + y_1 = 4$; if $i = 3$ then $cv(\hat{\Delta}) = (0,4,6,6,0,6,0,0)$; if $i = 4$ then $cv(\hat{\Delta}) = (0,0,6,6,0,6,0,0)$; if $i = 5$ then $cv(\hat{\Delta}) = (0,0,6,6,2,6,0,0)$; if $i = 6$ then $cv(\hat{\Delta}) = (0,0,6,6,2,6,0,0)$; if $i = 7$ then $cv(\hat{\Delta}) = (0,0,6,6,0,b_1,b_2,0)$; and if $i = 8$ then $d(u_7) = 3$ so $cv(\hat{\Delta}) = (0,0,6,6,0,6,0,3)$. Therefore $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{11\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly two vertices of degree greater than 3. If $d(u_1) = 3$ or $d(u_2) = 3$ then $cv(\hat{\Delta}) = (x_2, y_2, 6, 6, 2, b_1, b_2, 3)$ where, again by the above remark, $x_2 + y_2 = 5$ and $c^*(\hat{\Delta}) \leqslant -\pi + \pi = 0$. On the other hand if $d(u_1) > 3$ and $d(u_2) > 3$ then $cv(\hat{\Delta}) = (a_1, 4, 6, 6, 0, 6, 0, a_2)$ and $c^*(\hat{\Delta}) \leqslant -\pi + \frac{29\pi}{30} < 0$.

Case 11. Let $\hat{\Delta}$ be given by Figure 44(xi). If $c(u_1, u_2) > \frac{2\pi}{15}$ then $d(u_1) = 3$ and $c(u_8, u_1) = 0$; if $c(u_8, u_1) > \frac{2\pi}{15}$ then $c(u_1, u_2) = 0$; and if $c(u_6, u_7) > \frac{2\pi}{15}$ then $c(u_7, u_8) = 0$. Therefore $cv(\hat{\Delta}) = (b_1, 2, 6, 6, 4, a_1, a_2, b_2)$. It follows that $c^*(\hat{\Delta}) \leqslant c(\Delta) + \frac{11\pi}{10}$ and so if $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta}) < 0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see (from Figure 44(xi)) that $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{3\pi}{5} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. If $d(u_6) = d(u_7) = d(u_8) = 3$ then $c(u_5, u_6) = c(u_6, u_7) = c(u_7, u_8) = 0$; if $i = 6$ then $cv(\hat{\Delta}) = (6,0,6,6,4,1,0,0)$; if $i = 7$ then $cv(\hat{\Delta}) = (6,0,6,6,0,0,3,0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (6,0,6,6,0,0,3,0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{11\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly two vertices of degree greater than 3. If $d(u_6) = 3$ then $c(u_5, u_6) = 0$ and $cv(\hat{\Delta}) = (b_1, 2, 6, 6, 0, a_1, a_2, b_2)$; if $d(u_7) = 3$ then $cv(\hat{\Delta}) = (b_1, 2, 6, 6, x_2, y_2, 3, b_2)$ where again $x_2 + y_2 = 5$; if $d(u_6) > 3$ and $d(u_7) > 3$ then $cv(\hat{\Delta}) = (6, 0, 6, 6, 4, a_1, a_2, 0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\pi + \pi = 0$.

Case 12. Finally let $\hat{\Delta}$ be the region of Figure 44(xii). Suppose that $\hat{\Delta}$ has at least one vertex of degree greater than 4. Using a similar analysis as done for Case 1, it follows that $c^*(\hat{\Delta}) \leqslant -\frac{38\pi}{30} + \frac{36\pi}{30} < 0$. Indeed the maximum $\frac{36\pi}{30}$ can only be obtained when $cv(\hat{\Delta}) = (0, 6, 0, h_1, h_2, 6, 0, 6, 4)$. Suppose that $\hat{\Delta}$ has no vertices of degree greater than 4 and at least one vertex of degree 4. Then noting again that $\hat{\Delta}$ is not given by Figure 40(xvii), we see from Figure 40(xiv) that $c(u_i, u_j) = \frac{2\pi}{15}$ for $(i,j) \in \{(7,8), (8,9), (9,1), (1,2), (2,3), (3,4)\}$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{35\pi}{30} + \frac{32\pi}{30} < 0$, the maximum $\frac{32\pi}{30}$ being obtained when $cv(\hat{\Delta}) = (0, 4, 0, h_1, h_2, 6, 0, 4, 4)$. But if $\hat{\Delta}$ has no vertices of degree greater than 3 then $c(u_4, u_5) = 0$, $c(u_5, u_6) = \frac{\pi}{15}$ and, as in Case 1, for example either $c(u_1, u_2) = 0$ or $c(u_1, u_2) = \frac{2\pi}{15}$ and $c(u_2, u_3) = 0$. It follows that $cv(\hat{\Delta}) = (4,0,4,0,2,6,0,4,0)$ and $c^*(\hat{\Delta}) \leqslant -\pi + \frac{2\pi}{3} < 0$. This completes the proof. □

11. *Regions of type $\mathcal{B}$*

Let $\hat{\Delta}$ be a type $\mathcal{B}$ region as defined at the start of Section 9. Therefore $\hat{\Delta}$ is given by Figures 13(i), 14(i) and 31 or Figure 32(i), (ii), (iii) or (v); in particular $d(\hat{\Delta}) \geqslant 8$. A *b-segment* of $\hat{\Delta}$ of length k is a sequence of edges $e_1, \ldots, e_k$ of $\hat{\Delta}$ maximal with respect to each vertex having degree 3 with vertex label $a(a\lambda)(b^{-1}\mu) = axy^{-1}$ and which (up to inversion) contribute one of four possible alternating sequences to the corner labelling of $\hat{\Delta}$, namely: $x^{-1}, y^{-1}, \ldots, x^{-1}, y^{-1}$; $x^{-1}, y^{-1}, \ldots, y^{-1}, x^{-1}$; y^{-1}, x^{-1}; $y^{-1}, x^{-1}, \ldots, y^{-1}, x^{-1}$; $y^{-1}, x^{-1}, \ldots, x^{-1}, y^{-1}$. An example showing the first sequence is given in Figure 45(i) and so maximal in this case means that either $d(u_0) > 3$ or $d(u_0) = 3$ but does not extend the sequence to $\bar{y}, \bar{x}, \bar{y}, \ldots, \bar{x}, \bar{y}$; and that either $d(u_{k+2}) > 3$ or $d(u_{k+2}) = 3$ but does not extend the sequence to $\bar{x}, \bar{y}, \ldots, \bar{x}, \bar{y}, \bar{x}$. Since $\hat{\Delta}$ is of type $\mathcal{B}$, it must contain at least one b-segment in which at least one of the regions Δ_i $(1 \leqslant i \leqslant k)$ is given by the region Δ in Figures 13(i) and 14(i) and we will from now on call such a region Δ_i a *b-region*. Therefore a b-region contributes at most $\frac{\pi}{3}$ to $\hat{\Delta}$. (If Δ_i is

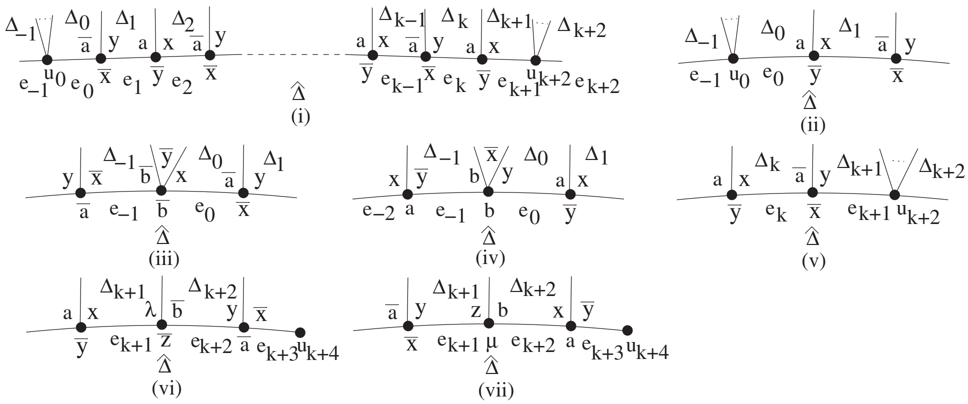

FIGURE 45. *b-segments.*

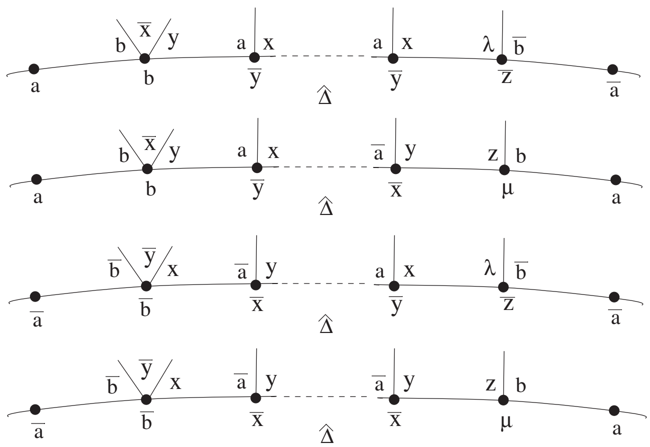

FIGURE 46. *Exceptional b-segments.*

not a b-region then, as shown in Figure 35, it contributes at most $\frac{\pi}{5}$ to $\hat{\Delta}$.) *The absorption rules for edges and vertices described in Section 10 apply also to* $\hat{\Delta}$. In Figure 31 $\hat{\Delta}$ receives $\frac{\pi}{5}, \frac{2\pi}{15}$ from Δ_1, Δ_2 so the vertex of degree 4 with label $b^{-1}b^{-1}y^{-1}x$ is used to absorb $\frac{\pi}{15}$; and in Figure 32(i) and (ii) $\hat{\Delta}$ receives $\frac{\pi}{5}$ across an edge, e say, but checking Figures 36– 38 shows that $\hat{\Delta}$ receives no curvature from $\hat{\Delta}_1$ across the neighbouring edge which is used to absorb $\frac{\pi}{15}$ noting from Figure 32(i) and (ii) that this is all the curvature that this edge will absorb (relative to curvature transferred to $\hat{\Delta}$).

It follows from the above paragraph and as in the proof of Lemma 10.1 that if the b-segments containing at least one b-region of $\hat{\Delta}$ contribute a total of n_1 edges to $\hat{\Delta}$ then putting $n = n_1 + n_2$,

$$c^*(\hat{\Delta}) \leqslant (2 - (n_1 + n_2))\pi + 2(n_1 + n_2)\frac{\pi}{3} + n_1\frac{\pi}{3} + n_2\frac{2\pi}{15} = \pi\left(2 - \frac{n_2}{5}\right). \qquad (\dagger)$$

Therefore if $n_2 \geqslant 10$ then $c^*(\hat{\Delta}) \leqslant 0$. The next result improves this bound slightly.

LEMMA 11.1. *If $n_2 \geqslant 9$ and $\hat{\Delta}$ is not given by Figure 46 (in which the b-segment contains at least one b-region) then $c^*(\hat{\Delta}) \leqslant 0$.*

Proof. We will show that the existence of a b-segment in which at least one Δ_i $(1 \leqslant i \leqslant k)$ is a b-region allows us to decrease the upper bound (†) for $c^*(\hat{\Delta})$ given above. First consider the region Δ_0 of Figure 45(i) or (ii). In each case if Δ_1 is not a b-region then $\hat{\Delta}$ receives at most $\frac{\pi}{5}$ from Δ_1 and the upper bound for $c^*(\hat{\Delta})$ is reduced by at least $\frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}$, so assume the Δ_1 is a b-region. In particular, according to the rules in Section 10 and at the start of this section, e_0 absorbs no positive curvature from Δ_1. Let $d(u_0) \geqslant 5$ and so u_0 can absorb at least $\frac{2\pi}{3} - \frac{2\pi}{5} = \frac{4\pi}{15}$. Since $\hat{\Delta}$ then receives at most $\frac{\pi}{15}$ from Δ_0 (see Figure 35(ii)) and since the maximum amount any vertex absorbs is $\frac{\pi}{6}$, in particular u_0 from Δ_{-1}, u_0 can absorb the $\frac{\pi}{15}$ crossing e_0 and so n_2 in (†) is reduced by 1, that is, $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$. Let $d(u_0) = 4$ and so u_0 can absorb $\frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$. If the total curvature $\hat{\Delta}$ receives across e_0 and e_{-1} is at most $\frac{3\pi}{10}$ then $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$, so assume otherwise. In particular $\hat{\Delta}$ must receive curvature from Δ_0 which forces $l(u_0)$ to be as shown in Figure 45(iii) and (iv) and so (see Figure 35(i)) $\hat{\Delta}$ receives at most $\frac{2\pi}{15}$ from Δ_0. To exceed a total of $\frac{3\pi}{10}$, therefore, it follows that $\hat{\Delta}$ must receive at least $\frac{\pi}{5}$ across e_{-1} and so (see Figure 40) $l(u_{-1})$ must be as shown in Figure 45(iii) and (iv) and in these figures the maximum combination $\hat{\Delta}$ can receive across e_{-1}, e_0 is $\frac{7\pi}{30}$, $\frac{2\pi}{15}$ (see Figure 42), therefore $c^*(\hat{\Delta})$ is reduced by at least $\frac{\pi}{15}$. Let $d(u_0) = 3$. Note that we use the fact that $l(u_0) \neq axy^{-1}$ in Figure 45(i) or (ii) for otherwise the b-segment would be extended, a contradiction. Given this, $l(u_0) = b\mu z$ forces $d(\Delta_0) \geqslant 6$ and $d(\Delta_{-1}) \geqslant 6$ and checking Figures 36– 38 shows that $\hat{\Delta}$ does not receive curvature across e_0 and at most $\frac{2\pi}{15}$ across e_{-1} so $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$.

Now consider the region Δ_{k+1} of Figure 45(i) and (v). Again if Δ_k is not a b-region then $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$ so assume otherwise. In particular e_{k+1} absorbs no positive curvature from Δ_k. Moreover, if Δ_k is given by Δ_1 of Figure 32(iii) or (v) (Configurations E and F) then $c^*(\hat{\Delta})$ is again reduced by $\frac{\pi}{5}$, so assume otherwise, in particular u_{k+2} is not given by the corresponding vertex of Δ_2 of Figure 32(iii) or (v). Let $d(u_{k+2}) \geqslant 5$ and so u_{k+2} can absorb $\frac{4\pi}{15}$. Since $\hat{\Delta}$ then receives at most $\frac{\pi}{15}$ from Δ_{k+1} (see Figure 35(ii)) and since the maximum amount u_{k+1} absorbs from Δ_{k+2} is $\frac{\pi}{6}$, u_{k+2} can absorb the $\frac{\pi}{15}$ crossing e_{k+1} and so $c^*(\hat{\Delta})$ is reduced by at least $\frac{2\pi}{15}$. Let $d(u_{k+2}) = 4$ and so u_{k+2} can absorb $\frac{\pi}{6}$. If $\hat{\Delta}$ does not receive curvature from Δ_{k+1} then $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$; otherwise checking possible vertex labels for u_{k+2} shows $l(u_{k+2}) = aaz\mu$ and $\hat{\Delta}$ receives at most $\frac{7\pi}{30}$ across e_{k+1} and 0 across e_{k+2}, so $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$. Let $d(u_{k+2}) = 3$ and so using the maximality of the b-segment and the fact that u_{k+2} is not given by Figure 32(iii) or (v) it follows that $l(u_{k+2})$ must be as shown in Figure 45(vi) and (vii). Then $d(\Delta_{k+1}) \geqslant 6$ and checking Figures 36– 38 show that $\hat{\Delta}$ does not receive curvature from Δ_{k+1}. It follows that $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$ except possibly when $d(u_{k+3}) = 3$ and $\hat{\Delta}$ receives $\frac{\pi}{6}$ or $\frac{\pi}{5}$ from Δ_{k+2} (see Figure 40). There are four cases. Two (see Figure 40(i), (ii), (vi) and (xv)) are given by Figure 45(vi) and (vii) where $\hat{\Delta}$ can receive $\frac{\pi}{5}$ from Δ_{k+2} and $c^*(\hat{\Delta})$ is reduced by $\frac{\pi}{15}$; and two (see Figure 40(xiii) and (xiv)) are given by Figures 37(iii) and 38(iii) in which the region $\hat{\Delta}_2$, $\hat{\Delta}_1$, Δ_2 (respectively) plays the role of the region $\hat{\Delta}$, Δ_{k+2}, Δ_{k+3} (respectively) which implies $l(u_{k+4}) = bbx^{-1}y$ so, in particular, e_{k+3} does not absorb curvature from e_{k+4} (relative to $\hat{\Delta}$). In each of these last two cases $\hat{\Delta}$ receives $\frac{\pi}{6}$ from Δ_{k+2} and $\frac{\pi}{10}$ from Δ_{k+3}, and since $\hat{\Delta}$ does not receive curvature from Δ_{k+1} it follows that $c^*(\hat{\Delta})$ is reduced by $\frac{2\pi}{15}$.

It follows from the above that if the b-segment of Figure 45(i) is not given by Figure 46 then there is a reduction of at least $\frac{\pi}{15} + \frac{2\pi}{15} = \frac{3\pi}{15}$ to $c^*(\hat{\Delta})$ (if $e_{k+2} = e_0$ the reduction is also $\frac{3\pi}{35}$) therefore $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{n_2}{5}) - \frac{3\pi}{15}$ and so $n_2 \geqslant 9 \Rightarrow c^*(\hat{\Delta}) \leqslant 0$. □

Lemma 11.2. *Let $\hat{\Delta}$ be a type $\mathcal{B}$ region such that $d(\hat{\Delta}) \geqslant 10$.*

(i) *If $\hat{\Delta}$ has exactly three b-segments that contain a b-region then $n_2 \geqslant 8$.*

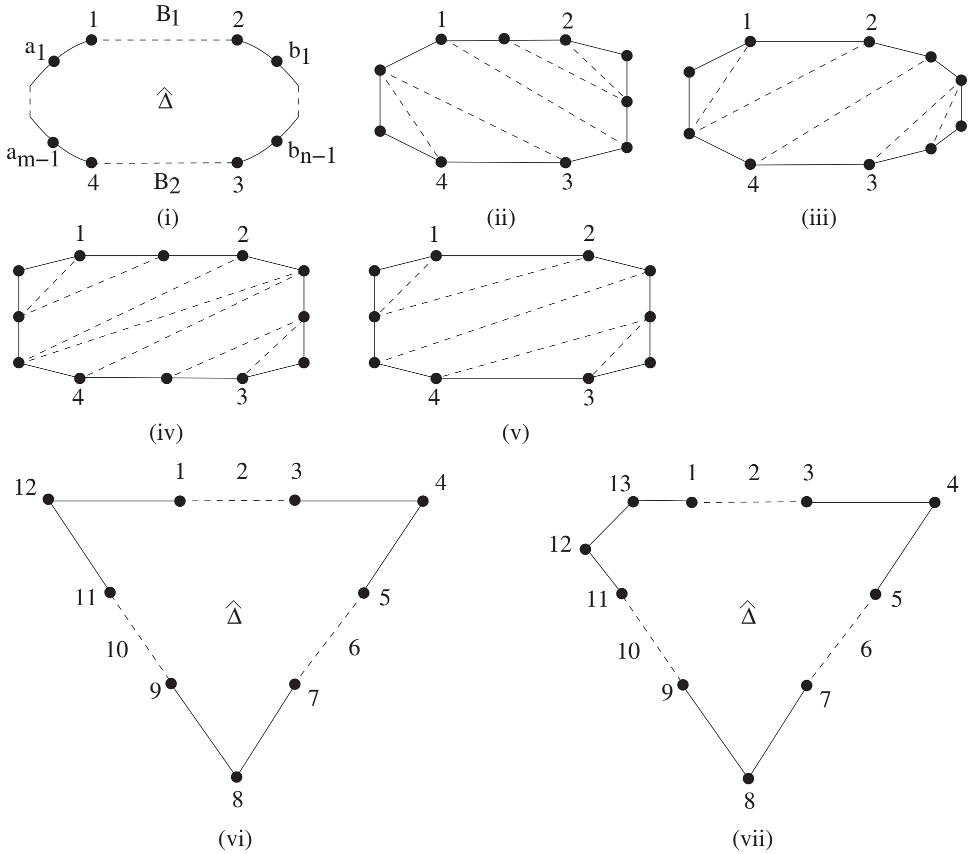

FIGURE 47. *Exactly 2 or 3 b-segments.*

Assume now that $\hat{\Delta}$ has exactly two b-segments B_1 and B_2 that contain a b-region as shown in Figure 47(i) and assume $(m, n) \in \{(2, j)\ (2 \leqslant j \leqslant 6), (3, 3), (3, 4), (3, 5), (4, 4)\}$ where m, n are given by Figure 47(i).

(ii) *$\hat{\Delta}$ must contain a shadow edge with an endpoint in B_1 and the other endpoint in B_2 except when $\hat{\Delta}$ is given by Figure 47(ii)–(v).*

(iii) *If $v \in \hat{\Delta}$ is a vertex of B_1 or B_2 and $(m, n) \neq (2, 6)$ then $i\deg(v) = 1$ where $i\deg(v)$ denotes the number of shadow edges in $\hat{\Delta}$ incident at v.*

(iv) *If $(m, n) \in \{(3, 3), (3, 4), (3, 5), (4, 4)\}$ and $\hat{\Delta}$ is not given by Figure 47(ii)–(v) there must be a shadow edge in $\hat{\Delta}$ either from 1 to B_2 or from 4 to B_1; and there must be a shadow edge in $\hat{\Delta}$ either from 2 to B_2 or from 3 to B_1.*

Finally assume that $\hat{\Delta}$ has exactly one b-segment containing a b-region.

(v) *If $n_2 \leqslant 8$ then $\hat{\Delta}$ is given by Figure 48.*

(vi) *If $n_2 = 9$ and $\hat{\Delta}$ is given by Figure 46 then $\hat{\Delta}$ is one of the regions of Figure 49.*

Proof. The proof is elementary but lengthy so we have omitted it. (Full details can be found at http://arxiv.org/abs/1708.01194.) As an illustration we give part of the proof of (i).

Let $\hat{\Delta}$ have exactly three b-segments and suppose by way of contradiction that $n_2 \leqslant 7$. Since there are at least two edges between any two b-segments it follows that $\hat{\Delta}$ is given by Figure 47(vi) ($n_2 = 6$) or 47(vii) ($n_2 = 7$) in which 2, 6, 10 refer to the (possibly empty) set

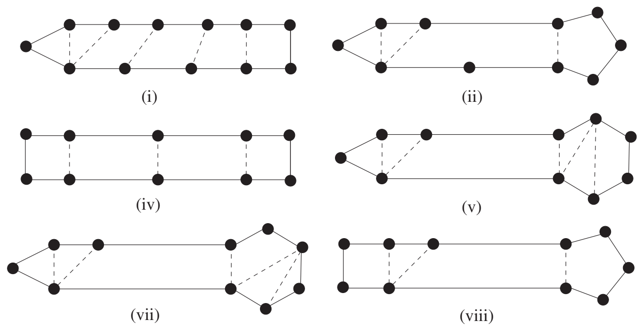

FIGURE 48. *At most* 8 *non b-segment edges.*

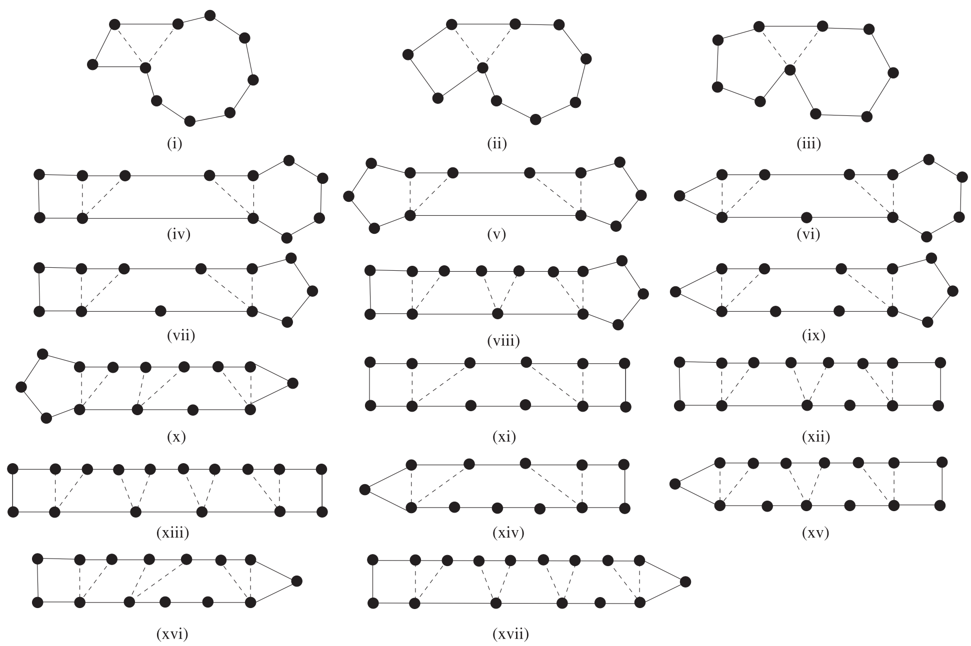

FIGURE 49. *Exactly* 9 *non b-segment edges.*

of vertices between vertices 1 and 3, 5 and 7, 9 and 11. We will consider the case $n_2 = 6$ only.

We remark here that if the corner label at a vertex v of $\hat{\Delta}$ is x or y then it follows from equations (3.1) in Section 3 that there must be an odd number of shadow edges in $\hat{\Delta}$ incident at v and it is clear that there are no shadow edges in $\hat{\Delta}$ connecting two vertices in the same b-segment. We write (ab) to indicate there is a shadow edge between vertices a and b with the understanding that if $a = 2$, for example, we mean a vertex belonging to a.

Consider Figure 47(vi). By the previous remark the number of (ab) involving each of 1, 3, 5, 7, 9 and 11 must be odd. It also follows that if $\{a, b\} \subseteq \{12, 1, 2, 3, 4\}$ or $\{4, 5, 6, 7, 8\}$ or $\{8, 9, 10, 11, 12\}$ then (ab) does not occur. Moreover (18) forces (19), (1 11) and this in turn forces a basic labelling contradiction (see Section 3), termed LAC. It follows that the only pairs involving 4, 8 or 12 are (4 10), (28) and (6 12). First assume that none of (35), (79) or (1 11) occur. Then since (15), (16) and (17) each forces (35), and (19), (1 10) each forces (1 11), we get a contradiction. Assume exactly one of (35), (79), (1 11) occurs — without any loss (79). Then again (15), (16) and (17) each force (35), and (19) and (1 10) each force (1 11), a contradiction. Assume exactly two of (35), (75), (1 11) occur — without any loss (35) and (79). Then (19) and (1 10) each forces (1 11), a contradiction; and (16) and (17) each forces a basic length contradiction at (35) (a shadow edge of length $n-1$) or forces either the pair (52), (52) or (52), (51) or (36), (36) or (36), (37) yielding LAC. This leaves (15). Since the number of (ab) involving 5 must be odd at least one of (59), (5 10) or (5 11) occurs. But (59) forces (11 5) and (5 10); (5 10) forces (11 5) and another (5 10); and (5 11) forces either a length contradiction at (79) or forces (95), (96) or (96), (96) or (7 10), (7 10) or (7 10), (7 11) yielding LAC in all cases.

Finally assume that (1 11), (35) and (79) occur. Since the length of each is $n-1$ we must have more pairs otherwise there is a length contradiction. Assume without any loss that 1 is involved in further pairs. Since (16) and (17) each forces either (36), (36) or (36)(37) or (52), (52) or (52)(51) yielding LAC it follows that at least two of (15), (19) and (1 10) occur. However (19), (1 10) and (1 10), (1 10) yield LAC and (15), (19) forces (59) and LAC. This leaves (15), (1 10) together with at least one of (25), (59), (5 10). But (25) yields LAC; (59) forces (19) or (69) and LAC; and finally (5 10) forces either a length contradiction or one of (7 10), (7 10) or (59)(69) or (69)(69) and LAC, our final contradiction. □

Notation. Throughout the following proofs we will use non-negative integers $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$ where: $a_1 + a_2 = 7$; $b_1 + b_2 = 8$; $c_1 + c_2 = 9$; $d_1 + d_2 = 10$; $e_1 + e_2 = 11$; and $f_1 + f_2 = 12$.

Proposition 11.3. *Let $\hat{\Delta}$ be a type $\mathcal{B}$ region. If $d(\hat{\Delta}) < 10$ then $c^*(\hat{\Delta}) \leqslant 0$.*

Proof. If $d(\hat{\Delta}) < 10$ then by Lemma 10.3 $\hat{\Delta}$ is given by Figure 44(viii), (x) or (xi).

Case 1. Let $\hat{\Delta}$ be given by Figure 44(viii) in which it is now assumed $d(u_3) = d(u_4) = 3$. Observe from Figures 41 and 42 that $c(u_6, u_7) + c(u_7, u_8) \leqslant \frac{11\pi}{30}$. Therefore $cv(\hat{\Delta}) = (4, 4, 10, 2, 2, e_1, e_2, 2)$ so $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + \frac{7\pi}{6}$ and if $\hat{\Delta}$ has at least three vertices of degree at least 4 then $c^*(\hat{\Delta}) \leqslant 0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, 0, 2, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{2\pi}{5} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. If $i = 1$ then $cv(\hat{\Delta}) = (4, 0, 10, 0, 0, 0, 2, 2)$; if $i = 2$ then $cv(\hat{\Delta}) = (4, 4, 10, 0, 0, 0, 2, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (0, 0, 10, 2, 2, 0, 2, 0)$; if $i = 6$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 2, 4, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, e_1, e_2, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, 0, 9, 2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{21\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_7) = 3$ then $cv(\hat{\Delta}) = (4, 4, 10, 2, 2, 4, 2, 2)$ and $c^*(\hat{\Delta}) \leqslant -\pi + \pi = 0$ so assume $i = 7$. If $j = 1$ then $cv(\hat{\Delta}) = (4, 0, 10, 0, 0, e_1, e_2, 2)$; if $j = 2$ then $cv(\hat{\Delta}) = (4, 4, 10, 0, 0, e_1, e_2, 0)$; if $j = 5$ then $cv(\hat{\Delta}) = (0, 0, 10, 2, 2, e_1, e_2, 0)$; if $j = 6$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 2, e_1, e_2, 0)$; and if $j = 8$ then $cv(\hat{\Delta}) = (0, 0, 10, 0, 0, e_1, e_2, 2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\pi + \frac{29\pi}{30} < 0$.

Remark 1. If $\hat{\Delta}$ is given by Figure 44(x) or (xi) then it is now assumed $d(u_4) = 3$, at least one of $d(u_3)$, $d(u_5)$ equals 3 and $d(u) > 3$ Note that in both figures if $d(u) > 4$ and $d(u_3) = 3$ then $c(u_3, u_4) = \frac{7\pi}{30}$; if $d(u) > 4$ and $d(u_3) > 3$ then $c(u_3, u_4) = \frac{\pi}{5}$; if $d(u) > 4$ and $d(u_5) = 3$ then $c(u_4, u_5) = \frac{7\pi}{30}$; and if $d(u) > 4$ and $d(u_5) > 3$ then $c(u_4, u_5) = \frac{\pi}{5}$. Note also that if $d(u) = 4, d(u_5) = d(u_6) = 3$ in Figure 44(x) or $d(u_2) = d(u_3) = 3$ in Figure 44(xi)and $\hat{\Delta}$ receives more than $\frac{2\pi}{15}$ across the (u_4, u_5)-edge, (u_3, u_4)-edge (respectively) then according to Configuration E in Figure 32(iii), Configuration F in Figure 32(v) (respectively) the surplus of at most $\frac{\pi}{5}$ is distributed out of $\hat{\Delta}$.

Remark 2. In Figure 44(x) if $d(u_1) = 3$ then, by the remark immediately preceding Proposition 10.4, $c(u_1, u_2) + c(u_2, u_3) \leqslant \frac{\pi}{6}$ and this bound can only be attained when $c(u_1, u_2) = \frac{\pi}{30}$,

$c(u_2, u_3) = \frac{2\pi}{15}$, $\hat{\Delta} = \hat{\Delta}_2$ of Figure 37(iv) and checking shows that Δ_6 of Figure 37(iv) must then be $\hat{\Delta}_4$ of Figure 18(ii); in particular, the vertices u_2, u_3 and u_8 of $\hat{\Delta}$ have degree greater than 3. Similarly if $d(u_7) = 3$ in Figure 44(xi) then $c(u_5, u_6) + c(u_6, u_7) = \frac{\pi}{6}$ forces the vertices u_5, u_6 and u_8 of $\hat{\Delta}$ to have degree greater than 3 (see Figure 38(iv)).

Case 2. Let $\hat{\Delta}$ be given by Figure 44(x) in which case (see Proposition 10.4 and Case 10) $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 2, b_1, b_2, a_1)$, so $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + \frac{41\pi}{30}$ and if $\hat{\Delta}$ has at least five vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant 0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then it follows by Remark 1 that either $d(u) = 4$, $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{13\pi}{15} - \frac{\pi}{5} = 0$ or $d(u) > 4$, $cv(\hat{\Delta}) = (0, 0, 7, 7, 0, 6, 0, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{2\pi}{3} = 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3 and assume that $d(u) = 4$. If $i = 1$ then $l(u_2) = b\mu z$ implies $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 3)$; if $i = 2$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 0, 6, 0, 0)$ where $x_1 + y_1 = 4$ by Remark 2; if $i = 3$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, 6, 0, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 0)$; if $i = 6$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 0, 0, 0)$; if $i = 6$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 2, 0, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, b_1, b_2, 0)$; and if $i = 8$ then $l(u_7) = \lambda b^{-1} z^{-1}$ implies $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 3)$. It follows that if $d(u_5) = d(u_6) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \pi - \frac{\pi}{5} < 0$; otherwise $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{24\pi}{30} < 0$. If now $d(u) > 4$ then each $cv(\hat{\Delta})$ is altered by replacing each 10 by 7 and it follows that $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{24\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3 and assume $d(u) = 4$. If $(i, j) = (1, 2)$ then $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 0, 6, 0, a_1)$ and so if $d(u_1) > 4$ or $d(u_2) > 4$ then $c^*(\hat{\Delta}) \leqslant -\frac{11\pi}{10} + \frac{37\pi}{30} - \frac{\pi}{5} < 0$; and if $d(u_1) = d(u_2) = 4$ then $c(u_8, u_1) = 0$ and, moreover, $c(u_1, u_2) > \frac{2\pi}{15}$ and $l(u_3) = axy^{-1}$ together imply (see Figure 40(x)) $c(u_2, u_3) = 0$ so $cv(\hat{\Delta}) = (b_1, b_2, 10, 10, 0, 6, 0, 0)$ and $c^*(\hat{\Delta} \leqslant -\pi + \frac{17\pi}{15} - \frac{\pi}{5} < 0$. If $(i, j) = (1, 3)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, 6, 0, 3)$; if $(i, j) = (1, 5)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 3)$; if $(i, j) = (1, 6)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 0, 0, 3)$; if $(i, j) = (1, 6)$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 2, 0, 3)$; if $(i, j) = (1, 7)$ then (see Proposition 10.4 and Case 10) $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, b_1, b_2, 3)$; if $(i, j) = (1, 8)$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, 6, 0, 3)$; if $(i, j) = (2, 3)$ then $cv(\hat{\Delta}) = (x_1, y_1, 6, 10, 0, 6, 0, 0)$; if $(i, j) = (2, 5)$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 6, 2, 6, 0, 0)$; if $(i, j) = (2, 6)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 2, 0, 0, 0)$; if $(i, j) = (2, 6)$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 2, 2, 0, 0)$; if $(i, j) = (2, 7)$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 0, b_1, b_2, 0)$; if $(i, j) = (2, 8)$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 0, 6, 0, 3)$; if $(i, j) = (3, 6)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 2, 6, 0, 0)$; if $(i, j) = (3, 7)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, b_1, b_2, 0)$; if $(i, j) = (3, 8)$ then $cv(\hat{\Delta}) = (0, 0, 6, 10, 0, 6, 0, 3)$; if $(i, j) = (5, 6)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 0)$; if $(i, j) = (5, 7)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, b_1, b_2, 0)$; if $(i, j) = (5, 8)$ then $cv(\hat{\Delta}) = (0, 0, 10, 6, 2, 6, 0, 3)$; if $(i, j) = (6, 7)$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, b_1, b_2, 0)$; if $(i, j) = (6, 8)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 0, 0, 3)$; if $(i, j) = (6, 8)$ and $d(u_6) > 4$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, 2, 0, 3)$; and if $(i, j) = (7, 8)$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 0, b_1, b_2, 3)$. It follows that if $(i, j) \neq (1, 2)$ and if $d(u_5) = d(u_6) = 3$ then $c^*(\hat{\Delta}) \leqslant -\pi + \frac{11\pi}{10} - \frac{\pi}{5} < 0$; or if $d(u_5) > 3$ or $d(u_6) > 3$ then $c^*(\hat{\Delta}) \leqslant -\pi + \pi = 0$. If now $d(u) > 4$ then, as before, replacing each 10 by 7 in the above yields $c^*(\hat{\Delta}) \leqslant -\pi + \frac{28\pi}{30} < 0$ except when $(i, j) = (1, 2)$ and either $d(u_1) > 4$ or $d(u_2) > 4$ and $c^*(\hat{\Delta}) \leqslant -\frac{11\pi}{10} + \frac{31\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (0, 0, 10, 10, 2, b_1, b_2, 3)$ and $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$; or if $d(u_5) = d(u_6) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{41\pi}{30} - \frac{\pi}{5} = 0$, so assume otherwise. If $d(u_3) = 4$ then $(d(u_4) = 3$ implies) $c(u_3, u_4) = 0$ and if $d(u_3) \geqslant 5$ then $c(u_3, u_4) = \frac{\pi}{15}$, and in both cases $c^*(\hat{\Delta} \leqslant 0$. Similarly if $d(u_5) \neq 3$ then $c^*(\hat{\Delta}) \leqslant 0$, so it can be assumed $d(u_3) = d(u_5) = 3$. If $(i, j, k) = (1, 2, 6)$ and $d(u_6) = 4$ then $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 2, 0, 0, a_1)$ and $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$; or if $d(u_6) > 4$ then $cv(\hat{\Delta}) = (a_2, 4, 10, 10, 2, 2, 0, a_1)$ and $c^*(\hat{\Delta}) \leqslant -\frac{19\pi}{15} + \frac{7\pi}{6} < 0$. If $(i, j, k) = (2, 6, 7)$ then $cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 2, b_1, b_2, 0)$ (by Remark 2); and if $(i, j, k) = (2, 6, 8)$ then

$cv(\hat{\Delta}) = (x_1, y_1, 10, 10, 2, 6, 0, 3)$. In both cases $c^*(\hat{\Delta}) \leqslant 0$. Finally let $\hat{\Delta}$ have exactly four vertices of degree greater than 3 and so $c(\hat{\Delta}) \leqslant -\frac{4\pi}{3}$. If any vertex has degree greater than 4 or if any of u_1, u_2, u_6 or u_7 has degree 3 then clearly $c^*(\hat{\Delta}) \leqslant 0$, so assume otherwise. But then $d(u_1) = d(u_2) = 4$ and $d(u_3) = 3$ together imply either $c(u_1, u_2) = 0$ or $c(u_2, u_3) = 0$ and $c^*(\hat{\Delta}) < 0$.

Case 3. Let $\hat{\Delta}$ be given by Figure 44(xi) in which case (see Proposition 10.4 and Case 11) $cv(\hat{\Delta}) = (b_2, 2, 10, 10, 4, a_1, a_2, b_1) = \frac{41\pi}{30}$ so if $\hat{\Delta}$ has at least five vertices of degree at least 4 then $c^*(\hat{\Delta}) \leqslant 0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then by Remark 1 preceding Case 2 either $d(u) = 4$, $cv(\hat{\Delta}) = (6, 0, 10, 10, 0, 0, 0, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{13\pi}{15} - \frac{\pi}{5} = 0$ or $d(u) > 4$, $cv(\hat{\Delta}) = (6, 0, 7, 7, 0, 0, 0, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{2\pi}{3} + \frac{2\pi}{3} - 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3 and assume $d(u) = 4$. If $i = 1$ then $cv(\hat{\Delta}) = (b_2, 0, 10, 10, 0, 0, 0, b_1)$; if $i = 2$ and $d(u_2) = 4$ then $l(u_1) = z^{-1}\lambda b^{-1}$ forces $c(u_1, u_2) = 0$ and $cv(\hat{\Delta}) = (0, 2, 10, 10, 0, 0, 0, 0)$; if $i = 2$ and $d(u_2) > 4$ then $cv(\hat{\Delta}) = (2, 2, 10, 10, 0, 0, 0, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (6, 2, 6, 10, 0, 0, 0, 0)$; if $i = 5$ then ($l(u_6) = b\mu z$ and so) $cv(\hat{\Delta}) = (6, 0, 10, 6, 0, 0, 0, 0)$; if $i = 6$ then $cv(\hat{\Delta}) = (6, 0, 10, 10, x_1, y_1, 0, 0)$ (where $x_1 + y_1 = 4$ by Remark 2); if $i = 7$ then $cv(\hat{\Delta}) = (6, 0, 10, 10, 0, 0, 3, 0)$; if $i = 8$ and $d(u_8) = 4$ then $cv(\hat{\Delta}) = (6, 0, 10, 10, 0, 0, 3, 0)$; and if $i = 8$ and $d(u_8) > 4$ then $cv(\hat{\Delta}) = (6, 0, 10, 10, 0, 0, 2, 2)$. Therefore $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{24\pi}{30}$ when $(d(u_2), d(u_3)) \neq (3, 3)$; and if $d(u_2) = d(u_3) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \pi - \frac{\pi}{5} < 0$. If now $d(u) > 4$ then replacing each 10 by 7 in the above yields $c^*(\hat{\Delta}) \leqslant -\frac{5\pi}{6} + \frac{24\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3 and assume $d(u) = 4$. If $(i, j) = (1, 2)$ then $cv(\hat{\Delta}) = (b_2, 2, 10, 10, 0, 0, 0, b_1)$; if $(i, j) = (1, 3)$ then $cv(\hat{\Delta}) = (b_2, 2, 6, 10, 0, 0, 0, b_1)$; if $(i, j) = (1, 5)$ then $cv(\hat{\Delta}) = (b_2, 0, 10, 6, 0, 0, 0, b_1)$; if $(i, j) = (1, 6)$ then $cv(\hat{\Delta}) = (b_2, 0, 10, 10, x_1, y_1, 0, b_1)$; if $(i, j) = (1, 7)$ then $cv(\hat{\Delta}) = (b_2, 0, 10, 10, 0, 0, 3, b_1)$; if $(i, j) = (1, 8)$ then $cv(\hat{\Delta}) = (b_2, 0, 10, 10, 0, 0, 3, b_1)$; if $(i, j) = (2, 3)$ then $cv(\hat{\Delta}) = (6, 2, 6, 10, 0, 0, 0, 0)$; if $(i, j) = (2, 5)$ then $cv(\hat{\Delta}) = (6, 2, 10, 6, 0, 0, 0, 0)$; if $(i, j) = (2, 6)$ and $d(u_2) = 4$ then $cv(\hat{\Delta}) = (0, 2, 10, 10, x_1, y_1, 0, 0)$; if $(i, j) = (2, 6)$ and $d(u_2) > 4$ then $cv(\hat{\Delta}) = (2, 2, 10, 10, x_1, y_1, 0, 0)$; if $(i, j) = (2, 7)$ and $d(u_2) = 4$ then $cv(\hat{\Delta}) = c(0, 2, 10, 10, 0, 0, 3, 0)$; if $(i, j) = (2, 7)$ and $d(u_2) > 4$ then $cv(\hat{\Delta}) = c(2, 2, 10, 10, 0, 0, 3, 0)$; if $(i, j) = (2, 8)$ and $d(u_2) = 4$ then $cv(\hat{\Delta}) = (0, 2, 10, 10, 0, 0, 3, 0)$; if $(i, j) = (2, 8)$ and $d(u_2) > 4$ then $cv(\hat{\Delta}) = (2, 2, 10, 10, 0, 0, 3, 0)$; if $(i, j) = (3, 6)$ then $cv(\hat{\Delta}) = (6, 2, 6, 10, x_1, y_1, 0, 0)$; if $(i, j) = (3, 7)$ then $cv(\hat{\Delta}) = (6, 2, 6, 10, 0, 0, 3, 0)$; if $(i, j) = (3, 8)$ then $cv(\hat{\Delta}) = (6, 2, 6, 10, 0, 0, 3, 0)$; if $(i, j) = (5, 6)$ then $cv(\hat{\Delta}) = (6, 0, 10, 6, x_1, y_1, 0, 0)$; if $(i, j) = (5, 7)$ then $cv(\hat{\Delta}) = (6, 0, 10, 6, 0, 0, 3, 0)$; if $(i, j) = (5, 8)$ then $cv(\hat{\Delta}) = (6, 0, 10, 6, 0, 0, 3, 0)$; if $(i, j) = (6, 7)$ then $cv(\hat{\Delta}) = (6, 0, 10, 10, 4, a_1, a_2, 0)$ and so if $d(u_6) > 4$ or $d(u_7) > 4$ then $c^*(\hat{\Delta}) \leqslant -\frac{11\pi}{10} + \frac{37\pi}{30} - \frac{\pi}{5} < 0$, or if $d(u_6) = d(u_7) = 4$ then $c(u_7, u_8) = 0$ and, moreover, $c(u_6, u_7) \geqslant \frac{2\pi}{15}$ and $l(u_5) = axy^{-1}$ together imply (see Figure 40(ix)) $c(u_5, u_6) = 0$ so $cv(\hat{\Delta}) = (6, 0, 10, 10, b_1, b_2, 0,$ and $c^*(\hat{\Delta} \leqslant -\pi + \frac{17\pi}{15} - \frac{\pi}{5} < 0$; if $(i, j) = (6, 8)$ then $cv(\hat{\Delta}) = (6, 0, 10, 10, x_1, y_1, 3, 0)$; and if $(i, j) = (7, 8)$ then $cv(\hat{\Delta}) = (6, 0, 10, 10, 0, 0, 3, 0)$. It follows that if $(i, j) \neq (6, 7)$ and if $d(u_2) = d(u_3) = 3$ then $c^*(\hat{\Delta}) \leqslant -\pi + \frac{11\pi}{10} - \frac{\pi}{5} < 0$; or if $d(u_2) > 3$ or $d(u_3) > 3$ then $c^*(\hat{\Delta}) \leqslant -\pi + \pi = 0$. If now $d(u) > 4$ then replacing each 10 by 7 in the above yields $c^*(\hat{\Delta}) \leqslant -\pi + \frac{14\pi}{15} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_6) = 3$ then $cv(\hat{\Delta}) = (b_2, 2, 10, 10, 0, 0, 3, b_1)$ and $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{11\pi}{10} < 0$; or if $d(u_2) = d(u_3) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{7\pi}{6} + \frac{41\pi}{30} - \frac{\pi}{5} = 0$, so assume otherwise. If $d(u_3) = 4$, $d(u_5) = 4$ (respectively) then $d(u_4) = 3$ implies $c(u_3, u_4) = 0$, $c(u_4, u_5) = 0$ (respectively) or if $d(u_3) \geqslant 5$, $d(u_5) \geqslant 5$ (respectively) then $c(u_3, u_4) = \frac{\pi}{15}$, $c(u_4, u_5) = \frac{\pi}{15}$ (respectively)and in each case $c^*(\hat{\Delta} \leqslant 0$. So it can be assumed $d(u_3) = d(u_5) = 3$. If $(i, j, k) = (6, 2, 7)$ and $d(u_2) = 4$ then $cv(\hat{\Delta}) = (0, 2, 10, 10, 4, a_1, a_2, 0)$; if $(i, j, k) = (6, 2, 7)$ and $d(u_2) > 4$ then $cv(\hat{\Delta}) = (2, 2, 10, 10, 4, a_1, a_2, 0)$; if $(i, j, k) = (6, 2, 8)$ then $cv(\hat{\Delta}) = (2, 2, 10, 10, x_1, y_1, 3, 0)$

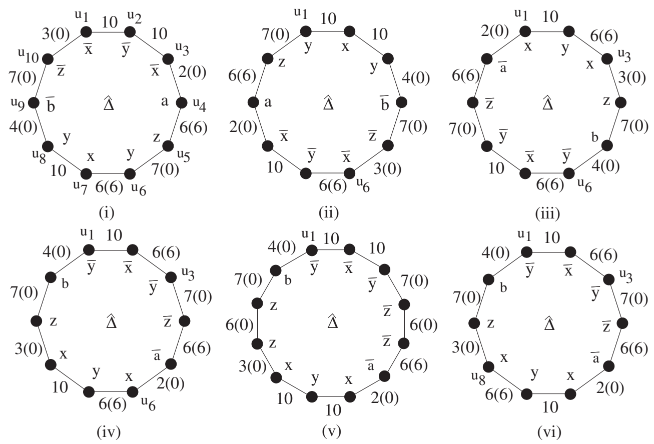

FIGURE 50. *Regions for Figure 47(ii)–(v).*

(by Remark 2); or if $(i,j,k) = (6,2,1)$ then $cv(\hat{\Delta}) = (b_2, 2, 10, 10, x_1, y_1, 0, b_1)$. In each case $c^*(\hat{\Delta}) \leqslant 0$ so assume that $\hat{\Delta}$ has exactly four vertices of degree greater than 3. Then $c(\hat{\Delta}) \leqslant -\frac{4\pi}{3}$. If $d(u_1) = 3$ then $cv(\hat{\Delta}) = (6, 2, 10, 10, 4, a_1, a_2, 0) = \frac{39\pi}{30}$; or if $d(u_1) = 4$ then $cv(\hat{\Delta}) = (0, 2, 10, 10, 4, a_1, a_2, 7) = \frac{40\pi}{30}$ and in both cases $c^*(\hat{\Delta}) \leqslant 0$. On the other hand if $d(u_1) \geqslant 5$ then $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + cv(\hat{\Delta}) \leqslant -\frac{43\pi}{30} + \frac{41\pi}{30} < 0$. □

LEMMA 11.4. *Let $\hat{\Delta}$ be a type $\mathcal{B}$ region. If $\hat{\Delta}$ is given by Figure 47(ii)–(v), 48 or 49 then* $c^*(\hat{\Delta}) \leqslant 0$.

Proof. Let $\hat{\Delta}$ be given by Figure 47(ii)–(v). Then (up to cyclic permutation and inversion) there are two ways to label each of (ii) and (iii); and one way to label each of (iv) and (v) and so $\hat{\Delta}$ is given by Figure 50. There are six *a-cases*. As usual we rely heavily on Figures 35–38 and 40–42.

Case a1. Let $\hat{\Delta}$ be given by Figure 50(i) in which (it can be seen from Figure 47(ii) that) $d(u_1) = d(u_2) = d(u_3) = d(u_7) = d(u_8) = 3$ and $d(u_6) > 3$. Then (see Figure 40) $cv(\hat{\Delta}) = (10, 10, 2, d_1, d_2, 6, 10, 4, a_1, a_2)$ so $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + \frac{59\pi}{30}$. Let $\hat{\Delta}$ have exactly one vertex u_6 of degree greater than 3. Then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 10, 0, 0, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{42\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_6, u_i of degree greater than 3. If $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 2, 6, 0, 6, 10, 0, 0, 0)$; if $i = 5$ then $cv(\hat{\Delta}) = (10, 10, 0, d_1, d_2, 6, 10, 0, 0, 0)$; if $i = 9$ then (using $d(u_5) = d(u_{10}) = 3$) $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 10, 4, 2, 0)$; and if $i = 10$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 10, 0, a_1, a_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_6, u_i, u_j of degree greater than 3. If $d(u_5) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, 6, 0, 6, 10, 4, a_1, a_2)$; if $d(u_{10}) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, d_1, d_2, 6, 10, 4, 2, 0)$; and if $(i,j) = (5, 10)$ then $cv(\hat{\Delta}) = (10, 10, 0, d_1, d_2, 6, 10, 0, a_1, a_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. If $\hat{\Delta}$ has more than three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{60\pi}{30} + \frac{59\pi}{30} < 0$.

Case a2. Let $\hat{\Delta}$ be given by Figure 50(ii) in which $d(u_1) = d(u_2) = d(u_3) = d(u_7) = d(u_8) = 3$ and $d(u_6) > 3$. Then $cv(\hat{\Delta}) = (10, 10, 4, a_1, a_2, 6, 10, 2, d_1, d_2)$ so $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + \frac{59\pi}{30}$. Let $\hat{\Delta}$ have exactly one vertex u_6 of degree greater than 3. Then $cv(\hat{\Delta}) = (10, 10, 0, 0, 3, 6, 10, 0, 6, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{45\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly two vertices u_6, u_i of degree greater than 3. If $i = 4$ and $d(u_6) = 4$ then ($l(u_6)$ together with $l(u_7)$ force) $cv(\hat{\Delta}) = (10, 10, 4, 2, 3, 0, 10, 0, 6, 0)$; if $i = 4$ and $d(u_6) > 4$ then (see Figure 35(ii)) $cv(\hat{\Delta}) = (10, 10, 4, 2, 2, 2, 10, 0, 6, 0)$; if $i = 5$

then $cv(\hat{\Delta}) = (10,10,0,a_1,a_2,6,10,0,6,0)$; if $i = 9$ then $cv(\hat{\Delta}) = (10,10,0,0,3,6,10,2,6,0)$; and if $i = 10$ then $cv(\hat{\Delta}) = (10,10,0,0,3,6,10,0,d_1,d_2)$. It follows that either $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$ or $c^*(\hat{\Delta}) \leqslant -\frac{53\pi}{30} + \frac{46\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_6, u_i, u_j of degree greater than 3. If $d(u_{10}) = 3$ then $cv(\hat{\Delta}) = (10,10,4,a_1,a_2,6,10,2,6,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$, so assume $i = 10$ and $j \in \{4,5,9\}$. If $j = 4$ then $cv(\hat{\Delta}) = (10,10,4,2,3,6,10,0,d_1,d_2)$; if $j = 5$ then $cv(\hat{\Delta}) = (10,10,0,a_1,a_2,6,10,0,d_1,d_2)$; and if $j = 9$ then $cv(\hat{\Delta}) = (10,10,0,0,3,6,10,2,d_1,d_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. If $\hat{\Delta}$ has more than three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{60\pi}{30} + \frac{59\pi}{30} = 0$.

Case a3. Let $\hat{\Delta}$ be given by Figure 50(iii) in which (see Figure 47(iii)) $d(u_1) = d(u_2) = d(u_7) = d(u_8) = 3$, $d(u_3) > 3$ and $d(u_6) > 3$. Then $cv(\hat{\Delta}) = (10,6,a_1,a_2,4,6,10,d_1,d_2,2)$ and $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + \frac{55\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. This leaves the case $d(u_3) > 3$ and $d(u_6) > 3$ only. Then $cv(\hat{\Delta}) = (10,6,3,0,4,6,10,0,6,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{45\pi}{30} < 0$.

Case a4. Let $\hat{\Delta}$ be given by Figure 50(iv) in which $d(u_1) = d(u_2) = d(u_7) = d(u_8) = 3$, $d(u_3) > 3$ and $d(u_6) > 3$. Then $cv(\hat{\Delta}) = (10,6,d_1,d_2,2,6,10,a_1,a_2,4)$ and $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + \frac{55\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. This leaves the case $d(u_3) > 3$ and $d(u_6) > 3$ only. Then $cv(\hat{\Delta}) = (10,6,7,6,2,6,10,0,0,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{47\pi}{30} < 0$.

Case a5. Let $\hat{\Delta}$ be given by Figure 50(v) in which (see Figure 47(iv)) $d(u_1) = d(u_2) = d(u_3) = d(u_7) = d(u_8) = d(u_9) = 3$. Then $cv(\hat{\Delta}) = (10,10,d_1,d_2,6,2,10,10,3,d_1,d_2,4)$ so $c^*(\hat{\Delta}) \leqslant c(\hat{\Delta}) + \frac{75\pi}{30}$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then $cv(\hat{\Delta}) = (10,10,0,0,6,0,10,10,0,0,0,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{60\pi}{30} + \frac{46\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. If $i = 4$ then $cv(\hat{\Delta}) = (10,10,d_1,d_2,6,0,10,10,0,0,0,0)$; if $i = 5$ then $cv(\hat{\Delta}) = (10,10,0,6,6,0,10,10,0,0,0,0)$; if $i = 6$ then $cv(\hat{\Delta}) = (10,10,0,0,6,2,10,10,0,0,0,0)$; if $i = 10$ then $cv(\hat{\Delta}) = (10,10,0,0,6,0,10,10,3,6,0,0)$; if $i = 11$ then $cv(\hat{\Delta}) = (10,10,0,0,6,0,10,10,0,d_1,d_2,0)$; and if $i = 12$ then $cv(\hat{\Delta}) = (10,10,0,0,6,0,10,10,0,0,7,4)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{65\pi}{30} + \frac{57\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_4) = d(u_5) = 3$ then $cv(\hat{\Delta}) = (10,10,0,0,6,2,10,10,3,d_1,d_2,4)$; if $d(u_{10}) = d(u_{11}) = 3$ then $cv(\hat{\Delta}) = (10,10,d_1,d_2,6,2,10,10,3,0,2,4)$; and if $d(u_{12}) = 3$ then $cv(\hat{\Delta}) = (10,10,d_1,d_2,6,2,10,10,3,6,0,0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{70\pi}{30} + \frac{67\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{75\pi}{30} + \frac{75\pi}{30} = 0$.

Case a6. Let $\hat{\Delta}$ be given by Figure 50(vi) in which (see Figure 47(v)) $d(u_1) = d(u_2) = d(u_6) = d(u_7) = 3$, $d(u_3) > 3$ and $d(u_8) > 3$. Then $cv(\hat{\Delta}) = (10,6,d_1,d_2,2,10,6,a_1,a_2,4)$ and $c^*(\Delta) \leqslant c(\hat{\Delta}) + \frac{55\pi}{30}$. If $\hat{\Delta}$ has at least three vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$. This leaves the case $d(u_3) > 3$ and $d(u_8) > 3$ only. Then $cv(\hat{\Delta}) = (10,6,0,6,0,10,6,3,0,0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{41\pi}{30} < 0$.

Now let $\hat{\Delta}$ be one of the regions of Figure 48. It turns out that (up to cyclic permutation and inversion) there are two ways to label each of Figure 48(i), (ii), (iii) and (iv); four ways to label (v); two ways to label each of (vi) and (vii); and four ways to label (viii). However the labelled regions produced by (vii) already appear in those produced by (vi); and two of the labelled regions produced by (viii) already appear in those produced by (ii), leaving a total of sixteen regions and $\hat{\Delta}$ is given by Figure 51. Table 4 gives $c(u_i, u_{i+1})$ $(1 \leqslant i \leqslant 8)$ in multiples of $\frac{\pi}{30}$ for each of the sixteen regions of Figure 51 with the total plus the contribution made via the b-segment in the final column. We note here that Lemma 9.2 is used for the bounds e_1, e_2

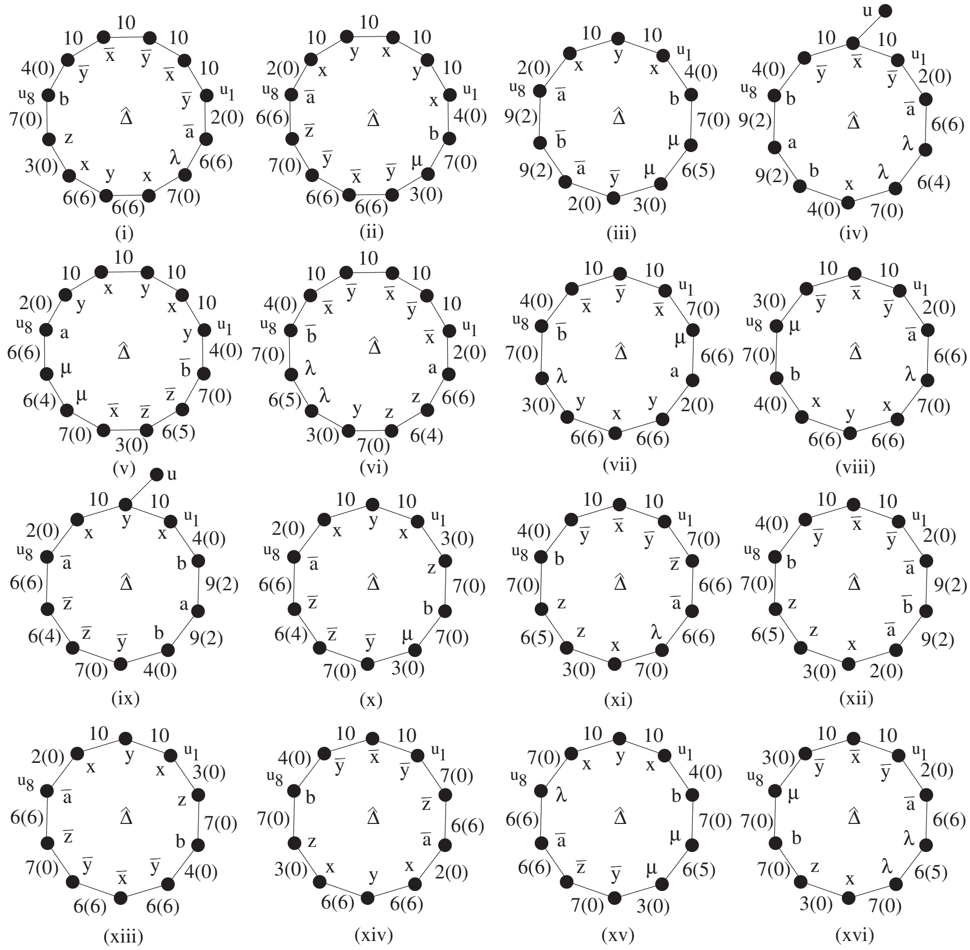

FIGURE 51. *Regions for Figure 48.*

and f_1, f_2 in rows (iii), (iv), (ix) and (xii); and that Figure 40(iv), (x), (xiii) and (xviii) is used to obtain the other bounds a_1, a_2, b_1, b_2 and d_1, d_2 in the table.

The regions in Figure 51(i), (ii), (v) and (vi) each have degree 12 and so $c(\hat{\Delta}) \leqslant (2-12)\pi + \frac{24\pi}{3} = -2\pi$, whereas the rest have degree 10 and in these cases $c(\hat{\Delta}) \leqslant -\frac{4\pi}{3}$. It follows from Table 4 that if $\hat{\Delta}$ has at least two vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant 0$ for (x); if at least three then $c^*(\hat{\Delta}) \leqslant 0$ for (i), (ii), (iii), (v), (vi), (vii), (viii), (xi), (xii), (xiii), (xiv) and (xvi); and if at least four then $c^*(\hat{\Delta}) \leqslant -\frac{40\pi}{30} + \frac{40\pi}{30} = 0$.

If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see from Figure 51 that $c^*(\hat{\Delta}) \leqslant -\frac{20\pi}{30} + \frac{18\pi}{30} < 0$.

We consider each of the sixteen *b-cases* in turn.

Case b1. Let $\hat{\Delta}$ be given by Figure 51(i). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_3) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 2, 6, 0, 6, 6, 3, 0, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 0, d_1, d_2, 6, 6, 0, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 0, 6, 0, 6, 6, 0, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{65\pi}{30} + \frac{64\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_3) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 2, 6, 0, 6, 6, 3, 2, 4)$; if $d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 2, d_1, d_2, 6, 6, 3, 0, 0)$;

TABLE 4. $c(u_i, u_{i+1})$ *for Figure* 51.

(i)	2	d_1	d_2	6	6	a_1	a_2	4	$35+40=75$
(ii)	4	a_1	a_2	6	6	d_1	d_2	2	$35+40=75$
(iii)	4	d_1	d_2	3	2	f_1	f_2	2	$33+20=53$
(iv)	2	6	d_1	d_2	e_1	e_2	e_1	e_2	$40+20=60$
(v)	4	b_1	b_2	3	d_1	d_2	6	2	$33+40=73$
(vi)	2	6	d_1	d_2	3	d_1	d_2	4	$35+40=75$
(vii)	d_1	d_2	2	6	6	a_1	a_2	4	$35+20=55$
(viii)	2	d_1	d_2	6	6	4	a_1	a_2	$35+20=55$
(ix)	e_1	e_2	e_1	e_2	b_1	b_2	6	2	$38+20=58$
(x)	a_1	a_2	a_1	a_2	b_1	b_2	6	2	$30+20=50$
(xi)	d_1	d_2	d_1	d_2	3	b_1	b_2	4	$35+20=55$
(xii)	2	f_1	f_2	2	3	b_1	b_2	4	$31+20=51$
(xiii)	a_1	a_2	4	6	6	d_1	d_2	2	$35+20=55$
(xiv)	d_1	d_2	2	6	6	a_1	a_2	4	$35+20=55$
(xv)	4	d_1	d_2	3	d_1	d_2	d_1	d_2	$37+20=57$
(xvi)	2	6	d_1	d_2	a_1	a_2	a_1	a_2	$32+20=52$

if $(i,j)=(3,8)$ then $c^*(\hat{\Delta})=(10,10,10,10,0,d_1,d_2,6,6,0,2,4)$; and if $(i,j)=(7,8)$ then $cv(\hat{\Delta})=(10,10,10,10,0,6,0,6,6,a_1,a_2,4)$. It follows that $c^*(\hat{\Delta})\leqslant -\frac{70\pi}{30}+\frac{69\pi}{30}<0$.

Case b2. Let $\hat{\Delta}$ be given by Figure 51(ii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2)=d(u_7)=3$ then $cv(\hat{\Delta})=(10,10,10,10,0,0,3,6,6,0,6,2)$; if $i=2$ then $cv(\hat{\Delta})=(10,10,10,10,4,2,0,6,6,0,6,0)$; and if $i=7$ then $cv(\hat{\Delta})=(10,10,10,10,0,0,0,6,6,d_1,d_2,2)$. It follows that $c^*(\hat{\Delta})\leqslant -\frac{65\pi}{30}+\frac{64\pi}{30}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_2)=3$ then $cv(\hat{\Delta})=(10,10,10,10,0,0,3,6,6,d_1,d_2,2)$; if $d(u_3)=d(u_7)=3$ then $cv(\hat{\Delta})=(10,10,10,10,4,2,3,6,6,0,6,2)$; if $(i,j)=(2,3)$ then $cv(\hat{\Delta})=(10,10,10,10,4,a_1,a_2,6,6,0,6,0)$; and if $(i,j)=(2,7)$ then $cv(\hat{\Delta})=(10,10,10,10,4,2,0,6,6,d_1,d_2,0)$. It follows that $c^*(\hat{\Delta})\leqslant -\frac{70\pi}{30}+\frac{69\pi}{30}<0$.

Case b3. Let $\hat{\Delta}$ be given by Figure 51(iii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_7)=3$ then $cv(\hat{\Delta})=(10,10,4,d_1,d_2,3,2,2,2,2)$; if $i=7$ then (see Lemma 9.2) $cv(\hat{\Delta})=(10,10,0,0,5,0,0,f_1,f_2,0)$. It follows that $c^*(\hat{\Delta})\leqslant -\frac{45\pi}{30}+\frac{45\pi}{30}$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_7)=3$ then $c^*(\hat{\Delta})<0$; if $d(u_2)=d(u_8)=3$ then $cv(\hat{\Delta})=(10,10,0,0,6,3,2,f_1,f_2,0)$; if $(i,j)=(7,2)$ then $cv(\hat{\Delta})=(10,10,4,2,5,0,0,f_1,f_2,0)$; and if $(i,j)=(7,8)$ then $cv(\hat{\Delta})=(10,10,0,0,5,0,0,f_1,f_2,2)$. It follows that $c^*(\hat{\Delta})\leqslant -\frac{50\pi}{30}+\frac{43\pi}{30}<0$.

Case b4. Let $\hat{\Delta}$ be given by Figure 51(iv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_4)=d(u_6)=d(u_8)=3$ then $cv(\hat{\Delta})=c(10,10,2,6,6,0,0,2,2,0)$; if $i=4$ then (see Figure 36) $cv(\hat{\Delta})=(10,10,0,6,0,7,0,2,2,0)$; if $i=6$ then $cv(\hat{\Delta})=(10,10,0,6,4,0,e_1,e_2,2,0)$; and if $i=8$ then $cv(\hat{\Delta})=(10,10,0,6,4,0,0,2,e_1,e_2)$. Therefore $c^*(\hat{\Delta})\leqslant -\frac{45\pi}{30}+\frac{43\pi}{30}$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_2)=d(u_6)=3$ then $cv(\hat{\Delta})=(10,10,0,6,d_1,d_2,0,2,e_1,e_2)$; if $d(u_4)=d(u_8)=3$ then $cv(\hat{\Delta})=(10,10,2,d_1,d_2,0,e_1,e_2,2,0)$; if $(i,j)=(2,4)$ then $cv(\hat{\Delta})=(10,10,2,6,0,7,0,2,2,0)$;

if $(i,j)=(2,8)$ then $cv(\hat{\Delta})=(10,10,2,6,4,0,0,2,e_1,e_2)$; if $(i,j)=(6,4)$ then $cv(\hat{\Delta})=(10,10,0,6,0,7,e_1,e_2,2,0)$; and if $(i,j)=(6,8)$ then $cv(\hat{\Delta})=(10,10,0,6,4,0,e_1,e_2,e_1,e_2)$. It follows that either $c^*(\hat{\Delta})\leqslant-\frac{50\pi}{30}+\frac{49\pi}{30}<0$ or $(i,j)=(6,8)$, but here either $d(u)=4$ and $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ across the (u_1,u_2) edge according to Configuration E of Figure 32(iii) and so $c^*(\hat{\Delta})\leqslant-\frac{50\pi}{30}+\frac{53\pi}{30}-\frac{\pi}{5}<0$ or $d(u)>4$, $cv(\hat{\Delta})=(7,7,0,6,4,0,e_1,e_2,e_1,e_2)$ and so $c^*(\hat{\Delta})\leqslant-\frac{50\pi}{30}+\frac{46\pi}{30}<0$. Let $\hat{\Delta}$ have exactly three vertices u_i,u_j,u_k of degree greater than 3. If $d(u_4)=3$ then (see Figure 40(xiv) and (xvii)) $cv(\hat{\Delta})=(10,10,2,d_1,d_2,0,e_1,e_2,e_1,e_2)$ and $c^*(\hat{\Delta})\leqslant-\frac{55\pi}{30}+\frac{54\pi}{30}<0$; or if $d(u_6)=3$ or $d(u_8)=3$ then $c^*(\hat{\Delta})\leqslant-\frac{55\pi}{30}+\frac{51\pi}{30}<0$; and if $(i,j,k)=(4,6,8)$ then $cv(\hat{\Delta})=(10,10,0,6,0,7,e_1,e_2,e_1,e_2)$ and $c^*(\hat{\Delta})\leqslant-\frac{55\pi}{30}+\frac{55\pi}{30}=0$.

Case b5. Let $\hat{\Delta}$ be given by Figure 51(v). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2)=d(u_6)=3$ then $cv(\hat{\Delta})=(10,10,10,10,0,0,6,3,0,d_1,d_2,2)$; if $i=2$ then $cv(\hat{\Delta})=(10,10,10,10,4,2,5,0,0,4,6,0)$; and if $i=6$ then $cv(\hat{\Delta})=(10,10,10,10,0,0,5,0,7,0,6,0)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{65\pi}{30}+\frac{61\pi}{30}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_2)=3$ or $d(u_6)=3$ then in each case $c^*(\hat{\Delta})\leqslant-\frac{70\pi}{30}+\frac{67\pi}{30}<0$; and if $(i,j)=(2,6)$ then $cv(\hat{\Delta})=(10,10,10,10,4,2,5,0,7,0,6,0)$ and $c^*(\hat{\Delta})<0$.

Case b6. Let $\hat{\Delta}$ be given by Figure 51(vi). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_4)=d(u_8)=3$ then $cv(\hat{\Delta})=(10,10,10,10,2,6,6,0,3,6,0,0)$; if $i=4$ then $cv(\hat{\Delta})=(10,10,10,10,0,6,d_1,d_2,0,5,0,0)$; and if $i=8$ then $cv(\hat{\Delta})=(10,10,10,10,0,6,4,0,0,5,2,4)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{65\pi}{30}+\frac{63\pi}{30}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_8)=3$ then $cv(\hat{\Delta})=(10,10,10,10,2,6,d_1,d_2,3,6,0,0)$; if $d(u_2)=d(u_4)=3$ then $cv(\hat{\Delta})=(10,10,10,10,0,6,6,0,3,d_1,d_2,4)$; if $(i,j)=(8,2)$ then $cv(\hat{\Delta})=(10,10,10,10,2,6,4,0,0,5,2,4)$; and if $(i,j)=(8,4)$ then $cv(\hat{\Delta})=(10,10,10,10,0,6,d_1,d_2,0,5,2,4)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{70\pi}{30}+\frac{69\pi}{30}<0$.

Case b7. Let $\hat{\Delta}$ be given by Figure 51(vii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2)=d(u_8)=3$ then $cv(\hat{\Delta})=(10,10,0,6,2,6,6,3,0,0)$; if $i=2$ then $cv(\hat{\Delta})=(10,10,d_1,d_2,0,6,6,0,0,0)$; and if $i=8$ then $cv(\hat{\Delta})=(10,10,0,6,0,6,6,0,2,4)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{45\pi}{30}+\frac{44\pi}{30}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_8)=3$ then $cv(\hat{\Delta})=(10,10,d_1,d_2,2,6,6,3,0,0)$; if $d(u_2)=d(u_7)=3$ then $cv(\hat{\Delta})=(10,10,0,6,2,6,6,3,2,4)$; if $(i,j)=(8,2)$ then $cv(\hat{\Delta})=(10,10,d_1,d_2,0,6,6,0,2,4)$; and if $(i,j)=(8,7)$ then $cv(\hat{\Delta})=(10,10,0,6,0,6,6,a_1,a_2,4)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{50\pi}{30}+\frac{49\pi}{30}<0$.

Case b8. Let $\hat{\Delta}$ be given by Figure 51(viii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_3)=d(u_7)=3$ then $cv(\hat{\Delta})=(10,10,2,6,0,6,6,0,0,3)$; if $i=3$ then $cv(\hat{\Delta})=(10,10,0,d_1,d_2,6,6,0,0,0)$; and if $i=7$ then $cv(\hat{\Delta})=(10,10,0,6,0,6,6,4,2,0)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{45\pi}{30}+\frac{44\pi}{30}<0$. Let $\hat{\Delta}$ have exactly two vertices u_i,u_j of degree greater than 3. If $d(u_7)=3$ then $cv(\hat{\Delta})=(10,10,2,d_1,d_2,6,6,0,0,3)$; if $d(u_2)=d(u_3)=3$ then $cv(\hat{\Delta})=(10,10,0,6,0,6,6,4,a_1,a_2)$; if $(i,j)=(7,2)$ then $cv(\hat{\Delta})=(10,10,2,6,0,6,6,4,2,0)$; and if $(i,j)=(7,3)$ then $cv(\hat{\Delta})=(10,10,0,d_1,d_2,6,6,4,2,0)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{50\pi}{30}+\frac{49\pi}{30}<0$.

Case b9. Let $\hat{\Delta}$ be given by Figure 51(ix). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2)=d(u_4)=3$ then $cv(\hat{\Delta})=(10,10,0,2,2,0,b_1,b_2,6,2)$; if $i=2$ then $cv(\hat{\Delta})=(10,10,e_1,e_2,2,0,0,4,6,0)$; and if $i=4$ then $cv(\hat{\Delta})=(10,10,0,2,e_1,e_2,0,4,6,0)$. It follows that $c^*(\hat{\Delta})\leqslant-\frac{45\pi}{30}+\frac{43\pi}{30}<0$.

Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = d(u_4) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, e_1, e_2, b_1, b_2, 6, 0)$; if $d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, 2, 0, b_1, b_2, 6, 0)$; if $(i, j) = (2, 4)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 0, 4, 6, 0)$; if $(i, j) = (2, 8)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, 2, 0, 0, 4, 6, 2)$; and if $(i, j) = (4, 8)$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, e_1, e_2, 0, 4, 6, 2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{47\pi}{30} < 0$ except for $(i, j) = (2, 4)$, in which case either $d(u) = 4$ and $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ across the (u_8, u_9) edge according to Configuration F of Figure 32(v) and $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{52\pi}{30} - \frac{\pi}{5} < 0$ or $d(u) > 4$, $cv(\hat{\Delta}) = (7, 7, e_1, e_2, e_1, e_2, 0, 4, 6, 0)$ and $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{46\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ or $d(u_4) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_6) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 0, 6, 6, 0)$; if $(i, j, k) = (2, 4, 6)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 7, 0, 6, 0)$; and if $(i, j, k) = (2, 4, 8)$ then $cv(\hat{\Delta}) = (10, 10, e_1, e_2, e_1, e_2, 0, 4, 6, 2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{55\pi}{30} = 0$.

Case b10. Let $\hat{\Delta}$ be given by Figure 51(x). Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. If $d(u_3) = d(u_6) = 3$ then $cv(\hat{\Delta}) = c(10, 10, 3, 0, 0, 3, 0, 6, 6, 2)$; if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, 2, 0, 0, 4, 6, 0)$; and if $i = 6$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 0, 0, 7, 0, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{40\pi}{30} < 0$.

Case b11. Let $\hat{\Delta}$ be given by Figure 51(xi). Let $\hat{\Delta}$ have exactly one vertex u_i of degree greater than 3. If $d(u_2) = d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 6, 0, 3, 6, 2, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 0, 0, 5, 0, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, 0, 5, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 6, 0, 0, 5, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 6, 0, 3, b_1, b_2, 4)$; if $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, 3, 6, 0, 0)$; if $d(u_4) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 0, 3, 6, 0, 0)$; if $(i, j) = (2, 4)$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, d_1, d_2, 0, 5, 0, 0)$; if $(i, j) = (2, 8)$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 6, 0, 0, 5, 2, 4)$; and if $(i, j) = (4, 8)$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, 0, 5, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{47\pi}{30} < 0$.

Case b12. Let $\hat{\Delta}$ be given by Figure 51(xii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_3) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, 2, 2, 2, 3, b_1, b_2, 4)$; and if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, f_1, f_2, 0, 0, 5, 0, 0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_3, u_j of degree greater than 3. If $d(u_7) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 2, f_1, f_2, 2, 3, 6, 0, 0)$; if $j = 7$ then $cv(\hat{\Delta}) = (10, 10, 0, f_1, f_2, 0, 0, 6, 0, 0)$; and if $j = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, f_1, f_2, 0, 0, 5, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{45\pi}{30} < 0$.

Case b13. Let $\hat{\Delta}$ be given by Figure 51(xiii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2) = d(u_3) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 0, 6, 6, d_1, d_2, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 3, 0, 0, 6, 6, 0, 6, 0)$; if $i = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, 4, 6, 6, 0, 6, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 0, 6, 6, 0, 6, 2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{44\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 2, 4, 6, 6, d_1, d_2, 2)$; if $d(u_3) = 3$ then $cv(\hat{\Delta}) = (10, 10, 3, 0, 0, 6, 6, d_1, d_2, 2)$; and if $(i, j) = (2, 3)$ then $cv(\hat{\Delta}) = (10, 10, a_1, a_2, 4, 6, 6, 0, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{50\pi}{30} = 0$.

Case b14. Let $\hat{\Delta}$ be given by Figure 51(xiv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 2, 6, 6, 3, 0, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 0, 6, 6, 0, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, 0, 2, 4)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{44\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 2, 6, 6, 3, 0, 0)$; if $d(u_6) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, d_1, d_2, 2, 6, 6, 0, 2, 4)$; if $(i, j) = (8, 6)$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, 3, 2, 4)$;

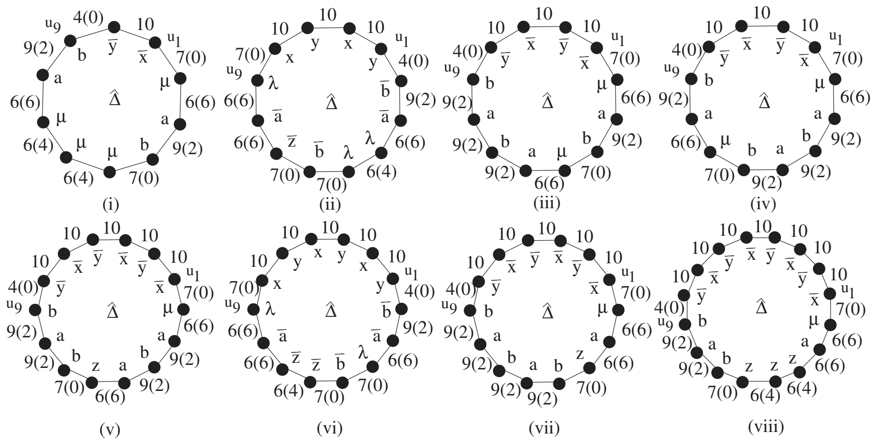

FIGURE 52. *Regions for Figure* 49.

and if $(i,j) = (8,7)$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 0, 6, 6, a_1, a_2, 4)$. It follows that $cv(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{50\pi}{30} = 0$.

Case b15. Let $\hat{\Delta}$ be given by Figure 51(xv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2) = d(u_6) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 3, 0, 6, d_1, d_2)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, 0, 6, 6, 0)$; and if $i = 6$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 5, 0, d_1, d_2, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{45\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10, 10, 0, 0, 6, 3, d_1, d_2, d_1, d_2)$; if $d(u_6) = d(u_8) = 3$ then $cv(\hat{\Delta}) = (10, 10, 4, d_1, d_2, 3, 0, 6, 6, 0)$, if $(i,j) = (2,6)$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, d_1, d_2, 6, 0)$; and if $(i,j) = (2,8)$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, 0, 6, d_1, d_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{49\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly three vertices of degree greater than 3. If $d(u_2) = 3$ or $d(u_6) = 3$ or $d(u_8) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{53\pi}{30} < 0$; and if $(i,j,k) = (2,6,8)$ then $cv(\hat{\Delta}) = (10, 10, 4, 2, 5, 0, d_1, d_2, d_1, d_2)$ and $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{51\pi}{30} < 0$.

Case b16. Let $\hat{\Delta}$ be given by Figure 51(xvi). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_4) = d(u_8) = 3$ then $c^*(\hat{\Delta}) = (10, 10, 2, 6, 6, 0, a_1, a_2, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, 0, 0, 0, 0)$; and if $i = 8$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, 5, 0, 0, 0, 2, 3)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_4) = 3$ or $d(u_8) = 3$ then $c^*(\hat{\Delta}) < 0$; and if $(i,j) = (4,8)$ then $cv(\hat{\Delta}) = (10, 10, 0, 6, d_1, d_2, a_1, a_2, a_1, a_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{50\pi}{30}$.

Finally let $\hat{\Delta}$ be given by one of the regions of Figure 49. It turns out that (up to cyclic permutations and inversion) there is one way to label each of Figure 49(i), (ii), (iii), (v) and (vi); two ways to label (iv); five ways to label (vii) or (viii); three ways to label (ix) or (x); seven ways to label (xi) or (xii) or (xiii); and seven ways to label (xiv) or (xv) or (xvi) or (xvii). This yields a total of 29 regions. There are however several coincidences amongst these regions resulting in $\hat{\Delta}$ being one of the eight regions given by Figure 52. Table 5 gives $c(u_i, u_{i+1})$ $(1 \leqslant i \leqslant 9)$ in multiples of $\pi/30$ for each of the eight regions of Figure 52 with the total plus the contribution via the b-segment in the final column.

We claim that $x_1 + y_1 + z_1 = 15$ in Table 5. To see this let $\hat{\Delta}$ be given by Figure 52(i). If $c(u_5, u_6) = 0$ then $x_1 + y_1 + z_1 = 14$, so assume otherwise, in which case $c(u_4, u_5) = \frac{2\pi}{15}$ (Figure 40(ix)). If now $c(u_5, u_6) = \frac{2\pi}{15}$ then $x_1 + y_1 + z_1 = 15$ by Lemma 9.2. On the other hand if $c(u_5, u_6) > \frac{2\pi}{15}$ then $d(u_5) = 3$ (see Figure 40) forcing $c(u_4, u_5) = \frac{\pi}{30}$ and $x_1 + y_1 + z_1 = 15$. Note that we use here and below the fact that labelling prevents $\hat{\Delta} = \hat{\Delta}_2$ of Figure 38. The

TABLE 5. $c(u_i, u_{i+1})$ *for Figure* 52.

(i)	d_1	d_2	x_1	y_1	z_1	6	6	e_1	e_2	$48+10=58$
(ii)	e_1	e_2	6	d_1	d_2	d_1	d_2	d_1	d_2	$47+30=77$
(iii)	d_1	d_2	x_1	y_1	z_1	f_1	f_2	e_1	e_2	$48+30=78$
(iv)	d_1	d_2	f_1	f_2	x_1	y_1	z_1	e_1	e_2	$48+30=78$
(v)	d_1	d_2	f_1	f_2	x_1	y_1	z_1	e_1	e_2	$48+50=98$
(vi)	e_1	e_2	d_1	d_2	d_1	d_2	6	d_1	d_2	$47+50=97$
(vii)	d_1	d_2	x_1	y_1	z_1	f_1	f_2	e_1	e_2	$48+50=98$
(viii)	d_1	d_2	6	6	x_1	y_1	z_1	e_1	e_2	$48+70=118$

arguments for $\hat{\Delta}$ of Figure 52(iii), (iv), (v), (vii) and (viii) are similar although for (v), (vii) and (viii) we use the fact, again both here and below, that $\hat{\Delta} \neq \hat{\Delta}_2$ of Figure 37.

Observe that $d(\hat{\Delta}) = 10$ in (i); $d(\hat{\Delta}) = 12$ in (ii)–(iv); $d(\hat{\Delta}) = 14$ in (v)–(vii); and $d(\hat{\Delta}) = 16$ in (viii). It follows that if $\hat{\Delta}$ has at least four vertices of degree greater than 3 then $c^*(\hat{\Delta}) \leqslant 0$. If $\hat{\Delta}$ has no vertices of degree greater than 3 then we see from Figure 52 that $c^*(\hat{\Delta}) \leqslant -7\pi + \frac{18\pi}{3} + \frac{24\pi}{30} < 0$.

We deal with each of the eight *c-cases* in turn.

Case c1. Let $\hat{\Delta}$ be given by Figure 52(i). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_4) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, d_1, d_2, 2, 0, 6, 6, 6, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 0, 6, c_1, c_2, 4, 4, 6, 2, 0)$ (the c_1, c_2 follows from $\hat{\Delta} \neq \hat{\Delta}_2$ of Figure 38(iv)); and if $i = 9$ then $cv(\hat{\Delta}) = (10, 0, 6, 2, 0, 4, 4, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{45\pi}{30} + \frac{43\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 0, 6, 2, 0, 6, 6, 6, e_1, e_2)$; if $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 0, 6, x_1, y_1, z_1, 6, 6, 2, 0)$; if $d(u_4) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, d_1, d_2, 2, 0, 6, 6, 6, 2, 0)$; if $(i, j) = (2, 4)$ then $cv(\hat{\Delta}) = (10, d_1, d_2, c_1, c_2, 4, 4, 6, 2, 0)$; if $(i, j) = (2, 9)$ then $cv(\hat{\Delta}) = (10, d_1, d_2, 2, 0, 4, 4, 6, e_1, e_2)$; if $(i, j) = (4, 9)$ then $cv(\hat{\Delta}) = (10, 0, 6, c_1, c_2, 4, 4, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{50\pi}{30} + \frac{50\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_4) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) < 0$; if $d(u_2) = d(u_7) = 3$ then (see Figure 40(xiv)) $cv(\hat{\Delta}) = (10, 0, 6, x_1, y_1, z_1, 4, 6, e_1, e_2)$; if $(i, j, k) = (4, 9, 2)$ then $cv(\hat{\Delta}) = (10, d_1, d_2, c_1, c_2, 4, 4, 6, e_1, e_2)$; and if $(i, j, k) = (4, 9, 7)$ then $cv(\hat{\Delta}) = (10, 0, 6, c_1, c_2, 4, 6, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{55\pi}{30} + \frac{54\pi}{30} < 0$.

Case c2. Let $\hat{\Delta}$ be given by Figure 52(ii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 2, 6, d_1, d_2, d_1, d_2, 6, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 10, e_1, e_2, 6, 4, 0, 0, 6, 6, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 2, 6, 4, 0, 0, 6, d_1, d_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{65\pi}{30} + \frac{64\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 2, 6, d_1, d_2, d_1, d_2, d_1, d_2)$; if $d(u_6) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, e_1, e_2, 6, 6, 0, 0, 6, d_1, d_2)$; and if $(i, j) = (2, 6)$ then $cv(\hat{\Delta}) = (10, 10, 10, e_1, e_2, 6, 4, 2, 2, 6, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{70\pi}{30} + \frac{69\pi}{30}$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ or $d(u_6) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{75\pi}{30} + \frac{73\pi}{30} < 0$; and if $(i, j, k) = (2, 6, 9)$ then $cv(\hat{\Delta}) =$(10, 10, 10, e_1, e_2, 6, 4, 2, 2, 6, d_1, d_2) and $c^*(\hat{\Delta}) \leqslant -\frac{75\pi}{30} + \frac{71\pi}{30} < 0$.

Case c3. Let $\hat{\Delta}$ be given by Figure 52(iii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_7) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2,$

$x_1, y_1, z_1, 2, 2, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 0, 6, f_1, f_2, 2, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 0, 6, 2, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{65\pi}{30} + \frac{61\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, x_1, y_1, z_1, 2, 2, e_1, e_2)$; if $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, x_1, y_1, z_1, f_1, f_2, 2, 0)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 0, 6, 2, 2, e_1, e_2)$; and if $(i, j) = (9, 7)$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 0, 6, f_1, f_2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{70\pi}{30} + \frac{69\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ or $d(u_7) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{75\pi}{30} + \frac{74\pi}{30} < 0$; and if $(i, j, k) = (2, 7, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 0, 6, f_1, f_2, e_1, e_2)$ and $c^*(\hat{\Delta}) \leqslant -\frac{75\pi}{30} + \frac{71\pi}{30} < 0$.

Case c4. Let $\hat{\Delta}$ be given by Figure 52(iv). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_4) = d(u_9) = 3$ then $c^*(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 2, x_1, y_1, z_1, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, f_1, f_2, 2, 0, 6, 2, 0)$; and if $i = 9$ then $c^*(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 2, 2, 0, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{65\pi}{30} + \frac{61\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, f_1, f_2, x_1, y_1, z_1, 2, 0)$; if $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, 2, 2, x_1, y_1, z_1, e_1, e_2)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, 2, 2, 2, 0, 6, e_1, e_2)$; and if $(i, j) = (9, 4)$ then $cv(\hat{\Delta}) = (10, 10, 10, 0, 6, f_1, f_2, 2, 0, 6, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{70\pi}{30} + \frac{69\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ or $d(u_4) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{75\pi}{30} + \frac{74\pi}{30} < 0$; and if $(i, j, k) = (2, 4, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, d_1, d_2, f_1, f_2, 2, 0, 6, e_1, e_2)$ and $c^*(\hat{\Delta}) \leqslant -\frac{75\pi}{30} + \frac{71\pi}{30} < 0$.

Case c5. Let $\hat{\Delta}$ be given by Figure 52(v). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_4) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 2, 2, x_1, y_1, z_1, 2, 0)$; if $i = 4$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, f_1, f_2, 6, 0, 2, 2, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 2, 2, 6, 0, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{85\pi}{30} + \frac{81\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, f_1, f_2, x_1, y_1, z_1, 2, 0)$; if $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 2, 2, x_1, y_1, z_1, e_1, e_2)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 2, 2, 6, 0, 2, e_1, e_2)$; and if $(i, j) = (9, 4)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, f_1, f_2, 6, 0, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{90\pi}{30} + \frac{89\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ or $d(u_4) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{95\pi}{30} + \frac{94\pi}{30} < 0$; and if $(i, j, k) = (2, 4, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, f_1, f_2, 6, 0, 2, e_1, e_2)$ and $c^*(\hat{\Delta}) \leqslant -\frac{95\pi}{30} + \frac{91\pi}{30} < 0$.

Case c6. Let $\hat{\Delta}$ be given by Figure 52(vi). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 2, d_1, d_2, d_1, d_2, 6, 6, 0)$; if $i = 2$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, e_1, e_2, 6, 0, 0, 4, 6, 6, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 2, 6, 0, 0, 4, 6, d_1, d_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{85\pi}{30} + \frac{84\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_2) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 2, d_1, d_2, d_1, d_2, 6, d_1, d_2)$; if $d(u_5) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, e_1, e_2, 6, 0, 0, 6, 6, d_1, d_2)$; and if $(i, j) = (2, 5)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, e_1, e_2, 6, 2, 2, 4, 6, 6, 0)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{90\pi}{30} + \frac{89\pi}{30}$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ or $d(u_5) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{95\pi}{30} + \frac{93\pi}{30} < 0$; and if $(i, j, k) = (2, 5, 9)$ then $cv(\hat{\Delta}) =(10, 10, 10, 10, 10, e_1, e_2, 6, 2, 2, 4, 6, d_1, d_2)$ and $c^*(\hat{\Delta}) \leqslant -\frac{95\pi}{30} + \frac{91\pi}{30} < 0$.

Case c7. Let $\hat{\Delta}$ be given by Figure 52(vii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_7) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, x_1, y_1, z_1, 2, 2, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 6, 0, 2, f_1, f_2, 2, 0)$; and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 6, 0, 2, 2, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{85\pi}{30} + \frac{81\pi}{30} < 0$. Let $\hat{\Delta}$ have exactly two vertices u_i, u_j of degree greater than 3. If $d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, x_1, y_1, z_1, f_1, f_2, 2, 0)$; if $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, x_1, y_1, z_1, 2, 2, e_1, e_2)$; if $(i, j) = (9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 6, 0, 2, 2, 2, e_1, e_2)$; and if $(i, j) = (9, 7)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 0, 6, 6, 0, 2, f_1, f_2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{90\pi}{30} + \frac{89\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_2) = 3$ or $d(u_7) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{95\pi}{30} + \frac{94\pi}{30} < 0$; if $(i, j, k) = (2, 7, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, d_1, d_2, 6, 0, 2, f_1, f_2, e_1, e_2)$ and $c^*(\hat{\Delta}) \leqslant -\frac{95\pi}{30} + \frac{91\pi}{30} < 0$.

Case c8. Let $\hat{\Delta}$ be given by Figure 52(viii). Suppose that $\hat{\Delta}$ has exactly one vertex u_i of degree greater than 3. If $d(u_7) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 6, 6, 0, 2, 2, 0)$; if $i = 7$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, 4, c_1, c_2, 2, 0)$ (the c_1, c_2 follows from $\hat{\Delta} \neq \hat{\Delta}_2$ of Figure 37(iv)); and if $i = 9$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, 4, 0, 2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{105\pi}{30} + \frac{103\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly two vertices, u_i, u_j of degree greater than 3. If $d(u_2) = d(u_7) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 6, 6, 0, 2, e_1, e_2)$; if $d(u_2) = d(u_9) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 6, x_1, y_1, z_1, 2, 0)$; if $d(u_7) = d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{110\pi}{30} + \frac{102\pi}{30}$; if $(i, j) = (2, 7)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 4, 4, c_1, c_2, 2, 0)$; if $(i, j) = (2, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 4, 4, 0, 2, e_1, e_2)$; and if $(i, j) = (7, 9)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, 4, c_1, c_2, e_1, e_2)$. It follows that $c^*(\hat{\Delta}) \leqslant -\frac{110\pi}{30} + \frac{110\pi}{30} = 0$. Let $\hat{\Delta}$ have exactly three vertices u_i, u_j, u_k of degree greater than 3. If $d(u_7) = 3$ or $d(u_9) = 3$ then $c^*(\hat{\Delta}) \leqslant -\frac{115\pi}{30} + \frac{111\pi}{30} < 0$; if $d(u_2) = d(u_4) = 3$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 4, x_1, y_1, z_1, e_1, e_2)$; if $(i, j, k) = (7, 9, 2)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, d_1, d_2, 6, 4, 4, c_1, c_2, e_1, e_2)$; and if $(i, j, k) = (7, 9, 4)$ then $cv(\hat{\Delta}) = (10, 10, 10, 10, 10, 10, 10, 0, 6, 6, 6, 4, c_1, c_2, e_1, e_2)$. It follows that $cv(\hat{\Delta}) \leqslant -\frac{115\pi}{30} + \frac{114\pi}{30} < 0$. □

The next result completes the proof of Proposition 4.3.

PROPOSITION 11.5. *If $\hat{\Delta}$ is a type $\mathcal{B}$ region and $d(\hat{\Delta}) \geqslant 10$ then $c^*(\hat{\Delta}) \leqslant 0$.*

Proof. It can be assumed $d(\hat{\Delta}) \geqslant 10$ and that $\hat{\Delta}$ is not one of the regions of Figures 47(ii)–(v), 48 or 49, otherwise Lemma 11.4 applies. Moreover if $n_2 \geqslant 10$ then $c^*(\hat{\Delta}) \leqslant 0$ so assume that $n_2 \leqslant 9$. It follows from the proof of Lemma 11.1 that the upper bound (†) immediately preceding Lemma 11.1 is reduced by at least $\frac{2\pi}{15}$ for each gap between two b-segments that contain b-regions so if there are at least three such b-segments then $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{n_2}{5}) - 3(\frac{2\pi}{15})$ implying $c^*(\hat{\Delta}) \leqslant 0$ for $n_2 \geqslant 8$. Since there are at least two edges between b-segments it follows that if $\hat{\Delta}$ contains more than three such b-segments then $c^*(\hat{\Delta}) \leqslant 0$ or if exactly three then $n_2 \geqslant 8$ by Lemma 11.2(i) and again $c^*(\hat{\Delta}) \leqslant 0$. If $\hat{\Delta}$ has exactly one b-segment that contains a b-region then $c^*(\hat{\Delta}) \leqslant 0$ by Lemma 11.4 together with Lemma 11.2(v) and (vi) so suppose from now on that $\hat{\Delta}$ contains exactly two such segments. Then $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{n_2}{5}) - 2(\frac{2\pi}{15})$ which implies $c^*(\hat{\Delta}) \leqslant 0$ for $n_2 \geqslant 9$, so assume $n_2 \leqslant 8$ in which case $\hat{\Delta}$ is given by Figure 47(i) where $(m, n) \in \{(2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (4, 4)\}$. Applying Lemmas 11.4 and 11.2(ii) shows it can be assumed that there is at least one shadow edge in $\hat{\Delta}$ between the two b-segments.

Let $m = 2$. It follows from the statement at the end of the above paragraph that $\hat{\Delta}$ contains the shadow edge (14) (of length $n - 1$) and $\hat{\Delta}$ is given by Figure 53(i) and (ii). If $(m, n) \neq (2, 6)$

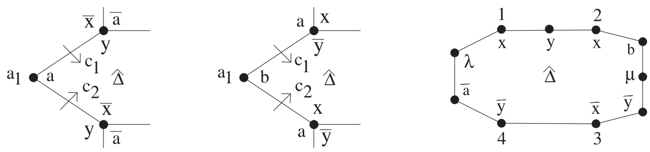

FIGURE 53. $m = 2$.

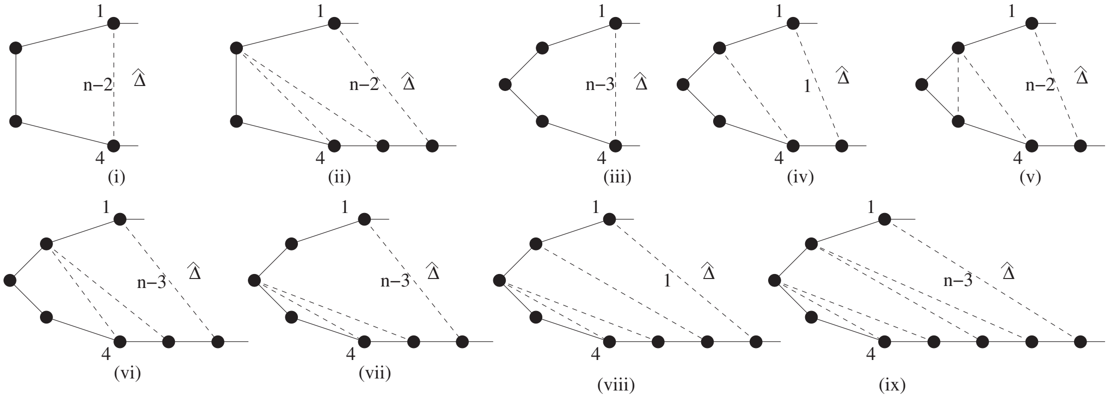

FIGURE 54. $m = 3$ or 4.

then $i\deg(1) = i\deg(4) = 1$ by Lemma 11.2(iii) and this leads to a length contradiction so let $(m,n) = (2,6)$. We claim that there is a reduction to (†) of $\frac{4\pi}{15}$ between vertices 1 and 4. Given this and the fact that there is a reduction of $\frac{2\pi}{15}$ between 2 and 3 we obtain $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{n_2}{5}) - \frac{6\pi}{15}$ and $c^*(\hat{\Delta}) \leqslant 0$ for $n_2 \geqslant 8$, in particular when $(m,n) = (2,6)$. To prove the claim observe that if $d(a_1) = 3$ in Figure 53(i) or (ii) then $c_1 = c_2 = 0$; and if $d(a_1) \geqslant 4$ then $c_1 + c_2 \leqslant \frac{2\pi}{15}$ (see Figure 35). In the first instance there is a deficit of at least $(\frac{2\pi}{3} + 2(\frac{2\pi}{15})) - \frac{2\pi}{3} = \frac{4\pi}{15}$; and in the second case the deficit is at least $(\frac{2\pi}{3} + 2(\frac{2\pi}{15})) - (\frac{2\pi}{4} + \frac{2\pi}{15}) = \frac{3\pi}{10}$.

Let $m = 3$ or 4. Applying Lemma 11.2(ii)–(iv) and Lemma 11.4 it can be assumed that $\hat{\Delta}$ is given by Figure 54 with the understanding that the segment of $\hat{\Delta}$ between vertices 2 and 3 is also one of these nine possibilities. (Note that in Figure 54 the length of the shadow edge incident at vertex 1 is shown.) We claim that if $m = 3$ then the edges between 1 and 4 produce a deficit of at least $\frac{2\pi}{5}$; and if $m = 4$ then the reduction is at least $\frac{\pi}{5}$. Given this, if $(m,n) = (3,3)$ then $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{6}{5}) - \frac{4\pi}{5} = 0$; if $(m,n) = (3,4)$ then $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{7}{5}) - \frac{3\pi}{5} = 0$; if $(m,n) = (3,5)$ then $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{8}{5}) - (\frac{2\pi}{5} + \frac{2\pi}{15}) < 0$; and if $(m,n) = (4,4)$ then $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{8}{5}) - 2(\frac{\pi}{5}) = 0$, so it remains to prove the claim for the possible labellings of the regions of Figure 54 and these are shown in Figure 55(i)–(xx). Indeed there are four ways to label each of Figure 54(iv) and (vi); and two ways to label each of the others. However the labelling obtained from Figure 54(vii) already appears in Figure 54(vi).

Let $m = 3$. Then Tables 6–9 give maximum values for κ_1, κ_2 and κ_3 of Figure 55(i)–(iii) as multiples of $\frac{\pi}{30}$. Also indicated in each case as a multiple of $\frac{\pi}{30}$ is the deficit $= \pi(\frac{2}{3} - \frac{2}{d(v_1)}) + \pi(\frac{2}{3} - \frac{2}{d(v_2)}) + (\frac{\pi}{30})(12 - (\kappa_1 + \kappa_2 + \kappa_3))$. The entries in each final column show that the deficit in each case is $\frac{2\pi}{5}$, as required, except for $d(v_1) = d(v_2) = 3$ and $d(u) > 4$ in Figure 55(i) and we consider this below. Note that in Tables 6 and 8 when $d(u) = 4$ in Figure 55(i) and (iii) and either $d(v_1) = d(v_2) = 3$ or $d(v_1) = 3$, $d(v_2) = 4$, $\frac{\pi}{5}$ is distributed from $\hat{\Delta}$ according to Configurations E and F of Figure 32(iii) and (v) resulting in deficits of 12 and 16 as shown. Note that in Table 7 when $d(v_1) = 3$ and $d(v_2) = 4$ the region $\hat{\Delta}$ cannot be $\hat{\Delta}_2$ of Figure 38(iv)

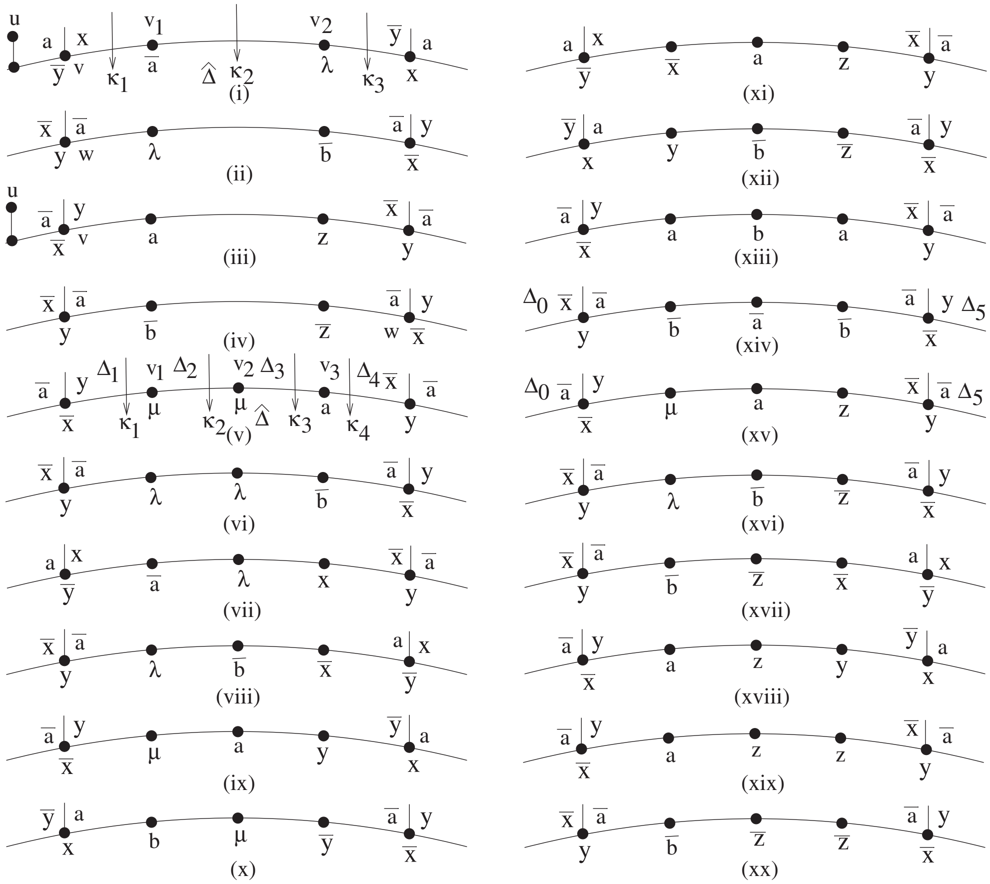

FIGURE 55. *Labelling for $m = 3$ or 4.*

TABLE 6. *Deficit for Figure 55(i).*

	$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	Deficit	
(i)	3	3	0	6	0	12 (9)	(Note)
(i)	4	3	2	0	0	15	
(i)	3	4	0	0	7	16 (13)	(Note)
(i)	5+	3	2	2	0	16	
(i)	3	5+	0	2	2	16	
(i)	4	4	2	0	7	13	
(i)	4	5+	2	4	2	17	
(i)	5+	4	2	4	7	12	
(i)	5+	5+	2	2	2	22	

because $d(w) = 3$ in Figure 55(ii) but the corresponding vertex in Figure 38(ii) has degree 4, and so $\kappa_2 = 1$ by Figure 36(x). Similarly in Table 9 when $d(v_1) = 4$ and $d(v_2) = (3)$ the region $\hat{\Delta}$ cannot be $\hat{\Delta}_2$ of Figure 37(iv); and so $\kappa_2 = 1$ by Figure 36(i).

Suppose $d(u) > 4$ in Figure 55(i) in which the vertex v corresponds to the vertex 4 of Figure 54(i). If there are at least two regions in the b-segment between vertices 4 and 3

Table 7. *Deficit for Figure 55*(ii).

	$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	Deficit	
(ii)	3	3	0	0	0	12	
(ii)	4	3	0	0	0	17	
(ii)	3	4	0	1	4	12	(Note)
(ii)	5+	3	2	2	0	16	
(ii)	3	5+	0	2	2	16	
(ii)	4	4	0	7	0	15	
(ii)	4	4	0	0	4	18	
(ii)	4	5+	0	4	2	19	
(ii)	5+	4	2	4	4	15	
(ii)	5+	5+	2	2	2	22	

Table 8. *Deficit for Figure 55*(iii).

	$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	Deficit	
(iii)	3	3	0	6	0	12	(Note)
(iii)	4	3	2	0	0	15	
(iii)	3	4	0	0	7	16	(Note)
(iii)	5+	3	2	2	0	16	
(iii)	3	5+	0	2	2	16	
(iii)	4	4	2	0	7	13	
(iii)	4	5+	2	4	2	17	
(iii)	5+	4	2	4	7	12	
(iii)	5+	5+	2	2	2	22	

then $2(\frac{10\pi}{30} - \frac{7\pi}{30}) = \frac{\pi}{5}$ is contributed to the deficit and so we obtain the totals $\frac{12\pi}{30}$ when $d(v_1) = d(v_2) = 3$ and $\frac{16\pi}{30}$ when $d(v_1) = 3$, $d(v_2) = 4$ as shown in Table 6. If however there is exactly one region in the b-segment then only $\frac{10\pi}{30} - \frac{7\pi}{30} = \frac{\pi}{10}$ is contributed to the deficit and so the total is $\frac{9\pi}{30}$ when $d(v_1) = d(v_2) = 3$ and $\frac{13\pi}{30}$ when $d(v_1) = 3$, $d(v_2) = 4$ as shown in parentheses in Table 6. If $(m, n) = (3, 5)$ then $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{8}{5}) - (\frac{9\pi}{30} + \frac{2\pi}{15}) < 0$ so it can be assumed $n \in \{3, 4\}$. But given that there are no vertices between 4 and 3, it follows immediately from length considerations that (i) of Figure 54 can only be combined with (iv) or (viii), and so, in particular, $n = 4$. Any attempt at labelling shows that (i) with (viii) is impossible and the unique region $\hat{\Delta}$ obtained from (i) with (iv) is given by Figure 53(iii) in which the segment of vertices from 2 to 3 corresponds to Figure 55(x). We show below that for Figure 55(x), the deficit is at least $\frac{\pi}{3}$ and so $c^*(\hat{\Delta}) \leqslant \pi(2 - \frac{7}{5}) - \frac{19\pi}{30} < 0$. If $d(u) > 4$ in Figure 55(iii) in which the vertex v corresponds to the vertex 4 of Figure 54(ii), then since there are at least two regions in the b-segment between 4 and 3 it follows that, as in the above case, the total deficit is 12 and 16 as shown in Table 8.

Now let $m = 4$ and consider Figure 55. (Recall that it remains to show that there is a deficit of at least $\frac{\pi}{5}$ in all cases except for Figure 55(x) where we must show that there is a deficit of at least $\frac{\pi}{3}$.) Checking Figures 35–38, 40 and 41 and Lemma 9.1 shows $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leqslant \frac{11\pi}{15}$ for (xiv); and $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leqslant \frac{9\pi}{15}$ in all other cases. Indeed the upper bounds are shown in Table 10. Note that $\kappa_4 \leqslant 2$ in (vii)–(x), (xvii) and (xviii) follows from the fact that $d(v_3) \geqslant 4$

TABLE 9. *Deficit for Figure 55*(iv).

	$d(v_1)$	$d(v_2)$	κ_1	κ_2	κ_3	Deficit	
(iv)	3	3	0	0	0	12	
(iv)	4	3	4	1	0	12	(Note)
(iv)	3	4	0	0	0	17	
(iv)	5+	3	2	2	0	16	
(iv)	3	5+	0	2	2	16	
(iv)	4	4	4	0	0	18	
(iv)	4	4	0	7	0	15	
(iv)	4	5+	4	4	2	15	
(iv)	5+	4	2	4	0	19	
(iv)	5+	5+	2	4	2	20	

TABLE 10. *Upper bounds for $m = 4$.*

	κ_1	κ_2	κ_3	κ_4			κ_1	κ_2	κ_3	κ_4	
(v)	b_1	b_2	6	2	16	(xiii)	x_1	x_2	x_3	x_4	18
(vi)	3	d_1	d_2	4	17	(xiv)	e_1	e_2	e_1	e_2	22
(vii)	2	6	7	2	17	(xv)	y_1	y_2	y_3	y_4	17
(viii)	3	7	4	2	16	(xvi)	a_1	a_2	a_1	a_2	14
(ix)	7	6	2	2	17	(xvii)	4	7	3	2	16
(x)	4	7	3	2	16	(xviii)	2	6	7	2	17
(xi)	2	2	6	7	17	(xix)	2	6	d_1	d_2	18
(xii)	2	4	7	3	16	(xx)	4	d_1	d_2	3	17

and if $d(v_3) = 4$ then $\kappa_4 = 0$; $\kappa_1 \leqslant 2$ in (xi) and (xii) follows from the fact that $d(v_1) \geqslant 4$ and if $d(v_1) = 4$ then $\kappa_1 = 0$. Note further that in (v) $\kappa_1 > 4$ implies $\kappa_2 = 0$ and $\kappa_2 > 4$ implies $\kappa_1 = 0$; in (vi) that $\kappa_3 > 4$ implies $\kappa_2 = 0$; that $x_1 + x_2 + x_3 + x_4 \leqslant 18$ in (xiii) follows from the fact that $d(v_1) = 3$ implies $\kappa_1 = 0$, $d(v_1) > 3$ implies $\kappa_2 = 5$, $d(v_3) = 3$ implies $\kappa_4 = 0$ and $d(v_3) > 3$ implies $\kappa_3 = 5$; that in (xiv) $\kappa_2 = 9$ or 8 forces $\kappa_1 = 0$ or 2, that $\kappa_2 \leqslant 4$ and that similar statements hold for κ_3 and κ_4; in (xv) the fact that $\kappa_1 > 4$ implies $\kappa_2 = 0$, $\kappa_2 > 4$ implies $\kappa_1 = 0$ or $\kappa_3 = 0$, $\kappa_3 > 4$ implies $\kappa_2 = 0$ or $\kappa_4 = 0$ and $\kappa_4 > 4$ implies $\kappa_3 = 0$ forces $y_1 + y_2 + y_3 + y_4 \leqslant 17$; in (xvi) that $\kappa_2 > 4$ implies $\kappa_1 = 0$ and $\kappa_3 > 4$ implies $\kappa_4 = 0$; in (xix) $\kappa_4 > 4$ implies $\kappa_3 = 0$; and in (xx) $\kappa_2 > 4$ implies $\kappa_3 = 0$. All other numerical entries for the upper bounds in Table 10 can be read directly from Figure 35.

In the cases where $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leqslant \frac{9\pi}{15}$ if at least one of v_1, v_2, v_3 has degree at least 5 then there is a deficit of at least $(\frac{2\pi}{3} + \frac{8\pi}{15}) - (\frac{2\pi}{5} + \frac{9\pi}{15}) = \frac{\pi}{5}$; and if at least two have degree at least 4 then the deficit is at least $(\frac{4\pi}{3} + \frac{8\pi}{15}) - (\pi + \frac{9\pi}{15}) = \frac{4\pi}{15}$, so it can be assumed $(d(v_1), d(v_2), d(v_3)) \in \{(3,3,3), (4,3,3), (3,4,3), (3,3,4)\}$. For cases (vii)–(x) and (xvii)–(xviii), $d(v_3) = 4$ which forces $\kappa_4 = 0$ and so $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leqslant \frac{\pi}{2}$ which gives a deficit of at least $\frac{\pi}{5}$. For (xi) and (xii), $d(v_1) = 4$ which forces $\kappa_1 = 0$ and so $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leqslant \frac{\pi}{2}$ which gives a deficit of at least $\frac{\pi}{5}$. In fact for case (x), $d(v_3) \geqslant 4$ and so if at least one of v_1 or v_2 has degree at least 4 then the deficit is at least $\frac{\pi}{3}$; or if $d(v_1) = d(v_2) = 3$ then $\kappa_1 = \kappa_2 = 0$ and we see from Table 10 that the deficit is at least $\frac{8\pi}{15}$, as required.

Table 11 shows the deficit for (v), (vi), (xiii), (xv), (xvi), (xix) and (xx).

TABLE 11. *Deficit for Figure* 55(v), (vi), (xiii), (xv), (xvi), (xix) *and* (xx).

	$d(v_1)$	$d(v_2)$	$d(v_3)$	κ_1	κ_2	κ_3	κ_4	Deficit	
(v)	3	3	3	0	4	6	0	6	
(v)	4	3	3	7	0	6	0	8	
(v)	3	4	3	0	0	0	0	21	
(v)	3	3	4	0	4	0	2	15	
(vi)	3	3	3	0	5	0	0	11	
(vi)	4	3	3	0	0	0	0	21	
(vi)	3	4	3	0	0	0	0	21	
(vi)	3	3	4	0	5	2	4	10	
(xiii)	3	3	3	0	2	2	0	12	
(xiii)	4	3	3	2	0	2	0	17	
(xiii)	3	4	3	0	9	0	0	12	
(xiii)	3	4	3	0	0	9	0	12	
(xiii)	3	3	4	0	2	0	2	17	
(xv)	3	3	3	0	6	6	0	6	(Note)
(xv)	4	3	3	7	0	6	0	8	
(xv)	3	4	3	0	0	0	0	21	
(xv)	3	3	4	0	6	0	7	8	
(xvi)	3	3	3	0	0	0	0	16	
(xvi)	4	3	3	0	0	0	0	21	
(xvi)	3	4	3	0	2	2	0	17	
(xvi)	3	3	4	0	0	0	0	21	
(xix)	3	3	3	0	6	4	0	6	
(xix)	4	3	3	2	0	4	0	15	
(xix)	3	4	3	0	0	0	0	21	
(xix)	3	3	4	0	6	0	7	8	
(xx)	3	3	3	0	0	5	0	11	
(xx)	4	3	3	4	2	5	0	10	
(xx)	3	4	3	0	0	0	0	21	
(xx)	3	3	4	0	0	0	0	21	

As can be seen from Table 11 for these cases it remains to explain the first row for (xv). Consider (xv) with $d(v_1) = d(v_2) = d(v_3) = 3$. Then $\kappa_1 = \kappa_4 = 0$, $\kappa_2 \leqslant \frac{\pi}{5}$ and $\kappa_3 \leqslant \frac{\pi}{5}$. If $\kappa_1 + \kappa_2 \leqslant \frac{\pi}{3}$ then the deficit is at least $\frac{\pi}{5}$ so assume otherwise. If $\hat{\Delta}$ receives less than $\frac{\pi}{5}$ from each of Δ_2 and Δ_3 then deficit $\geqslant \frac{\pi}{5}$, so assume otherwise. If $\hat{\Delta}$ receives $\frac{\pi}{5}$ from Δ_2 then Δ_2 is given by Δ of Figure 7(iii) or Figure 8(iv). But if Δ_2 is Δ of Figure 8(iv) then $\hat{\Delta}$ does not receive any curvature from Δ_3 and we are done; and if Δ_2 is Δ of Figure 7(iii) then according to Configuration D of Figure 32(ii), $\hat{\Delta}$ receives $\frac{3\pi}{10}$ from Δ_0 and the deficit is increased by $\frac{\pi}{30}$. If $\hat{\Delta}$ receives $\frac{\pi}{5}$ from Δ_3 then Δ_3 is given by Figure 7(iii) or Figure 10(i) and (ii). But if Δ_3 is Δ of Figure 10(i) and (ii) then $\hat{\Delta}$ does not receive any curvature from Δ_2 and we are done; and if Δ_3 is Δ of Figure 7(iii) then according to Configuration C of Figure 32(i), $\hat{\Delta}$ receives $\frac{3\pi}{10}$ from Δ_5 and the deficit is increased by $\frac{\pi}{30}$. It follows that the deficit is at least $\frac{2\pi}{15} + \frac{2\pi}{30} = \frac{\pi}{5}$.

Table 12. *Deficit for Figure 55*(xiv).

	$d(v_1)$	$d(v_2)$	$d(v_3)$	κ_1	κ_2	κ_3	κ_4	Deficit	
(xiv)	3	3	3	0	2	2	0	12	
(xiv)	4	3	3	e_1	e_2	2	0	8	
(xiv)	3	4	3	0	0	0	0	21	
(xiv)	3	3	4	0	2	e_1	e_2	8	
(xiv)	5+	3	3	2	2	2	0	18	
(xiv)	3	5+	3	0	2	2	0	20	
(xiv)	3	3	5+	0	2	2	2	18	
(xiv)	4	4	3	x_1	x_2	0	0	17	
(xiv)	4	3	4	e_1	e_2	e_1	e_2	4(6)	(Note)
(xiv)	3	4	4	0	0	x_1	x_2	17	

Finally the case Figure 55(xiv) is given by Table 12. Note that $d(v_1) = d(v_2) = 4$ implies $\kappa_1 = 0$ or $\kappa_2 = 0$ and that $d(v_3) = d(v_4) = 4$ implies $\kappa_3 = 0$ or $\kappa_4 = 0$ which implies $x_1 + x_2 = 5$ in Table 12; and the e_1, e_2 entries are explained by Figure 42 and Lemma 9.2(iv). Since (see Table 10) $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 \leqslant \frac{11\pi}{15}$, if there is a vertex of degree at least 4 and one of degree at least 5 it follows that the deficit is at least $\frac{7\pi}{30}$; and if there are at least three vertices of degree at least 4 then the deficit $\geqslant \frac{3\pi}{10}$ and so we see from Table 12 that to complete the proof the penultimate row for (xiv), that is, case (xiv) with $d(v_1) = d(v_3) = 4$ and $d(v_2) = 3$ must be considered. Note that for this subcase the deficit is at least $\frac{2\pi}{15}$ and so it remains to show that the deficit is in fact at least $\frac{\pi}{5}$. If $\kappa_1 + \kappa_2 \leqslant \frac{\pi}{3}$ and $\kappa_3 + \kappa_4 \leqslant \frac{\pi}{3}$ then the deficit $\geqslant \frac{\pi}{5}$, so assume otherwise. If $\kappa_1 + \kappa_2 > \frac{\pi}{3}$ then the only way this can occur (see Figure 42) is if $\kappa_1 = \frac{2\pi}{15}$ and $\kappa_2 = \frac{7\pi}{30}$ forcing Δ_1 to be given by Δ of Figure 20(v) and Δ_2 to be given by Δ of Figure 16(iii). But either this gives Configuration B of Figure 31(v), a contradiction since then $\kappa_2 = \frac{\pi}{5}$ only, or $d(u) > 4$ in Figure 31(v) and the deficit is at least $\frac{2\pi}{15} + (\frac{\pi}{3} - \frac{7\pi}{30}) = \frac{7\pi}{30}$. If $\kappa_3 + \kappa_4 > \frac{\pi}{3}$ then the only way this can occur is if $\kappa_3 = \frac{7\pi}{30}$ and $\kappa_4 = \frac{2\pi}{15}$ forcing Δ_3 to be given by Δ of Figure 20(vi) and Δ_4 to be given by Δ of Figure 16(ii). But either this gives Configuration A of Figure 31(i), a contradiction since then $\kappa_3 = \frac{\pi}{5}$ only, or $d(u) > 4$ in Figure 31(i) and again the deficit is at least $\frac{7\pi}{30}$. □

The proof of Proposition 4.3 follows from 10.2, 10.4, 11.3 and 11.5.

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Martin Edjvet
School of Mathematical Sciences
The University of Nottingham
University Park
Nottingham NG7 2RD
United Kingdom

martin.edjvet@nottingham.ac.uk

Arye Juhász
Department of Mathematics
Technion–Israel Institute of Technology
Haifa 32000
Israel

arju@techunix.tech