

# Supplementary Material for “Estimation of Tail Risk based on Extreme Expectiles”

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Further simulation results are discussed in Section A. The proofs of all theoretical results in the main paper and additional technical results are provided in Section B.

## A Additional simulations

The aim of this section is to explore some additional features that were briefly mentioned in Section 6. We will illustrate the following points:

(A.1) Bias and MSE estimates.

(A.2) Quality of asymptotic approximations.

Let us first comment on some implementation details. We used in all our simulations the Hill estimator of  $\gamma$ , the extreme level  $\tau'_n = 0.995$  for  $n = 100$  and  $\tau'_n = 0.9994$  for  $n = 1000$ . The corresponding true extreme expectiles  $\xi_{\tau'_n}$  can be calculated by the existing function “`et( $\tau'_n, df$ )`” in the R package ‘`expectreg`’. In what concerns the intermediate levels  $\tau_n$  involved in both estimators  $\tilde{\xi}_{\tau'_n}^* \equiv \tilde{\xi}_{\tau'_n}^*(\tau_n)$  and  $\hat{\xi}_{\tau'_n}^* \equiv \hat{\xi}_{\tau'_n}^*(\tau_n)$ , we used the same considerations as in Ferreira et al. (2003). Namely, they always considered  $\tau_n = 1 - \frac{k}{n}$  with the range of intermediate integers  $k$ , say, from  $\log(n^{1-\varepsilon})$  to  $n/\log(n^{1-\varepsilon})$ , where  $\varepsilon = 0.1$  [this restriction allows to reject too small values or those very near  $n^{1-\varepsilon}$ ]. The value  $k$  can actually be viewed as the effective sample size for tail extrapolation. A larger  $k$  leads to estimators with more bias, while smaller  $k$  results in higher variance.

### A.1 Bias and MSE estimates

Figures 1 and 2 (respectively, Figures 3 and 4) give the root-MSE estimates computed over 10,000 replications for samples of size 100 and 1000 simulated from the Student (respectively, positive Student)  $t$ -models, while Figures 5 and 6 (respectively, Figures 7 and 8) give the bias estimates for the same models. Each figure displays the evolution of the obtained Monte-Carlo results, for the two normalized estimators  $\tilde{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  and  $\hat{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$ , as functions of the sample fraction  $k$ . Tables 1 and 2 report the root-MSE and bias estimates obtained by using for each estimator the optimal value of  $k$  minimizing its MSE.

As regards the Student distributions which correspond to real-valued profit-loss variables, our tentative conclusion from Figures 1-2 and Figures 5-6 is that the indirect estimator  $\hat{\xi}_{\tau'_n}^*$  has a harder time with small samples, and this can be compensated by taking larger samples. Indeed, for  $n = 100$ , the direct estimator  $\tilde{\xi}_{\tau'_n}^*$  performs better than  $\hat{\xi}_{\tau'_n}^*$  in terms of both MSE and bias, whatever the thickness of the tails. Also, in contrast to the direct estimator’s plot, the indirect one exhibits more volatility. In what concerns  $n = 1000$ , it seems that  $\hat{\xi}_{\tau'_n}^*$  is superior to  $\tilde{\xi}_{\tau'_n}^*$  only in terms of MSE for slightly heavy tails (*i.e.*  $df = 7, 9$ ), whereas the accuracy of  $\tilde{\xi}_{\tau'_n}^*$  is more respectable for heavier tails (*i.e.*  $df = 3, 5$ ),

as can be seen from Table 1. It should be, however, clear that even in the favorable case to  $\widehat{\xi}_{\tau'_n}^*$ , where  $n = 1000$  and  $df \in \{7, 9\}$ , the estimator  $\widetilde{\xi}_{\tau'_n}^*$  has actually almost overall a smaller MSE except for a very small zone of values of  $k$ , as can be seen from Figure 2 (bottom panels). Due to the tightness of that zone, the detection of the optimal  $k$  which minimizes the MSE of  $\widehat{\xi}_{\tau'_n}^*$  is hard to manage in practice.

By contrast, in the case of the positive Student distributions which correspond to non-negative loss variables, it can be seen from Figures 3-4 and Figures 7-8 as well as Table 2 that the indirect estimator  $\widehat{\xi}_{\tau'_n}^*$  is superior to the direct estimator  $\widetilde{\xi}_{\tau'_n}^*$  in all scenarios except for the single case  $n = 100$  and  $df = 3$ . We repeated this kind of exercise with the Fréchet distribution  $F(y) = e^{-y^{-1/\gamma}}$ ,  $y > 0$ , and Pareto distribution  $F(y) = 1 - y^{-1/\gamma}$ ,  $y > 1$ , and arrived at the same tentative conclusion.

It may also be seen that most of the error is due to variance, the squared bias being much smaller in all cases. It is interesting that in almost all cases the bias was positive. This may be explained by the sensitivity of high expectiles to the magnitude of heavy tails, since they are based on “squared” error loss minimization.

$n = 100$					$n = 1000$				
$df$	RMSE		BIAS		$df$	RMSE		BIAS	
	$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$	$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$		$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$	$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$
3	1.5010	47.9486	0.4888	1.7107	3	0.4809	0.5403	0.2080	0.2599
5	0.5963	2.9132	0.1253	0.4139	5	0.2867	0.2981	0.0816	0.1088
7	0.4385	0.8001	0.0797	0.2486	7	0.2172	0.2119	0.0666	0.0629
9	0.3753	0.6200	0.0579	0.1685	9	0.1908	0.1781	0.0271	0.0440

Table 1: Monte-Carlo results obtained for the Student  $t_3$ ,  $t_5$ ,  $t_7$  and  $t_9$ -distributions, using the optimal sample fraction  $k$  minimizing the MSE of each estimator.

$n = 100$					$n = 1000$				
$df$	RMSE		BIAS		$df$	RMSE		BIAS	
	$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$	$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$		$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$	$\widetilde{\xi}_{\tau'_n}^*$	$\widehat{\xi}_{\tau'_n}^*$
3	0.9848	1.0833	0.4923	0.4423	3	0.5089	0.4023	0.2810	0.1932
5	0.5762	0.5511	0.2098	0.1959	5	0.3016	0.2297	0.1269	0.0697
7	0.4122	0.3786	0.1392	0.0685	7	0.2219	0.1639	0.0800	0.0280
9	0.3509	0.3033	0.1181	0.0315	9	0.1934	0.1393	0.0742	0.0107

Table 2: Monte-Carlo results obtained for the positive Student  $t_3$ ,  $t_5$ ,  $t_7$  and  $t_9$ -distributions, using the optimal sample fraction  $k$  minimizing the MSE of each estimator.

## A.2 Quality of asymptotic approximations

We first investigate the normality of the estimators  $\widehat{\xi}_{\tau'_n}^*$  and  $\widetilde{\xi}_{\tau'_n}^*$ . The asymptotic normality of  $\widehat{\xi}_{\tau'_n}^*/\xi_{\tau'_n}$  in Corollary 3 can be expressed as  $r_n \log(\widehat{\xi}_{\tau'_n}^*/\xi_{\tau'_n}) \xrightarrow{d} \Gamma$ , with  $r_n = \frac{\sqrt{k}}{\log[k/(n(1-\tau'_n))]}$ . Likewise, the asymptotic normality of  $\widetilde{\xi}_{\tau'_n}^*/\xi_{\tau'_n}$  in Corollary 4 can be expressed as  $r_n \log(\widetilde{\xi}_{\tau'_n}^*/\xi_{\tau'_n}) \xrightarrow{d} \Gamma$ . The limit distribution  $\Gamma$  of the Hill estimator is  $\mathcal{N}(\lambda_2/(1-\rho), \gamma^2)$ , as pointed out below Theorem 1. It can be shown that the Student  $t_\nu$  distributions satisfy the conditions of the two aforementioned corollaries, with  $\gamma = 1/\nu$ ,  $\rho = -2/\nu$  and

$$A(t) \sim \frac{\nu + 1}{\nu + 2} (c_\nu t)^{-2/\nu}, \quad c_\nu = \frac{2\Gamma((\nu + 1)/2)\nu^{(\nu-1)/2}}{\sqrt{\nu\pi}\Gamma(\nu/2)}.$$

Hence, we can compare the distributions of

$$\widehat{W}_n := \left[ r_n \log(\widehat{\xi}_{\tau'_n}^*/\xi_{\tau'_n}) - \lambda_2/(1 - \rho) \right] / \gamma \quad \text{and} \quad \widetilde{W}_n := \left[ r_n \log(\widetilde{\xi}_{\tau'_n}^*/\xi_{\tau'_n}) - \lambda_2/(1 - \rho) \right] / \gamma$$

with the limit distribution  $\mathcal{N}(0, 1)$ , with  $\lambda_2 = \sqrt{k}A(\frac{n}{k})$ . The Q–Q-plots in Figures 9 and 10 present, respectively, the sample quantiles of  $\widehat{W}_n$  and  $\widetilde{W}_n$ , based on 10,000 simulated samples of size  $n = 1000$ , versus the theoretical standard normal quantiles. For each estimator, we used the optimal value of  $k$  that minimizes its MSE as in Table 1. It may be seen that the scatters for the Student  $t_\nu$  distributions, with  $\nu = 3, 5, 7, 9$  displayed respectively from top to bottom and from left to right, are quite encouraging especially for the LAWS estimator  $\widetilde{\xi}_{\tau'_n}^*$  (Figure 10). Likewise, we conclude from the scatters for the positive Student  $t_\nu$  distributions, displayed in Figures 11 and 12, that the limit Theorem 3 and its Corollaries 3 and 4 provide adequate approximations for finite sample sizes.

Next, we investigate the normality of the estimators  $\widehat{\text{XMES}}^*(\tau'_n)$  and  $\widetilde{\text{XMES}}^*(\tau'_n)$  by comparing the distributions of

$$\widetilde{W}_n = \left[ r_n \log \left( \frac{\widetilde{\text{XMES}}^*(\tau'_n)}{\text{XMES}} \right) - \frac{\lambda_2}{(1 - \rho_X)} \right] / \gamma_X \quad \text{and} \quad \widehat{W}_n = \left[ r_n \log \left( \frac{\widehat{\text{XMES}}^*(\tau'_n)}{\text{XMES}} \right) - \frac{\lambda_2}{(1 - \rho_X)} \right] / \gamma_X$$

with the limit distribution  $\mathcal{N}(0, 1)$ , where  $r_n = \sqrt{k}/\log[k/(n(1 - \tau'_n))]$  and  $\lambda_2 = \sqrt{k}A_X(n/k)$ . The scatters in Figures 13 and 14 present, respectively, the sample quantiles of  $\widehat{W}_n$  and  $\widetilde{W}_n$ , based on 10,000 simulated samples of size  $n = 1000$ , versus the theoretical standard normal quantiles. For each estimator, we used the optimal value of  $k$  that minimizes its MSE. The obtained Q–Q-plots for the Student  $t_\nu$ -distributions on  $(0, \infty)^2$ , with  $\nu = 3, 5, 7, 9$ , indicate that the limit Theorems 4 and 5 provide adequate approximations for finite sample sizes, with a slight advantage for the estimator  $\widetilde{\text{XMES}}^*(\tau'_n)$  in Figure 13.

## B Proofs

For notational simplicity, let  $\overline{F} = \overline{F}_Y$  be the survival function of  $Y$ . It is a consequence of Theorem 2.3.9 in de Haan and Ferreira (2006, p.48) that condition  $\mathcal{C}_2(\gamma, \rho, A)$  entails the following second-order condition for the related survival function  $\overline{F}$ :

$$\forall x > 0, \lim_{t \rightarrow \infty} \frac{1}{A(1/\overline{F}(t))} \left[ \frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-1/\gamma} \right] = x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho}. \quad (\text{B.1})$$

**Proof of Proposition 1.** We start by noticing that the equation

$$\xi_\tau - \mathbb{E}(Y) = \frac{2\tau - 1}{1 - \tau} \mathbb{E}[(Y - \xi_\tau)_+] \quad (\text{B.2})$$

entails, for  $\tau$  sufficiently large so that  $\xi_\tau > 0$ ,

$$1 - \frac{\mathbb{E}(Y)}{\xi_\tau} = \frac{2\tau - 1}{1 - \tau} \mathbb{E} \left( \left( \left[ \frac{Y}{\xi_\tau} - 1 \right] \mathbb{I}\{Y/\xi_\tau \geq 1\} \right) \right). \quad (\text{B.3})$$

An integration by parts yields

$$\begin{aligned} \mathbb{E} \left( \left[ \frac{Y}{\xi_\tau} - 1 \right] \mathbb{1}\{Y/\xi_\tau \geq 1\} \right) &= \int_1^\infty \bar{F}(\xi_\tau x) dx \\ &= \bar{F}(\xi_\tau) \left( \frac{\gamma}{1-\gamma} + \int_1^{+\infty} \left[ \frac{\bar{F}(\xi_\tau x)}{\bar{F}(\xi_\tau)} - x^{-1/\gamma} \right] dx \right). \end{aligned}$$

Recall that since  $Y$  has an infinite right endpoint,  $\xi_\tau \rightarrow \infty$  as  $\tau \uparrow 1$ ; using together Equation (B.1), Theorem 2.3.9 in de Haan and Ferreira (2006) and a uniform inequality such as Theorem B.3.10 in de Haan and Ferreira (2006) applied to the function  $\bar{F}$ , we get after some easy computations

$$\mathbb{E} \left( \left[ \frac{Y}{\xi_\tau} - 1 \right] \mathbb{1}\{Y/\xi_\tau \geq 1\} \right) = \bar{F}(\xi_\tau) \left( \frac{\gamma}{1-\gamma} + A \left( \frac{1}{\bar{F}(\xi_\tau)} \right) \frac{1 + o(1)}{(1-\gamma)(1-\rho-\gamma)} \right). \quad (\text{B.4})$$

Plugging this equality into (B.3), we thus get

$$\frac{\bar{F}(\xi_\tau)}{1-\tau} = (\gamma^{-1} - 1) \left( 1 - \frac{\mathbb{E}(Y)}{\xi_\tau} \right) \frac{1}{2\tau - 1} \left( 1 + A \left( \frac{1}{\bar{F}(\xi_\tau)} \right) \frac{1}{\gamma(1-\rho-\gamma)} (1 + o(1)) \right)^{-1}$$

and therefore

$$\frac{\bar{F}(\xi_\tau)}{1-\tau} = (\gamma^{-1} - 1) \left( 1 - \frac{\mathbb{E}(Y)}{\xi_\tau} (1 + o(1)) + 2(1-\tau)(1 + o(1)) - A \left( \frac{1}{\bar{F}(\xi_\tau)} \right) \frac{1}{\gamma(1-\rho-\gamma)} (1 + o(1)) \right).$$

In particular, as noted in Bellini *et al.* (2014):

$$\frac{\bar{F}(\xi_\tau)}{1-\tau} \rightarrow (\gamma^{-1} - 1) \quad \text{and thus} \quad \xi_\tau = (\gamma^{-1} - 1)^{-\gamma} q_\tau (1 + o(1)) \quad (\text{B.5})$$

as  $\tau \uparrow 1$ . Because  $\gamma < 1$ , a consequence of this is that  $(1-\tau)\xi_\tau = O((1-\tau)q_\tau) \rightarrow 0$  as  $\tau \uparrow 1$  and so

$$\frac{\bar{F}(\xi_\tau)}{1-\tau} = (\gamma^{-1} - 1) \left( 1 - \frac{(\gamma^{-1} - 1)^\gamma}{q_\tau} (\mathbb{E}(Y) + o(1)) - \frac{(\gamma^{-1} - 1)^{-\rho}}{\gamma(1-\rho-\gamma)} A((1-\tau)^{-1})(1 + o(1)) \right)$$

where the regular variation property of  $|A|$  was used. This completes the proof.  $\blacksquare$

The key element in the proof of Corollary 1 is to apply Proposition 1 in conjunction with the following generic result.

**Lemma 1.** *Assume that  $v, V$  are such that  $v(\tau) \uparrow \infty$  and  $V(\tau) \downarrow 0$ , as  $\tau \uparrow 1$ , and there exists  $B > 0$  such that*

$$\frac{V(\tau)}{\bar{F}(v(\tau))} = B(1 + e(\tau))$$

where  $e(\tau) \rightarrow 0$  as  $\tau \uparrow 1$ . If condition  $\mathcal{C}_2(\gamma, \rho, A)$  holds, with  $\gamma > 0$  and  $F$  strictly increasing, then

$$\frac{v(\tau)}{U(1/V(\tau))} = B^\gamma \left( 1 + \gamma e(\tau)(1 + o(1)) + A(1/V(\tau)) \left[ \frac{B^\rho - 1}{\rho} + o(1) \right] \right) \quad \text{as } \tau \uparrow 1.$$

**Proof of Lemma 1.** Apply the function  $U$  to get

$$\frac{v(\tau)}{U(1/V(\tau))} - B^\gamma = \frac{U(B[1 + e(\tau)]/V(\tau))}{U(1/V(\tau))} - B^\gamma.$$

By Theorem 2.3.9 in de Haan and Ferreira (2006), we may find a function  $A_0$ , equivalent to  $A$  at infinity, such that for any  $\varepsilon > 0$ , there is  $t_0(\varepsilon) > 1$  such that for  $t, tx \geq t_0(\varepsilon)$ ,

$$\left| \frac{1}{A_0(t)} \left( \frac{U(tx)}{U(t)} - x^\gamma \right) - x^\gamma \frac{x^\rho - 1}{\rho} \right| \leq \frac{\varepsilon}{[(2B)^{\gamma+\rho} + (B/2)^{\gamma+\rho}][(2B)^\varepsilon + (B/2)^{-\varepsilon}]} x^{\gamma+\rho} \max(x^\varepsilon, x^{-\varepsilon}).$$

Thus, for  $\tau$  sufficiently close to 1, using this inequality with  $t = 1/V(\tau)$  and  $x = B[1 + e(\tau)]$  gives that

$$\left| \frac{1}{A_0(1/V(\tau))} \left( \frac{U(B[1 + e(\tau)]/V(\tau))}{U(1/V(\tau))} - B^\gamma(1 + e(\tau))^\gamma \right) - B^\gamma(1 + e(\tau))^\gamma \frac{B^\rho(1 + e(\tau))^\rho - 1}{\rho} \right| \leq \varepsilon$$

and therefore

$$\frac{1}{A_0(1/V(\tau))} \left( \frac{U(B[1 + e(\tau)]/V(\tau))}{U(1/V(\tau))} - B^\gamma(1 + e(\tau))^\gamma \right) \rightarrow B^\gamma \frac{B^\rho - 1}{\rho} \text{ as } \tau \uparrow 1.$$

The desired result follows by a simple first-order Taylor expansion. ■

**Proof of Corollary 1.** We have in view of Proposition 1 that

$$\frac{1 - \tau}{\overline{F}(\xi_\tau)} = (\gamma^{-1} - 1)^{-1}(1 + e(\tau))$$

with

$$e(\tau) = \frac{(\gamma^{-1} - 1)^\gamma}{q_\tau} (\mathbb{E}(Y) + o(1)) + \frac{(\gamma^{-1} - 1)^{-\rho}}{\gamma(1 - \rho - \gamma)} A((1 - \tau)^{-1})(1 + o(1)) \text{ as } \tau \uparrow 1.$$

Using Lemma 1 and recalling that  $U(1/(1 - \tau)) = q_\tau$  gives the result. ■

**Proof of Theorem 1.** The consistency statement is an immediate consequence of the convergence

$$\frac{Y_{n-[n(1-\tau_n)],n}}{q_{\tau_n}} = \frac{Y_{n-[n(1-\tau_n)],n}}{U((1-\tau_n)^{-1})} = \frac{Y_{n-[n(1-\tau_n)],n}}{U(n/[n(1-\tau_n)])} (1 + o(1)) \xrightarrow{\mathbb{P}} 1$$

which follows from the regular variation of  $U$  and Corollary 2.2.2 in de Haan and Ferreira (2006, p.41). The asymptotic distribution is obtained by writing

$$\frac{\widehat{\xi}_{\tau_n}}{\xi_{\tau_n}} - 1 = \left( \frac{(\widehat{\gamma}^{-1} - 1)^{-\widehat{\gamma}}}{(\gamma^{-1} - 1)^{-\gamma}} - 1 \right) + \left( \frac{\widehat{q}_{\tau_n}}{q_{\tau_n}} - 1 \right) (1 + o_{\mathbb{P}}(1)) - r(\tau_n)(1 + o_{\mathbb{P}}(1)),$$

where  $\sqrt{n(1 - \tau_n)}r(\tau_n) \rightarrow \lambda$  in view of Corollary 1. Since

$$\forall x \in (0, 1), \frac{d}{dx} ((x^{-1} - 1)^{-x}) = (x^{-1} - 1)^{-x} \{ (1 - x)^{-1} - \log(x^{-1} - 1) \},$$

the delta-method entails

$$\sqrt{n(1 - \tau_n)} \left( \frac{(\widehat{\gamma}^{-1} - 1)^{-\widehat{\gamma}}}{(\gamma^{-1} - 1)^{-\gamma}} - 1 \right) \xrightarrow{d} [(1 - \gamma)^{-1} - \log(\gamma^{-1} - 1)]\Gamma = m(\gamma)\Gamma, \quad (\text{B.6})$$

from which the result easily follows. ■

Before moving to the proof of Theorem 2, we shall show a handful of useful preliminary results. The next two lemmas are entirely based on non-probabilistic arguments. In the first one, we use the fact that  $\eta_\tau(y)/2$  is continuously differentiable with derivative

$$\varphi_\tau(y) := |\tau - \mathbb{I}\{y \leq 0\}|y.$$

**Lemma 2.** *For all  $x, y \in \mathbb{R}$  and  $\tau \in (0, 1)$ ,*

$$\frac{1}{2}(\eta_\tau(x - y) - \eta_\tau(x)) = -y\varphi_\tau(x) - \int_0^y (\varphi_\tau(x - t) - \varphi_\tau(x))dt.$$

**Proof of Lemma 2.** The result is a simple consequence of the equality

$$\frac{1}{2}(\eta_\tau(x - y) - \eta_\tau(x)) = \int_x^{x-y} \varphi_\tau(s)ds = - \int_0^y \varphi_\tau(x - t)dt$$

obtained by the change of variables  $s = x - t$ . ■

The next result gives a Lipschitz property for the derivative  $\varphi_\tau$ .

**Lemma 3.** *For all  $x, h \in \mathbb{R}$  and  $\tau \in (0, 1)$ , we have*

$$\varphi_\tau(x - h) - \varphi_\tau(x) = -h|\tau - \mathbb{I}\{x \leq 0\}| + (1 - 2\tau)(x - h)(\mathbb{I}\{x \leq h\} - \mathbb{I}\{x \leq 0\}),$$

and in particular  $|\varphi_\tau(x - h) - \varphi_\tau(x)| \leq |h|(1 - \tau + 2\mathbb{I}\{x > \min(h, 0)\})$ .

**Proof of Lemma 3.** Write

$$\varphi_\tau(x - h) - \varphi_\tau(x) = -h|\tau - \mathbb{I}\{x \leq 0\}| + (x - h)(|\tau - \mathbb{I}\{x \leq h\}| - |\tau - \mathbb{I}\{x \leq 0\}|).$$

Besides,

$$\begin{aligned} & |\tau - \mathbb{I}\{x \leq h\}| - |\tau - \mathbb{I}\{x \leq 0\}| \\ &= (1 - \tau)(\mathbb{I}\{x \leq h\} - \mathbb{I}\{x \leq 0\}) + \tau(\mathbb{I}\{x > h\} - \mathbb{I}\{x > 0\}) \\ &= (1 - 2\tau)(\mathbb{I}\{x \leq h\} - \mathbb{I}\{x \leq 0\}), \end{aligned}$$

from which the desired equality follows. The required bound on  $|\varphi_\tau(x - h) - \varphi_\tau(x)|$  is then obtained by noting that

$$|\tau - \mathbb{I}\{x \leq 0\}| = \tau \mathbb{I}\{x > 0\} + (1 - \tau) \mathbb{I}\{x \leq 0\} \leq 1 - \tau + \mathbb{I}\{x > 0\} \tag{B.7}$$

and

$$|x - h| |\mathbb{I}\{x \leq h\} - \mathbb{I}\{x \leq 0\}| \leq |h| |\mathbb{I}\{x \leq h\} - \mathbb{I}\{x \leq 0\}| \leq |h| \mathbb{I}\{x > \min(h, 0)\}. \tag{B.8}$$

Combining (B.7) and (B.8) completes the proof. ■

The last result will be useful to derive the limit distribution of the objective function  $\psi_n(u)$  described in (8).

**Lemma 4.** *Pick  $a > 1$  and assume that  $\mathbb{E}|Y_-|^a < \infty$  and  $0 < \gamma < 1/a$ . Then*

$$\mathbb{E}(|\varphi_\tau(Y - \xi_\tau)|^a) = a\xi_\tau^a(1 - \tau)(\gamma^{-1} - 1)B(a, \gamma^{-1} - a)(1 + o(1)) \quad \text{as } \tau \uparrow 1,$$

where  $B(s, t) = \int_0^1 u^{s-1}(1 - u)^{t-1} du$  is the Beta function evaluated at  $(s, t)$ .

**Proof of Lemma 4.** As a first step, write

$$\mathbb{E}(|\varphi_\tau(Y - \xi_\tau)|^a) = (1 - \tau)^a \mathbb{E}([\xi_\tau - Y]^a \mathbb{I}\{Y \leq \xi_\tau\}) + \tau^a \mathbb{E}([Y - \xi_\tau]^a \mathbb{I}\{Y > \xi_\tau\}). \quad (\text{B.9})$$

Furthermore, for any  $x, y$  such that  $x < y$ ,  $(y - x)^a \leq 2^{a-1}(|x|^a + |y|^a)$  by Hölder's inequality, so that

$$\mathbb{E}([\xi_\tau - Y]^a \mathbb{I}\{Y \leq \xi_\tau\}) \leq 2^{a-1} \mathbb{E}([\xi_\tau|^a + |Y|^a] \mathbb{I}\{Y \leq \xi_\tau\}).$$

The condition  $\gamma < 1/a$  ensures that  $\mathbb{E}|Y|^a < \infty$ . Recall that  $\xi_\tau \uparrow \infty$  as  $\tau \uparrow 1$  and use the dominated convergence theorem to get

$$\mathbb{E}([\xi_\tau - Y]^a \mathbb{I}\{Y \leq \xi_\tau\}) = O(\xi_\tau^a) \quad \text{as } \tau \uparrow 1. \quad (\text{B.10})$$

Besides, an integration by parts and a change of variables entail

$$\begin{aligned} \mathbb{E}([Y - \xi_\tau]^a \mathbb{I}\{Y > \xi_\tau\}) &= a\xi_\tau^{a-1} \int_{\xi_\tau}^{\infty} \left(\frac{x}{\xi_\tau} - 1\right)^{a-1} \bar{F}(x) dx \\ &= a\xi_\tau^a \bar{F}(\xi_\tau) \int_1^{\infty} (v - 1)^{a-1} \frac{\bar{F}(\xi_\tau v)}{\bar{F}(\xi_\tau)} dv. \end{aligned}$$

Using a uniform convergence theorem such as Proposition B.1.10 in de Haan and Ferreira (2006, p.360) gives

$$\mathbb{E}([Y - \xi_\tau]^a \mathbb{I}\{Y > \xi_\tau\}) = a\xi_\tau^a \bar{F}(\xi_\tau) \int_1^{\infty} (v - 1)^{a-1} v^{-1/\gamma} dv (1 + o(1)) \quad \text{as } \tau \uparrow 1.$$

Combining this equality with (B.5) yields

$$\mathbb{E}([Y - \xi_\tau]^a \mathbb{I}\{Y > \xi_\tau\}) = a\xi_\tau^a(1 - \tau)(\gamma^{-1} - 1) \int_1^{\infty} (v - 1)^{a-1} v^{-1/\gamma} dv (1 + o(1)) \quad \text{as } \tau \uparrow 1. \quad (\text{B.11})$$

Combining (B.9), (B.10), (B.11) and using the change of variables  $u = 1 - v^{-1}$  gives the desired result. ■

**Proof of Theorem 2.** Use Lemma 2 to write, for any  $u$ ,

$$\psi_n(u) = -uT_{1,n} + T_{2,n}(u) \quad (\text{B.12})$$

$$\text{with } T_{1,n} := \frac{1}{\sqrt{n(1 - \tau_n)}} \sum_{i=1}^n \frac{1}{\xi_{\tau_n}} \varphi_{\tau_n}(Y_i - \xi_{\tau_n}) =: \sum_{i=1}^n S_{n,i}$$

$$\text{and } T_{2,n}(u) := -\frac{1}{\xi_{\tau_n}^2} \sum_{i=1}^n \int_0^{u\xi_{\tau_n}/\sqrt{n(1-\tau_n)}} (\varphi_{\tau_n}(Y_i - \xi_{\tau_n} - t) - \varphi_{\tau_n}(Y_i - \xi_{\tau_n})) dt.$$

The random variables  $S_{n,i}$  are independent, identically distributed, and centered since

$$\xi_{\tau_n} = \operatorname{argmin}_{u \in \mathbb{R}} \mathbb{E}(\eta_{\tau_n}(Y_i - u) - \eta_{\tau_n}(Y_i)) \Rightarrow \mathbb{E}(\varphi_{\tau_n}(Y_i - \xi_{\tau_n})) = 0$$

(where a differentiation under the expectation sign was used). We shall prove that

$$\frac{T_{1,n}}{\sqrt{\operatorname{Var}(T_{1,n})}} \xrightarrow{d} \mathcal{N}(0, 1) \quad (\text{B.13})$$

for which it is sufficient to show that for some  $\delta > 0$ ,

$$\frac{n \mathbb{E}|S_{n,1}|^{2+\delta}}{[n \operatorname{Var}(S_{n,1})]^{1+\delta/2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and use Lyapunov's criterion. Choose  $\delta > 0$  small enough so that  $\gamma < 1/(2 + \delta)$  and apply Lemma 4 to get

$$\frac{n \mathbb{E}|S_{n,1}|^{2+\delta}}{[n \operatorname{Var}(S_{n,1})]^{1+\delta/2}} = O([n(1 - \tau_n)]^{-\delta/2}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Convergence (B.13) follows and especially, Lemma 4 entails

$$T_{1,n} \xrightarrow{d} \mathcal{N}\left(0, \frac{2\gamma}{1 - 2\gamma}\right). \quad (\text{B.14})$$

We now turn to the control of the second term  $T_{2,n}(u)$ . Write

$$T_{2,n}(u) = T_{3,n}(u) - \frac{n}{\xi_{\tau_n}^2} \int_0^{u\xi_{\tau_n}/\sqrt{n(1-\tau_n)}} [\mathbb{E}(\varphi_{\tau_n}(Y - \xi_{\tau_n} - t)) - \mathbb{E}(\varphi_{\tau_n}(Y - \xi_{\tau_n}))] dt. \quad (\text{B.15})$$

The random term  $T_{3,n}(u)$  is a sum of independent, identically distributed and centered random variables, which we shall examine after having controlled first the nonrandom term on the right-hand side of (B.15). By Lemma 3, we obtain

$$\begin{aligned} & \mathbb{E}(\varphi_{\tau_n}(Y - \xi_{\tau_n} - t)) - \mathbb{E}(\varphi_{\tau_n}(Y - \xi_{\tau_n})) \\ &= (1 - 2\tau_n) \mathbb{E}((Y - \xi_{\tau_n} - t)(\mathbb{I}\{Y \leq \xi_{\tau_n} + t\} - \mathbb{I}\{Y \leq \xi_{\tau_n}\})) \\ &- t \mathbb{E}(|\tau_n - \mathbb{I}\{Y \leq \xi_{\tau_n}\}|). \end{aligned} \quad (\text{B.16})$$

Clearly

$$\mathbb{E}(|\tau_n - \mathbb{I}\{Y \leq \xi_{\tau_n}\}|) = \tau_n \bar{F}(\xi_{\tau_n}) + (1 - \tau_n) F(\xi_{\tau_n}).$$

It therefore follows from (B.5) that

$$\mathbb{E}(|\tau_n - \mathbb{I}\{Y \leq \xi_{\tau_n}\}|) = \gamma^{-1}(1 - \tau_n)(1 + o(1)) \quad (\text{B.17})$$

as  $n \rightarrow \infty$ . Let further  $\psi(t) := \mathbb{E}((Y - t)\mathbb{I}\{Y > t\})$  and observe that

$$\begin{aligned} \mathbb{E}((Y - \xi_{\tau_n} - t)(\mathbb{I}\{Y \leq \xi_{\tau_n} + t\} - \mathbb{I}\{Y \leq \xi_{\tau_n}\})) &= \mathbb{E}((Y - \xi_{\tau_n} - t)(\mathbb{I}\{Y > \xi_{\tau_n}\} - \mathbb{I}\{Y > \xi_{\tau_n} + t\})) \\ &= \psi(\xi_{\tau_n}) - \psi(\xi_{\tau_n} + t) - t \bar{F}(\xi_{\tau_n}). \end{aligned}$$



Integrating by parts entails

$$\psi(\xi_{\tau_n}) - \psi(\xi_{\tau_n} + t) = \int_{\xi_{\tau_n}}^{\xi_{\tau_n} + t} \bar{F}(x) dx = \xi_{\tau_n} \bar{F}(\xi_{\tau_n}) \int_1^{1+t/\xi_{\tau_n}} \frac{\bar{F}(\xi_{\tau_n} v)}{\bar{F}(\xi_{\tau_n})} dv$$

from which we deduce that

$$\mathbb{E}((Y - \xi_{\tau_n} - t)(\mathbb{I}\{Y \leq \xi_{\tau_n} + t\} - \mathbb{I}\{Y \leq \xi_{\tau_n}\})) = t \bar{F}(\xi_{\tau_n}) \left( \frac{\xi_{\tau_n}}{t} \int_1^{1+t/\xi_{\tau_n}} \frac{\bar{F}(\xi_{\tau_n} v)}{\bar{F}(\xi_{\tau_n})} dv - 1 \right).$$

We now bound the term into brackets as follows: let  $I_n(u) = [0, |u|\xi_{\tau_n}/\sqrt{n(1-\tau_n)}]$  and write

$$\begin{aligned} & \sup_{|t| \in I_n(u)} \left| \frac{\xi_{\tau_n}}{t} \int_1^{1+t/\xi_{\tau_n}} \frac{\bar{F}(\xi_{\tau_n} v)}{\bar{F}(\xi_{\tau_n})} dv - 1 \right| \\ & \leq \sup_{|t| \in I_n(u)} \frac{\xi_{\tau_n}}{|t|} \left| \int_1^{1+t/\xi_{\tau_n}} \left[ \frac{\bar{F}(\xi_{\tau_n} v)}{\bar{F}(\xi_{\tau_n})} - v^{-1/\gamma} \right] dv \right| + o(1) \\ & = o(1) \end{aligned}$$

by the uniform convergence theorem for regularly varying functions [see Theorem 1.5.2 in Bingham *et al.* (1987), p.22], the continuity of  $v \mapsto v^{-1/\gamma}$  at 1 and the convergence  $n(1-\tau_n) \rightarrow \infty$ . As a consequence, by (B.5), the equality

$$\mathbb{E}((Y - \xi_{\tau_n} - t)(\mathbb{I}\{Y \leq \xi_{\tau_n} + t\} - \mathbb{I}\{Y \leq \xi_{\tau_n}\})) = t(1 - \tau_n)r_n(t) \quad (\text{B.18})$$

holds with  $r_n(t) \rightarrow 0$  uniformly in  $t$  such that  $|t| \in I_n(u)$ . Combine (B.15), (B.16), (B.17) and (B.18) to get

$$T_{2,n}(u) = \frac{u^2}{2\gamma}(1 + o(1)) + T_{3,n}(u), \quad (\text{B.19})$$

$$\text{with } T_{3,n}(u) := -\frac{1}{\xi_{\tau_n}^2} \sum_{i=1}^n \int_0^{u\xi_{\tau_n}/\sqrt{n(1-\tau_n)}} [\mathcal{S}_{n,i}(\xi_{\tau_n} + t) - \mathcal{S}_{n,i}(\xi_{\tau_n})] dt$$

where the  $\mathcal{S}_{n,i}(v) := \varphi_{\tau_n}(Y_i - v) - \mathbb{E}(\varphi_{\tau_n}(Y_i - v))$  are independent copies of  $\mathcal{S}_n(v) := \varphi_{\tau_n}(Y - v) - \mathbb{E}(\varphi_{\tau_n}(Y - v))$ . Thus

$$\text{Var}(T_{3,n}(u)) = \frac{n}{\xi_{\tau_n}^4} \text{Var} \left( \int_0^{u\xi_{\tau_n}/\sqrt{n(1-\tau_n)}} [\mathcal{S}_n(\xi_{\tau_n} + t) - \mathcal{S}_n(\xi_{\tau_n})] dt \right).$$

We now notice that for any  $v$ ,  $\mathcal{S}_n(v)$  is centered and thus

$$\text{Var}(T_{3,n}(u)) = \frac{n}{\xi_{\tau_n}^4} \int_{[0, u\xi_{\tau_n}/\sqrt{n(1-\tau_n)}]^2} \mathbb{E}([\mathcal{S}_n(\xi_{\tau_n} + s) - \mathcal{S}_n(\xi_{\tau_n})][\mathcal{S}_n(\xi_{\tau_n} + t) - \mathcal{S}_n(\xi_{\tau_n})]) ds dt$$

(where the integrability properties of  $Y$  were used to switch integrals and expectation). By the Cauchy-Schwarz inequality,

$$\text{Var}(T_{3,n}(u)) \leq \frac{n}{\xi_{\tau_n}^4} \left( \int_0^{u\xi_{\tau_n}/\sqrt{n(1-\tau_n)}} \sqrt{\mathbb{E}(|\mathcal{S}_n(\xi_{\tau_n} + t) - \mathcal{S}_n(\xi_{\tau_n})|^2)} dt \right)^2. \quad (\text{B.20})$$

Applying Lemma 3, we get for any  $t$

$$|\mathcal{S}_n(\xi_{\tau_n} + t) - \mathcal{S}_n(\xi_{\tau_n})| \leq 2|t|[1 - \tau_n + \mathbb{I}\{Y > \xi_{\tau_n} + \min(t, 0)\} + \bar{F}(\xi_{\tau_n} + \min(t, 0))].$$

Using the inequality  $|a + b + c|^2 \leq 3(a^2 + b^2 + c^2)$  yields

$$\mathbb{E}(|\mathcal{S}_n(\xi_{\tau_n} + t) - \mathcal{S}_n(\xi_{\tau_n})|^2) \leq 12t^2[(1 - \tau_n)^2 + \bar{F}(\xi_{\tau_n} + \min(t, 0))(1 + \bar{F}(\xi_{\tau_n} + \min(t, 0)))]. \quad (\text{B.21})$$

Finally, using again the regular variation property of  $\bar{F}$  and the convergence  $n(1 - \tau_n) \rightarrow \infty$ ,

$$\sup_{|s| \in I_n(u)} |\bar{F}(\xi_{\tau_n} + s) - \bar{F}(\xi_{\tau_n})| = \bar{F}(\xi_{\tau_n}) \sup_{|s| \in I_n(u)} \left| \frac{\bar{F}(\xi_{\tau_n} + s)}{\bar{F}(\xi_{\tau_n})} - 1 \right| = o(\bar{F}(\xi_{\tau_n})) = o(1 - \tau_n) \quad (\text{B.22})$$

in view of (B.5). Using (B.5) once again and combining (B.20), (B.21) and (B.22) yields

$$\text{Var}(T_{3,n}(u)) = O\left(\frac{n}{\xi_{\tau_n}^4}(1 - \tau_n) \left| \int_0^{u\xi_{\tau_n}/\sqrt{n(1-\tau_n)}} |t| dt \right|^2\right) = O\left(\frac{1}{n(1 - \tau_n)}\right) \rightarrow 0$$

as  $n \rightarrow \infty$ . Whence the convergence  $T_{3,n}(u) \xrightarrow{\mathbb{P}} 0$ ; combining (B.12), (B.14) and (B.19) entails

$$\psi_n(u) \xrightarrow{d} -uZ\sqrt{\frac{2\gamma}{1-2\gamma}} + \frac{u^2}{2\gamma} \text{ as } n \rightarrow \infty$$

(with  $Z$  being standard Gaussian) in the sense of finite-dimensional convergence. As a function of  $u$ , this limit is almost surely finite and defines a convex function which has a unique minimum at

$$u^* = \gamma\sqrt{\frac{2\gamma}{1-2\gamma}}Z \stackrel{d}{=} \mathcal{N}\left(0, \gamma^2\frac{2\gamma}{1-2\gamma}\right).$$

Applying Theorem 5 in Knight (1999) completes the proof. ■

**Proof of Theorem 3.** By the equalities (9) and (10), we have

$$\log\left(\frac{\bar{\xi}_{\tau'_n}^*}{\xi_{\tau'_n}}\right) = \log\left(\frac{\hat{q}_{\tau'_n}^*}{q_{\tau'_n}}\right) + \log\left(\frac{\bar{\xi}_{\tau_n}}{\xi_{\tau_n}}\right) - \log\left(\frac{\hat{q}_{\tau_n}}{q_{\tau_n}}\right) + \log\left(\frac{\xi_{\tau_n}}{q_{\tau_n}}\right) - \log\left(\frac{\xi_{\tau'_n}}{q_{\tau'_n}}\right).$$

Furthermore, the convergence  $\log[(1 - \tau_n)/(1 - \tau'_n)] \rightarrow \infty$  entails

$$\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \log \left( \frac{\hat{q}_{\tau'_n}^*}{q_{\tau'_n}} \right) \xrightarrow{d} \Gamma, \quad (\text{B.23})$$

$$\begin{aligned} \frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \log \left( \frac{\bar{\xi}_{\tau_n}}{\xi_{\tau_n}} \right) &= O_{\mathbb{P}}(1/\log[(1 - \tau_n)/(1 - \tau'_n)]) \\ &= o_{\mathbb{P}}(1), \end{aligned} \quad (\text{B.24})$$

$$\begin{aligned} \frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \log \left( \frac{\hat{q}_{\tau_n}}{q_{\tau_n}} \right) &= O_{\mathbb{P}}(1/\log[(1 - \tau_n)/(1 - \tau'_n)]) \\ &= o_{\mathbb{P}}(1), \end{aligned} \quad (\text{B.25})$$

$$\begin{aligned} \frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \left[ \log \left( \frac{\xi_{\tau_n}}{q_{\tau_n}} \right) - \log \left( \frac{\xi_{\tau'_n}}{q_{\tau'_n}} \right) \right] &= O \left( \frac{\sqrt{n(1 - \tau_n)}[r(\tau_n) + r(\tau'_n)]}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \right) \\ &= O \left( \frac{\sqrt{n(1 - \tau_n)}r(\tau_n)}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \right) \\ &= o(1). \end{aligned} \quad (\text{B.26})$$

Here, Theorem 4.3.8 in de Haan and Ferreira (2006, p.138) was used to show (B.23), while (B.24) and (B.25) follow from the distributional convergence assumption on  $\bar{\xi}_{\tau_n}$  and from Theorem 2.4.1 in de Haan and Ferreira (2006, p.50), respectively. Convergence (B.26) is a consequence of Corollary 1 and, in what concerns the relationship  $r(\tau'_n) = O(r(\tau_n))$ , of the regular variation of  $s \mapsto q_{1-s^{-1}}$  and  $|A|$ . A combination of these convergence results and a use of the delta-method give the desired conclusion. ■

**Proof of Proposition 2.** We shall actually prove the more general statement

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}(X|Y > t)}{U_X(1/\bar{F}_Y(t))} = \int_0^\infty R(x^{-1/\gamma_X}, 1) dx \quad (\text{B.27})$$

which contains both Proposition 1 in Cai *et al.* (2015) and our desired result (13), because  $\xi_{Y,\tau} \rightarrow \infty$  as  $\tau \uparrow 1$ . For any  $t > 0$ ,

$$\begin{aligned} \frac{\mathbb{E}(X|Y > t)}{U_X(1/\bar{F}_Y(t))} &= \frac{1}{U_X(1/\bar{F}_Y(t))} \int_0^\infty \frac{\mathbb{P}(X > s, Y > t)}{\bar{F}_Y(t)} ds \\ &= \frac{1}{U_X(1/\bar{F}_Y(t))} \int_0^\infty \frac{\mathbb{P}(\bar{F}_X(X) \leq \bar{F}_X(s), \bar{F}_Y(Y) \leq \bar{F}_Y(t))}{\bar{F}_Y(t)} ds \\ &= \int_0^\infty \frac{\mathbb{P}(\bar{F}_X(X) \leq \bar{F}_X(U_X(1/\bar{F}_Y(t))x), \bar{F}_Y(Y) \leq \bar{F}_Y(t))}{\bar{F}_Y(t)} dx. \end{aligned} \quad (\text{B.28})$$

Note now that because  $X$  is heavy-tailed,  $\bar{F}_X(Tx) \sim x^{-1/\gamma_X} \bar{F}_X(T)$  as  $T \rightarrow \infty$  and we have that:

$$\forall x > 0, \bar{F}_X(U_X(1/\bar{F}_Y(t))x) \sim x^{-1/\gamma_X} \bar{F}_Y(t) \text{ as } t \rightarrow \infty.$$

Thus, by condition  $\mathcal{JC}(R)$ :

$$\forall x > 0, \lim_{t \rightarrow \infty} \frac{\mathbb{P}(\overline{F}_X(X) \leq \overline{F}_X(U_X(1/\overline{F}_Y(t))x), \overline{F}_Y(Y) \leq \overline{F}_Y(t))}{\overline{F}_Y(t)} = R(x^{-1/\gamma_X}, 1). \quad (\text{B.29})$$

It only remains to show that the integral in (B.28) and the limit in (B.29) can be interchanged, and this can be done exactly in the same way as in the proof of Proposition 1 of Cai *et al.* (2015), so we omit the details.

To show the second convergence result (14), we apply (B.27) successively to  $t = \xi_{Y,\tau}$  and  $t = q_{Y,\tau}$  in conjunction with (4) to get

$$\lim_{\tau \uparrow 1} \frac{\text{XMES}(\tau)}{\text{QMES}(\tau)} = \lim_{\tau \uparrow 1} \frac{U_X(1/\overline{F}_Y(\xi_{Y,\tau}))}{U_X(1/\overline{F}_Y(q_{Y,\tau}))} = \lim_{\tau \uparrow 1} \left( \frac{\overline{F}_Y(q_{Y,\tau})}{\overline{F}_Y(\xi_{Y,\tau})} \right)^{\gamma_X} = (\gamma_Y^{-1} - 1)^{-\gamma_X}.$$

■

**Proof of Theorem 4.** We start by the case when  $X > 0$  almost surely. In this situation,

$$\widetilde{\text{XMES}}(\tau_n) = \frac{\sum_{i=1}^n X_i \mathbb{I}\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}{\sum_{i=1}^n \mathbb{I}\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}.$$

Write then

$$\begin{aligned} \log \left( \frac{\widetilde{\text{XMES}}^*(\tau'_n)}{\text{XMES}(\tau'_n)} \right) &= \log \left( \frac{\widetilde{\text{XMES}}(\tau_n)}{\text{XMES}(\tau_n)} \right) + \log \left( \frac{\text{XMES}(\tau_n)}{\text{XMES}(\tau'_n)} \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\gamma_X} \right) \\ &+ (\hat{\gamma}_X - \gamma_X) \log \left( \frac{1 - \tau_n}{1 - \tau'_n} \right). \end{aligned}$$

Using the delta-method, the proof shall then be complete provided that

$$\sqrt{n(1 - \tau_n)} \left( \frac{\widetilde{\text{XMES}}(\tau_n)}{\text{XMES}(\tau_n)} - 1 \right) = \text{O}_{\mathbb{P}}(1) \quad (\text{B.30})$$

and

$$\sqrt{n(1 - \tau_n)} \left( \frac{\text{XMES}(\tau_n)}{\text{XMES}(\tau'_n)} \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\gamma_X} - 1 \right) = \text{O}(1). \quad (\text{B.31})$$

To show (B.30), write

$$\frac{\widetilde{\text{XMES}}(\tau_n)}{\text{XMES}(\tau_n)} = \frac{\mathbb{E}(\mathbb{I}_{\{Y > \xi_{Y,\tau_n}\}})}{\frac{1}{n} \sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} \frac{\frac{1}{n} \sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}}{\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y,\tau_n}\}})} \frac{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}}{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}} \frac{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}}{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}}. \quad (\text{B.32})$$

Firstly,

$$\sqrt{n(1 - \tau_n)} \left( \frac{\frac{1}{n} \sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}}{\mathbb{E}(\mathbb{I}_{\{Y > \xi_{Y,\tau_n}\}})} - 1 \right) = \text{O}_{\mathbb{P}}(1) \quad (\text{B.33})$$

because the variance of the term on the left-hand side is bounded in view of Proposition 1. Secondly,

$$\begin{aligned} \text{Var} \left[ \sqrt{n(1-\tau_n)} \left( \frac{\frac{1}{n} \sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}\}}}{\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y, \tau_n}\}})} - 1 \right) \right] &\leq \frac{(1-\tau_n) \mathbb{E}(X^2 \mathbb{I}_{\{Y > \xi_{Y, \tau_n}\}})}{[\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y, \tau_n}\}})]^2} \\ &= \frac{(1-\tau_n)}{\mathbb{P}(Y > \xi_{Y, \tau_n})} \frac{\mathbb{E}(X^2 | Y > \xi_{Y, \tau_n})}{[\mathbb{E}(X | Y > \xi_{Y, \tau_n})]^2}. \end{aligned}$$

Applying Proposition 1 and then (13) in Proposition 2,

$$\text{Var} \left[ \sqrt{n(1-\tau_n)} \left( \frac{\frac{1}{n} \sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}\}}}{\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y, \tau_n}\}})} - 1 \right) \right] = O \left( \frac{\mathbb{E}(X^2 | Y > \xi_{Y, \tau_n})}{[U_X(1/\bar{F}_Y(\xi_{Y, \tau_n}))]^2} \right).$$

Notice then that condition  $\mathcal{JC}(R)$  is equivalent, for all  $x$  and  $y$  which are not both infinite, to

$$\lim_{t \rightarrow \infty} t \mathbb{P}(X \geq U_X(t/x), Y \geq U_Y(t/y)) = R(x, y).$$

Since  $(U_X)^2 = U_{X^2}$  (because  $X > 0$ ), this entails

$$\lim_{t \rightarrow \infty} t \mathbb{P}(X^2 \geq U_{X^2}(t/x), Y \geq U_Y(t/y)) = R(x, y).$$

Hence,  $(X^2, Y)$  also satisfies condition  $\mathcal{JC}(R)$ . Thus, by (13) in Proposition 2,

$$\frac{\mathbb{E}(X^2 | Y > \xi_{Y, \tau_n})}{[U_X(1/\bar{F}_Y(\xi_{Y, \tau_n}))]^2} = \frac{\mathbb{E}(X^2 | Y > \xi_{Y, \tau_n})}{U_{X^2}(1/\bar{F}_Y(\xi_{Y, \tau_n}))} = O(1)$$

which entails

$$\text{Var} \left[ \sqrt{n(1-\tau_n)} \left( \frac{\frac{1}{n} \sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}\}}}{\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y, \tau_n}\}})} - 1 \right) \right] = O(1)$$

and therefore

$$\sqrt{n(1-\tau_n)} \left( \frac{\frac{1}{n} \sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}\}}}{\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y, \tau_n}\}})} - 1 \right) = O_{\mathbb{P}}(1). \quad (\text{B.34})$$

Thirdly, by Theorem 2,  $\tilde{\xi}_{Y, \tau_n}$  is  $\sqrt{n(1-\tau_n)}$ -relatively consistent, so that for any  $\varepsilon > 0$ , we may find  $K > 0$  such that

$$\left| \frac{\tilde{\xi}_{Y, \tau_n}}{\xi_{Y, \tau_n}} - 1 \right| \leq \frac{K}{\sqrt{n(1-\tau_n)}}$$

with probability larger than  $1 - \varepsilon$  eventually. In what follows we assume that  $K$  is chosen so that this is the case. With probability larger than  $1 - \varepsilon$  eventually, we then have

$$\begin{aligned} &\left| \frac{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \tilde{\xi}_{Y, \tau_n}\}}}{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}\}}} - 1 \right| \\ &\leq \max \left( \left| \frac{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}(1+K/\sqrt{n(1-\tau_n)})\}}}{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}\}}} - 1 \right|, \left| \frac{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}(1-K/\sqrt{n(1-\tau_n)})\}}}{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y, \tau_n}\}}} - 1 \right| \right). \end{aligned}$$

By straightforward variance calculations,

$$\begin{aligned} & \sqrt{n(1-\tau_n)} \left( \frac{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)})\}}}{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right) \\ &= O_{\mathbb{P}} \left( \sqrt{n(1-\tau_n)} \left| \frac{\mathbb{P}(Y > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)})}{\mathbb{P}(Y > \xi_{Y,\tau_n})} - 1 \right| \right). \end{aligned}$$

By a uniform inequality such as Theorem B.3.10 in de Haan and Ferreira (2006) applied to the function  $\bar{F}_Y$ , we get

$$\frac{\mathbb{P}(Y > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)})}{\mathbb{P}(Y > \xi_{Y,\tau_n})} - 1 = O \left( \frac{1}{\sqrt{n(1-\tau_n)}} \right) \quad (\text{B.35})$$

and therefore

$$\sqrt{n(1-\tau_n)} \left( \frac{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}}{\sum_{i=1}^n \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right) = O_{\mathbb{P}}(1). \quad (\text{B.36})$$

Lastly, write with probability larger than  $1 - \varepsilon$  eventually:

$$\begin{aligned} & \left| \frac{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}}{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right| \\ & \leq \max \left( \left| \frac{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}(1+K/\sqrt{n(1-\tau_n)})\}}}{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right|, \left| \frac{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}(1-K/\sqrt{n(1-\tau_n)})\}}}{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right| \right). \end{aligned}$$

By a straightforward modification of (B.36),

$$\begin{aligned} & \sqrt{n(1-\tau_n)} \left( \frac{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)})\}}}{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right) \\ &= O_{\mathbb{P}} \left( \sqrt{n(1-\tau_n)} \left| \frac{\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)})\}})}{\mathbb{E}(X \mathbb{I}_{\{Y > \xi_{Y,\tau_n}\}})} - 1 \right| \right). \end{aligned}$$

Applying (B.35) and (13) in Proposition 2, we obtain

$$\begin{aligned} & \sqrt{n(1-\tau_n)} \left( \frac{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)})\}}}{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right) \\ &= O_{\mathbb{P}} \left( \sqrt{n(1-\tau_n)} \left| \frac{\mathbb{E}(X|Y > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)})}{\mathbb{E}(X|Y > \xi_{Y,\tau_n})} - 1 \right| \right) \\ &= O_{\mathbb{P}} \left( \sqrt{n(1-\tau_n)} \left| \frac{\mathbb{E}(X|Y > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)}) - \mathbb{E}(X|Y > \xi_{Y,\tau_n}))}{U_X(1/\bar{F}_Y(\xi_{Y,\tau_n}))} \right| \right). \end{aligned}$$

It is therefore enough to show that

$$\frac{|\mathbb{E}(X|Y > \xi_{Y,\tau_n}(1 \pm K/\sqrt{n(1-\tau_n)}) - \mathbb{E}(X|Y > \xi_{Y,\tau_n}))|}{U_X(1/\bar{F}_Y(\xi_{Y,\tau_n}))} = O_{\mathbb{P}} \left( \frac{1}{\sqrt{n(1-\tau_n)}} \right).$$

Because expectiles and quantiles are asymptotically proportional in view of Corollary 1:

$$\xi_{Y,\tau_n} = (\gamma_Y^{-1} - 1)^{-\gamma_Y} q_{Y,\tau_n} \left( 1 + O \left( \frac{1}{\sqrt{n(1 - \tau_n)}} \right) \right),$$

this can be achieved by using condition  $\mathcal{JC}_2(R, \beta, \kappa)$  in the same way followed by Cai *et al.* (2015) to examine the convergence of the term  $J_2$  that they introduce in the proof of their Proposition 3, see pp.438-439 therein. We then get

$$\sqrt{n(1 - \tau_n)} \left( \frac{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}}{\sum_{i=1}^n X_i \mathbb{I}_{\{Y_i > \xi_{Y,\tau_n}\}}} - 1 \right) = O_{\mathbb{P}}(1). \quad (\text{B.37})$$

Combining (B.32), (B.33), (B.34), (B.36) and (B.37) concludes the proof of (B.30).

Now, to prove (B.31), notice first that

$$\sqrt{n(1 - \tau_n)} \left( \frac{\text{XMES}(\tau_n)}{U_X(1/\bar{F}_Y(\xi_{Y,\tau_n}))} - \int_0^\infty R(x^{-1/\gamma_X}, 1) dx \right) = O(1) \quad (\text{B.38})$$

$$\text{and } \sqrt{n(1 - \tau_n)} \left( \frac{\text{XMES}(\tau'_n)}{U_X(1/\bar{F}_Y(\xi_{Y,\tau'_n}))} - \int_0^\infty R(x^{-1/\gamma_X}, 1) dx \right) = O(1); \quad (\text{B.39})$$

this can be verified using condition  $\mathcal{JC}_2(R, \beta, \kappa)$  along the lines of proof of Lemma 3 and (28) in Cai *et al.* (2015), because expectiles and quantiles are asymptotically proportional. Besides, by Proposition 1,

$$\begin{aligned} \sqrt{n(1 - \tau_n)} \left( \frac{\bar{F}_Y(\xi_{Y,\tau_n})}{1 - \tau_n} - (\gamma_Y^{-1} - 1) \right) &= O(1) \\ \text{and } \sqrt{n(1 - \tau_n)} \left( \frac{\bar{F}_Y(\xi_{Y,\tau'_n})}{1 - \tau'_n} - (\gamma_Y^{-1} - 1) \right) &= O(1) \end{aligned}$$

so that by condition  $\mathcal{C}_2(\gamma_X, \rho_X, A_X)$  and convergence  $\sqrt{n(1 - \tau_n)} A_X ((1 - \tau_n)^{-1}) \rightarrow \lambda_2 \in \mathbb{R}$ ,

$$\sqrt{n(1 - \tau_n)} \left( \frac{U_X(1/\bar{F}_Y(\xi_{Y,\tau_n}))}{U_X(1/\bar{F}_Y(\xi_{Y,\tau'_n}))} \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\gamma_X} - 1 \right) = O(1). \quad (\text{B.40})$$

Combining (B.38), (B.39) and (B.40) completes the proof of (B.31).

We now show how the condition that  $X > 0$  almost surely can be dropped in our framework. Define  $X_+ = \max(X, 0)$  and

$$\text{XMES}^+(\tau'_n) := \mathbb{E}(X_+ | Y > \xi_{Y,\tau'_n}),$$

*i.e.*  $\text{XMES}^+$  is the marginal expected shortfall of the positive part of  $X$ , and write

$$\frac{\widetilde{\text{XMES}}^*(\tau'_n)}{\text{XMES}(\tau'_n)} = \frac{\widetilde{\text{XMES}}^*(\tau'_n) \text{XMES}^+(\tau'_n)}{\text{XMES}^+(\tau'_n) \text{XMES}(\tau'_n)}. \quad (\text{B.41})$$

The first part of the proof of Theorem 2 in Cai *et al.* (2015), see pp.440–441 and in particular condition (35) there, shows that  $X_+$  satisfies condition  $\mathcal{JC}_2(R, \beta, \kappa)$ . As a consequence, we may apply the result we have just shown to the random variable  $X_+$  to get

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]} \left( \frac{\widehat{\text{XMES}}^*(\tau'_n)}{\text{XMES}^+(\tau'_n)} - 1 \right) \xrightarrow{d} \Gamma. \quad (\text{B.42})$$

Let now  $X_- = X - X_+$  so that

$$\frac{\text{XMES}(\tau'_n)}{\text{XMES}^+(\tau'_n)} = 1 + \frac{\mathbb{E}(X_- | Y > \xi_{Y, \tau'_n})}{\text{XMES}^+(\tau'_n)}. \quad (\text{B.43})$$

Since  $\xi_{Y, \tau'_n} \uparrow \infty$ , we have for  $n$  large enough that

$$\frac{\text{XMES}^+(\tau'_n)}{U_X(1/\bar{F}_Y(\xi_{Y, \tau'_n}))} = \frac{\text{XMES}^+(\tau'_n)}{U_{X^+}(1/\bar{F}_Y(\xi_{Y, \tau'_n}))}$$

so that

$$\frac{\text{XMES}^+(\tau'_n)}{U_X(1/\bar{F}_Y(\xi_{Y, \tau'_n}))} \rightarrow \int_0^\infty R(x^{-1/\gamma_X}, 1) dx$$

as  $n \rightarrow \infty$  and therefore

$$\frac{\text{XMES}(\tau'_n)}{\text{XMES}^+(\tau'_n)} = 1 + \text{O} \left( \frac{\mathbb{E}(X_- | Y > \xi_{Y, \tau'_n})}{U_X(1/\bar{F}_Y(\xi_{Y, \tau'_n}))} \right). \quad (\text{B.44})$$

Since extreme expectiles and extreme quantiles are asymptotically proportional, we have as in Cai *et al.* (2015) that

$$|\mathbb{E}(X_- | Y > \xi_{Y, \tau'_n})| = \text{O}((1-\tau'_n)^{-1+(1-\kappa)(1-\gamma_X)}) \quad \text{and} \quad \frac{1}{U_X(1/\bar{F}_Y(\xi_{Y, \tau'_n}))} = \text{O}((1-\tau'_n)^{\gamma_X}). \quad (\text{B.45})$$

Plugging this into (B.44) entails

$$\frac{\text{XMES}(\tau'_n)}{\text{XMES}^+(\tau'_n)} = 1 + \text{O}((1-\tau'_n)^{-\kappa(1-\gamma_X)}) = 1 + \text{o} \left( \frac{1}{\sqrt{n(1-\tau_n)}} \right). \quad (\text{B.46})$$

Plugging (B.46) and (B.42) into (B.41) concludes the proof.  $\blacksquare$

**Proof of Theorem 5.** Write

$$\log \left( \frac{\widehat{\text{XMES}}^*(\tau'_n)}{\text{XMES}(\tau'_n)} \right) = \log \left( \frac{\widehat{\text{QMES}}^*(\tau'_n)}{\text{QMES}(\tau'_n)} \right) + \log \left( \frac{(\hat{\gamma}_Y^{-1} - 1)^{-\hat{\gamma}_X}}{(\gamma_Y^{-1} - 1)^{-\gamma_X}} \right) - \log \left( (\gamma_Y^{-1} - 1)^{\gamma_X} \frac{\text{XMES}(\tau'_n)}{\text{QMES}(\tau'_n)} \right).$$

Firstly, by Theorem 2 in Cai *et al.* (2015),

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]} \log \left( \frac{\widehat{\text{QMES}}^*(\tau'_n)}{\text{QMES}(\tau'_n)} \right) \xrightarrow{d} \Gamma. \quad (\text{B.47})$$



Secondly, writing

$$\log \left( \frac{(\widehat{\gamma}_Y^{-1} - 1)^{-\widehat{\gamma}_X}}{(\gamma_Y^{-1} - 1)^{-\gamma_X}} \right) = - [(\widehat{\gamma}_X - \gamma_X) \log(\widehat{\gamma}_Y^{-1} - 1) + \gamma_X (\log(\widehat{\gamma}_Y^{-1} - 1) - \log(\gamma_Y^{-1} - 1))]$$

we get

$$\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \log \left( \frac{(\widehat{\gamma}_Y^{-1} - 1)^{-\widehat{\gamma}_X}}{(\gamma_Y^{-1} - 1)^{-\gamma_X}} \right) \xrightarrow{\mathbb{P}} 0. \quad (\text{B.48})$$

Thirdly, by (B.39), which is

$$\sqrt{n(1 - \tau_n)} \left( \frac{\text{XMES}^+(\tau'_n)}{U_{X^+}(1/\overline{F}_Y(\xi_{Y,\tau'_n}))} - \int_0^\infty R(x^{-1/\gamma_X}, 1) dx \right) = O(1)$$

together with (B.46) and the equality  $U_{X^+} = U_X$  in a neighborhood of infinity, we obtain

$$\sqrt{n(1 - \tau_n)} \left( \frac{\text{XMES}(\tau'_n)}{U_X(1/\overline{F}_Y(\xi_{Y,\tau'_n}))} - \int_0^\infty R(x^{-1/\gamma_X}, 1) dx \right) = O(1).$$

The similar relationship (Cai *et al.*, 2015)

$$\sqrt{n(1 - \tau_n)} \left( \frac{\text{QMES}(\tau'_n)}{U_X(1/\overline{F}_Y(q_{Y,\tau'_n}))} - \int_0^\infty R(x^{-1/\gamma_X}, 1) dx \right) = O(1)$$

then yields

$$\log \left( (\gamma_Y^{-1} - 1)^{\gamma_X} \frac{\text{XMES}(\tau'_n)}{\text{QMES}(\tau'_n)} \right) = \log \left( (\gamma_Y^{-1} - 1)^{\gamma_X} \frac{U_X(1/\overline{F}_Y(\xi_{Y,\tau'_n}))}{U_X(1/\overline{F}_Y(q_{Y,\tau'_n}))} \right) + O \left( \frac{1}{\sqrt{n(1 - \tau_n)}} \right). \quad (\text{B.49})$$

Now, by Proposition 1,

$$\sqrt{n(1 - \tau_n)} \left( \frac{\overline{F}_Y(\xi_{Y,\tau'_n})}{1 - \tau'_n} - (\gamma_Y^{-1} - 1) \right) = O(1)$$

and  $\overline{F}_Y(q_{Y,\tau'_n}) = 1 - \tau'_n$  by continuity of  $F_Y$ , so that by condition  $\mathcal{C}_2(\gamma_X, \rho_X, A_X)$  and convergence  $\sqrt{n(1 - \tau_n)} A_X ((1 - \tau_n)^{-1}) \rightarrow 0$ ,

$$\sqrt{n(1 - \tau_n)} \left( (\gamma_Y^{-1} - 1)^{\gamma_X} \frac{U_X(1/\overline{F}_Y(\xi_{Y,\tau'_n}))}{U_X(1/\overline{F}_Y(q_{Y,\tau'_n}))} - 1 \right) = O(1).$$

In conjunction with (B.49), this entails

$$\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \log \left( (\gamma_Y^{-1} - 1)^{\gamma_X} \frac{\text{XMES}(\tau'_n)}{\text{QMES}(\tau'_n)} \right) \rightarrow 0. \quad (\text{B.50})$$

A combination of (B.47), (B.48), (B.50) and the delta-method completes the proof.  $\blacksquare$

**Proof of Proposition 3.** We start by obtaining an equivalent for the numerator of  $1 - \tau'_n(\alpha_n)$ , which is equal to

$$q_{\alpha_n} \mathbb{E} \left( \left[ \frac{Y}{q_{\alpha_n}} - 1 \right] \mathbb{I}\{Y/q_{\alpha_n} > 1\} \right).$$

Just as in the proof of Proposition 1, we integrate by parts to obtain

$$\mathbb{E} \left( \left[ \frac{Y}{q_{\alpha_n}} - 1 \right] \mathbb{I}\{Y/q_{\alpha_n} > 1\} \right) = \bar{F}(q_{\alpha_n}) \left( \frac{\gamma}{1 - \gamma} + \int_1^{+\infty} \left[ \frac{\bar{F}(q_{\alpha_n}x)}{\bar{F}(q_{\alpha_n})} - x^{-1/\gamma} \right] dx \right).$$

Since  $q_{\alpha_n} \rightarrow \infty$  as  $n \rightarrow \infty$ , we can apply Proposition B.1.10 in de Haan and Ferreira (2006) to the function  $\bar{F}$  to get

$$\mathbb{E} \left( \left[ \frac{Y}{q_{\alpha_n}} - 1 \right] \mathbb{I}\{Y/q_{\alpha_n} > 1\} \right) = \bar{F}(q_{\alpha_n}) \left( \frac{\gamma}{1 - \gamma} + o(1) \right) = (1 - \alpha_n) \frac{\gamma}{1 - \gamma} (1 + o(1)).$$

To obtain an equivalent of the denominator, we note that

$$\mathbb{E} |Y - q_{\alpha_n}| = q_{\alpha_n} \mathbb{E} \left| \frac{Y}{q_{\alpha_n}} - 1 \right| = q_{\alpha_n} (1 + o(1))$$

where we used the dominated convergence theorem together with the fact that  $q_{\alpha_n} \rightarrow \infty$ . Wrapping up, we obtain

$$\frac{\mathbb{E} (|Y - q_{\alpha_n}| \mathbb{I}\{Y > q_{\alpha_n}\})}{\mathbb{E} |Y - q_{\alpha_n}|} = (1 - \alpha_n) \frac{\gamma}{1 - \gamma} (1 + o(1))$$

which is the desired result. ■

**Proof of Theorem 6.** Our first goal is to show that

$$\frac{1 - \hat{\tau}'_n(\alpha_n)}{1 - \tau'_n(\alpha_n)} - 1 = O_{\mathbb{P}}(1). \tag{B.51}$$

To this end, we write

$$\frac{1 - \hat{\tau}'_n(\alpha_n)}{1 - \tau'_n(\alpha_n)} - 1 = \frac{\hat{\gamma}}{\gamma} \times \frac{1 - \gamma}{1 - \hat{\gamma}} \times \frac{(1 - \alpha_n) \frac{\gamma}{1 - \gamma}}{1 - \tau'_n(\alpha_n)} - 1. \tag{B.52}$$

The delta-method yields

$$\sqrt{n(1 - \tau_n)} \left( \frac{\gamma}{\hat{\gamma}} \times \frac{1 - \hat{\gamma}}{1 - \gamma} - 1 \right) = O_{\mathbb{P}}(1). \tag{B.53}$$

Recall now (B.4) in the proof of Proposition 1 which here translates into

$$\begin{aligned} \frac{(1 - \alpha_n) \frac{\gamma}{1 - \gamma}}{\mathbb{E} \left[ \left[ \frac{Y}{q_{\alpha_n}} - 1 \right] \mathbb{I}\{Y > q_{\alpha_n}\} \right]} - 1 &= O[A(1/\bar{F}(q_{\alpha_n}))] = O[A((1 - \alpha_n)^{-1})] \\ &= O(1/\sqrt{n(1 - \tau_n)}), \end{aligned} \tag{B.54}$$

because  $q_{\alpha_n} = \xi_{\tau'_n(\alpha_n)}$  and using the regular variation property of  $A$ . Write further

$$\begin{aligned}
\mathbb{E} \left| \frac{Y}{q_{\alpha_n}} - 1 \right| - 1 &= \mathbb{E} \left[ \left| \frac{Y}{q_{\alpha_n}} - 1 \right| \mathbb{I}_{\{Y > q_{\alpha_n}\}} \right] + \mathbb{E} \left[ \left( 1 - \frac{Y}{q_{\alpha_n}} \right) \mathbb{I}_{\{Y \leq q_{\alpha_n}\}} \right] - 1 \\
&= \mathbb{E} \left[ \left| \frac{Y}{q_{\alpha_n}} - 1 \right| \mathbb{I}_{\{Y > q_{\alpha_n}\}} \right] - \frac{\mathbb{E}(Y \mathbb{I}_{\{Y \leq q_{\alpha_n}\}})}{q_{\alpha_n}} - \bar{F}(q_{\alpha_n}) \\
&= O(\max\{1 - \alpha_n, 1/q_{\alpha_n}\}) = O(1/q_{\alpha_n}) = o(1/\sqrt{n(1 - \tau_n)}) \tag{B.55}
\end{aligned}$$

where we successively used (B.54), the dominated convergence theorem, the relationship  $1 - \alpha_n = o(1/q_{\alpha_n})$  valid because  $0 < \gamma < 1$ , and the regular variation property of  $t \mapsto q_{1-t}$ . Combining (B.52), (B.53), (B.54) and (B.55) with the definition

$$1 - \tau'_n(\alpha_n) = \frac{\mathbb{E}\{|Y - q_{\alpha_n}| \mathbb{I}(Y > q_{\alpha_n})\}}{\mathbb{E}|Y - q_{\alpha_n}|}$$

results in (B.51).

The idea to prove (i) is now to write

$$\widehat{\xi}_{\widehat{\tau}'_n(\alpha_n)}^* = \left( \frac{1 - \widehat{\tau}'_n(\alpha_n)}{1 - \tau_n} \right)^{-\widehat{\gamma}} \widehat{\xi}_{\tau_n} = \left( \frac{1 - \widehat{\tau}'_n(\alpha_n)}{1 - \tau'_n(\alpha_n)} \right)^{-\widehat{\gamma}} \times \left\{ \left( \frac{1 - \tau'_n(\alpha_n)}{1 - \tau_n} \right)^{-\widehat{\gamma}} \widehat{\xi}_{\tau_n} \right\}. \tag{B.56}$$

We have

$$\begin{aligned}
\left( \frac{1 - \widehat{\tau}'_n(\alpha_n)}{1 - \tau'_n(\alpha_n)} \right)^{-\widehat{\gamma}} &= \exp \left( -\widehat{\gamma} \log \left[ \frac{1 - \widehat{\tau}'_n(\alpha_n)}{1 - \tau'_n(\alpha_n)} \right] \right) \\
&= \exp \left( - \left[ \gamma + O_{\mathbb{P}} \left( \frac{1}{\sqrt{n(1 - \tau_n)}} \right) \right] \times O_{\mathbb{P}} \left( \frac{1}{\sqrt{n(1 - \tau_n)}} \right) \right) \\
&= 1 + O_{\mathbb{P}} \left( \frac{1}{\sqrt{n(1 - \tau_n)}} \right) \tag{B.57}
\end{aligned}$$

by a Taylor expansion. Furthermore

$$\left( \frac{1 - \tau'_n(\alpha_n)}{1 - \tau_n} \right)^{-\widehat{\gamma}} \widehat{\xi}_{\tau_n} = \widehat{\xi}_{\tau'_n(\alpha_n)}^*$$

by definition of the class of estimators  $\widehat{\xi}^*$ . From Proposition 3, we conclude that the conditions of Corollary 3 are satisfied if the parameter  $\tau'_n$  there is replaced by  $\tau'_n(\alpha_n)$ . By Corollary 3 then:

$$\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n(\alpha_n))]} \left( \frac{\widehat{\xi}_{\tau'_n(\alpha_n)}^*}{\xi_{\tau'_n(\alpha_n)}} - 1 \right) \xrightarrow{d} \Gamma.$$

Finally

$$\log \left[ \frac{1 - \tau_n}{1 - \tau'_n(\alpha_n)} \right] = \log \left[ \frac{1 - \tau_n}{1 - \alpha_n} \right] + \log \left[ \frac{1 - \alpha_n}{1 - \tau'_n(\alpha_n)} \right]$$

and the first term above tends to infinity, while the second term converges to a finite constant in view of Proposition 3. Consequently

$$\log \left[ \frac{1 - \tau_n}{1 - \tau'_n(\alpha_n)} \right] = \log \left[ \frac{1 - \tau_n}{1 - \alpha_n} \right] (1 + o(1)).$$

Together with the equality  $\xi_{\tau'_n(\alpha_n)} = q_{\alpha_n}$  which is true by definition of  $\tau'_n(\alpha_n)$ , this entails

$$\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \alpha_n)]} \left( \frac{\widehat{\xi}_{\tau'_n(\alpha_n)}^*}{q_{\alpha_n}} - 1 \right) \xrightarrow{d} \Gamma. \quad (\text{B.58})$$

Combining (B.56), (B.57) and (B.58) completes the proof of (i). The proof of (ii) is similar (just apply Corollary 4 instead of Corollary 3 when needed) and is therefore omitted. ■

**Proof of Theorem 7.** Again, we only show how to prove (i), the proof of (ii) being similar. Write

$$\begin{aligned} \widetilde{\text{XMES}}^*(\widehat{\tau}'_n(\alpha_n)) &= \left( \frac{1 - \widehat{\tau}'_n(\alpha_n)}{1 - \tau_n} \right)^{-\widehat{\gamma}} \widetilde{\text{XMES}}(\tau_n) \\ &= \left( \frac{1 - \widehat{\tau}'_n(\alpha_n)}{1 - \tau'_n(\alpha_n)} \right)^{-\widehat{\gamma}} \times \left\{ \left( \frac{1 - \tau'_n(\alpha_n)}{1 - \tau_n} \right)^{-\widehat{\gamma}} \widetilde{\text{XMES}}(\tau_n) \right\}. \end{aligned}$$

The first term is controlled by using (B.51), and the second one is handled by arguing just as in the proof of Theorem 6, with  $\widehat{\xi}$  replaced by  $\widetilde{\text{XMES}}$  throughout and by applying Theorem 4 instead of Corollary 3. We omit the details. ■

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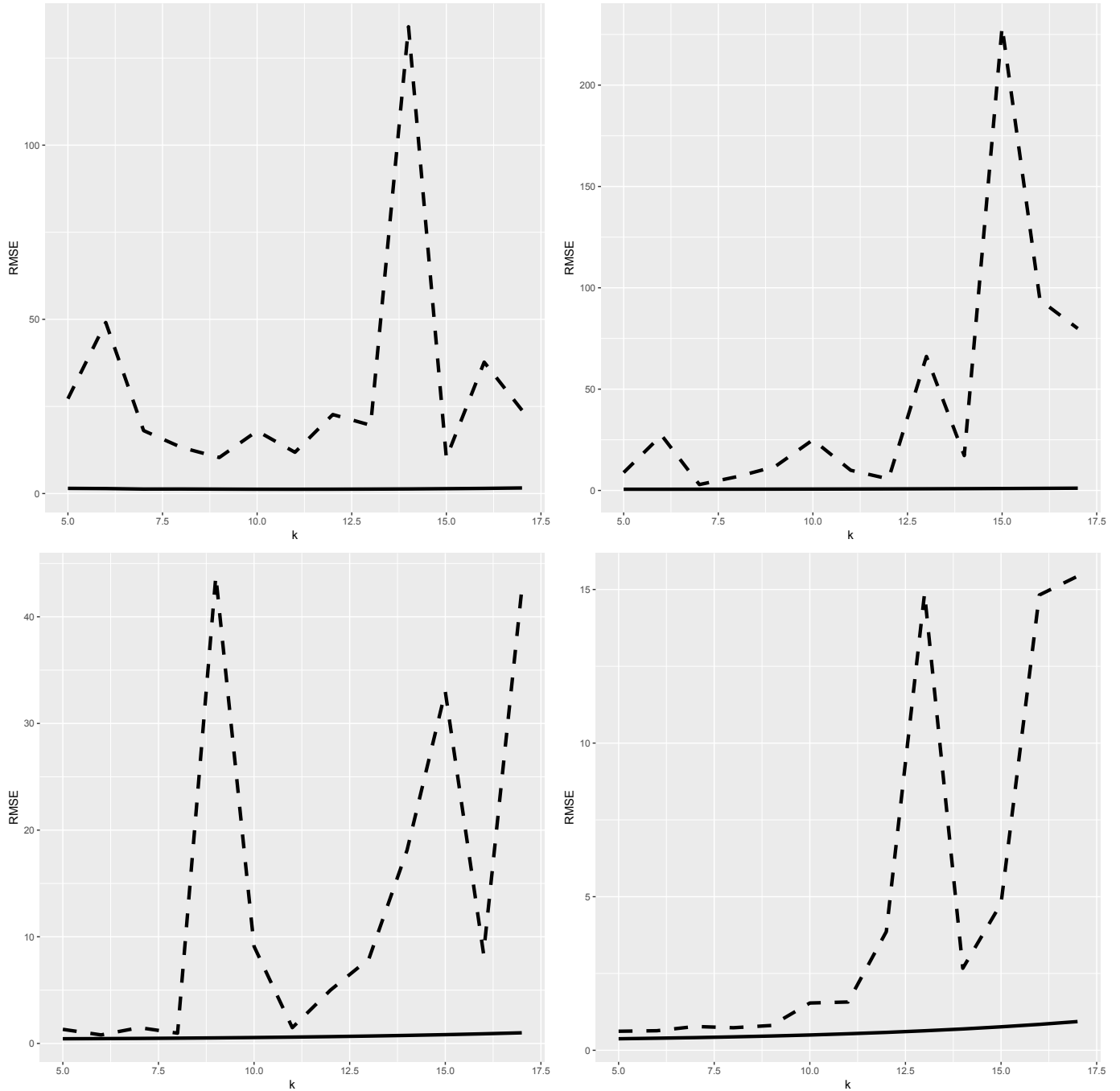


Figure 1: Root MSE estimates of  $\tilde{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (solid line) and  $\hat{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (dashed line), as functions of  $k$ , for the  $t_3$ ,  $t_5$ ,  $t_7$  and  $t_9$ -distributions, respectively, from top to bottom and from left to right. Results for the sample size  $n = 100$ .

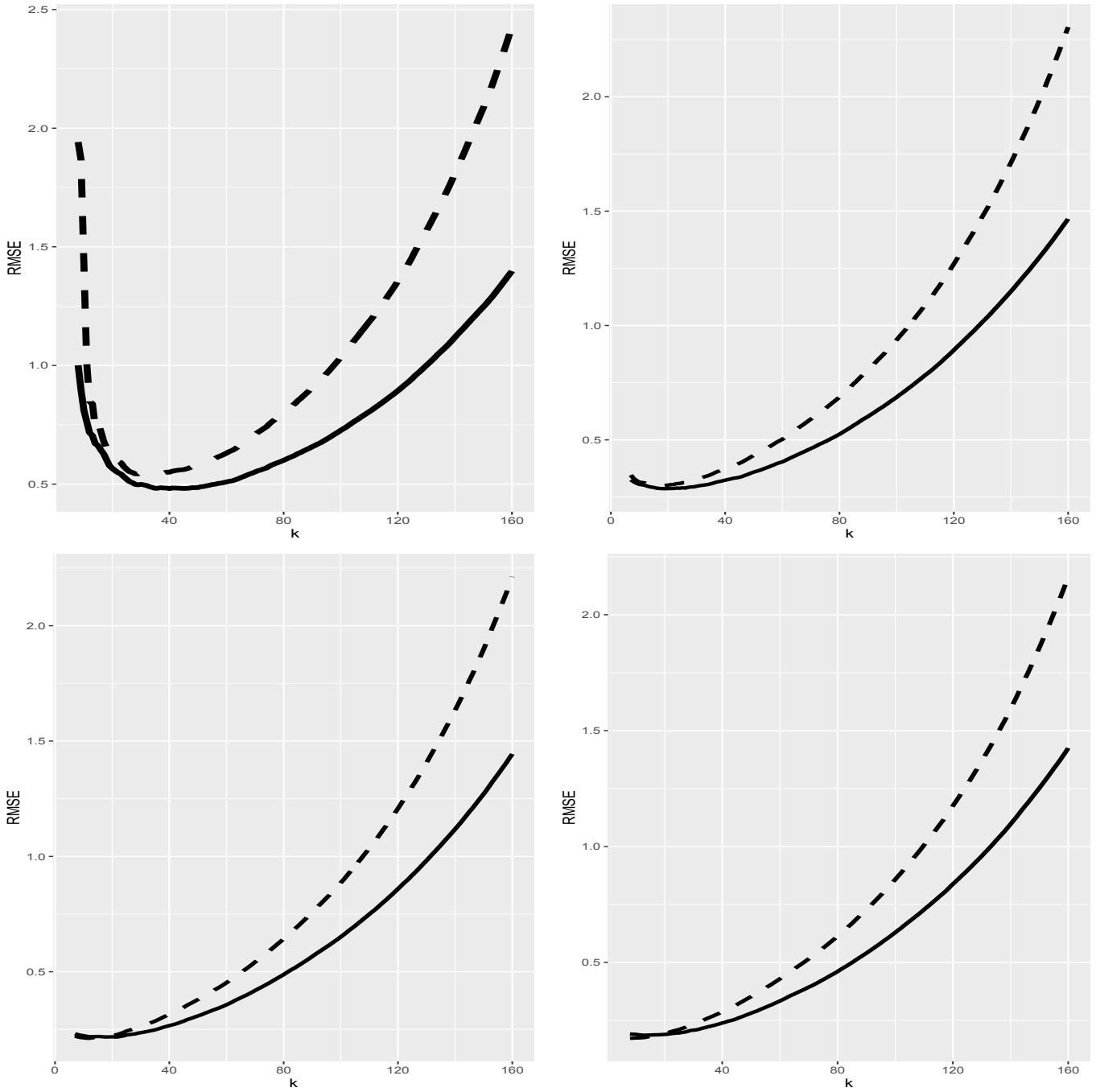


Figure 2: *As before*—Results for the sample size  $n = 1000$ .

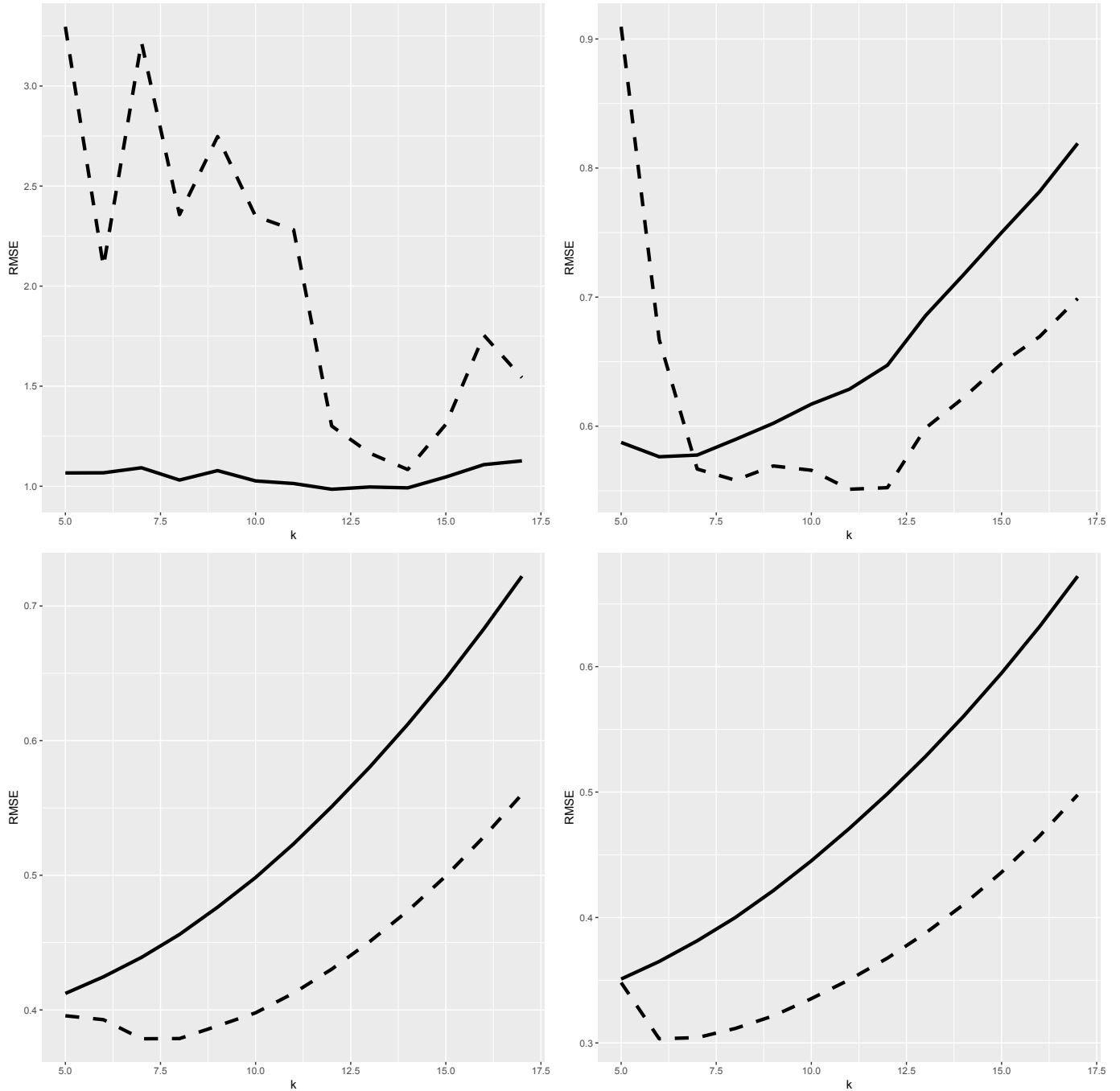


Figure 3: Root MSE estimates of  $\tilde{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (solid line) and  $\hat{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (dashed line), as functions of  $k$ , for the positive Student  $t_3$ ,  $t_5$ ,  $t_7$  and  $t_9$ -distributions, respectively, from top to bottom and from left to right. Results for the sample size  $n = 100$ .



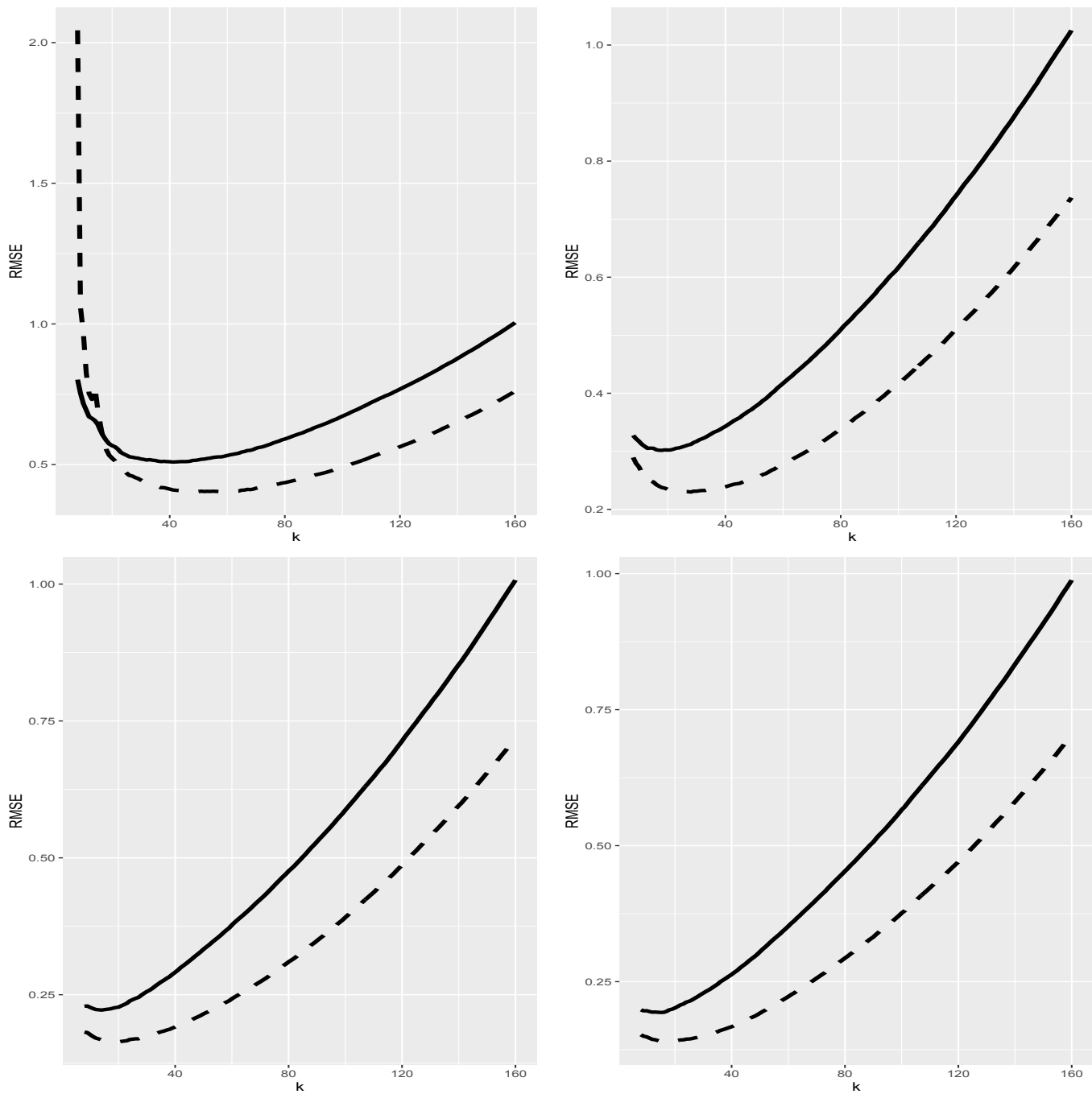


Figure 4: *As before*—Results for the sample size  $n = 1000$ .

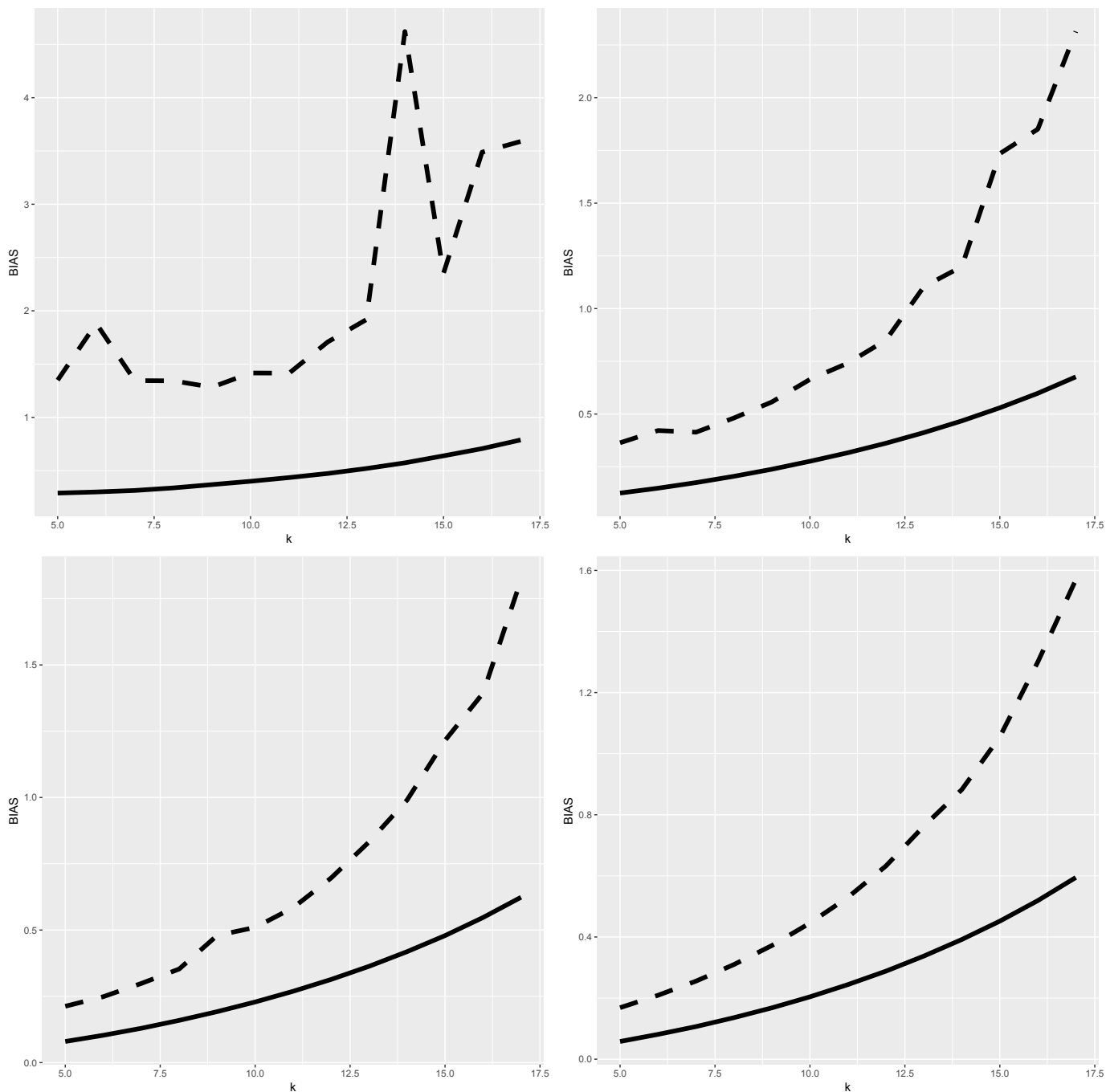


Figure 5: Bias estimates of  $\tilde{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (solid line) and  $\hat{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (dashed line), as functions of  $k$ , for the  $t_3$ ,  $t_5$ ,  $t_7$  and  $t_9$ -distributions, respectively, from top to bottom and from left to right. Results for the sample size  $n = 100$ .

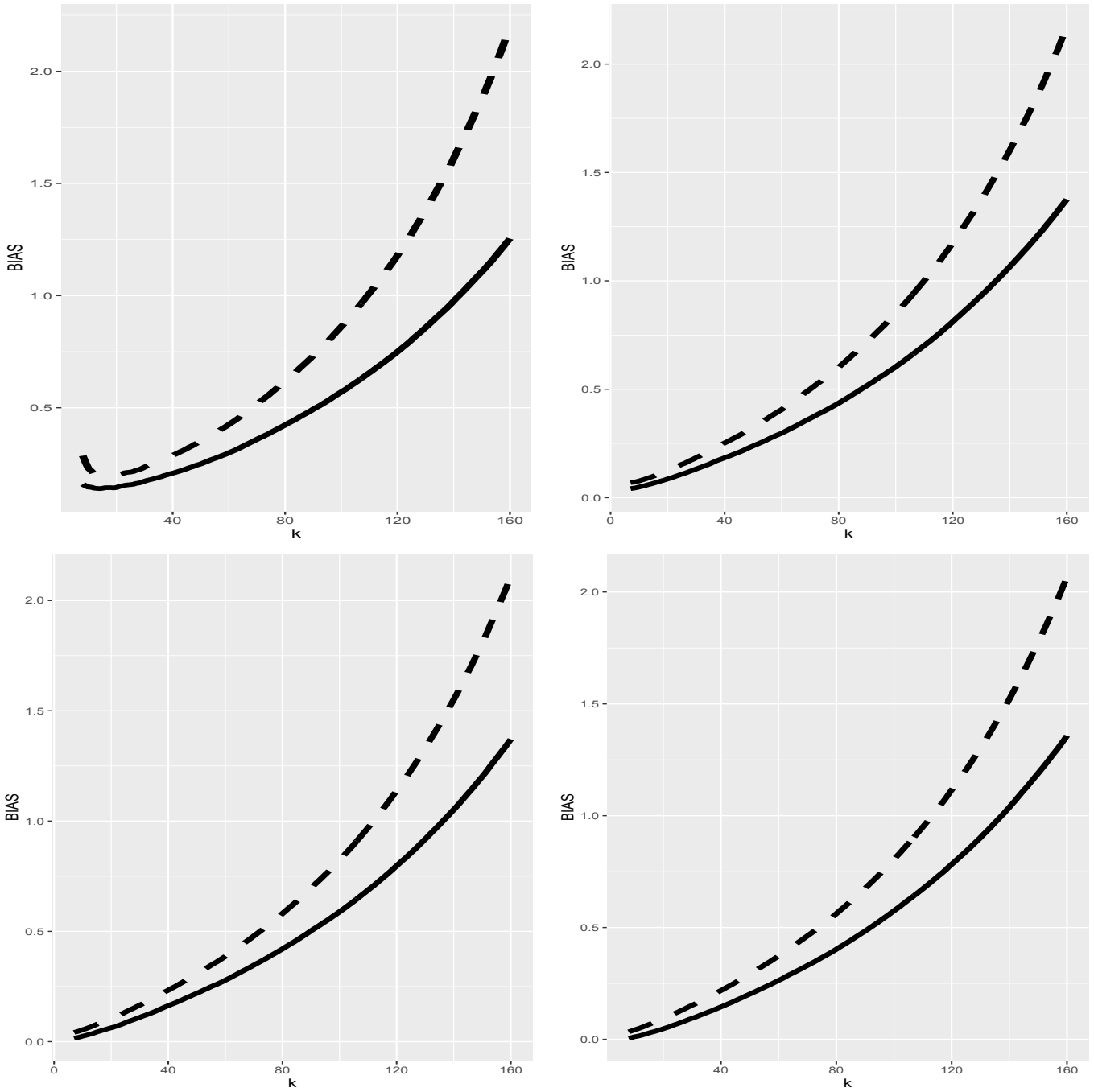


Figure 6: *As before*—Results for the sample size  $n = 1000$ .

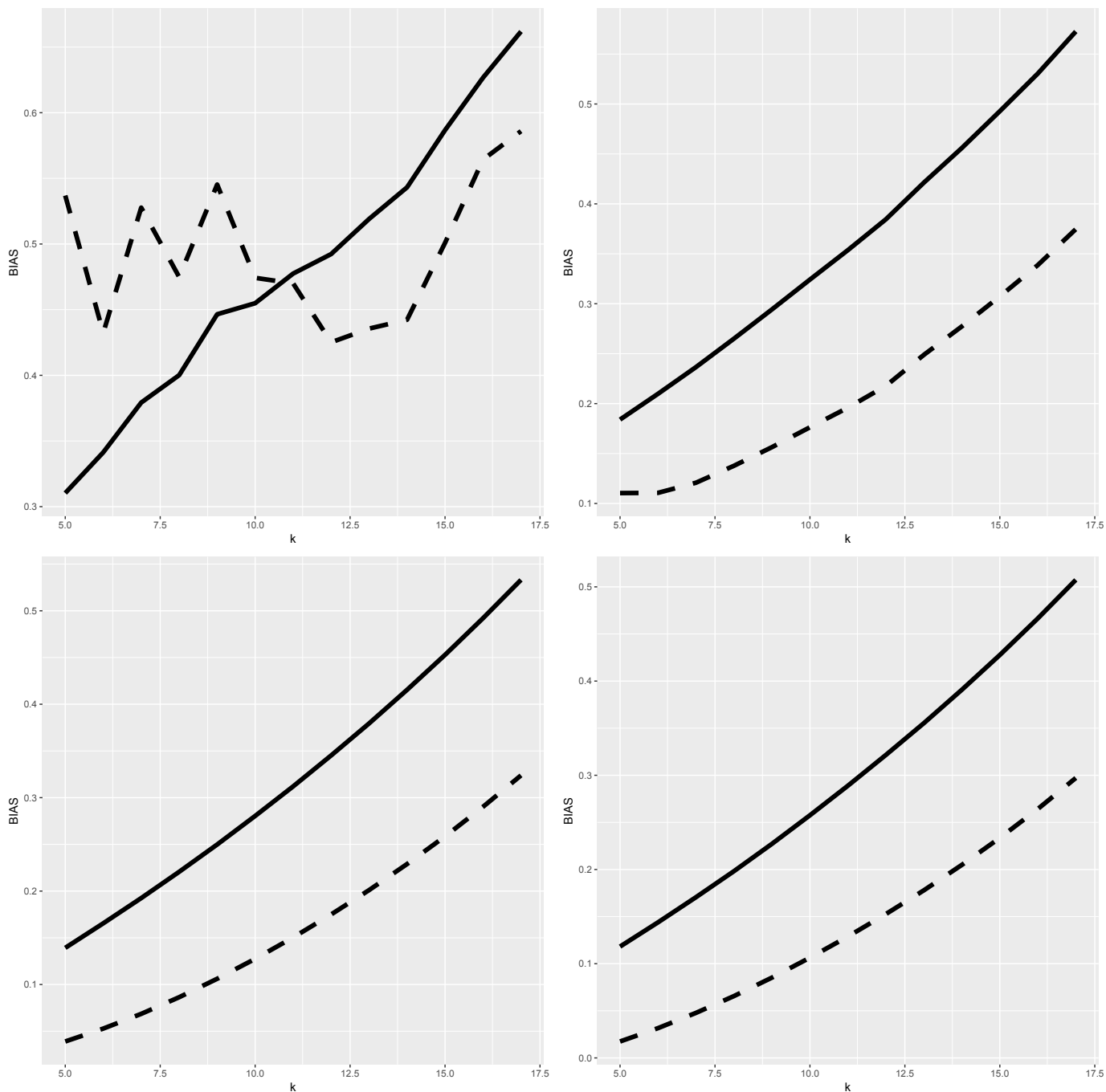


Figure 7: Bias estimates of  $\tilde{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (solid line) and  $\hat{\xi}_{\tau'_n}^*(k)/\xi_{\tau'_n}$  (dashed line), as functions of  $k$ , for the positive Student  $t_3$ ,  $t_5$ ,  $t_7$  and  $t_9$ -distributions, respectively, from top to bottom and from left to right. Results for the sample size  $n = 100$ .

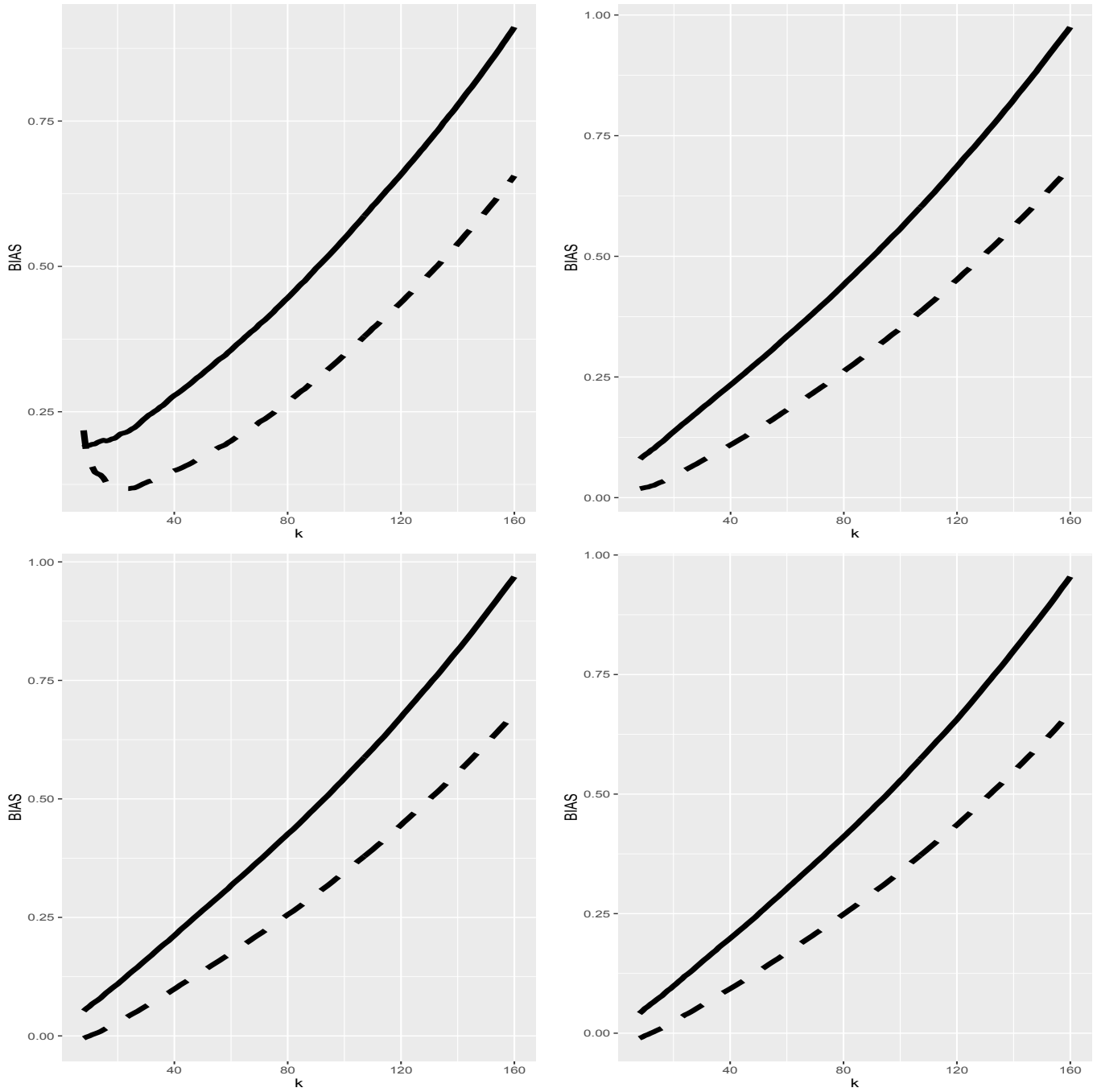


Figure 8: *As before*—Results for the sample size  $n = 1000$ .

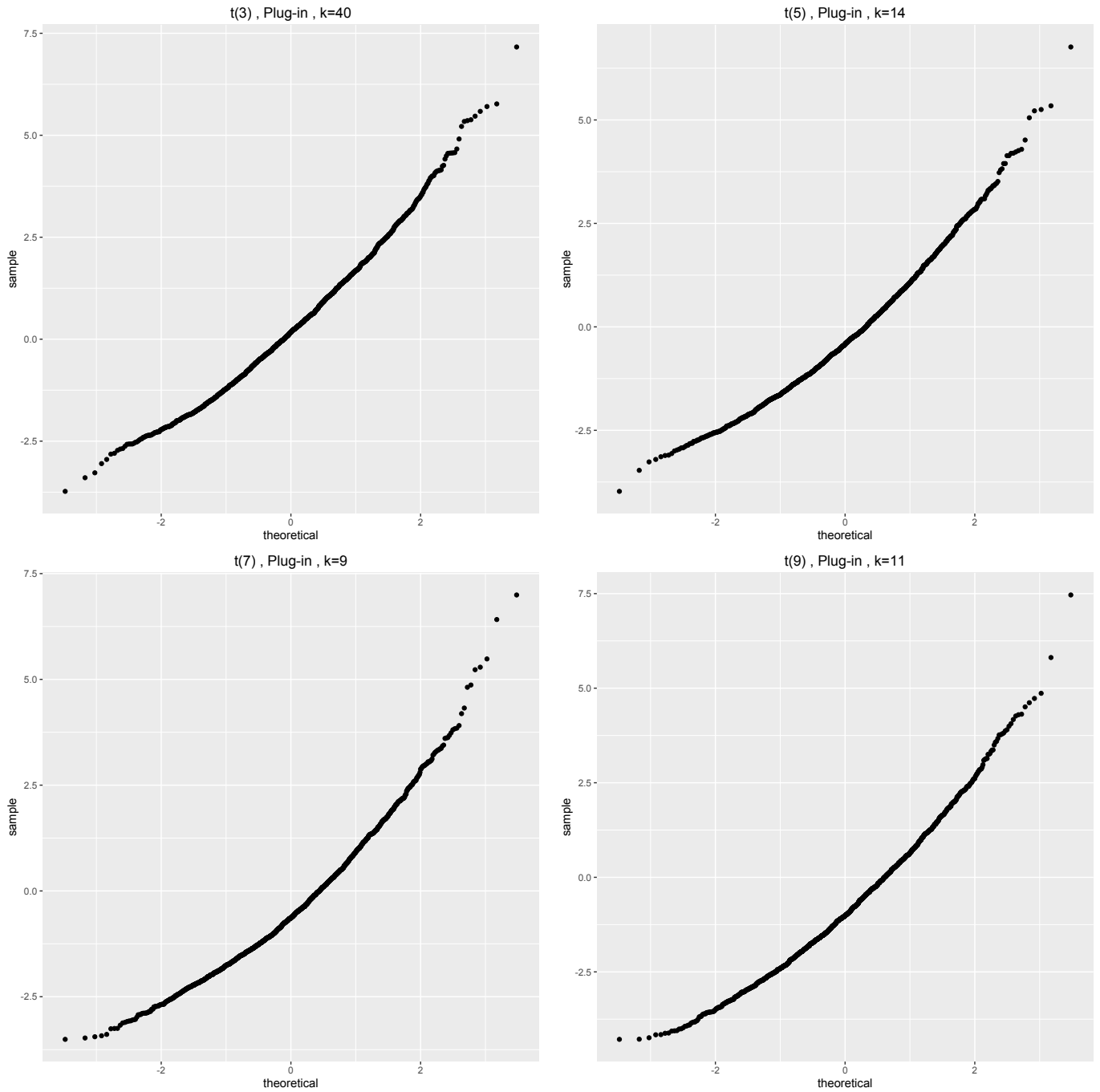


Figure 9:  $Q$ - $Q$ -plots on quality of asymptotic approximations. Each plot shows the sample quantiles of  $\widehat{W}_n$  versus the theoretical standard normal quantiles, based on 10,000 samples of size  $n = 1000$ . Data are simulated from the Student  $t_\nu$  with  $\nu = 3, 5, 7, 9$ , respectively, from top to bottom and from left to right.

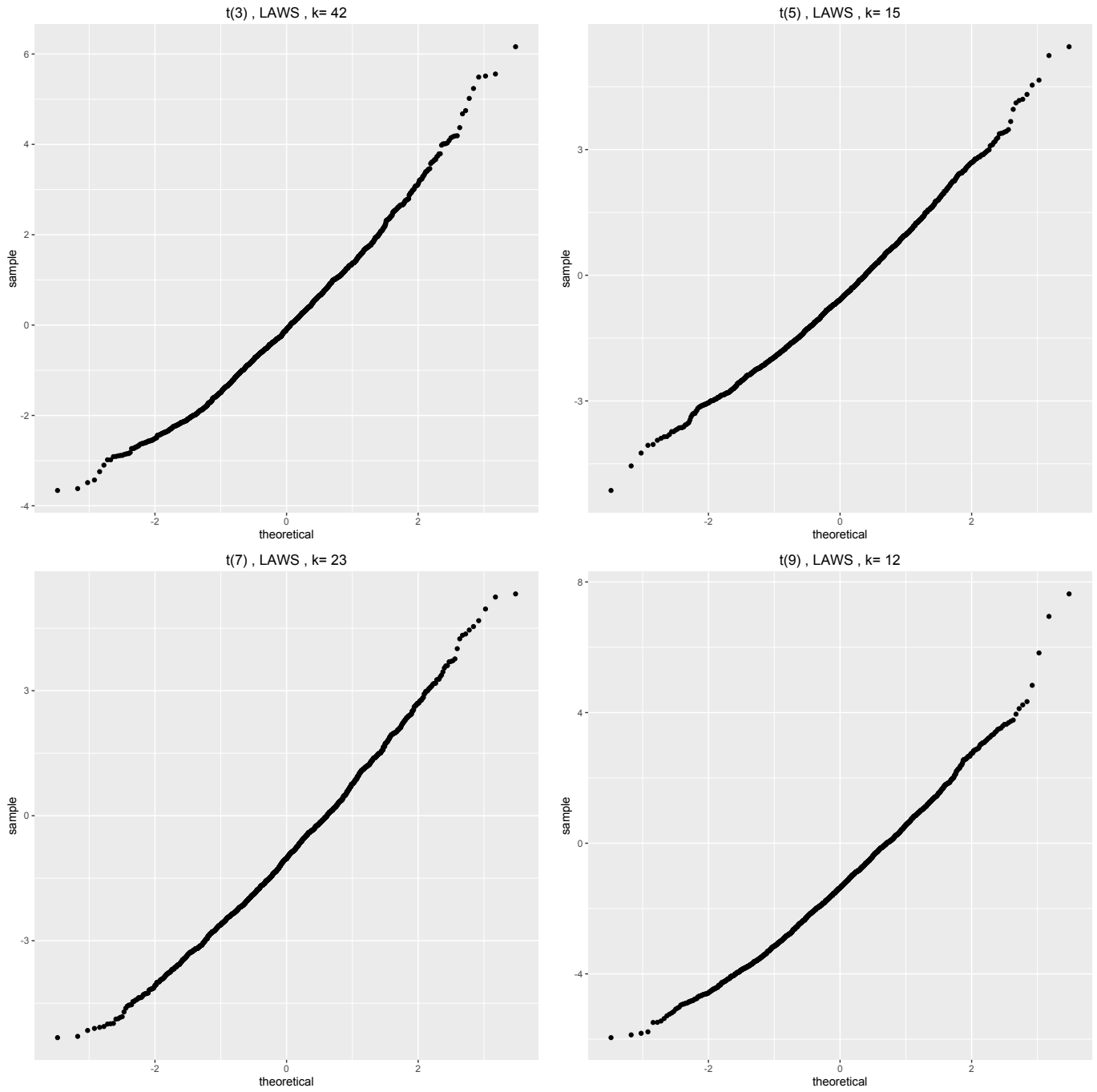


Figure 10: *As before—Scatters for  $\widetilde{W}_n$ .*

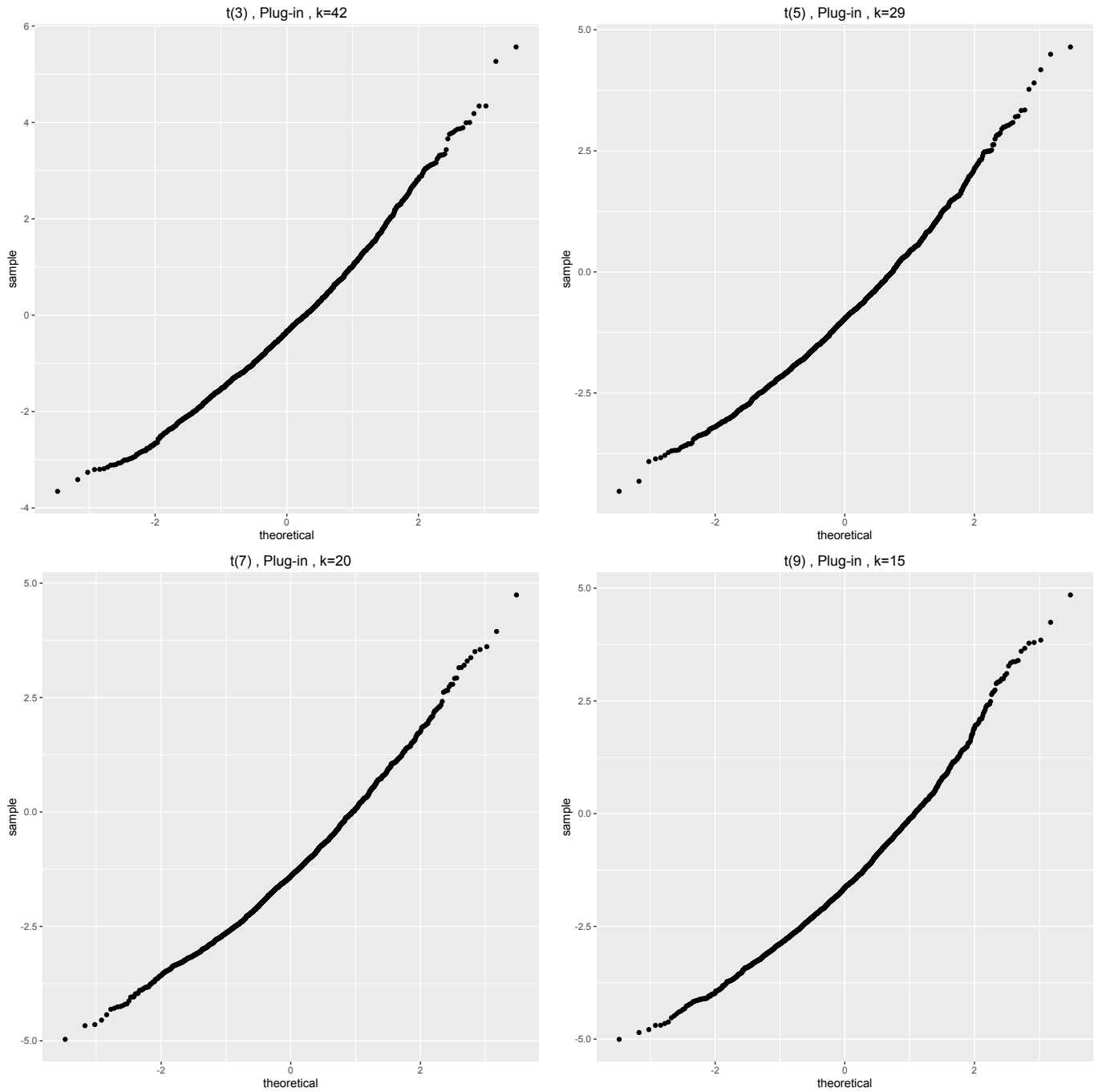


Figure 11:  $Q$ - $Q$ -plots on quality of asymptotic approximations. Each plot shows the sample quantiles of  $\widehat{W}_n$  versus the theoretical standard normal quantiles, based on 10,000 samples of size  $n = 1000$ . Data are simulated from the positive Student  $t_\nu$  with  $\nu = 3, 5, 7, 9$ , respectively, from top to bottom and from left to right.



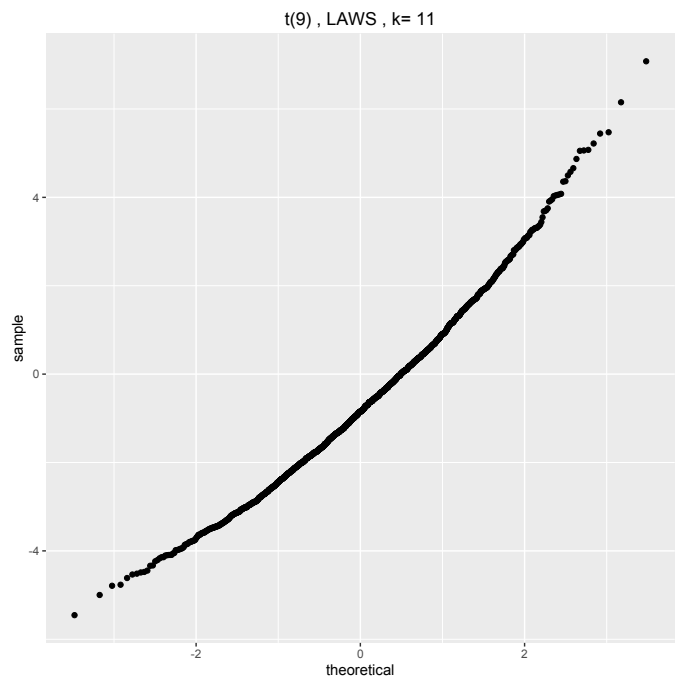
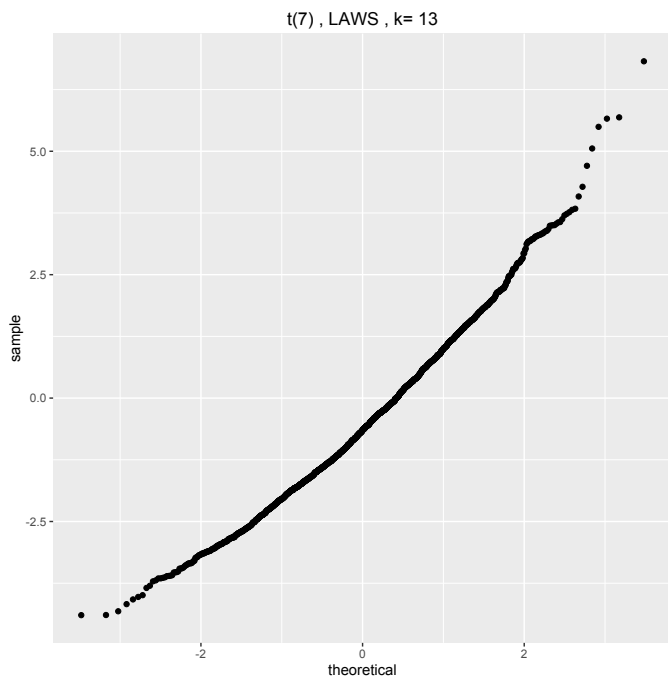
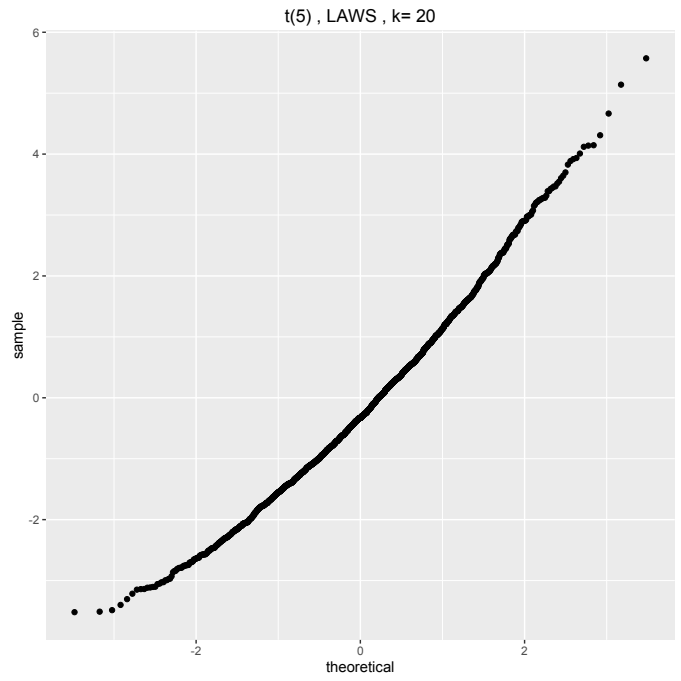
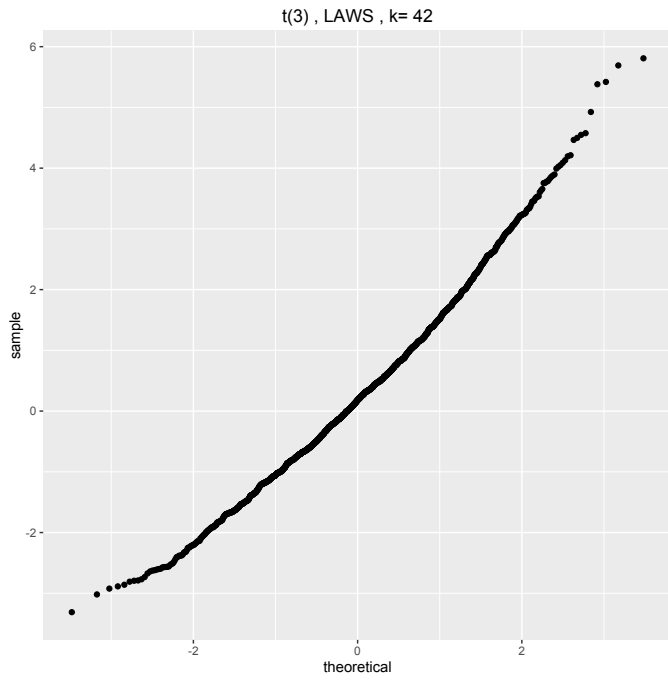


Figure 12: *As before—Scatters for  $\widetilde{W}_n$ .*

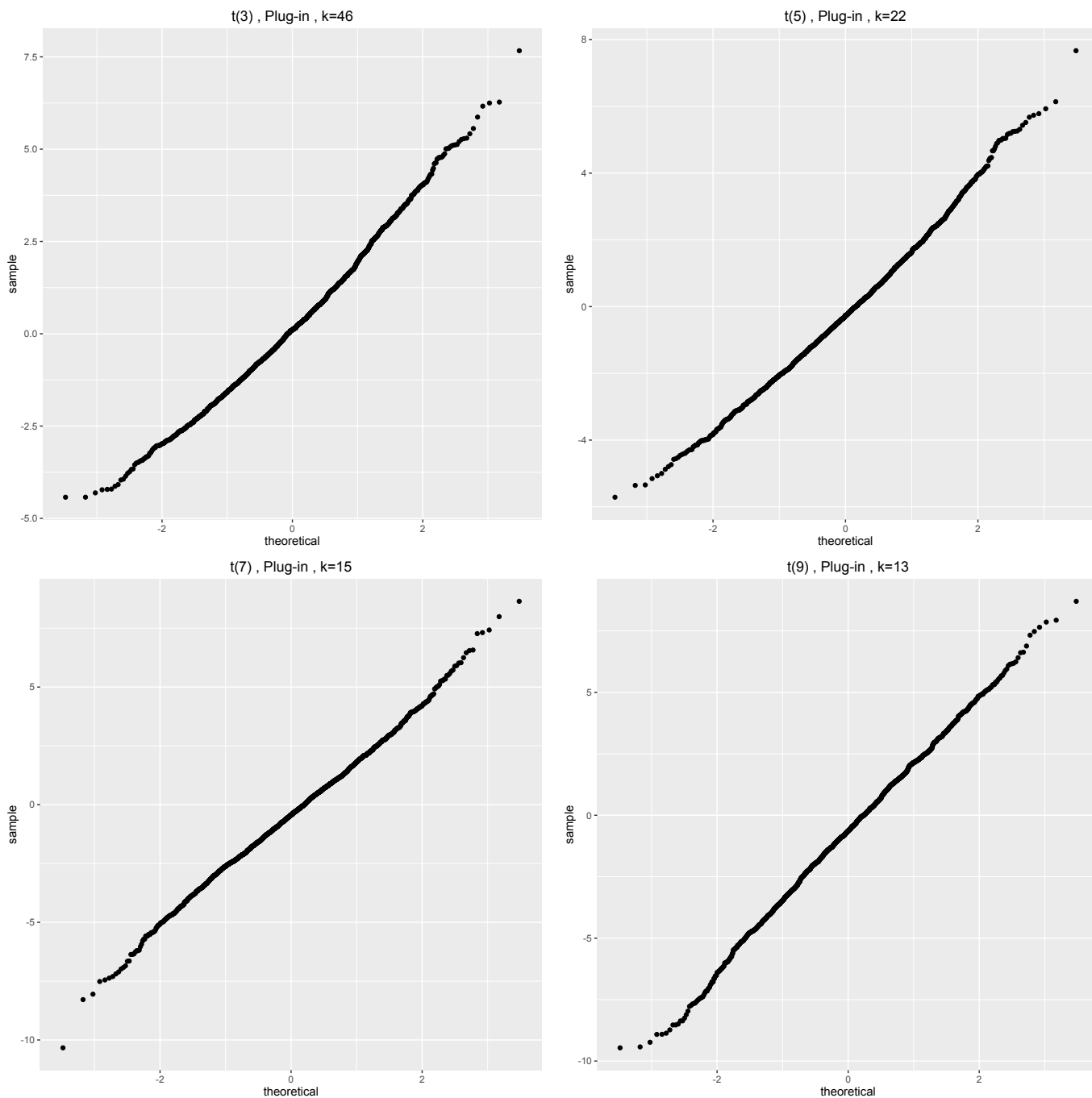


Figure 13:  $Q$ - $Q$ -plots of the sample quantiles of  $\widehat{W}_n$  versus the theoretical standard normal quantiles, based on 10,000 samples of size  $n = 1000$ . Data are simulated from the Student  $t_\nu$ -distribution on  $(0, \infty)^2$  with  $\nu = 3, 5, 7, 9$ , respectively, from top to bottom and from left to right.

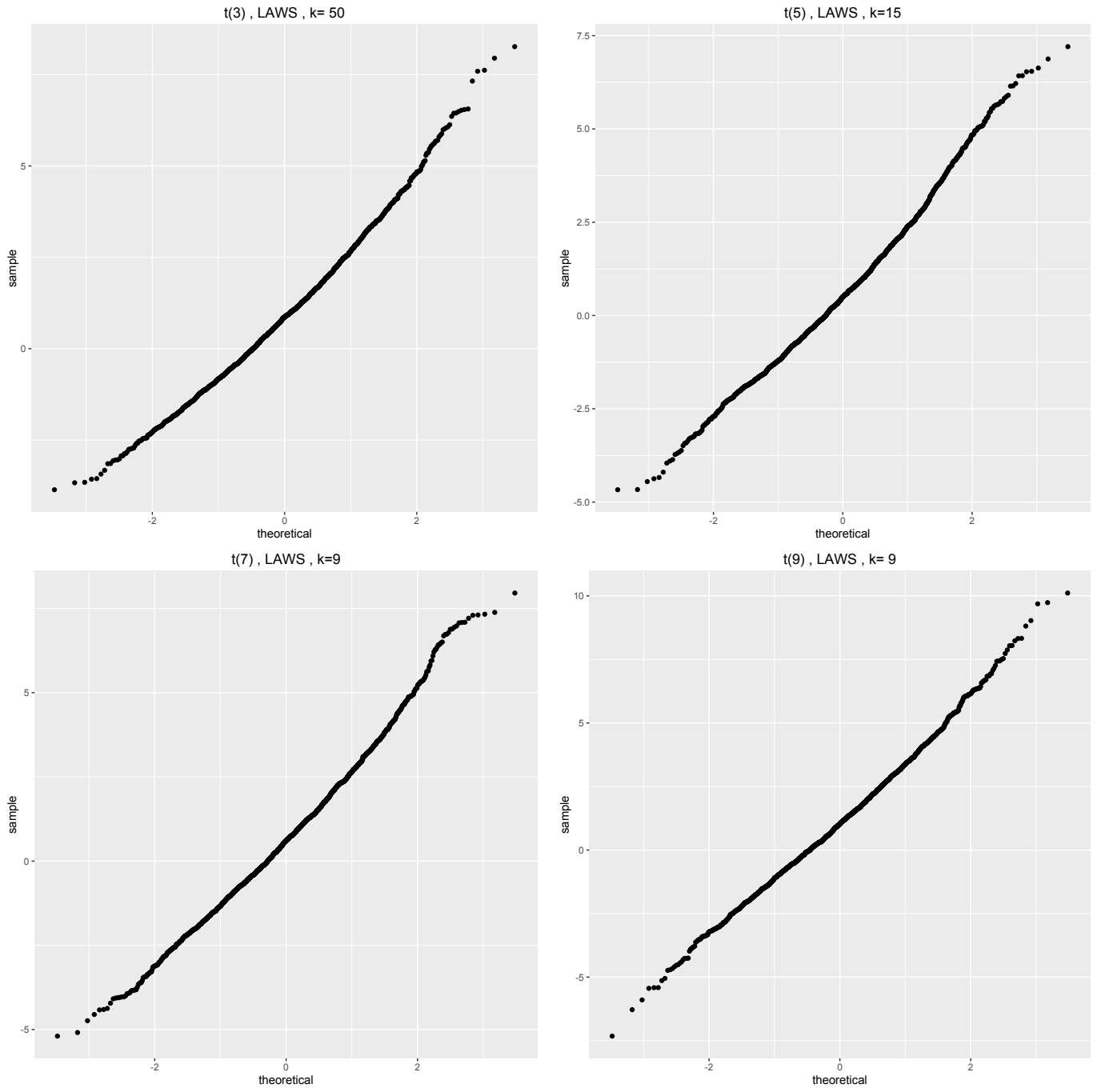


Figure 14: *As before—Scatters for  $\widetilde{W}_n$ .*

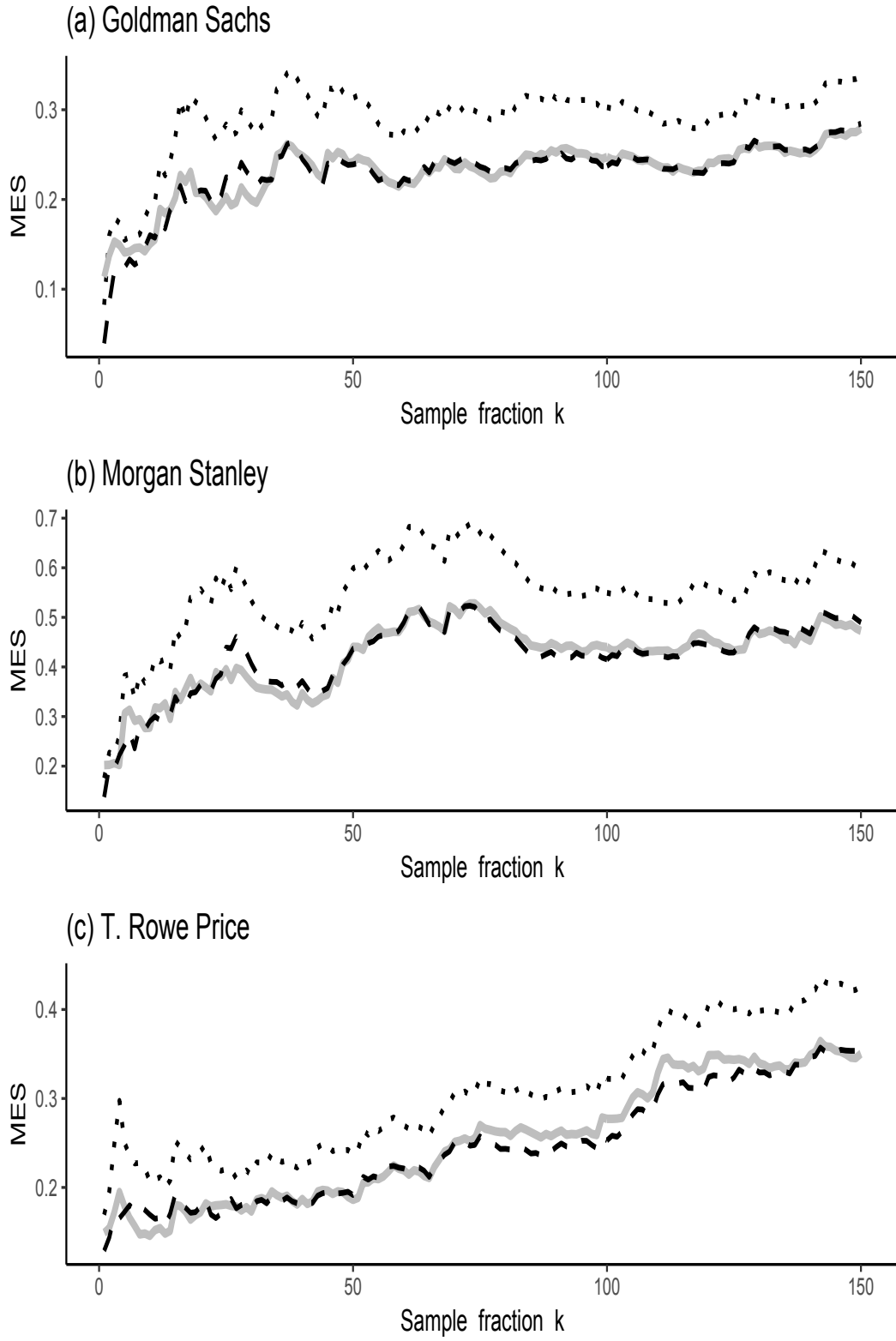


Figure 15: The estimates  $\widehat{XME S}^*$  (dashed),  $\widetilde{XME S}^*$  (solid) and  $\widehat{QME S}^*$  (dotted) for the three banks.

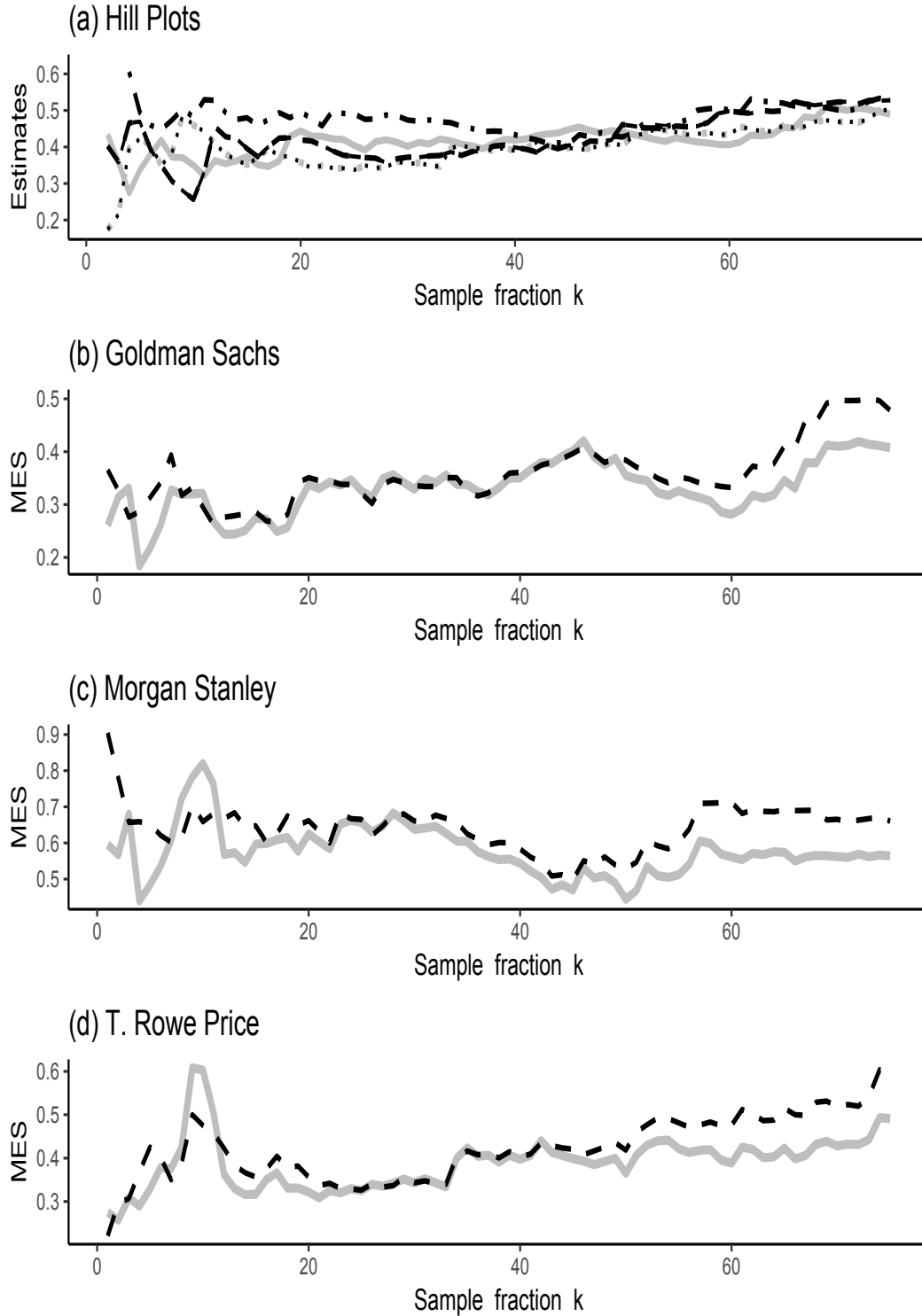


Figure 16: (a) Hill estimates  $\hat{\gamma}_Y$  based on weekly loss returns of market index (dashed), along with  $\hat{\gamma}_X$  based on weekly loss returns of three investment banks: Goldman Sachs (solid), Morgan Stanley (dashed-dotted), T. Rowe Price (dotted). (b)-(d) The estimates  $\widehat{QMES}^*(\alpha_n)$  in dashed line and  $\widehat{XMES}^*(\hat{\gamma}'_n(\alpha_n))$  in solid line for the three banks, with  $n = 522$  and  $\alpha_n = 1 - 1/n$ .