Determining Firing Strengths Through A Novel Similarity Measure to Enhance Uncertainty Handling in Non-Singleton Fuzzy Logic Systems

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Abstract: Non-Singleton Fuzzy Logic Systems (NSFLSs) have the potential to tackle uncertainty within the design of fuzzy systems. The inference process has a major role in determining results, being partly based on the interaction of input and antecedent fuzzy sets (in generating firing levels). Recent studies have shown that the standard technique for determining firing strengths risks substantial information loss in terms of the interaction of the input and antecedents. To address this issue, alternative approaches, which employ the centroid of intersections (*cen*-NS) and similarity measures (*sim*-NS), have been developed. More recently, a novel similarity measure for fuzzy sets has been introduced, but as yet this has not been used for NSFLSs. This paper focuses on exploring the potential of this new similarity measure in combination with the *sim*-NS approach to generate a more suitable firing level for non-singleton input. Experiments are presented for fuzzy systems trained using both noisy and noise-free time series. The prediction results of NSFLSs for the novel similarity measure and the current approaches are compared. Analysis of the results shows that the novel similarity measure, used within the *sim*-NS approach, can be a more stable and suitable method suitable to be used in real world applications.

1 INTRODUCTION

Most real world applications contain a variety of sources of uncertainty that depend on different circumstances, and hence the ability to handle uncertainties becomes an indispensable component in decision making applications. Fuzzy logic systems (FLSs) are considered as a robust systems for handling decision making under uncertainty (Zadeh, 1965). FLSs have been successfully utilised in a variety of areas, including data mining, pattern recognitions and time series predictions (Mendel, 2001)

FLSs processes are completed in three essential steps; fuzzification, inferencing and defuzzification. In fuzzification, crisp input values are transformed into fuzzy sets (FSs). This transformation can be implemented as singleton (SFLSs) or non-singleton (NSFLSs). Due to simplicity and lower computational cost of SFLSs, it is the most commonly used design in literature; however, studies show that NSFLSs have the potential to provide better results than SFLSs for the same number of rules (Balazinski et al., 1993; Hayashi et al., 1993; Larsen, 1980; Pedrycz, 1992; Sahab and Hagras, 2010).

In inferencing, the firing strength of the rule is defined based on the interaction between input FSs and antecedent FSs. As the most used standard composition-based technique, the maximum membership degree grade of the intersection between the input FS and antecedent FS is determined as the firing strength. However, recent work, including Pourabdollah et al. (2015) and Wagner et al. (2016) showed that adopting the maximum point of the intersection to determine the firing strength risks substantial information loss in terms of the interaction of the input and antecedent FSs. To address this issue, they introduced alternatives which employ the centroid of the intersection (*cen*-NS) and similarity measures (*sim*-NS), between input and antecedent FSs, respectively. While Wagner et al. proposed a generic application of any similarity measure (e.g., Jaccard, Dice), they focused on the Jaccard measure (1908). Yet the Jaccard similarity measure is not highly sensitive to the width of FSs or the size of the intersection when one interval is a subset of another (Kabir et al., 2017). Therefore, employing a new similarity measure may have the potential to be a more stable approach in determining firing levels in the inference step of FLSs.

More recently, Kabir et al. (2017) introduced a novel similarity measure intended to enable more comprehensive capture of the similarity between sets, while also being bounded by the Dice and Jaccard similarity measures. However, to date, this new similarity measure has not been applied in the context of NSFLSs. This paper therefore focuses on exploring the potential of this new similarity measure in combination with the *sim*-NS approach. To enable a systematic comparison to alternative previously introduced NSFLS approaches, the paper follows the experimental strategy of Pourabdollah et al. (2015) and Wagner et al. (2016), showing the performance for all different NSFLSs for a series of time-series prediction experiments.

The structure of this paper is as follows. Section II provides background information on the standard (singleton) composition method, *cen*-NS, *sim*-NS using Jaccard, and the novel similarity measure. Also, Mackey-Glass time series generation with noise adding process are introduced. In Section III, experimental environment and the results are discussed. In Section IV, conclusion of the experiments and possible future work directions are provided.

2 BACKGROUND

In this section, the background material for singleton and non-singleton FLSs, and the various techniques for determining firing strength (standard composition, *cen*-NS and *sim*-NS) will be introduced. Lastly, Mackey-Glass time series generating and noise adding procedures will be presented.

2.1 Singleton and Non-singleton Fuzzy Logic Systems

In standard singleton fuzzification, a given crisp input x is transformed into an input fuzzy set I, represented by a membership function $\mu_I(x)$ that takes values in the interval [0,1], formulated as;

$$I = \{x, (\mu_I(x)) \mid \forall x \in X\} \quad (1)$$

While singleton sets are characterised by a single point in A having the value 1, non-singleton sets are characterised depending on the design choice. A pictorial demonstration of singleton and non-singleton Gaussian input can be seen in Fig 1. (Note that in practice, singleton fuzzification is often done implicitly, immediately determining the firing strength by simply calculating $\mu_A(x)$ for the given value of x.)

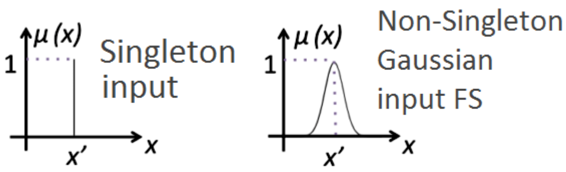

Figure 1: Singleton and non-singleton Gaussian FSs

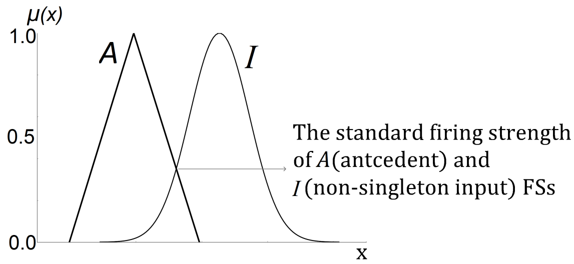

Figure 2: An illustration of the standard approach. The *max* degree membership of the intersection between A (antecedent) and I (input) FSs is determined as firing strength

2.2 Non-Singleton Fuzzy Logic Systems with Standard Composition-based Inference

In Mamdani NSFLSs, the firing levels are defined according to the interaction of non-singleton input and antecedent sets (Mamdani and Assilian, 1975; Mendel, 2001). In the standard composition based inference approach, the maximum membership degree of the intersection (between the input and antecedent sets) is determined as the firing level. An illustration of this firing level determining approach (between a triangular antecedent and a Gaussian non-singleton input FS) can be seen in Fig. 2.

Even though the standard composition based technique has been extensively studied, the most important limitation lies in the fact that different input FSs (e.g. with different standard deviations) may intersect antecedent at the same membership grade, resulting in the same firing level, despite the fact that those input FSs are clearly different (see Fig 3).

2.3 Centroid Based Approach

The centroid-based inferencing approach, known as *cen*-NS, focuses on the area of intersection between input and antecedent FSs (Pourabdollah et al., 2015). Firstly, the centroid of intersection between input FS (I) and antecedent FS (A) is calculated;

$$x_{cen}(I \cap A) = \frac{\sum_i^n x_i \mu(x_i)}{\sum_i^n \mu(x_i)} \quad (2)$$

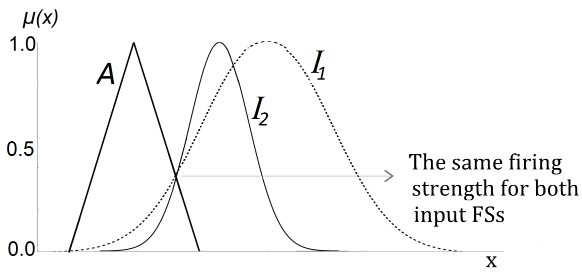

Figure 3: An illustration of two distinct fuzzy sets having the same intersection level with A

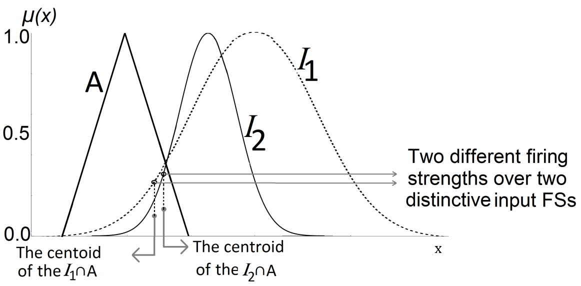

Figure 4: An illustration of the centroid-based firing strength technique (cen-NS). The centroid of intersection is calculated and the corresponding membership degree at the position of the centroid is defined as the firing strength.

where n is the number of dicretisation levels in the intersection between the input FS (I) and the antecedent FS (A)

Then, the corresponding membership degree of the centroid ($x_{cen}(I \cap A)$) is defined to be the firing strength;

$$\mu_{I \cap A}(x_{cen}(I \cap A)) \tag{3}$$

An illustration of the *cen*-NS technique can be seen in Fig. 4. The centroid of intersection for two distinct input FSs (I_1 and I_2) and the antecedent (A) are calculated respectively. Then the calculated centroids are projected to the intersection ($A \cap I$) to produce firing strengths.

In the experiment of Pourabdollah et al. (2015), two different time series datasets (Mackey-Glass and Lorenz) were used and two different noise levels (10dB and 5dB) were added to those time series. The Wang-Mendel (1992) method was utilised to create rules from either noise-free or noisy time series in the training of the FLS. The MSE results obtained showed that the *cen*-NS technique outperforms the standard composition method by between 7% and 17%.

Wagner et al. (2016) suggested that, whilst an interesting development, one possible issue with the *cen*-NS technique is that similar input and antecedent FSs generate high firing levels simply because their intersection may have high membership grades at their centroids, rather than because the input FS actually strongly matching the antecedent FS.

2.4 Similarity Based Approach

A similarity measure on fuzzy sets is a function that determines to what degree (in the interval of [0,1]) two fuzzy sets contain the same values with the same degree of membership (McCulloch and Wagner, 2016).

Wagner et al. (2016) have proposed that similarity ratios, between input and antecedent FSs, can be utilised to determine firing levels. As a sample of this approach, the Jaccard similarity ratio (1908) was focused to determine firing strengths in their study.

2.4.1 The Jaccard Similarity Measure

The Jaccard similarity ratio (Jaccard, 1908), which is in the interval [0,1], is determined for discrete FLSs as follow;

$$S(I,A) = \frac{\sum_i^t min(\mu_A(x_i), \mu_I(x_i))}{\sum_i^t max(\mu_A(x_i), \mu_I(x_i))} \tag{4}$$

where t is the discretisation levels over the both input FS (I) and the antecedent FS (A).

Wagner et al. (2016) utilised the same experimental procedures as the Pourabdollah et al. (2015) study, and the experimental results showed that the Jaccard ratio based inference system can improve MSE values by between 23% and 31%.

Yet the Jaccard ratio is not highly sensitive to changes in the widths of FSs, such as in the case that one interval is a subset of another (Kabir et al., 2017). For instance, when an antecedent and input sets have their centres at the same location (see Fig. 5), the firing level of that intersection is presumed to be one, normally. However, the Jaccard ratio produces non-intuitive firing strength results, in that as the inner set shown in Fig. 5 is narrowed, a lower Jaccard ratio would be generated and, as that narrowing is continued, the Jaccard ratio would get closer to zero. However, when the inner FS continues to narrow to eventually be a singleton FS, the Jaccard ratio would spike to one. Because of this inconsistent behaviour of the Jaccard ratio, it may not produce the most appropriate firing levels in such situations. Hence, the Jaccard ratio may not the best option to be used in the inference step of NSFLSs.

2.5 The Novel Similarity Measure

Kabir's similarity measure (Kabir et al., 2017) focuses on the features;

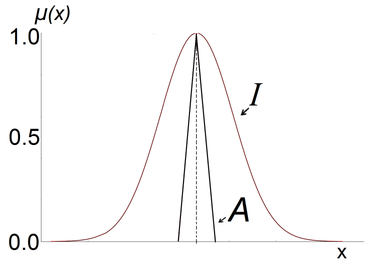

Figure 5: An Input FS (I) excessively covers an antecedent FS (A)

- Sensitivity to changes in the width of intervals
- Sensitivity to the size of the intersection when one interval is a subset of another

The proposed similarity measure focuses on the overlapping ratios which is bounded [0,1] and is formulated as follow;

$$S_{OR}(I,A) = min\left(\frac{\sum_i^p min(\mu_A(x_i),\mu_I(x_i))}{\sum_i^p \mu_A(x_i)}, \frac{\sum_i^k min(\mu_A(x_i),\mu_I(x_i))}{\sum_i^k \mu_I(x_i)}\right) \quad (5)$$

where p is the discretisation levels in the input FS (I) and k is discretisation levels in the antecedent FS (A).

2.6 The Time Series

Since adding noise to Mackey-Glass (MG) time series is an easily manageable procedure, it is commonly chosen to be studied. The generating procedures of MG is performed by using the following formula (Mackey et al., 1977; Mouzouris and Mendel, 1997);

$$\frac{dx(t)}{dx)} = \frac{ax(t-\tau)}{1+x^{10}(t-\tau)} - bx(t) \quad (6)$$

The noise in the MG time series is measured by the signal-to-noise-ratio (SNR) and the noise adding operation is performed as follows.

Firstly σ_{noise} value is calculated by using σ of the noise free set;

$$\sigma_{noise} = \frac{\sigma_{noise \quad free \quad dataset}}{10^{(\frac{SNR}{20})}} \quad (7)$$

Noise values are found by using a uniform random variable with zero mean in the interval of $[-\delta,\delta]$, where $[\delta = \sqrt{3\sigma_{noise}}]$, and then the noise values determined (δ) are added to the noise free dataset to obtain noisy sets.

3 EXPERIMENTS AND RESULTS

In this section, all implemented procedures in the presented study experiments will be explained, and the results obtained are presented.

3.1 Time Series

The Mackey Glass time series is chosen to be used in our experiment and the generation was performed by using (6). In order to provide a chaotic behaviour in MG, τ is set to 30, while $a = 0.2$ and $b = 0.1$. $x(t)$ is calculated for 2000 time points ($t = [-999 : 1000]$) and due to the fluctuation tendency in the initial part of the time series, the last 1000 points are taken to be used in our experiment. While the initial 700 points ($t = 1$ to $t = 700$) of the generated time series are used to train the FLS, the remaining 300 points are used in the testing process of the FLS. Six different noise levels (0,2,3,5,10 and 20 dB) were added to the time series to be used in different variations of the experiment.

3.2 Training and Testing

The rule creation in the training phase was performed using the Wang-Mendel (1992) one-pass method, as follows.

- Seven equally distributed triangular FSs (see Fig. 6) are created as antecedents, where each antecedent interval was defined as follows:
  - Firstly, the min (x_{min}) and max (x_{max}) point of the training time series is obtained and the mean point of each triangular antecedent is calculated;

$$\mu_i = a_{min} + \frac{(i-1)(x_{max}-x_{min})}{t-1} \quad (8)$$

    where i is the current number of antecedents and t is the total number of antecedents (seven in our experiments).
  - After calculating μ_i value of each antecedent, the interval (left and right points of each triangular set) were determined;

$$left = \mu_i - \frac{(x_{max}-x_{min})}{t-1} \quad (9)$$

$$right = \mu_i + \frac{(x_{max}-x_{min})}{t-1} \quad (10)$$

    Where t is 7.
- Nine past points were used as inputs and projected to the corresponding triangular antecedents.

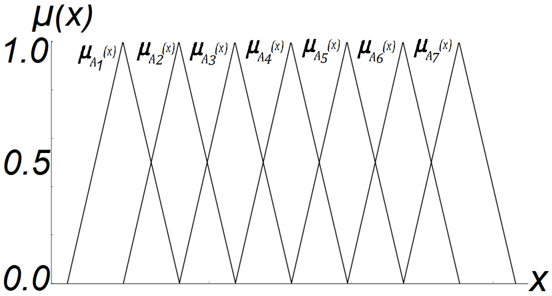

Figure 6: An illustration of the used 7 triangular antecedent FSs in the Wang-Mendel (Wang and Mendel, 1992) rule creation procedures

- The following (10^{th}) point was designated as the output and the window sliding procedure applied until reaching the end of training set.

$$\begin{aligned} x^1 &= [x_1, x_2 ... x_9] \quad output = x_{10} \\ x^2 &= [x_2, x_3 ... x_{10}] \quad output = x_{11} \\ &\quad . \\ &\quad . \\ x^{691} &= [x_{691}, x_{692} ... x_{699}] \quad output = x_{700} \end{aligned} \tag{11}$$

We carried out two main experiments:

- **Experiment 1:** The standard deviations of input FSs were adjusted according to the known noise level in the testing data.
- **Experiment 2:** The noise levels are assumed to be unknown, and the standard deviation of input FSs were fixed for each of the six noise levels (not adjusted according to corresponding noise levels in testing one at a time). Two different fixed standard deviations were used (Experiment 2a and 2b).

As a first phase of the each experiment, training of the FLS was done by using the first 700 points of the noise-free time series and the testing was implemented by using six different noisy time series in turn (**noise free training**). After noise free training and testing was completed, as a second phase of each experiment, training was done by using the 700 points from noisy times series and the testing was implemented on the remained 300 points from the corresponding noisy sets (**noisy training**). The two procedures above (noise-free training and noisy training) were repeated for each variation of experiments.

3.3 Design of the Fuzzy Logic System

Four different FLSs were created: a standard NS-FLS, which employs standard technique (between non-singleton inputs and antecedents) to generate firing strengths, *cen*-NS, *sim*-NS using Jaccard, and *sim*-NS using the novel similarity measure (termed *Kab*-NS). As practised in (Pourabdollah et al., 2015) and (Wagner et al., 2016), Mamdani inference with centroid defuzzification was used with the *min* and *max* operators for the t-norm and t-conorm respectively. The discretisation level (100 steps) is used for all fuzzy sets in FLSs. The input sets in NSFLSs are designed as Gaussian distributions which was centred on the crisp input. In Experiment 1, the standard-deviation of input sets was determined by means of (7) and all training-testing procedures were repeated under six different *SNR* values (0,2,3,5,10 and 20 dB). In Experiment 2, the standard deviation of input sets was fixed to be 5 dB noise (0.1613) (7) and 0 dB noise (0.2869) respectively and a FLS was implemented for both noise-free training and noisy training procedures, each using six different noisy time series.

The MSE over the 300 testing points was utilised to measure the overall error of each FLS. In order to mitigate the effect of randomness in the noise addition process, each experiment was repeated 30 times for all case scenarios and the average of generated MSEs were calculated.

3.4 Results

3.4.1 Experiment 1: The Corresponding Standard Deviations of Gaussian Input Fuzzy Sets

Firstly, the noise-free data set ($t = 1$ to $t = 700$) was used in training of the FLS , which resulted in 184 rules. After rule creation was completed, the previously generated six different noisy time series (between $t = 700$ to $t = 1000$) were used to test the FLS in turn. As mentioned in the previous section, the standard deviations of the Gaussian input sets were adjusted according to the noise levels as used in the noisy time series. In comparison with the standard approach, *Kab*-NS (similarity based input using the novel similarity measure) reduced MSE results by 31%, 21%, 15%, 11%, 10% and 7% under 20dB, 10dB, 5dB, 3db, 2dB and 0 dB, respectively for noise-free training scenarios (left side of the Fig. 7).

After noise-free training procedures were completed, training was repeated by using noisy time series ($t = 1$ to $t = 700$), and the remaining 300 points from the same noisy sets were used in testing. As before, the standard deviation of the Gaussian input FSs was adjusted to the level used in the corresponding noise level each time. When the noisy training and noisy testing (right side of the Fig.7) cases are scrutinised for *Kab*-NS technique, a similar tendency of improvement (24%,16%,13%,10%,11% and 7%) can be recognised compared to the standard composition method.

3.4.2 Experiment 2: The Non-Corresponding Standard Deviations of Gaussian Input Fuzzy Sets

These experiments were then modified to examine the behaviour of the all approaches under unknown noise levels. In these versions of the experiment, the same procedures from Experiment 1 ('noise-free training, noisy testing' and 'noisy training, noisy testing') were repeated. However, it was assumed that the noise levels in time series are unknown and hence the standard deviation of Gaussian input sets was not adjusted under each different noise level. Rather the noise in the input sets was fixed to two different levels.

Experiment 2a: The standard deviation was fixed to be 5dB noise (0.161) and all procedures from Experiment 1 were implemented without adjusting input FSs. All the noise-free and noisy training procedures results can be seen in Fig.8.

Experiment 2b: This time the standard deviation was fixed to be 0dB noise (0.286) and again all operations were repeated without adjusting standard deviations of input FSs. The experimental result can be seen in Fig. 9.

3.5 Discussion

When the noise free training of Experiment 1 (left hand side of the Fig. 7) is analysed, it can be seen that the novel similarity measure (*Kab*-NS) outperforms both the standard and centroid (*cen*-NS) techniques significantly under low noise levels. Under very noisy conditions (as the MSE values get closer for all approaches), the *cen*-NS technique shows slightly better performance. Comparing the MSE results from *sim*-NS and *Kab*-NS, we can see that the results for both techniques are the same under almost all conditions. In summary, for noise-free training *Kab*-NS outperforms standard technique significantly for all six different conditions. Also, *Kab*-NS shows either the same or better results than *cen*-NS in five cases out of six, and also it has the same MSE results with *sim*-NS for five out of six cases. When the noisy training of the Experiment 1 (right hand side of the Fig. 7) is analysed, *Kab*-NS outperforms both standard and *cen*-NS techniques significantly under all conditions (all six cases) and again it shows the same MSE results as *sim*-NS in five cases out of six.

As mentioned before, in Experiment 2a the standard deviations were fixed at the 5 dB noise level. The noise-free training of this experiment (left hand side of Fig. 8) shows that *Kab*-NS has better results than standard technique in five cases out of six, and it shows the same result under the 0 dB noisy testing. In comparison to *cen*-NS, the new approach shows lower MSE result in three out of six cases and it has the same result with *cen*-NS for the remained three cases (under 3, 2 and 0 dBs). When we compare *sim*-NS and *Kab*-NS, the two approaches shows quite similar MSE results (*Kab*-NS is better for two cases and *sim*-NS is better for another two cases, while the remaining two cases have similar results). In the noisy training of Experiment 2a (right hand side of the Fig. 8), *Kab*-NS again outperforms standard and *cen*-NS approaches in five cases out of six, whereas the standard technique shows the best result under 0 dB noise conditions. When we compare MSE results of *sim*-NS and *Kab*-NS, it can be seen that *Kab*-NS has better results under almost all conditions except the case of 5 db noise in testing.

When the standard deviations were fixed as 0 dB noise, in the Experiment 2b, in all the noise-free and noisy training instances (both sides of Fig. 9), *Kab*-NS outperformed the standard technique and *sim*-NS under all 12 scenarios (both noise-free and noisy training). However it should be mentioned that the *cen*-NS has the lowest MSE results among all four variants in 7 cases out of 12.

To recapitulate, when all the 36 cases are examined, *Kab*-NS outperforms standard techniques significantly in 35 cases and also it outperforms *cen*-NS in 26 cases out of 36. However, the majority of those in which worse performance is observed (10 cases) occurs when the standard deviation was fixed at 0 dB noise, which corresponds to the highest noise condition. Therefore further research should be investigated under very noisy conditions. Both *sim*-NS and *Kab*-NS have generally similar or the same average MSE results. It is worthwhile noting that the goal of this work is not specifically to achieve the best performance in applications which use different approaches for generating firing levels but to study and compare the various approaches to try to discover the most reliable approach to be used under different conditions.

This is particularly relevant in situations in which the noise level cannot easily be known in advance, which is often the case in the real-world. Situations might include when the FLS must be designed and fixed in advance of implementation in the real world, or in situations where the noise level itself is varying in an unpredictable manner.

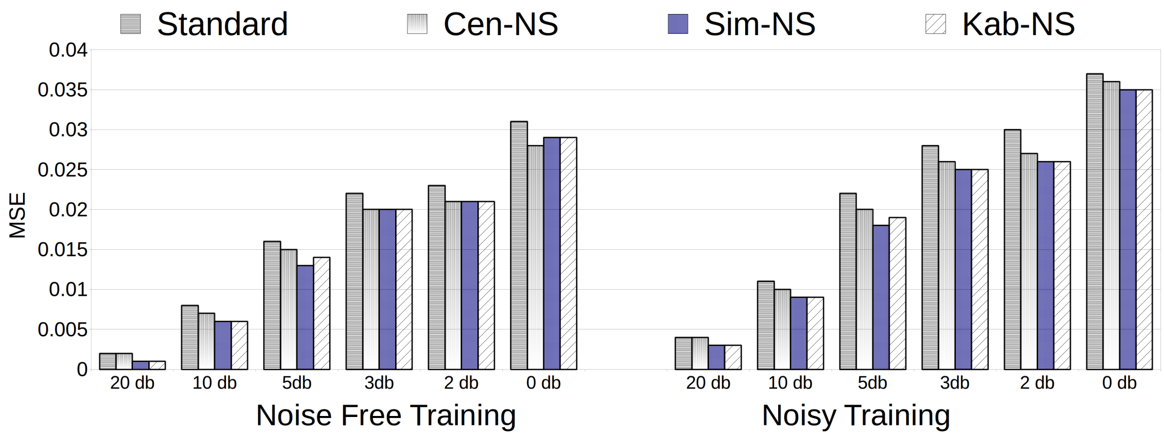

Figure 7: The NSFLS Prediction performance comparison produced by different inference based approaches. Each standard deviation of input FSs is set to the corresponding noise level

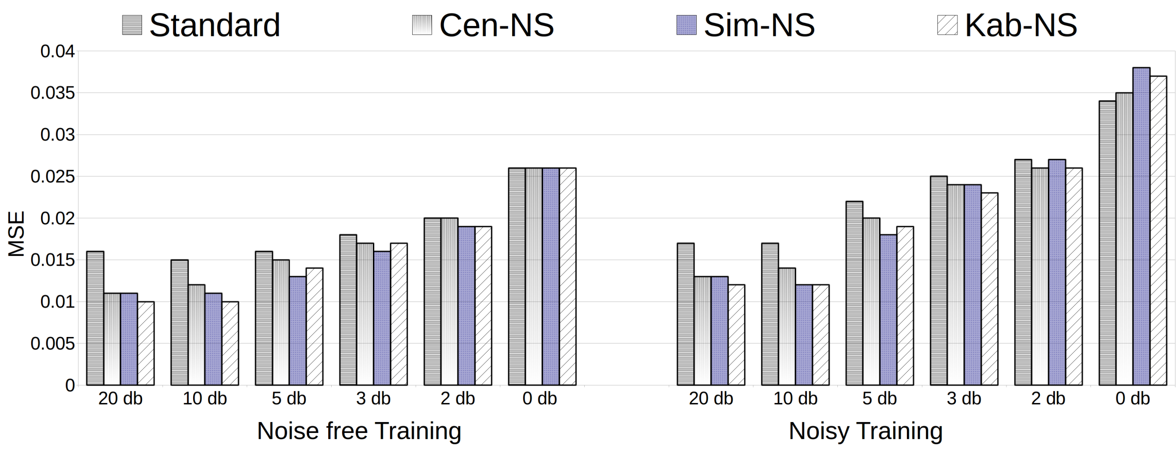

Figure 8: The NSFLS Prediction performance comparison. Each standard deviation of input FSs is set to 5 dB σ_{noise}

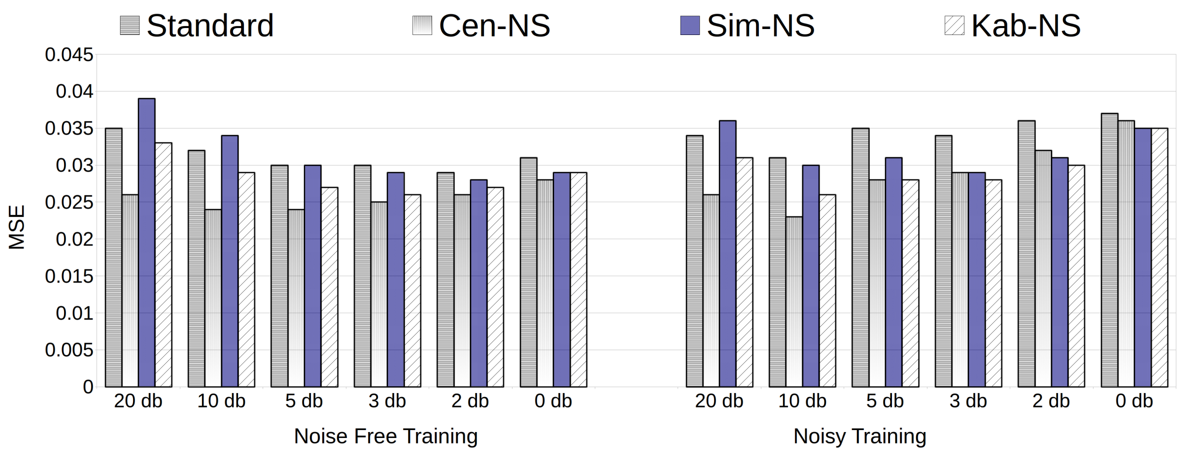

Figure 9: The NSFLS Prediction performance comparison. Each standard deviation of input FSs is set to 0 dB σ_{noise}

4 CONCLUSION and FUTURE WORK

We have implemented and compared different inference based approaches (Standard, *cen*-NS, *sim*-NS using the Jaccard similarity ratio, and *Kab*-NS using the novel similarity measure). Because of the limitations and issues observed in current approaches, this paper has focused on exploring the potential of a new novel similarity measure in combination with the *sim*-NS approach. Kabir's similarity measure (Kabir et al., 2017) is sensitive both to changes in the width of FSs and to the case in which one FS is a subset of another. Considering these features, it has now been used for the first time to define firing levels in FLSs. The evidence from this study points towards the idea that *sim*-NS with the Kabir's similarity measure could indeed be a suitable approach to be used in FLSs, especially under unknown noise conditions of real world cases. However, this is a tentative finding, and more work needs to be carried out on different data sets under a wider range of conditions to further evaluate this.

Future work will concentrate on different interesting aspects. The *sim*-NS will be implemented by using different similarity measures (e.g. Dice similarity) between antecedents and input FSs. Alternative time series datasets (for example, the Lorenz time series) will be used in FLS. Different design types for antecedent and input FSs will be implemented and the results will be examined. Lastly, due to the increased modelling capabilities of type-2 fuzzy logic in handling uncertainty, different type-2 designs will be explored.

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