Extreme M-quantiles as risk measures: From L^1 to L^p optimization

Supplementary Material

Abdelaati Daouia[(1)], Stéphane Girard[(2)] & Gilles Stupfler[(3)]

[(1)] Toulouse School of Economics, University of Toulouse Capitole, France

[(2)] Team Mistis, Inria Grenoble Rhône-Alpes & LJK, Inovallée, 655, av. de l'Europe, Montbonnot, 38334 Saint-Ismier cedex, France

[(3)] School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom

Supplement A contains the proofs of all theoretical results in the main paper. Supplement B provides additional technical results and lemmas. Further simulation results are discussed in Supplement C. An application to medical insurance data is given in Supplement D.

A Main results and proofs

Proof of Proposition 1. The starting point is that $q_\tau(p)$ is a solution of the equation

$$(1-\tau)\mathbb{E}((q-X)^{p-1}\mathbb{1}_{\{X<q\}}) = \tau\mathbb{E}((X-q)^{p-1}\mathbb{1}_{\{X>q\}}), \tag{A.1}$$

which is equivalent to:

$$\begin{aligned} I_1(q;p) &= (1-\tau)I_2(q;p) \\ \text{with } I_1(q;p) &= \mathbb{E}\left(\left[\frac{X}{q}-1\right]^{p-1}\mathbb{1}_{\{X>q\}}\right) \text{ and } I_2(q;p) = \mathbb{E}\left(\left|\frac{X}{q}-1\right|^{p-1}\right). \end{aligned} \tag{A.2}$$

We now claim that $\tau\mapsto q_\tau(p)$ is an increasing function on $(0,1)$, tending to $+\infty$ as $\tau\uparrow 1$. If indeed $\tau\mapsto q_\tau(p)$ were not an increasing function, one could find $0<\tau_1<\tau_2<1$ with $q_{\tau_1}(p)\geqslant q_{\tau_2}(p)$. But then, since the maps $q\mapsto\mathbb{E}((q-X)^{p-1}\mathbb{1}_{\{X<q\}})$ and $q\mapsto\mathbb{E}((X-q)^{p-1}\mathbb{1}_{\{X>q\}})$ are respectively nondecreasing and nonincreasing, one would get thanks to (A.1) that:

$$\begin{aligned} (1-\tau_2)\mathbb{E}((q_{\tau_2}(p)-X)^{p-1}\mathbb{1}_{\{X<q_{\tau_2}(p)\}}) &< (1-\tau_1)\mathbb{E}((q_{\tau_1}(p)-X)^{p-1}\mathbb{1}_{\{X<q_{\tau_1}(p)\}}) \\ &= \tau_1\mathbb{E}((X-q_{\tau_1}(p))^{p-1}\mathbb{1}_{\{X>q_{\tau_1}(p)\}}) < \tau_2\mathbb{E}((X-q_{\tau_2}(p))^{p-1}\mathbb{1}_{\{X>q_{\tau_2}(p)\}}). \end{aligned}$$

This is certainly a contradiction because of (A.1) again. Now, if $q_\tau(p)$ did not tend to $+\infty$ as $\tau\uparrow 1$, then it would converge to some finite q^* due to the function $\tau\mapsto q_\tau(p)$ being increasing. The functions $q\mapsto\mathbb{E}((q-X)^{p-1}\mathbb{1}_{\{X<q\}})$ and $q\mapsto\mathbb{E}((X-q)^{p-1}\mathbb{1}_{\{X>q\}})$ being continuous on $\mathbb{R}$ by the dominated convergence theorem, this entails by letting

$\tau \uparrow 1$ in (A.1) with $q \equiv q_\tau(p)$ that $\mathbb{E}((X - q^*)^{p-1} \mathbb{1}_{\{X > q^*\}}) = 0$. Consequently $X \leqslant q^*$ with probability 1, which is a contradiction since X has a heavy right-tail and thus an infinite right endpoint.

The idea is then to compute asymptotic equivalents of both the expectations $I_1(q;p)$ and $I_2(q;p)$ as $q \to +\infty$ and then solve equation (A.2) by replacing these terms by the aforementioned equivalents with $q_\tau(p)$ substituted in place of q.

We start by computing an asymptotic equivalent of $I_1(q;p)$. Write

$$I_1(q;p) = \mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(1)\right] \mathbb{1}_{\{X>q\}}\right)$$

with $H(x) = (x-1)^{p-1} \mathbb{1}_{\{x \geqslant 1\}}$, and apply Lemma 1(i) with $b = 1$ to get

$$I_1(q;p) = \overline{F}(q) \int_1^{+\infty} (p-1)(x-1)^{p-2} x^{-1/\gamma} dx (1 + \mathrm{o}(1)).$$

An integration by parts and the change of variables $y = 1/x$ entail

$$\begin{aligned} I_1(q;p) &= \frac{\overline{F}(q)}{\gamma} \int_1^{+\infty} (x-1)^{p-1} x^{-1/\gamma - 1} dx (1 + \mathrm{o}(1)) \\ &= \frac{\overline{F}(q)}{\gamma} \int_0^1 (1-y)^{p-1} y^{1/\gamma - p} dy (1 + \mathrm{o}(1)) \\ &= \frac{\overline{F}(q)}{\gamma} B(p, \gamma^{-1} - p + 1)(1 + \mathrm{o}(1)) \end{aligned} \tag{A.3}$$

as $q \to +\infty$.

We now examine $I_2(q;p)$. Write

$$I_2(q;p) = I_1(q;p) + \mathbb{E}\left(\left|\frac{X}{q} - 1\right|^{p-1} \mathbb{1}_{\{X<-q\}}\right) + \mathbb{E}\left(\left|\frac{X}{q} - 1\right|^{p-1} \mathbb{1}_{\{|X| \leqslant q\}}\right). \tag{A.4}$$

By (A.3), the first term on the right-hand side above converges to 0 as $q \to +\infty$. The second one is controlled by writing

$$\mathbb{E}\left(\left|\frac{X}{q} - 1\right|^{p-1} \mathbb{1}_{\{X<-q\}}\right) \leqslant \left(\frac{2}{q}\right)^{p-1} \mathbb{E}\left(X_-^{p-1} \mathbb{1}_{\{X<-q\}}\right) = \mathrm{o}(q^{-(p-1)}) = \mathrm{o}(1) \tag{A.5}$$

as $q \to +\infty$, while the asymptotic behavior of the third term is obtained by noting that the integrand converges almost surely to 1 and is bounded by 2^{p-1} which, by the dominated convergence theorem, entails:

$$\mathbb{E}\left(\left|\frac{X}{q} - 1\right|^{p-1} \mathbb{1}_{\{|X| \leqslant q\}}\right) \to 1 \quad \text{as} \quad q \to +\infty. \tag{A.6}$$

Combining (A.3), (A.4), (A.5) and (A.6), we arrive at

$$I_2(q;p) \to 1 \quad \text{as} \quad q \to +\infty. \tag{A.7}$$

Using (A.3) and (A.7), equation (A.2) thus yields

$$1 - \tau = \overline{F}(q_\tau(p)) \frac{B(p, \gamma^{-1} - p + 1)}{\gamma} (1 + \mathrm{o}(1))$$

as $\tau \uparrow 1$, which is the desired result. ■

Proof of Proposition 2. As in the proof of Proposition 1, the starting point is the fact that $q_\tau(p)$ is the unique solution of equation (A.2). Let us provide an asymptotic expansion of both sides of this equation as $\tau \uparrow 1$.

The left-hand side of (A.2) is the easiest part: we use Lemma 1(ii) with $H(x) = (x-1)^{p-1}\mathbb{1}_{\{x\geqslant 1\}}$ and $b = 1$ to get, as $q \to +\infty$,

$$\frac{I_1(q;p)}{\overline{F}(q)} - \frac{B(p, \gamma_r^{-1} - p + 1)}{\gamma_r} = A\left(\frac{1}{\overline{F}(q)}\right) \int_1^{+\infty} (p-1)(x-1)^{p-2} x^{-1/\gamma_r} \frac{x^{\rho/\gamma_r} - 1}{\gamma_r \rho} dx(1 + \mathrm{o}(1)). \tag{A.8}$$

When $\rho < 0$, an integration by parts and the change of variables $y = 1/x$ entail

$$\begin{aligned} &\frac{I_1(q;p)}{\overline{F}(q)} - \frac{B(p, \gamma_r^{-1} - p + 1)}{\gamma_r} \\ = \;& A\left(\frac{1}{\overline{F}(q)}\right) \frac{1}{\gamma_r \rho} \left[\frac{1-\rho}{\gamma_r} B(p, (1-\rho)\gamma_r^{-1} - p + 1) - \frac{1}{\gamma_r} B(p, \gamma_r^{-1} - p + 1)\right] (1 + \mathrm{o}(1)) \end{aligned} \tag{A.9}$$

as $q \to +\infty$.

Let us now turn to the right-hand side of (A.2), which we break down as:

$$\mathbb{E}\left(\left|\frac{X}{q} - 1\right|^{p-1}\right) = I_1(q;p) + \mathbb{E}\left(\left[1 - \frac{X}{q}\right]^{p-1} \mathbb{1}_{\{X \leqslant q\}}\right). \tag{A.10}$$

An equivalent of $I_1(q;p)$ is already known by (A.3):

$$I_1(q;p) = \frac{B(p, \gamma_r^{-1} - p + 1)}{\gamma_r} \overline{F}(q)(1 + \mathrm{o}(1))$$

as $q \to +\infty$. The second term in (A.10) can be decomposed itself as follows:

$$\begin{aligned} \mathbb{E}\left(\left[1 - \frac{X}{q}\right]^{p-1} \mathbb{1}_{\{X \leqslant q\}}\right) &= \mathbb{E}\left(\left\{\left[1 - \frac{X}{q}\right]^{p-1} - 1\right\} \mathbb{1}_{\{X \leqslant q\}}\right) + F(q) \\ &= J_1(q;p) + J_2(q;p) + 1 - \overline{F}(q) \qquad \text{(A.11)} \\ \text{with } J_1(q;p) &= \mathbb{E}\left(\left\{\left[1 - \frac{X}{q}\right]^{p-1} - 1\right\} \mathbb{1}_{\{0 < X \leqslant q\}}\right) \\ \text{and } J_2(q;p) &= \mathbb{E}\left(\left\{\left[1 - \frac{X}{q}\right]^{p-1} - 1\right\} \mathbb{1}_{\{X < 0\}}\right). \end{aligned}$$

We start by examining the asymptotic behavior of $J_1(q;p)$. Let $H(x) = -(p-1)^{-1}(1-x)^{p-1}\mathbb{1}_{\{0 \leqslant x \leqslant 1\}}$ and apply Lemma 1(iii), (iv) and (v) to obtain:

$$\begin{aligned} J_1(q;p) &= -(p-1)\mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(0)\right] \mathbb{1}_{\{X>0\}}\right) \\ &= -(p-1) \begin{cases} \dfrac{\mathbb{E}(X\mathbb{1}_{\{X>0\}})}{q}(1 + \mathrm{o}(1)) & \text{if } \gamma_r < 1 \text{ or } \gamma_r = 1 \text{ and } \mathbb{E}(X_+) < \infty, \\ \dfrac{\mathbb{E}(X\mathbb{1}_{\{0<X<q\}})}{q}(1 + \mathrm{o}(1)) & \text{if } \gamma_r = 1 \text{ and } \mathbb{E}(X_+) = \infty, \\ \overline{F}(q) B(p-1, 1 - \gamma_r^{-1})(1 + \mathrm{o}(1)) & \text{if } \gamma_r > 1, \end{cases} \end{aligned} \tag{A.12}$$

as $q \to +\infty$. To control $J_2(q;p)$, notice first that

$$J_2(q;p) = \mathbb{E}\left(\left\{\left[1 - \frac{X}{q}\right]^{p-1} - 1\right\} \mathbb{1}_{\{X<0\}}\right) = \mathbb{E}\left(\left\{\left[1 + \frac{X_-}{q}\right]^{p-1} - 1\right\} \mathbb{1}_{\{X_->0\}}\right)$$

and apply Lemma 1(iii), (iv) and (v) with $H(x) = (p-1)^{-1}(1+x)^{p-1}$ to get

$$
\begin{aligned}
J_2(q;p) &= (p-1)\mathbb{E}\left(\left[H\left(\frac{X_-}{q}\right) - H(0)\right] \mathbb{1}_{\{X_->0\}}\right) \\
&= (p-1)\begin{cases}
-\dfrac{\mathbb{E}(X\mathbb{1}_{\{X<0\}})}{q}(1+\mathrm{o}(1)) & \text{if } \gamma_\ell < 1 \text{ or } \gamma_\ell = 1 \text{ and } \mathbb{E}(X_-) < \infty \text{ or } \overline{F}_- \text{ is light-tailed,} \\
-\dfrac{\mathbb{E}(X\mathbb{1}_{\{-q<X<0\}})}{q}(1+\mathrm{o}(1)) & \text{if } \gamma_\ell = 1 \text{ and } \mathbb{E}(X_-) = \infty, \\
F(-q)B(\gamma_\ell^{-1} - p + 1, 1 - \gamma_\ell^{-1})(1+\mathrm{o}(1)) & \text{if } \gamma_\ell > 1.
\end{cases}
\end{aligned}
\tag{A.13}
$$

This is obtained by noticing that, in the case $\gamma_\ell = 1$, we have $\mathbb{E}(X_- \mathbb{1}_{\{0<X_-<q\}}) = -\mathbb{E}(X\mathbb{1}_{\{-q<X<0\}})$, and in the case $\gamma_\ell > 1$, the change of variables $u = x/(1+x)$, or equivalently $x = u/(1-u)$, yields

$$
\int_0^{+\infty} (1+x)^{p-2} x^{-1/\gamma_\ell} dx = \int_0^1 (1-u)^{1/\gamma_\ell - p} u^{-1/\gamma_\ell} du = B(\gamma_\ell^{-1} - p + 1, 1 - \gamma_\ell^{-1}).
$$

Finally, notice that the regular variation property of A (see Theorem 2.3.3 in de Haan and Ferreira, 2006) and Proposition 1 entail

$$
A\left(\frac{1}{\overline{F}(q_\tau(p))}\right) = \left[\frac{B(p, \gamma_r^{-1} - p + 1)}{\gamma_r}\right]^\rho A\left(\frac{1}{1-\tau}\right)(1+\mathrm{o}(1)). \tag{A.14}
$$

Combining (A.2), (A.9)–(A.14) and replacing q by $q_\tau(p)$ shows that

$$
\begin{aligned}
&\overline{F}(q_\tau(p))\left(\frac{B(p, \gamma_r^{-1} - p + 1)}{\gamma_r} + A\left(\frac{1}{1-\tau}\right) K(p, \gamma_r, \rho)(1+\mathrm{o}(1))\right) \\
&= (1-\tau)\left(1 - \overline{F}(q_\tau(p)) - (p-1)[R_r(q_\tau(p), p, \gamma_r) - R_\ell(q_\tau(p), p, \gamma_\ell)]\right).
\end{aligned}
\tag{A.15}
$$

Using Corollary 1 and the regular variation of the functions $\overline{F}$ and $\overline{F}_-$ (when it is heavy-tailed), we get

$$
\begin{aligned}
R_r(q_\tau(p), p, \gamma_r) &= \begin{cases}
\dfrac{\mathbb{E}(X\mathbb{1}_{\{X>0\}})}{q_\tau(p)}(1+\mathrm{o}(1)) & \text{if } \gamma_r < 1 \text{ or } \gamma_r = 1 \text{ and } \mathbb{E}(X_+) < \infty, \\
\dfrac{\mathbb{E}(X\mathbb{1}_{\{0<X<q_\tau(p)\}})}{q_\tau(p)}(1+\mathrm{o}(1)) & \text{if } \gamma_r = 1 \text{ and } \mathbb{E}(X_+) = \infty, \\
\overline{F}(q_\tau(p))B(p-1, 1-\gamma_r^{-1})(1+\mathrm{o}(1)) & \text{if } \gamma_r > 1
\end{cases} \\
&= \begin{cases}
\left[\dfrac{\gamma_r}{B(p, \gamma_r^{-1} - p + 1)}\right]^{\gamma_r} \dfrac{\mathbb{E}(X\mathbb{1}_{\{X>0\}})}{q_\tau(1)}(1+\mathrm{o}(1)) & \text{if } \gamma_r < 1 \text{ or } \gamma_r = 1 \text{ and } \mathbb{E}(X_+) < \infty, \\
\left[\dfrac{\gamma_r}{B(p, \gamma_r^{-1} - p + 1)}\right]^{\gamma_r} \dfrac{\mathbb{E}(X\mathbb{1}_{\{0<X<q_\tau(1)\}})}{q_\tau(1)}(1+\mathrm{o}(1)) & \text{if } \gamma_r = 1 \text{ and } \mathbb{E}(X_+) = \infty, \\
\dfrac{\gamma_r}{B(p, \gamma_r^{-1} - p + 1)}\overline{F}(q_\tau(1))B(p-1, 1-\gamma_r^{-1})(1+\mathrm{o}(1)) & \text{if } \gamma_r > 1
\end{cases} \\
&\sim \left[\frac{\gamma_r}{B(p, \gamma_r^{-1} - p + 1)}\right]^{\min(\gamma_r, 1)} R_r(q_\tau(1), p, \gamma_r)
\end{aligned}
$$

and

$$
R_\ell(q_\tau(p),p,\gamma_\ell) = \begin{cases} -\dfrac{\mathbb{E}(X\mathbb{1}_{\{X<0\}})}{q_\tau(p)}(1+\mathrm{o}(1)) & \text{if } \gamma_\ell<1 \text{ or } \gamma_\ell=1 \text{ and } \mathbb{E}(X_-)<\infty \text{ or } \overline{F}_- \text{ is light-tailed,} \\ -\dfrac{\mathbb{E}(X\mathbb{1}_{\{-q_\tau(p)<X<0\}})}{q_\tau(p)}(1+\mathrm{o}(1)) & \text{if } \gamma_\ell=1 \text{ and } \mathbb{E}(X_-)=\infty, \\ F(-q_\tau(p))B(\gamma_\ell^{-1}-p+1,1-\gamma_\ell^{-1})(1+\mathrm{o}(1)) & \text{if } \gamma_\ell>1 \end{cases}
$$

$$
= \begin{cases} -\left[\dfrac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\gamma_r}\dfrac{\mathbb{E}(X\mathbb{1}_{\{X<0\}})}{q_\tau(1)}(1+\mathrm{o}(1)) & \text{if } \gamma_\ell<1 \text{ or } \gamma_\ell=1 \text{ and } \mathbb{E}(X_-)<\infty \text{ or } \overline{F}_- \text{ is light-tailed,} \\ -\left[\dfrac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\gamma_r}\dfrac{\mathbb{E}(X\mathbb{1}_{\{-q_\tau(1)<X<0\}})}{q_\tau(1)}(1+\mathrm{o}(1)) & \text{if } \gamma_\ell=1 \text{ and } \mathbb{E}(X_-)=\infty, \\ \left[\dfrac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\gamma_r/\gamma_\ell} \times F(-q_\tau(1))B(\gamma_\ell^{-1}-p+1,1-\gamma_\ell^{-1})(1+\mathrm{o}(1)) & \text{if } \gamma_\ell>1 \end{cases}
$$

$$
\sim \left[\frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\gamma_r/\max(\gamma_\ell,1)} R_\ell(q_\tau(1),p,\gamma_\ell).
$$

Consequently, by Proposition 1,

$$
\begin{aligned}
&\overline{F}(q_\tau(p))+(p-1)[R_r(q_\tau(p),p,\gamma_r)-R_\ell(q_\tau(p),p,\gamma_\ell)] \\
&= \frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}(1-\tau)(1+\mathrm{o}(1)) \\
&+(p-1)\left(\left[\frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\min(\gamma_r,1)}R_r(q_\tau(1),p,\gamma_r)-\left[\frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\gamma_r/\max(\gamma_\ell,1)}R_\ell(q_\tau(1),p,\gamma_\ell)\right).
\end{aligned}
$$

Rearranging equation (A.15) yields

$$
\begin{aligned}
\frac{\overline{F}(q_\tau(p))}{1-\tau} &= \frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\left(1+A\left(\frac{1}{1-\tau}\right)\frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}K(p,\gamma_r,\rho)(1+\mathrm{o}(1))\right)^{-1} \\
&\times \left[1-\frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}(1-\tau)(1+\mathrm{o}(1))\right. \\
&\quad -(p-1)\left(\left[\frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\min(\gamma_r,1)}R_r(q_\tau(1),p,\gamma_r)\right. \\
&\quad\quad \left.\left.-\left[\frac{\gamma_r}{B(p,\gamma_r^{-1}-p+1)}\right]^{\gamma_r/\max(\gamma_\ell,1)}R_\ell(q_\tau(1),p,\gamma_\ell)\right)\right].
\end{aligned}
$$

Using a straightforward Taylor expansion of the function $x\mapsto(1+x)^{-1}$ in a neighborhood of 0 completes the proof.

■

Proof of Proposition 3. By Proposition 2 and a Taylor expansion,

$$
\frac{1-\tau}{\overline{F}(q_\tau(p))} = \frac{B(p,\gamma_r^{-1}-p+1)}{\gamma_r}(1-R(\tau,p)(1+\mathrm{o}(1))).
$$

Because $U(1/(1-\tau))=q_\tau(1)$, the assertion is then a straightforward consequence of Lemma 2. ■

Proof of Theorem 1. Notice that $y \mapsto \eta_\tau(y;p)/p$ is continuously differentiable with derivative

$$\varphi_\tau(y;p) = |\tau - \mathbb{1}_{\{y\leqslant 0\}}||y|^{p-1}\operatorname{sign}(y).$$

Use Lemma 3 to write, for any u,

$$\begin{aligned}\psi_n(u;p) &= -uT_{1,n} + T_{2,n}(u) + T_{3,n}(u) \qquad (A.16)\\ \text{with } T_{1,n} &:= \frac{1}{\sqrt{n(1-\tau_n)}}\sum_{i=1}^n \frac{1}{[q_{\tau_n}(p)]^{p-1}}\varphi_{\tau_n}(X_i - q_{\tau_n}(p);p),\\ T_{2,n}(u) &:= -\frac{1}{[q_{\tau_n}(p)]^p}\sum_{i=1}^n \int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} [\mathbb{E}(\varphi_{\tau_n}(X_i - q_{\tau_n}(p) - t;p)) - \mathbb{E}(\varphi_{\tau_n}(X_i - q_{\tau_n}(p);p))]dt\\ \text{and } T_{3,n}(u) &:= -\frac{1}{[q_{\tau_n}(p)]^p}\sum_{i=1}^n \int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} [\mathcal{S}_{n,i}(q_{\tau_n}(p)+t) - \mathcal{S}_{n,i}(q_{\tau_n}(p))]dt\end{aligned}$$

where $\mathcal{S}_{n,i}(v) := \varphi_{\tau_n}(X_i - v;p) - \mathbb{E}(\varphi_{\tau_n}(X - v;p))$.

By Lemmas 8, 9 and 10, we get

$$\psi_n(u;p) \xrightarrow{d} -uZ\sqrt{V(\gamma;p)} + \frac{u^2}{2\gamma} \quad \text{as} \quad n \to \infty$$

(with Z being standard Gaussian) in the sense of finite-dimensional convergence. As a function of u, this limit is almost surely finite and defines a convex function which has a unique minimum at

$$u^* = \gamma\sqrt{V(\gamma;p)}Z \stackrel{d}{=} \mathcal{N}\left(0, \gamma^2 V(\gamma;p)\right).$$

Applying the convexity lemma of Geyer (1996) completes the proof. ■

Proof of Theorem 2. Write

$$\log\left(\frac{\widehat{q}^W_{\tau_n'}(p)}{q_{\tau_n'}(p)}\right) = (\widehat{\gamma}_n - \gamma)\log\left(\frac{1-\tau_n}{1-\tau_n'}\right) + \log\left(\frac{\widehat{q}_{\tau_n}(p)}{q_{\tau_n}(p)}\right) - \log\left(\left[\frac{1-\tau_n'}{1-\tau_n}\right]^\gamma \frac{q_{\tau_n'}(p)}{q_{\tau_n}(p)}\right).$$

The convergence $\log[(1-\tau_n)/(1-\tau_n')] \to \infty$ yields

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]}\log\left(\frac{\widehat{q}_{\tau_n}(p)}{q_{\tau_n}(p)}\right) = \mathrm{O}_{\mathbb{P}}\left(1/\log[(1-\tau_n)/(1-\tau_n')]\right) = \mathrm{o}_{\mathbb{P}}(1), \qquad (A.17)$$

$$\begin{aligned}\text{and } &\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]}\log\left(\left[\frac{1-\tau_n'}{1-\tau_n}\right]^\gamma \frac{q_{\tau_n'}(p)}{q_{\tau_n}(p)}\right)\\ &= \frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]}\left(\log\left(\frac{q_{\tau_n'}(p)}{q_{\tau_n'}(1)}\right) - \log\left(\frac{q_{\tau_n}(p)}{q_{\tau_n}(1)}\right) + \log\left(\left[\frac{1-\tau_n'}{1-\tau_n}\right]^\gamma \frac{q_{\tau_n'}(1)}{q_{\tau_n}(1)}\right)\right)\\ &= \mathrm{O}\left(\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]}[R(\tau_n,p) + |A((1-\tau_n)^{-1})| + R(\tau_n',p) + |A((1-\tau_n')^{-1})|]\right)\\ &= \mathrm{O}\left(\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]}[R(\tau_n,p) + |A((1-\tau_n)^{-1})|]\right)\\ &= \mathrm{o}(1). \qquad (A.18)\end{aligned}$$

Convergence (A.17) is a consequence of our Theorem 1. Convergence (A.18) follows from a combination of Proposition 3 and of Theorem 2.3.9 in de Haan and Ferreira (2006) and, in what concerns the relationship $R(\tau_n',p) = \mathrm{O}(R(\tau_n,p))$, from the regular variation of $\overline{F}$, $\overline{F}_-$, $s \mapsto U(s) = q_{1-s^{-1}}(1)$ and $|A|$. Combining these elements and using the Delta-method leads to the desired conclusion. ■

Proof of Theorem 3. We start by writing

$$\log\left(\frac{\widetilde{q}^{W}_{\tau'_n}(p)}{q_{\tau'_n}(p)}\right) = \log\left(\frac{\widehat{q}^{W}_{\tau'_n}(1)}{q_{\tau'_n}(1)}\right) + \log\left(\frac{C(\widehat{\gamma}_n;p)}{C(\gamma_r;p)}\right) - \log\left(\frac{q_{\tau'_n}(p)}{C(\gamma_r;p)q_{\tau'_n}(1)}\right). \tag{A.19}$$

To work on the first term on the right-hand side, note that

$$\log\left(\frac{\widehat{q}^{W}_{\tau'_n}(1)}{q_{\tau'_n}(1)}\right) = (\widehat{\gamma}_n - \gamma)\log\left(\frac{1-\tau_n}{1-\tau'_n}\right) + \log\left(\frac{\widehat{q}_{\tau_n}(1)}{q_{\tau_n}(1)}\right) - \log\left(\left[\frac{1-\tau'_n}{1-\tau_n}\right]^{\gamma}\frac{q_{\tau'_n}(1)}{q_{\tau_n}(1)}\right).$$

Since $\widehat{q}_{\tau_n}(1) = X_{n-\lfloor n(1-\tau_n)\rfloor,n}$, the convergence $\log[(1-\tau_n)/(1-\tau'_n)] \to \infty$ and a use of Theorem 2.3.9 of de Haan and Ferreira (2006) yield:

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}\log\left(\frac{\widehat{q}_{\tau_n}(1)}{q_{\tau_n}(1)}\right) = \mathrm{O}_{\mathbb{P}}\left(1/\log[(1-\tau_n)/(1-\tau'_n)]\right) = \mathrm{o}_{\mathbb{P}}(1),$$

$$\text{and}\quad \frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}\log\left(\left[\frac{1-\tau'_n}{1-\tau_n}\right]^{\gamma}\frac{q_{\tau'_n}(1)}{q_{\tau_n}(1)}\right) = \mathrm{O}\left(\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}|A((1-\tau_n)^{-1})|\right) = \mathrm{o}(1).$$

As a consequence:

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}\log\left(\frac{\widehat{q}^{W}_{\tau'_n}(1)}{q_{\tau'_n}(1)}\right) \xrightarrow{d} \zeta. \tag{A.20}$$

To conclude the proof, it is then enough to examine the behavior of the second and third term on the right-hand side of Equation (A.19). First,

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}\log\left(\frac{C(\widehat{\gamma}_n;p)}{C(\gamma_r;p)}\right) = \mathrm{O}_{\mathbb{P}}\left(1/\log[(1-\tau_n)/(1-\tau'_n)]\right) = \mathrm{o}_{\mathbb{P}}(1), \tag{A.21}$$

because of the $\sqrt{n(1-\tau_n)}-$convergence of $\widehat{\gamma}_n$ and of the differentiability of the mapping $x \mapsto \log C(x;p)$ at γ_r. Second,

$$\begin{aligned}
\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}\log\left(\frac{q_{\tau'_n}(p)}{C(\gamma_r;p)q_{\tau'_n}(1)}\right) &= \mathrm{O}_{\mathbb{P}}\left(\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}[R(\tau'_n,p) + |A((1-\tau'_n)^{-1})|]\right) \\
&= \mathrm{O}_{\mathbb{P}}\left(\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}[R(\tau_n,p) + |A((1-\tau_n)^{-1})|]\right) \\
&= \mathrm{o}_{\mathbb{P}}(1),
\end{aligned} \tag{A.22}$$

which follows from a combination of Proposition 3 and of Theorem 2.3.9 in de Haan and Ferreira (2006) and, in what concerns the relationship $R(\tau'_n,p) = \mathrm{O}(R(\tau_n,p))$, from the regular variation of $\overline{F}$, $\overline{F}_-$, $s \mapsto U(s) = q_{1-s^{-1}}(1)$ and $|A|$. Combining these elements and using the Delta-method leads to the desired conclusion. ■

Proof of Theorem 4. We write

$$\frac{1-\widehat{\tau}'_n(p,\alpha_n;1)}{1-\tau'_n(p,\alpha_n;1)} - 1 = \frac{\gamma_r}{\widehat{\gamma}_n} \times \frac{B\left(p, \frac{1}{\widehat{\gamma}_n} - p + 1\right)}{B\left(p, \frac{1}{\gamma_r} - p + 1\right)} \times \frac{(1-\alpha_n)\frac{1}{\gamma_r}B\left(p, \frac{1}{\gamma_r} - p + 1\right)}{1-\tau'_n(p,\alpha_n;1)} - 1. \tag{A.23}$$

Now

$$\sqrt{n(1-\tau_n)}\left(\frac{\gamma_r}{\widehat{\gamma}_n} - 1\right) = \frac{1}{\widehat{\gamma}_n} \times \sqrt{n(1-\tau_n)}(\gamma_r - \widehat{\gamma}_n) \xrightarrow{d} -\frac{\zeta}{\gamma_r} \tag{A.24}$$

by Slutsky's lemma. Moreover, using the relationship

$$\frac{\partial B}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\right) = B(x,y)\left(\Psi(y) - \Psi(x+y)\right)$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, we obtain

$$\frac{d}{dx}\left[B\left(p, \frac{1}{x} - p + 1\right)\right] = -\frac{1}{x^2} B\left(p, \frac{1}{x} - p + 1\right)\left[\Psi\left(\frac{1}{x} - p + 1\right) - \Psi\left(\frac{1}{x} + 1\right)\right].$$

The delta-method then yields

$$\begin{aligned}
&\sqrt{n(1-\tau_n)}\left(\frac{B\left(p, \dfrac{1}{\widehat{\gamma}_n} - p + 1\right)}{B\left(p, \dfrac{1}{\gamma_r} - p + 1\right)} - 1\right) \\
&= \frac{1}{B\left(p, \dfrac{1}{\gamma_r} - p + 1\right)} \times \sqrt{n(1-\tau_n)}\left[B\left(p, \frac{1}{\widehat{\gamma}_n} - p + 1\right) - B\left(p, \frac{1}{\gamma_r} - p + 1\right)\right] \\
&\xrightarrow{d} -\frac{\zeta}{\gamma_r^2}\left[\Psi\left(\frac{1}{\gamma_r} - p + 1\right) - \Psi\left(\frac{1}{\gamma_r} + 1\right)\right]. \qquad \text{(A.25)}
\end{aligned}$$

To complete the proof, we note that

$$\frac{(1-\alpha_n)\dfrac{1}{\gamma_r} B\left(p, \dfrac{1}{\gamma_r} - p + 1\right)}{1 - \tau_n'(p, \alpha_n; 1)} = \frac{(1-\alpha_n)\dfrac{1}{\gamma_r} B\left(p, \dfrac{1}{\gamma_r} - p + 1\right)}{\mathbb{E}\left[\left|\dfrac{X}{q_{\alpha_n}(1)} - 1\right|^{p-1} \mathbb{1}_{\{X > q_{\alpha_n}(1)\}}\right]} \mathbb{E}\left[\left|\frac{X}{q_{\alpha_n}(1)} - 1\right|^{p-1}\right].$$

Recall now (A.8) in the proof of Proposition 2 which here translates into

$$\begin{aligned}
\frac{(1-\alpha_n)\dfrac{1}{\gamma_r} B\left(p, \dfrac{1}{\gamma_r} - p + 1\right)}{\mathbb{E}\left[\left|\dfrac{X}{q_{\alpha_n}(1)} - 1\right|^{p-1} \mathbb{1}_{\{X > q_{\alpha_n}(1)\}}\right]} - 1 &= \frac{\overline{F}(q_{\alpha_n}(1))\dfrac{1}{\gamma_r} B\left(p, \dfrac{1}{\gamma_r} - p + 1\right)}{\mathbb{E}\left[\left|\dfrac{X}{q_{\alpha_n}(1)} - 1\right|^{p-1} \mathbb{1}_{\{X > q_{\alpha_n}(1)\}}\right]} - 1 \\
&= \mathrm{O}[A(1/\overline{F}(q_{\alpha_n}(1)))] \\
&= \mathrm{O}[A((1-\alpha_n)^{-1})].
\end{aligned}$$

Similarly, by (A.10)–(A.13) in the proof of Proposition 2, we get

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{X}{q_{\alpha_n}(1)} - 1\right|^{p-1}\right] - 1 &= \mathrm{O}[\max\{\overline{F}(q_{\alpha_n}(1)), R_r(q_{\alpha_n}(1), p, \gamma_r), R_\ell(q_{\alpha_n}(1), p, \gamma_\ell)\}] \\
&= \mathrm{O}[\max\{1-\alpha_n, R_r(q_{\alpha_n}(1), p, \gamma_r), R_\ell(q_{\alpha_n}(1), p, \gamma_\ell)\}].
\end{aligned}$$

Combine these two asymptotic bounds to obtain

$$\frac{(1-\alpha_n)\dfrac{1}{\gamma_r} B\left(p, \dfrac{1}{\gamma_r} - p + 1\right)}{1 - \tau_n'(p, \alpha_n; 1)} - 1 = \mathrm{O}[\max\{1-\alpha_n, A((1-\alpha_n)^{-1}), R_r(q_{\alpha_n}(1), p, \gamma_r), R_\ell(q_{\alpha_n}(1), p, \gamma_\ell)\}]. \quad \text{(A.26)}$$

Combining (A.23), (A.24), (A.25) and (A.26) leads to

$$\sqrt{n(1-\tau_n)}\left(\frac{1 - \widehat{\tau}_n'(p, \alpha_n; 1)}{1 - \tau_n'(p, \alpha_n; 1)} - 1\right) = -\left\{1 + \frac{1}{\gamma_r}\left[\Psi\left(\frac{1}{\gamma_r} - p + 1\right) - \Psi\left(\frac{1}{\gamma_r} + 1\right)\right]\right\}\frac{\zeta}{\gamma_r} + \mathrm{O}(1) = \mathrm{O}_{\mathbb{P}}(1)$$

proving the first statement. In the case when

$$\sqrt{n(1-\tau_n)}\max\{1-\alpha_n, A((1-\alpha_n)^{-1}), R_r(q_{\alpha_n}(1),p,\gamma_r), R_\ell(q_{\alpha_n}(1),p,\gamma_\ell)\} \to 0$$

the above equality becomes

$$\sqrt{n(1-\tau_n)}\left(\frac{1-\widehat{\tau}_n'(p,\alpha_n;1)}{1-\tau_n'(p,\alpha_n;1)}-1\right) = -\left\{1+\frac{1}{\gamma_r}\left[\Psi\left(\frac{1}{\gamma_r}-p+1\right)-\Psi\left(\frac{1}{\gamma_r}+1\right)\right]\right\}\frac{\zeta}{\gamma_r}+\mathrm{o}(1)$$

which implies the second statement and concludes the proof. ∎

Proof of Theorem 5. The key point is to write

$$\widehat{q}^W_{\widehat{\tau}_n'(p,\alpha_n;1)}(p) = \left(\frac{1-\widehat{\tau}_n'(p,\alpha_n;1)}{1-\tau_n}\right)^{-\widehat{\gamma}_n}\widehat{q}_{\tau_n}(p) = \left(\frac{1-\widehat{\tau}_n'(p,\alpha_n;1)}{1-\tau_n'(p,\alpha_n;1)}\right)^{-\widehat{\gamma}_n}\times\left\{\left(\frac{1-\tau_n'(p,\alpha_n;1)}{1-\tau_n}\right)^{-\widehat{\gamma}_n}\widehat{q}_{\tau_n}(p)\right\}. \quad (A.27)$$

Now, by Theorem 4,

$$\frac{1-\widehat{\tau}_n'(p,\alpha_n;1)}{1-\tau_n'(p,\alpha_n;1)} = 1+\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right)$$

and therefore

$$\begin{aligned}\left(\frac{1-\widehat{\tau}_n'(p,\alpha_n;1)}{1-\tau_n'(p,\alpha_n;1)}\right)^{-\widehat{\gamma}_n} &= \exp\left(-\widehat{\gamma}_n\log\left[\frac{1-\widehat{\tau}_n'(p,\alpha_n;1)}{1-\tau_n'(p,\alpha_n;1)}\right]\right)\\ &= \exp\left(-\left[\gamma+\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right)\right]\times\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right)\right)\\ &= 1+\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right) \quad (A.28)\end{aligned}$$

by a Taylor expansion. Furthermore, we have

$$\left(\frac{1-\tau_n'(p,\alpha_n;1)}{1-\tau_n}\right)^{-\widehat{\gamma}_n}\widehat{q}_{\tau_n}(p) = \widehat{q}^W_{\tau_n'(p,\alpha_n;1)}(p)$$

by definition of the extrapolated class of estimators $\widehat{q}^W(p)$. Using the asymptotic equivalent

$$1-\tau_n'(p,\alpha_n;1) \sim (1-\alpha_n)\frac{1}{\gamma_r}B\left(p,\frac{1}{\gamma_r}-p+1\right) \quad (A.29)$$

we conclude that the conditions of Theorem 2 are satisfied if the parameter τ_n' there is set equal to $\tau_n'(p,\alpha_n;1)$. By Theorem 2:

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n'(p,\alpha_n;1))]}\left(\frac{\widehat{q}^W_{\tau_n'(p,\alpha_n;1)}(p)}{q_{\tau_n'(p,\alpha_n;1)}(p)}-1\right)\xrightarrow{d}\zeta.$$

Now

$$\log\left[\frac{1-\tau_n}{1-\tau_n'(p,\alpha_n;1)}\right] = \log\left[\frac{1-\tau_n}{1-\alpha_n}\right]+\log\left[\frac{1-\alpha_n}{1-\tau_n'(p,\alpha_n;1)}\right]$$

and in the right-hand side of this identity, the first term tends to infinity, while the second term converges to a finite constant in view of (A.29). As a conclusion

$$\log\left[\frac{1-\tau_n}{1-\tau_n'(p,\alpha_n;1)}\right]\sim\log\left[\frac{1-\tau_n}{1-\alpha_n}\right].$$

Together with the equality $q_{\tau_n'(p,\alpha_n;1)}(p) = q_{\alpha_n}(1)$ which is true by definition of $\tau_n'(p,\alpha_n;1)$, this entails

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\alpha_n)]}\left(\frac{\widehat{q}^W_{\tau_n'(p,\alpha_n;1)}(p)}{q_{\alpha_n}(1)}-1\right)\xrightarrow{d}\zeta. \quad (A.30)$$

Combining (A.27), (A.28) and (A.30) completes the proof of the first convergence. ∎

Proof of Theorem 6. The proof of this result is similar to that of Theorem 5: just apply Theorem 3 instead of Theorem 2 in order to prove the required analogue of (A.30). ■

Proof of Theorem 7. The proof of this result is the same as that of Theorem 3, with $\widehat{q}^W_{\tau'_n}(1)$ being replaced by $\widehat{q}^W_{\tau'_n}(p)$ [thus applying Theorem 2 to obtain an analogue of (A.20)] and the mapping $x \mapsto \log C(x;p)$ being replaced by $x \mapsto \log[C(x;2)C^{-1}(x;p)]$. The details of the proof are therefore omitted. ■

Proof of Theorem 8. The proof of this result is entirely similar to that of Theorem 5 and is therefore omitted. ■

Proof of Theorem 9. The proof of this result is entirely similar to that of Theorem 6 and is therefore omitted. ■

B Auxiliary results and proofs

Lemma 1. *Let X be a random variable whose survival function $\overline{F}$ satisfies condition $\mathcal{C}_1(\gamma)$, and let H be an absolutely continuous function whose derivative h is nonnegative and is such that*

$$\exists a \geqslant 0,\ \exists \delta > 0,\ \forall b > a,\ \int_b^{+\infty} h(x) x^{-1/\gamma-\delta} dx < \infty.$$

(i) For any $b > a$, we have, as $q \to +\infty$:

$$\mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(b) \right] \mathbb{1}_{\{X > bq\}} \right) = \overline{F}(q) \int_b^{+\infty} h(x) x^{-1/\gamma} dx (1 + \mathrm{o}(1)).$$

(ii) If moreover $\overline{F}$ satisfies condition $\mathcal{C}_2(\gamma, \rho, A)$, then for any $b > a$, we have, as $q \to +\infty$:

$$\begin{aligned} & \mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(b) \right] \mathbb{1}_{\{X > bq\}} \right) \\ = \ & \overline{F}(q) \left(\int_b^{+\infty} h(x) x^{-1/\gamma} dx + A\left(\frac{1}{\overline{F}(q)}\right) \int_b^{+\infty} h(x) x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho} dx (1 + \mathrm{o}(1)) \right). \end{aligned}$$

Assume further that $a = 0$ and that h is right-continuous at 0 with $h(0) = 1$. Let $X_+ = \max(X, 0)$ denote the positive part of X.

(iii) If $\gamma < 1$, or $\gamma = 1$ and $\mathbb{E}(X_+) < \infty$, then, as $q \to +\infty$:

$$\mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(0) \right] \mathbb{1}_{\{X > 0\}} \right) = \frac{\mathbb{E}(X_+)}{q} (1 + \mathrm{o}(1)).$$

This result also holds true if the function $\overline{F}$ is actually light-tailed.

(iv) If $\gamma = 1$ and $\mathbb{E}(X_+) = \infty$, then the function $q \mapsto \mathbb{E}(X \mathbb{1}_{\{0 < X < q\}})$ is slowly varying and, as $q \to +\infty$:

$$\mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(0) \right] \mathbb{1}_{\{X > 0\}} \right) = \frac{\mathbb{E}(X \mathbb{1}_{\{0 < X < q\}})}{q} (1 + \mathrm{o}(1)).$$

(v) If $\gamma > 1$, then, as $q \to +\infty$:

$$\mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(0) \right] \mathbb{1}_{\{X > 0\}} \right) = \overline{F}(q) \int_0^{+\infty} h(x) x^{-1/\gamma} dx (1 + \mathrm{o}(1)).$$

Proof of Lemma 1. The basic idea of the proof is to note that an integration by parts entails, for $b \geqslant a$:

$$I(b;q) := \mathbb{E}\left(\left[H\left(\frac{X}{q}\right) - H(b)\right]\mathbb{1}_{\{X>bq\}}\right) = \int_b^{+\infty} h(x)\overline{F}(qx)dx.$$

To show (i), write

$$I(b;q) = \overline{F}(q)\left(\int_b^{+\infty} h(x)x^{-1/\gamma}dx + \int_b^{+\infty} h(x)\left[\frac{\overline{F}(qx)}{\overline{F}(q)} - x^{-1/\gamma}\right]dx\right) \tag{B.1}$$

and use a uniform bound such as Theorem B.2.18 in de Haan and Ferreira (2006) to get

$$I(b;q) = \overline{F}(q)\int_b^{+\infty} h(x)x^{-1/\gamma}dx(1+\mathrm{o}(1))$$

as $q \to +\infty$, which is (i).

Assertion (ii) is obtained in a similar way by using (B.1), the second-order condition $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$ and a uniform inequality such as Theorem B.3.10 in de Haan and Ferreira (2006) applied to the function $\overline{F}$.

The first step in order to show (iii), (iv) and (v) is to split $I(0;q)$ as

$$\begin{aligned} I(0;q) &= \int_0^{\varepsilon} h(x)\overline{F}(qx)dx + \int_{\varepsilon}^{+\infty} h(x)\overline{F}(qx)dx \\ &= \frac{1}{q}\int_0^{q\varepsilon} h\left(\frac{x}{q}\right)\overline{F}(x)dx + \int_{\varepsilon}^{+\infty} h(x)\overline{F}(qx)dx \end{aligned} \tag{B.2}$$

where ε is an arbitrary positive real number. To prove (iii), note that if $X \leqslant 0$ almost surely there is nothing to prove; otherwise, because

$$\mathbb{E}(X\mathbb{1}_{\{0<X<q\varepsilon\}}) = \int_0^{q\varepsilon} \overline{F}(x)dx,$$

we obtain:

$$I(0;q) - \frac{\mathbb{E}(X\mathbb{1}_{\{0<X<q\varepsilon\}})}{q} = \frac{1}{q}\int_0^{q\varepsilon}\left[h\left(\frac{x}{q}\right) - 1\right]\overline{F}(x)dx + \int_{\varepsilon}^{+\infty} h(x)\overline{F}(qx)dx.$$

Since $\mathbb{E}(X_+) < \infty$ the function $\overline{F}$ is nonincreasing and integrable in a neighborhood of infinity. This entails

$$x\overline{F}(x) \leqslant 2\int_{x/2}^{x} \overline{F}(t)dt \to 0 \quad \text{as} \quad x \to +\infty$$

and therefore that $\overline{F}(q) = \mathrm{o}(1/q)$ as $q \to +\infty$; this is of course also true if $\overline{F}$ is light-tailed. We thus obtain, by part (i) when $\overline{F}$ is regularly varying:

$$\forall \varepsilon > 0,\ I(0;q) - \frac{\mathbb{E}(X\mathbb{1}_{\{0<X<q\varepsilon\}})}{q} = \frac{1}{q}\int_0^{q\varepsilon}\left[h\left(\frac{x}{q}\right) - 1\right]\overline{F}(x)dx + \mathrm{o}\left(\frac{1}{q}\right).$$

By the dominated convergence theorem, $\mathbb{E}(X\mathbb{1}_{\{X\geqslant q\varepsilon\}}) \downarrow 0$ as $q \to +\infty$ and then:

$$\forall \varepsilon > 0,\ I(0;q) - \frac{\mathbb{E}(X_+)}{q} = \frac{1}{q}\int_0^{q\varepsilon}\left[h\left(\frac{x}{q}\right) - 1\right]\overline{F}(x)dx + \mathrm{o}\left(\frac{1}{q}\right).$$

For any $\alpha > 0$, choose now ε such that $|h(x) - 1| \leqslant \alpha/(1+\mathbb{E}(X_+))$ for all $x \in [0,\varepsilon]$; this yields

$$\left|I(0;q) - \frac{\mathbb{E}(X_+)}{q}\right| \leqslant \frac{\alpha}{1+\mathbb{E}(X_+)}\left\{\frac{\mathbb{E}(X\mathbb{1}_{\{0<X<q\varepsilon\}})}{q}\right\} + \frac{\alpha}{1+\mathbb{E}(X_+)}\frac{1}{q} \leqslant \frac{\alpha}{q}$$

for q large enough. Because α is arbitrary, this completes the proof of (iii).

To show (iv), use (B.2) to get for any $\varepsilon > 0$:

$$I(0;q) = \frac{1}{q}\int_0^1 h\left(\frac{x}{q}\right)\overline{F}(x)dx + \int_{1/q}^{\varepsilon} h(x)\overline{F}(qx)dx + \int_{\varepsilon}^{+\infty} h(x)\overline{F}(qx)dx$$

for q large enough. By the right-continuity of h at 0 and part (i) of the present Lemma, we get

$$I(0;q) = \int_{1/q}^{\varepsilon} h(x)\overline{F}(qx)dx + \mathrm{O}\left(\max\left[\frac{1}{q}, \overline{F}(q)\right]\right).$$

For an arbitrary $\alpha \in (0,1)$, choose now ε so small that $h(x) \in [1-\alpha/4, 1+\alpha/4]$ when $x \in [0,\varepsilon]$. We get

$$\left(1-\frac{\alpha}{4}\right)\int_{1/q}^{\varepsilon} \overline{F}(qx)dx \leqslant \int_{1/q}^{\varepsilon} h(x)\overline{F}(qx)dx \leqslant \left(1+\frac{\alpha}{4}\right)\int_{1/q}^{\varepsilon} \overline{F}(qx)dx$$

$$\Leftrightarrow \quad \left(1-\frac{\alpha}{4}\right)\frac{1}{q}\int_1^{q\varepsilon} \overline{F}(x)dx \leqslant \int_{1/q}^{\varepsilon} h(x)\overline{F}(qx)dx \leqslant \left(1+\frac{\alpha}{4}\right)\frac{1}{q}\int_1^{q\varepsilon} \overline{F}(x)dx.$$

By Proposition 1.5.9a in Bingham *et al.* (1987), the function $z \mapsto \int_1^z \overline{F}(x)dx = \int_1^z \{x\overline{F}(x)\}dx/x$ is slowly varying in a neighborhood of $+\infty$ (*i.e.* regularly varying with index 0) so that for q large enough,

$$\left(1-\frac{\alpha}{2}\right)\frac{1}{q}\int_1^{q} \overline{F}(x)dx \leqslant \int_{1/q}^{\varepsilon} h(x)\overline{F}(qx)dx \leqslant \left(1+\frac{\alpha}{2}\right)\frac{1}{q}\int_1^{q} \overline{F}(x)dx.$$

Finally, we have $\int_1^q \overline{F}(x)dx \uparrow \mathbb{E}(X\mathbb{1}_{\{X\geqslant 1\}}) = +\infty$ as $q \to +\infty$ and, by Proposition 1.5.9a in Bingham *et al.* (1987):

$$\frac{1}{\overline{F}(q)}\left\{\frac{1}{q}\int_1^q \overline{F}(x)dx\right\} = \frac{1}{q\overline{F}(q)}\int_1^q \{x\overline{F}(x)\}\frac{dx}{x} \to +\infty$$

as $q \to +\infty$. In other words, for q large enough,

$$(1-\alpha)\frac{1}{q}\int_1^q \overline{F}(x)dx \leqslant I(0;q) \leqslant (1+\alpha)\frac{1}{q}\int_1^q \overline{F}(x)dx.$$

Since α is arbitrary, this entails

$$I(0;q) = \frac{1}{q}\int_1^q \overline{F}(x)dx(1+\mathrm{o}(1)) = \frac{1}{q}\int_0^q \overline{F}(x)dx(1+\mathrm{o}(1)) = \frac{\mathbb{E}(X\mathbb{1}_{\{0<X<q\}})}{q}(1+\mathrm{o}(1))$$

as $q \to +\infty$: the proof of (iv) is then complete.

To show (v), let $\beta \in (0,1)$ be such that $1/\gamma < 1-\beta$ and use once again (B.2) to get:

$$I(0;q) = \frac{1}{q}\int_0^{q^\beta} h\left(\frac{x}{q}\right)\overline{F}(x)dx + \int_{q^{-(1-\beta)}}^{+\infty} h(x)\overline{F}(qx)dx.$$

By the right-continuity of h at 0 and the asymptotic relationship $q^{-(1-\beta)} = \mathrm{o}(\overline{F}(q))$ as $q \to +\infty$,

$$I(0;q) = \frac{1}{q}\int_0^{q^\beta} \overline{F}(x)dx(1+\mathrm{o}(1)) + \int_{q^{-(1-\beta)}}^{+\infty} h(x)\overline{F}(qx)dx = \int_{q^{-(1-\beta)}}^{+\infty} h(x)\overline{F}(qx)dx + \mathrm{o}(\overline{F}(q)).$$

In the spirit of the proof of (i), write now

$$I(0;q) = \overline{F}(q)\left(\int_{q^{-(1-\beta)}}^{+\infty} h(x)x^{-1/\gamma}dx + \int_{q^{-(1-\beta)}}^{+\infty} h(x)\left[\frac{\overline{F}(qx)}{\overline{F}(q)} - x^{-1/\gamma}\right]dx\right) + \mathrm{o}(\overline{F}(q)).$$

Since in the second integral we have $qx \geqslant q^\beta \to +\infty$, we may use again a uniform bound such as Theorem B.2.18 in de Haan and Ferreira (2006) to get

$$I(0;q) = \overline{F}(q)\int_{q^{-(1-\beta)}}^{+\infty} h(x)x^{-1/\gamma}dx(1+\mathrm{o}(1)).$$

Finally, since $-1/\gamma > -1$, the function $x \mapsto x^{-1/\gamma}$ is integrable in a neighborhood of 0, and thus

$$I(0; q) = \overline{F}(q) \int_0^{+\infty} h(x) x^{-1/\gamma} dx (1 + \mathrm{o}(1))$$

as $q \to +\infty$, which completes the proof of (v). $\square$

Lemma 2. *Assume that v, V are such that $v(\tau) \uparrow \infty$ and $V(\tau) \downarrow 0$, as $\tau \uparrow 1$, and there exists $B > 0$ such that*

$$\frac{V(\tau)}{\overline{F}(v(\tau))} = B(1 + e(\tau))$$

where $e(\tau) \to 0$ as $\tau \uparrow 1$. If condition $\mathcal{C}_2(\gamma, \rho, A)$ holds, with $\gamma > 0$ and F strictly increasing, then

$$\frac{v(\tau)}{U(1/V(\tau))} = B^\gamma \left(1 + \gamma e(\tau)(1 + \mathrm{o}(1)) + A(1/V(\tau)) \left[\frac{B^\rho - 1}{\rho} + \mathrm{o}(1)\right]\right) \quad \text{as} \quad \tau \uparrow 1.$$

Proof. Apply the function U to get

$$\frac{v(\tau)}{U(1/V(\tau))} - B^\gamma = \frac{U(B[1 + e(\tau)]/V(\tau))}{U(1/V(\tau))} - B^\gamma.$$

By Theorem 2.3.9 in de Haan and Ferreira (2006), we may find a function A_0, equivalent to A at infinity, such that for any $\varepsilon > 0$, there is $t_0(\varepsilon) > 1$ such that for t, $tx \geqslant t_0(\varepsilon)$,

$$\left| \frac{1}{A_0(t)} \left(\frac{U(tx)}{U(t)} - x^\gamma \right) - x^\gamma \frac{x^\rho - 1}{\rho} \right| \leqslant \frac{\varepsilon}{[(2B)^{\gamma+\rho} + (B/2)^{\gamma+\rho}][(2B)^\varepsilon + (B/2)^{-\varepsilon}]} x^{\gamma+\rho} \max(x^\varepsilon, x^{-\varepsilon}).$$

Thus, for τ sufficiently close to 1, using this inequality with $t = 1/V(\tau)$ and $x = B[1 + e(\tau)]$ gives that

$$\left| \frac{1}{A_0(1/V(\tau))} \left(\frac{U(B[1 + e(\tau)]/V(\tau))}{U(1/V(\tau))} - B^\gamma (1 + e(\tau))^\gamma \right) - B^\gamma (1 + e(\tau))^\gamma \frac{B^\rho (1 + e(\tau))^\rho - 1}{\rho} \right| \leqslant \varepsilon$$

and therefore

$$\frac{1}{A_0(1/V(\tau))} \left(\frac{U(B[1 + e(\tau)]/V(\tau))}{U(1/V(\tau))} - B^\gamma (1 + e(\tau))^\gamma \right) \to B^\gamma \frac{B^\rho - 1}{\rho} \quad \text{as} \quad \tau \uparrow 1.$$

The desired result follows by a simple first-order Taylor expansion. $\square$

In the next result we use the fact that $y \mapsto \eta_\tau(y; p)/p$ is continuously differentiable with derivative

$$\varphi_\tau(y; p) = |\tau - \mathbb{1}_{\{y \leqslant 0\}}| |y|^{p-1} \operatorname{sign}(y).$$

Lemma 3. *For all x, $y \in \mathbb{R}$ and $\tau \in (0, 1)$,*

$$\frac{1}{p}(\eta_\tau(x - y; p) - \eta_\tau(x; p)) = -y \varphi_\tau(x; p) - \int_0^y (\varphi_\tau(x - t; p) - \varphi_\tau(x; p)) dt.$$

Proof of Lemma 3. The result follows from the identity

$$\frac{1}{p}(\eta_\tau(x - y; p) - \eta_\tau(x; p)) = \int_x^{x-y} \varphi_\tau(s; p) ds = -\int_0^y \varphi_\tau(x - t; p) dt$$

obtained by the change of variables $s = x - t$. $\square$

The next lemma gives asymptotic equivalents for a number of moments that will be used in our examination of the convergence of the direct empirical estimator.

Lemma 4. *Assume that the survival function $\overline{F}$ satisfies condition $\boldsymbol{\mathcal{C}_1(\gamma)}$. Pick $a \geqslant 1$ and assume that $\gamma < 1/[a(p-1)]$ and $\mathbb{E}(X_-^{a(p-1)}) < \infty$. Then:*

(i) We have

$$\mathbb{E}(|\varphi_\tau(X-q_\tau(p);p)|^a \mathbb{1}_{\{X>q_\tau(p)\}}) = a(p-1)[q_\tau(p)]^{a(p-1)}(1-\tau)\frac{\gamma B(a(p-1),\gamma^{-1}-a(p-1))}{B(p,\gamma^{-1}-p+1)}(1+\mathrm{o}(1)) \quad \text{as } \tau \uparrow 1.$$

(ii) We have

$$\mathbb{E}(|\varphi_\tau(X-q_\tau(p);p)|^a \mathbb{1}_{\{X\leqslant q_\tau(p)\}}) = (1-\tau)^a[q_\tau(p)]^{a(p-1)}(1+\mathrm{o}(1)) \quad \text{as } \tau \uparrow 1.$$

(iii) When $a > 1$, we have

$$\mathbb{E}(|\varphi_\tau(X-q_\tau(p);p)|^a) = a(p-1)[q_\tau(p)]^{a(p-1)}(1-\tau)\frac{\gamma B(a(p-1),\gamma^{-1}-a(p-1))}{B(p,\gamma^{-1}-p+1)}(1+\mathrm{o}(1)) \quad \text{as } \tau \uparrow 1.$$

Proof of Lemma 4. Define $\theta = a(p-1)$. To show (i), note that

$$\mathbb{E}(|\varphi_\tau(X-q_\tau(p);p)|^a \mathbb{1}_{\{X>q_\tau(p)\}}) = \tau^a \mathbb{E}([X-q_\tau(p)]^\theta \mathbb{1}_{\{X>q_\tau(p)\}})$$

and apply Lemma 1(i) with $H(x) = (x-1)^\theta \mathbb{1}_{\{x\geqslant 1\}}$ and $b = 1$ to get

$$\mathbb{E}([X-q_\tau(p)]^\theta \mathbb{1}_{\{X>q_\tau(p)\}}) = \theta[q_\tau(p)]^\theta \overline{F}(q_\tau(p)) \int_1^\infty (v-1)^{\theta-1} v^{-1/\gamma} dv (1+\mathrm{o}(1)) \quad \text{as } \tau \uparrow 1.$$

Combining this equality with Proposition 1 and the change of variables $u = 1 - v^{-1}$, we obtain

$$\mathbb{E}([X-q_\tau(p)]^\theta \mathbb{1}_{\{X>q_\tau(p)\}}) = \theta[q_\tau(p)]^\theta (1-\tau)\frac{\gamma B(\theta,\gamma^{-1}-\theta)}{B(p,\gamma^{-1}-p+1)}(1+\mathrm{o}(1)) \quad \text{as } \tau \uparrow 1$$

which is (i). To show (ii), write

$$\mathbb{E}(|\varphi_\tau(X-q_\tau(p);p)|^a \mathbb{1}_{\{X\leqslant q_\tau(p)\}}) = (1-\tau)^a[q_\tau(p)]^\theta \mathbb{E}\left(\left[1-\frac{X}{q_\tau(p)}\right]^\theta \mathbb{1}_{\{X\leqslant q_\tau(p)\}}\right).$$

The conditions $\gamma < \theta^{-1}$ and $\mathbb{E}(X_-^\theta) < \infty$ ensure that $\mathbb{E}|X|^\theta < \infty$. Recall that $q_\tau(p) \uparrow +\infty$ as $\tau \uparrow 1$ and use the dominated convergence theorem to get

$$\mathbb{E}(|\varphi_\tau(X-q_\tau(p);p)|^a \mathbb{1}_{\{X\leqslant q_\tau(p)\}}) = (1-\tau)^a[q_\tau(p)]^\theta(1+\mathrm{o}(1))$$

as required. Finally, combining (i) and (ii) gives (iii) and concludes the proof. □

Lemma 5. *Let (x_n) be a positive sequence tending to infinity and $(h_{t,n})$, $t \in T_n$ be a class of functions such that*

$$\sup_{t\in T_n} \sup_{x\geqslant x_n} |h_{t,n}(x)| \to 0 \quad \text{as } n \to \infty.$$

(i) Assume that the survival function $\overline{F}$ satisfies condition $\boldsymbol{\mathcal{H}_1(\gamma)}$. Then:

$$\sup_{t\in T_n} \sup_{x\geqslant x_n} |h_{t,n}(x)|^{-1} \left| \frac{\overline{F}(x(1+h_{t,n}(x)))}{\overline{F}(x)} - \left(1 - \frac{h_{t,n}(x)}{\gamma}\right)\right| \to 0 \text{ as } n \to \infty.$$

(ii) Assume that the survival function $\overline{F}$ satisfies condition $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$. Then:

$$\sup_{t\in T_n} \sup_{x\geqslant x_n} \left[\max(|h_{t,n}(x)|, |A(1/\overline{F}(x))|)\right]^{-1} \left| \frac{\overline{F}(x(1+h_{t,n}(x)))}{\overline{F}(x)} - \left(1 - \frac{h_{t,n}(x)}{\gamma}\right)\right| \to 0 \text{ as } n \to \infty.$$

Proof of Lemma 5. We first prove (i). Write for any (h, x):

$$\frac{\overline{F}(x(1+h))}{\overline{F}(x)} = (1+h)^{-1/\gamma} \frac{c(x(1+h))}{c(x)} \exp\left(\int_x^{x(1+h)} \frac{\Delta(u)}{u} du \right). \tag{B.3}$$

By the mean value theorem, we have for n large enough

$$|c(x(1+h_{t,n}(x))) - c(x)| \leqslant |x h_{t,n}(x)| \max_{y \in [x, x(1+h_{t,n}(x))]} |c'(y)| \leqslant 2|h_{t,n}(x)| \max_{y \in [x_n/2, \infty)} |y c'(y)|$$

for all $t \in T_n$ and $x \geqslant x_n$, which entails

$$\sup_{t \in T_n} \sup_{x \geqslant x_n} \frac{1}{|h_{t,n}(x)|} |c(x(1+h_{t,n}(x))) - c(x)| \to 0 \text{ as } n \to \infty. \tag{B.4}$$

Furthermore

$$\frac{1}{|h_{t,n}(x)|} \left| \int_x^{x(1+h_{t,n}(x))} \frac{\Delta(u)}{u} du \right| \leqslant \left| \frac{\log(1+h_{t,n}(x))}{h_{t,n}(x)} \right| \max_{y \in [x, x(1+h_{t,n}(x))]} |\Delta(y)| \to 0$$

as $n \to \infty$ for all $t \in T_n$ and $x \geqslant x_n$, so that the inequality $|e^z - 1| \leqslant |z| e^{|z|}$ yields

$$\sup_{t \in T_n} \sup_{x \geqslant x_n} \frac{1}{|h_{t,n}(x)|} \left| \exp\left(\int_x^{x(1+h_{t,n}(x))} \frac{\Delta(u)}{u} du \right) - 1 \right| \to 0 \text{ as } n \to \infty. \tag{B.5}$$

Combine (B.3), (B.4) and (B.5) with the Taylor expansion $(1+h)^{-1/\gamma} = 1 - h/\gamma + \mathrm{o}(h)$ as $h \to 0$ to complete the proof of (i).

We now turn to the proof of (ii). Using a uniform inequality such as Theorem B.3.10 in de Haan and Ferreira (2006) applied to the function $\overline{F}$, we get that for any $\varepsilon > 0$ small enough there is $x_0 > 1$ such that for all $x \geqslant 2x_0$ and $s \in [1/2, 2]$:

$$\left| \frac{1}{A(1/\overline{F}(x))} \left[\frac{\overline{F}(sx)}{\overline{F}(x)} - s^{-1/\gamma} \right] \right| \leqslant \varepsilon.$$

Applying this to $s = 1 + h_{t,n}(x)$, $x \geqslant x_n$ and letting $\varepsilon \to 0$ we obtain:

$$\sup_{t \in T_n} \sup_{x \geqslant x_n} \left| \frac{1}{A(1/\overline{F}(x))} \left[\frac{\overline{F}(x(1+h_{t,n}(x)))}{\overline{F}(x)} - (1+h_{t,n}(x))^{-1/\gamma} \right] \right| = \mathrm{o}(1).$$

Using again the Taylor expansion $(1+h)^{-1/\gamma} = 1 - h/\gamma + \mathrm{o}(h)$ as $h \to 0$ completes the proof. □

The next result gives a Lipschitz property for the derivative φ_τ.

Lemma 6. *For all x, $h \in \mathbb{R}$ and $\tau \in (0,1)$, we have*

$$\begin{aligned} \varphi_\tau(x-h;p) - \varphi_\tau(x;p) &= |\tau - \mathbb{1}_{\{x \leqslant 0\}}| \left(|x-h|^{p-1} \operatorname{sign}(x-h) - |x|^{p-1} \operatorname{sign}(x) \right) \\ &+ (1-2\tau)(\mathbb{1}_{\{x \leqslant h\}} - \mathbb{1}_{\{x \leqslant 0\}}) |x-h|^{p-1} \operatorname{sign}(x-h). \end{aligned}$$

Especially,

$$|\varphi_\tau(x-h;p) - \varphi_\tau(x;p)| \leqslant |h|^{p-1} \mathbb{1}_{\{|x| \leqslant |h|\}} + (1 - \tau + \mathbb{1}_{\{x>0\}}) \begin{cases} 2|h|^{p-1} & \text{if } 1 < p < 2 \\ (p-1)(2^{p-2}+1)(|h|^{p-1} + |x|^{p-2}|h|) & \text{if } p \geqslant 2. \end{cases}$$

Proof of Lemma 6. The equality result is a straightforward consequence of the fact that

$$|\tau - \mathbb{1}_{\{x\leqslant h\}}| - |\tau - \mathbb{1}_{\{x\leqslant 0\}}| = (1-\tau)(\mathbb{1}_{\{x\leqslant h\}} - \mathbb{1}_{\{x\leqslant 0\}}) + \tau(\mathbb{1}_{\{x>h\}} - \mathbb{1}_{\{x>0\}}) = (1-2\tau)(\mathbb{1}_{\{x\leqslant h\}} - \mathbb{1}_{\{x\leqslant 0\}}).$$

To show the bound on the oscillation of φ_τ, note first that

$$\mathbb{1}_{\{x\leqslant h\}} - \mathbb{1}_{\{x\leqslant 0\}} = \begin{cases} \mathbb{1}_{\{0<x\leqslant h\}} & \text{if } h>0 \\ -\mathbb{1}_{\{h<x\leqslant 0\}} & \text{if } h<0 \end{cases}$$

and consequently

$$\begin{aligned} |(1-2\tau)(\mathbb{1}_{\{x\leqslant h\}} - \mathbb{1}_{\{x\leqslant 0\}})|x-h|^{p-1}\operatorname{sign}(x-h)| &\leqslant \begin{cases} |x-h|^{p-1}\mathbb{1}_{\{0<x\leqslant h\}} & \text{if } h>0 \\ |x-h|^{p-1}\mathbb{1}_{\{h<x\leqslant 0\}} & \text{if } h<0 \end{cases} \\ &\leqslant |h|^{p-1}\mathbb{1}_{\{|x|\leqslant |h|\}}. \end{aligned} \tag{B.6}$$

Next, when $1<p<2$, because $v\mapsto v^{p-2}$ is decreasing on $(0,\infty)$ it is clear that

$$\begin{aligned} ||x-h|^{p-1}\operatorname{sign}(x-h) - |x|^{p-1}\operatorname{sign}(x)| &= \left|\int_x^{x-h}(p-1)|v|^{p-2}dv\right| \\ &\leqslant (p-1)\int_{-|h|}^{|h|}|v|^{p-2}dv \\ &= 2|h|^{p-1}. \end{aligned} \tag{B.7}$$

When $p\geqslant 2$, write

$$\begin{aligned} ||x-h|^{p-1}\operatorname{sign}(x-h) - |x|^{p-1}\operatorname{sign}(x)| &= \left|\int_x^{x-h}(p-1)|v|^{p-2}dv\right| \\ &\leqslant (p-1)|h|\max_{v\in[x,x-h]}|v|^{p-2} \\ &\leqslant (p-1)|h|[|x-h|^{p-2}+|x|^{p-2}] \end{aligned}$$

by the monotonicity of $v\mapsto v^{p-2}$ on $[0,\infty)$, and therefore

$$\begin{aligned} ||x-h|^{p-1}\operatorname{sign}(x-h) - |x|^{p-1}\operatorname{sign}(x)| &\leqslant (p-1)|h|[(|x|+|h|)^{p-2}+|x|^{p-2}] \\ &\leqslant (p-1)(2^{p-2}+1)|h|[\max(|x|,|h|)]^{p-2} \\ &\leqslant (p-1)(2^{p-2}+1)(|h|^{p-1}+|x|^{p-2}|h|). \end{aligned} \tag{B.8}$$

Combining (B.6), (B.7) and (B.8) completes the proof. □

The lemma below is a useful convergence result for the variance of row-wise partial sums of a triangular array of strictly stationary, dependent and square-integrable random variables.

Lemma 7. *Let $(V_{i,j})$ be a triangular array of square-integrable random variables such that:*

- *for any positive integer n and any $k\leqslant n$, the random variable $V_{n,k}$ is $\sigma(X_k)-$measurable;*
- *for any positive integer n, the random variables $V_{n,k}$, $1\leqslant k\leqslant n$ are identically distributed.*

Then, if the sequence (X_n) is $\rho-$mixing with $\sum_{n=1}^{\infty} \rho(n) < \infty$, we have

$$\lim_{n\to\infty} \frac{1}{n\,\mathrm{Var}(V_{n,1})} \mathrm{Var}\left(\sum_{k=1}^{n} V_{n,k}\right) \quad \textit{exists and is finite.}$$

Proof of Lemma 7. Use the strict stationarity of the sequence to obtain

$$\begin{aligned}\mathrm{Var}\left(\sum_{k=1}^{n} V_{n,k}\right) &= n\,\mathrm{Var}(V_{n,1}) + 2\sum_{k=2}^{n}(n-k+1)\,\mathrm{Cov}(V_{n,1}, V_{n,k}) \\ &= n\,\mathrm{Var}(V_{n,1})\left(1 + 2\sum_{k=2}^{n}\frac{n-k+1}{n}\,\mathrm{corr}(V_{n,1}, V_{n,k})\right).\end{aligned}$$

It is therefore enough to show that the sequence (s_n) defined by

$$s_n := \sum_{k=2}^{n}\frac{n-k+1}{n}\,\mathrm{corr}(V_{n,1}, V_{n,k})$$

converges, or equivalently, that it is a Cauchy sequence. For this, we use the definition of the mixing coefficients $\rho(n)$ to obtain, for any positive integers p and q,

$$\begin{aligned}|s_p - s_{p+q}| &\leqslant \sum_{k=2}^{p}\left|\frac{p-k+1}{p} - \frac{p+q-k+1}{p+q}\right|\rho(k-1) + \sum_{k=p+1}^{p+q}\frac{p+q-k+1}{p+q}\rho(k-1) \\ &= \frac{q}{p+q}\sum_{k=2}^{p}\frac{k-1}{p}\rho(k-1) + \sum_{k=p+1}^{p+q}\frac{p+q-k+1}{p+q}\rho(k-1) \\ &\leqslant \frac{1}{p}\sum_{k=1}^{p-1}k\rho(k) + \sum_{k=p}^{p+q-1}\rho(k).\end{aligned}$$

Kronecker's lemma gives that the first sum above is arbitrarily small for p large enough due to the convergence of the series $\sum_{n=1}^{\infty}\rho(n)$; besides, the second term is less than a remainder of this convergent series starting at the pth term, and is therefore arbitrarily small as well for p large enough. Consequently $|s_p - s_{p+q}|$ is arbitrarily small if p is chosen large enough, which entails the convergence of (s_n) and concludes the proof. □

The last three results are the essential steps to the proof of Theorem 1.

Lemma 8. *Work under the conditions of Theorem 1. Let*

$$T_{1,n} = \frac{1}{\sqrt{n(1-\tau_n)}}\sum_{i=1}^{n}\frac{1}{[q_{\tau_n}(p)]^{p-1}}\varphi_{\tau_n}(X_i - q_{\tau_n}(p); p).$$

Then there is $\sigma^2 \in [0,\infty)$ such that

$$T_{1,n} \xrightarrow{d} \mathcal{N}\left(0, V(\gamma;p)(1+\sigma^2)\right) \quad \textit{as} \quad n\to\infty.$$

If moreover (X_n) is an independent sequence, then $\sigma^2 = 0$.

Proof of Lemma 8. Note that the random variables $\varphi_{\tau_n}(X_i - q_{\tau_n}(p); p)$, $1 \leqslant i \leqslant n$ are clearly centered because

$$q_{\tau_n}(p) = \arg\min_{u\in\mathbb{R}} \mathbb{E}(\eta_{\tau_n}(X_i - u; p) - \eta_{\tau_n}(X_i; p)) \Rightarrow \mathbb{E}(\varphi_{\tau_n}(X_i - q_{\tau_n}(p); p)) = 0$$

by differentiating under the expectation sign. Write then

$$\begin{aligned} T_{1,n} &= T_{1,1,n} + T_{1,2,n} \qquad \text{(B.9)} \\ \text{with } T_{1,1,n} &= \frac{1}{\sqrt{n(1-\tau_n)}} \sum_{i=1}^{n} \frac{1}{[q_{\tau_n}(p)]^{p-1}} \left[\varphi_{\tau_n}(X_i - q_{\tau_n}(p); p) \mathbb{1}_{\{X_i \leqslant q_{\tau_n}(p)\}} - \mathbb{E}\left(\varphi_{\tau_n}(X - q_{\tau_n}(p); p) \mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}} \right) \right] \\ \text{and } T_{1,2,n} &= \frac{1}{\sqrt{n(1-\tau_n)}} \sum_{i=1}^{n} \frac{1}{[q_{\tau_n}(p)]^{p-1}} \left[\varphi_{\tau_n}(X_i - q_{\tau_n}(p); p) \mathbb{1}_{\{X_i > q_{\tau_n}(p)\}} - \mathbb{E}\left(\varphi_{\tau_n}(X - q_{\tau_n}(p); p) \mathbb{1}_{\{X > q_{\tau_n}(p)\}} \right) \right]. \end{aligned}$$

Here $T_{1,1,n}$ and $T_{1,2,n}$ are again sums of centered variables, which we analyse separately. The first term $T_{1,1,n}$ is controlled by noting that since $\rho(n) \leqslant 2\sqrt{\phi(n)}$ (see Lemma 1.1 in Ibragimov, 1962), the series $\sum_{n=1}^{\infty} \rho(n)$ converges and we may use Lemma 7 to get

$$\operatorname{Var}(T_{1,1,n}) = \mathrm{O}\left(\frac{\operatorname{Var}\left(\varphi_{\tau_n}(X - q_{\tau_n}(p); p) \mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}} - \mathbb{E}\left(\varphi_{\tau_n}(X - q_{\tau_n}(p); p) \mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}} \right) \right)}{(1-\tau_n)[q_{\tau_n}(p)]^{2(p-1)}} \right).$$

Using Lemma 4(ii), we conclude that $\operatorname{Var}(T_{1,1,n}) = \mathrm{o}(1-\tau_n)$, proving that

$$T_{1,1,n} \xrightarrow{\mathbb{P}} 0. \qquad \text{(B.10)}$$

We now work on $T_{1,2,n}$. The essential step is to show that

$$\frac{T_{1,2,n}}{\sqrt{\operatorname{Var}(T_{1,2,n})}} \xrightarrow{d} \mathcal{N}(0,1). \qquad \text{(B.11)}$$

For this, we use the Lindeberg-type central limit theorem of Utev (1990): taking, with the notation therein, $j_n = 1$ and $k_n = n$, and setting

$$\begin{aligned} T_{1,2,n} &= \sum_{i=1}^{n} V_{n,i} \\ \text{with } V_{n,i} &:= \frac{1}{\sqrt{n(1-\tau_n)}} \frac{1}{[q_{\tau_n}(p)]^{p-1}} \left[\varphi_{\tau_n}(X_i - q_{\tau_n}(p); p) \mathbb{1}_{\{X_i > q_{\tau_n}(p)\}} - \mathbb{E}\left(\varphi_{\tau_n}(X - q_{\tau_n}(p); p) \mathbb{1}_{\{X > q_{\tau_n}(p)\}} \right) \right], \end{aligned}$$

it is enough to show that

$$\forall \varepsilon > 0, \ \frac{1}{\operatorname{Var}(T_{1,2,n})} \sum_{i=1}^{n} \mathbb{E}\left(V_{n,i}^2 \mathbb{1}_{\{|V_{n,i}| \geqslant \varepsilon \sqrt{\operatorname{Var}(T_{1,2,n})}\}} \right) \to 0 \ \text{ as } \ n \to \infty.$$

Because the $V_{n,i}$, $1 \leqslant i \leqslant n$ are identically distributed, by writing $V_{n,i}^2 = V_{n,i}^{2+\delta} V_{n,i}^{-\delta}$ it is easy to see that this convergence will be shown provided we prove that for some suitably small $\delta > 0$, the following Lyapunov condition holds:

$$\frac{n\mathbb{E}|V_{n,1}|^{2+\delta}}{[\operatorname{Var}(T_{1,2,n})]^{1+\delta/2}} \to 0 \ \text{ as } \ n \to \infty. \qquad \text{(B.12)}$$

To prove convergence (B.12), we first obtain an equivalent of the denominator. Apply Lemma 7 to get

$$\exists c \in [0, \infty), \ \lim_{n \to \infty} \frac{\operatorname{Var}(T_{1,2,n})}{n \operatorname{Var}(V_{n,1})} = c.$$

Note then that by strict stationarity,

$$\frac{\operatorname{Var}(T_{1,2,n})}{n \operatorname{Var}(V_{n,1})} = 1 + 2 \sum_{k=2}^{n} \frac{n-k+1}{n} \operatorname{corr}(V_{n,1}, V_{n,k}).$$

It follows that $c = 1$ in the case of independent observations; otherwise, the function $x \mapsto \varphi_{\tau_n}(x - q_{\tau_n}(p); p)\mathbb{1}_{\{x > q_{\tau_n}(p)\}}$ is increasing, so that the positive quadrant dependence of (X_1, X_k) implies that $\operatorname{corr}(V_{n,1}, V_{n,k})$ is nonnegative for any k and n, see Lehmann (1966). Consequently

$$\frac{\operatorname{Var}(T_{1,2,n})}{n \operatorname{Var}(V_{n,1})} = 1 + 2 \sum_{k=2}^{n} \frac{n-k+1}{n} \operatorname{corr}(V_{n,1}, V_{n,k}) \geqslant 1.$$

Letting $n \to \infty$ shows that $c \geqslant 1$ and therefore $c = 1 + \sigma^2$ for some $\sigma^2 \geqslant 0$, as required. Besides, using Lemma 4(i) entails

$$n \operatorname{Var}(V_{n,1}) \to 2\gamma(p-1) \frac{B(2p-2, \gamma^{-1} - 2p + 2)}{B(p, \gamma^{-1} - p + 1)} \quad \text{as} \quad n \to \infty.$$

The formulas $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ and $\Gamma(x + 1) = x\Gamma(x)$ now yield

$$2\gamma(p-1) \frac{B(2p-2, \gamma^{-1} - 2p + 2)}{B(p, \gamma^{-1} - p + 1)} = V(\gamma; p)$$

so that

$$\lim_{n\to\infty} \operatorname{Var}(T_{1,2,n}) = (1 + \sigma^2) \lim_{n\to\infty} n \operatorname{Var}(V_{n,1}) = V(\gamma; p)(1 + \sigma^2). \tag{B.13}$$

Using this convergence, it follows that (B.12) and therefore convergence (B.11) will be shown if for some $\delta > 0$, $n\mathbb{E}|V_{n,1}|^{2+\delta} \to 0$. Choose now $\delta > 0$ so small that $\gamma < 1/[(2 + \delta)(p - 1)]$ and $\mathbb{E}(X_-^{(2+\delta)(p-1)}) < \infty$: the convergence $n\mathbb{E}|V_{n,1}|^{2+\delta} \to 0$ is then a straightforward consequence of the Hölder inequality and Lemma 4(i). Hence (B.11), which recalling (B.13) is exactly

$$T_{1,2,n} \xrightarrow{d} \mathcal{N}\left(0, V(\gamma; p)(1 + \sigma^2)\right). \tag{B.14}$$

Combine (B.9), (B.10) and (B.14) to conclude the proof. □

Lemma 9. *Work under the conditions of Theorem 1. Let*

$$T_{2,n}(u) = -\frac{n}{[q_{\tau_n}(p)]^p} \int_0^{u q_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} [\mathbb{E}(\varphi_{\tau_n}(X - q_{\tau_n}(p) - t; p)) - \mathbb{E}(\varphi_{\tau_n}(X - q_{\tau_n}(p); p))]dt.$$

Then

$$T_{2,n}(u) \to \frac{u^2}{2\gamma} \quad \textit{as} \quad n \to \infty.$$

Proof of Lemma 9. By Lemma 6, we obtain

$$\begin{aligned}
& \mathbb{E}(\varphi_{\tau_n}(X - q_{\tau_n}(p) - t; p)) - \mathbb{E}(\varphi_{\tau_n}(X - q_{\tau_n}(p); p)) \\
= \;& (1 - 2\tau_n)\mathbb{E}(|X - q_{\tau_n}(p) - t|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p) - t)(\mathbb{1}_{\{X \leqslant q_{\tau_n}(p)+t\}} - \mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}})) \\
+ \;& \mathbb{E}\left(|\tau_n - \mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}}| \left(|X - q_{\tau_n}(p) - t|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p) - t) - |X - q_{\tau_n}(p)|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p))\right)\right),
\end{aligned}$$

that is:

$$\begin{aligned}
& \mathbb{E}(\varphi_{\tau_n}(X - q_{\tau_n}(p) - t; p)) - \mathbb{E}(\varphi_{\tau_n}(X - q_{\tau_n}(p); p)) \\
= \;& (1 - 2\tau_n)\mathbb{E}(|X - q_{\tau_n}(p) - t|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p) - t)(\mathbb{1}_{\{X > q_{\tau_n}(p)\}} - \mathbb{1}_{\{X > q_{\tau_n}(p)+t\}})) \\
+ \;& \tau_n\mathbb{E}\left(\left(|X - q_{\tau_n}(p) - t|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p) - t) - |X - q_{\tau_n}(p)|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p))\right) \mathbb{1}_{\{X > q_{\tau_n}(p)\}}\right) \\
+ \;& (1 - \tau_n)\mathbb{E}\left(\left(|X - q_{\tau_n}(p) - t|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p) - t) - |X - q_{\tau_n}(p)|^{p-1} \operatorname{sign}(X - q_{\tau_n}(p))\right) \mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}}\right) \\
=: \;& (1 - 2\tau_n)T_{2,1,n}(t) + \tau_n T_{2,2,n}(t) + (1 - \tau_n)T_{2,3,n}(t).
\end{aligned}$$

The idea is now to control the three terms appearing in the above representation. In all these terms, $|t|$ varies in the interval $I_n(u) = [0, |u|q_{\tau_n}(p)/\sqrt{n(1-\tau_n)}]$ which is such that

$$\sup_{|t|\in I_n(u)} \frac{|t|}{q_{\tau_n}(p)} \to 0 \quad \text{as} \quad n \to \infty. \tag{B.15}$$

In this proof, all $\mathrm{o}(\cdot)$ and $\mathrm{O}(\cdot)$ terms are to be understood as uniform in $|t| \in I_n(u)$. We also let $G(x) = |x|^{p-1}\,\mathrm{sign}(x)$, whose (Lebesgue) derivative is $g(x) = (p-1)|x|^{p-2}$ on $\mathbb{R}$.

First term $T_{2,1,n}(t)$: Writing

$$\mathbb{1}_{\{X>q_{\tau_n}(p)\}} - \mathbb{1}_{\{X>q_{\tau_n}(p)+t\}} = \begin{cases} \mathbb{1}_{\{q_{\tau_n}(p)<X\leqslant q_{\tau_n}(p)+t\}} & \text{if } t>0 \\ -\mathbb{1}_{\{q_{\tau_n}(p)+t<X\leqslant q_{\tau_n}(p)\}} & \text{if } t<0 \end{cases}$$

it follows that:

$$\begin{aligned} T_{2,1,n}(t) &= \begin{cases} \mathbb{E}(G(X-q_{\tau_n}(p)-t)\mathbb{1}_{\{q_{\tau_n}(p)<X\leqslant q_{\tau_n}(p)+t\}}) & \text{if } t>0 \\ -\mathbb{E}(G(X-q_{\tau_n}(p)-t)\mathbb{1}_{\{q_{\tau_n}(p)+t<X\leqslant q_{\tau_n}(p)\}}) & \text{if } t<0 \end{cases} \\ &= \begin{cases} \mathbb{E}([G(-t)+\int_{q_{\tau_n}(p)}^{X} g(v-q_{\tau_n}(p)-t)dv]\mathbb{1}_{\{q_{\tau_n}(p)<X\leqslant q_{\tau_n}(p)+t\}}) & \text{if } t>0 \\ -\mathbb{E}(\int_{q_{\tau_n}(p)+t}^{X} g(v-q_{\tau_n}(p)-t)dv\mathbb{1}_{\{q_{\tau_n}(p)+t<X\leqslant q_{\tau_n}(p)\}}) & \text{if } t<0 \end{cases} \\ &= \begin{cases} G(-t)\mathbb{P}(q_{\tau_n}(p)<X\leqslant q_{\tau_n}(p)+t) + \int_{q_{\tau_n}(p)}^{q_{\tau_n}(p)+t} g(v-q_{\tau_n}(p)-t)\mathbb{P}(v<X\leqslant q_{\tau_n}(p)+t)dv & \text{if } t>0 \\ -\int_{q_{\tau_n}(p)+t}^{q_{\tau_n}(p)} g(v-q_{\tau_n}(p)-t)\mathbb{P}(v<X\leqslant q_{\tau_n}(p))dv & \text{if } t<0 \end{cases} \end{aligned}$$

If condition $\boldsymbol{\mathcal{H}_1(\gamma)}$ holds, then by (B.15) and Lemma 5(i) we get when $t>0$:

$$\begin{aligned} &\int_{q_{\tau_n}(p)}^{q_{\tau_n}(p)+t} g(v-q_{\tau_n}(p)-t)\mathbb{P}(v<X\leqslant q_{\tau_n}(p)+t)dv \\ &= (p-1)\int_{q_{\tau_n}(p)}^{q_{\tau_n}(p)+t} (q_{\tau_n}(p)+t-v)^{p-2}\overline{F}(v)\frac{q_{\tau_n}(p)+t-v}{\gamma v}(1+\mathrm{o}(1))dv \\ &= (p-1)\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}\int_{q_{\tau_n}(p)}^{q_{\tau_n}(p)+t} (q_{\tau_n}(p)+t-v)^{p-1}(1+\mathrm{o}(1))dv \\ &= \frac{p-1}{p}t^p\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)). \end{aligned}$$

If now we work under condition $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$, we can use Lemma 5(ii) instead to obtain

$$\begin{aligned} &\int_{q_{\tau_n}(p)}^{q_{\tau_n}(p)+t} g(v-q_{\tau_n}(p)-t)\mathbb{P}(v<X\leqslant q_{\tau_n}(p)+t)dv \\ &= (p-1)\int_{q_{\tau_n}(p)}^{q_{\tau_n}(p)+t} (q_{\tau_n}(p)+t-v)^{p-2}\overline{F}(v)\left[\frac{q_{\tau_n}(p)+t-v}{\gamma v}(1+\mathrm{o}(1)) + \mathrm{o}\left(A\left(\frac{1}{\overline{F}(v)}\right)\right)\right]dv \\ &= \frac{p-1}{p}t^p\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)) + \mathrm{o}\left(\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\int_{q_{\tau_n}(p)}^{q_{\tau_n}(p)+t} (q_{\tau_n}(p)+t-v)^{p-2}dv\right) \\ &= \frac{p-1}{p}t^p\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)) + \mathrm{o}\left(\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)t^{p-1}\right). \end{aligned}$$

Likewise, when $t<0$ we have under condition $\boldsymbol{\mathcal{H}_1(\gamma)}$ that:

$$-\int_{q_{\tau_n}(p)+t}^{q_{\tau_n}(p)} g(v-q_{\tau_n}(p)-t)\mathbb{P}(v<X\leqslant q_{\tau_n}(p))dv = -\frac{1}{p}(-t)^p\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)),$$

and under condition $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$ that

$$
\begin{aligned}
-\int_{q_{\tau_n}(p)+t}^{q_{\tau_n}(p)} g(v-q_{\tau_n}(p)-t)\mathbb{P}(v<X\leqslant q_{\tau_n}(p))dv &= -\frac{1}{p}(-t)^p\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)) \\
&+ \mathrm{o}\left(\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)(-t)^{p-1}\right).
\end{aligned}
$$

Using Lemma 5(i) again and Proposition 1 we get, under condition $\boldsymbol{\mathcal{H}_1(\gamma)}$:

$$
\begin{aligned}
(1-2\tau_n)T_{2,1,n}(t)=\frac{1}{p}|t|^p\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)) &= \mathrm{o}\left(|t|[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))\right) \\
&= \mathrm{o}\left(|t|[q_{\tau_n}(p)]^{p-2}(1-\tau_n)\right). \qquad \text{(B.16)}
\end{aligned}
$$

Similarly, under condition $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$, we have by Lemma 5(ii) that

$$
\begin{aligned}
(1-2\tau_n)T_{2,1,n}(t) &= \frac{1}{p}|t|^p\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1))+\mathrm{o}\left(\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)|t|^{p-1}\right) \\
&= \mathrm{o}\left(|t|[q_{\tau_n}(p)]^{p-2}(1-\tau_n)\right)+\mathrm{o}\left([q_{\tau_n}(p)]^{p-1}n^{-1/2}\sqrt{1-\tau_n}\right). \qquad \text{(B.17)}
\end{aligned}
$$

Second term $T_{2,2,n}(t)$: In the same spirit, write

$$
\begin{aligned}
&T_{2,2,n}(t) \\
=\ & \mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{X>q_{\tau_n}(p)\}}dv\right) \\
=\ & \begin{cases} \mathbb{E}\left(\int_{\mathbb{R}} g(v)\mathbb{1}_{\{X-q_{\tau_n}(p)<v<X-q_{\tau_n}(p)-t,\ X>q_{\tau_n}(p)\}}dv\right) & \text{if } t<0 \\ -\mathbb{E}\left(\int_{\mathbb{R}} g(v)\mathbb{1}_{\{X-q_{\tau_n}(p)-t<v<X-q_{\tau_n}(p),\ X>q_{\tau_n}(p)\}}dv\right) & \text{if } t>0 \end{cases} \\
=\ & \begin{cases} \int_0^\infty g(v)\mathbb{P}(q_{\tau_n}(p)+\max(0,v+t)<X<q_{\tau_n}(p)+v)dv & \text{if } t<0 \\ -\int_{-t}^\infty g(v)\mathbb{P}(q_{\tau_n}(p)+\max(0,v)<X<q_{\tau_n}(p)+v+t)dv & \text{if } t>0 \end{cases} \\
=\ & \begin{cases} \int_0^{-t} g(v)\mathbb{P}(q_{\tau_n}(p)<X<q_{\tau_n}(p)+v)dv+\int_{-t}^\infty g(v)\mathbb{P}(q_{\tau_n}(p)+v+t<X<q_{\tau_n}(p)+v)dv & \text{if } t<0 \\ -\int_{-t}^0 g(v)\mathbb{P}(q_{\tau_n}(p)<X<q_{\tau_n}(p)+v+t)dv-\int_0^\infty g(v)\mathbb{P}(q_{\tau_n}(p)+v<X<q_{\tau_n}(p)+v+t)dv & \text{if } t>0. \end{cases}
\end{aligned}
$$

When $t<0$, we get by (B.15) and Lemma 5(i):

$$
\begin{aligned}
\int_0^{-t} g(v)\mathbb{P}(q_{\tau_n}(p)<X<q_{\tau_n}(p)+v)dv &= \int_0^{-t} g(v)\overline{F}(q_{\tau_n}(p))\frac{v}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1))dv \\
&= \frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(p-1)\int_0^{-t} v^{p-1}(1+\mathrm{o}(1))dv \\
&= \frac{(-t)^p}{p}(p-1)\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1))
\end{aligned}
$$

when $\boldsymbol{\mathcal{H}_1(\gamma)}$ holds. Working under $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$ instead and using Lemma 5(ii) entails

$$
\begin{aligned}
&\int_0^{-t} g(v)\mathbb{P}(q_{\tau_n}(p)<X<q_{\tau_n}(p)+v)dv \\
=\ & \int_0^{-t} g(v)\overline{F}(q_{\tau_n}(p))\left[\frac{v}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1))+\mathrm{o}\left(A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\right)\right]dv \\
=\ & \frac{(-t)^p}{p}(p-1)\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1))+\mathrm{o}\left(\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)(-t)^{p-1}\right).
\end{aligned}
$$

Furthermore, applying Lemma 5 again yields:

$$\int_{-t}^{\infty} g(v)\mathbb{P}(q_{\tau_n}(p)+v+t<X<q_{\tau_n}(p)+v)dv = \int_{-t}^{\infty} g(v)\overline{F}(q_{\tau_n}(p)+v+t)\frac{-t}{\gamma(q_{\tau_n}(p)+v+t)}(1+\mathrm{o}(1))dv$$

under $\boldsymbol{\mathcal{H}_1(\gamma)}$, and

$$\begin{aligned} &\int_{-t}^{\infty} g(v)\mathbb{P}(q_{\tau_n}(p)+v+t<X<q_{\tau_n}(p)+v)dv \\ = &\int_{-t}^{\infty} g(v)\overline{F}(q_{\tau_n}(p)+v+t)\left[\frac{-t}{\gamma(q_{\tau_n}(p)+v+t)}(1+\mathrm{o}(1))+\mathrm{o}\left(A\left(\frac{1}{\overline{F}(q_{\tau_n}(p)+v+t)}\right)\right)\right]dv \end{aligned}$$

under $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$. Let $\varepsilon>0$ be such that $2+\gamma^{-1}-p-\varepsilon>0$. By a uniform convergence theorem for regularly varying functions (see Theorem 1.5.2 in Bingham *et al.*, 1987) we obtain

$$\sup_{x\geqslant 1}\left|\frac{(qx)^{-\varepsilon+1/\gamma}\overline{F}(qx)}{q^{-\varepsilon+1/\gamma}\overline{F}(q)}-x^{-\varepsilon}\right|\to 0 \quad \text{as} \quad q\to+\infty.$$

As a consequence

$$\begin{aligned} &\int_{-t}^{\infty} g(v)\overline{F}(q_{\tau_n}(p)+v+t)\frac{-t}{\gamma(q_{\tau_n}(p)+v+t)}(1+\mathrm{o}(1))dv \\ = &\frac{-t}{\gamma}(p-1)[q_{\tau_n}(p)]^{1/\gamma}\overline{F}(q_{\tau_n}(p))\int_{-t}^{\infty} v^{p-2}(q_{\tau_n}(p)+v+t)^{-1-1/\gamma}dv \\ + &\mathrm{o}\left(-t[q_{\tau_n}(p)]^{1/\gamma-\varepsilon}\overline{F}(q_{\tau_n}(p))\int_{-t}^{\infty} v^{p-2}(q_{\tau_n}(p)+v+t)^{-1-1/\gamma+\varepsilon}dv\right). \end{aligned}$$

Now, for n large enough and all $w\geqslant 0$,

$$0\leqslant w^{p-2}\left(1+w+\frac{t}{q_{\tau_n}(p)}\right)^{-1-1/\gamma}\leqslant w^{p-2}\left(\frac{1}{2}+w\right)^{-1-1/\gamma}$$

where the right-hand side defines an integrable function on $(0,\infty)$. By the dominated convergence theorem, we get

$$\begin{aligned} \int_{-t}^{\infty} v^{p-2}(q_{\tau_n}(p)+v+t)^{-1-1/\gamma}dv &= [q_{\tau_n}(p)]^{p-2-1/\gamma}\int_{-t/q_{\tau_n}(p)}^{\infty} w^{p-2}\left(1+w+\frac{t}{q_{\tau_n}(p)}\right)^{-1-1/\gamma}dw \\ &= [q_{\tau_n}(p)]^{p-2-1/\gamma}\int_{0}^{\infty} w^{p-2}(1+w)^{-1-1/\gamma}dw(1+\mathrm{o}(1)). \end{aligned}$$

The change of variables $z=(1+w)^{-1}$ yields

$$\int_{0}^{\infty} w^{p-2}(1+w)^{-1-1/\gamma}dw=\int_{0}^{1}(1-z)^{p-2}z^{1+1/\gamma-p}dz=B(p-1,2+\gamma^{-1}-p).$$

Similarly

$$\int_{-t}^{\infty} v^{p-2}(q_{\tau_n}(p)+v+t)^{-1-1/\gamma+\varepsilon}dv=[q_{\tau_n}(p)]^{p-2-1/\gamma+\varepsilon}B(p-1,2+\gamma^{-1}-p-\varepsilon)(1+\mathrm{o}(1))$$

so that under $\boldsymbol{\mathcal{H}_1(\gamma)}$:

$$\int_{-t}^{\infty} g(v)\mathbb{P}(q_{\tau_n}(p)+v+t<X<q_{\tau_n}(p)+v)dv=\frac{-t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1)).$$

When $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$ holds, because the function $\overline{F}\times A\circ(1/\overline{F})$ is regularly varying with index $(\rho-1)/\gamma\leqslant-1/\gamma$ and therefore

$$p-2+\frac{\rho-1}{\gamma}\leqslant p-2-\frac{1}{\gamma}<p-2+(2-2p)=-p<-1,$$

we can argue along the same lines to obtain

$$\begin{aligned}
& \int_{-t}^{\infty} g(v)\overline{F}(q_{\tau_n}(p)+v+t)A\left(\frac{1}{\overline{F}(q_{\tau_n}(p)+v+t)}\right)dv \\
= \;& (p-1)[q_{\tau_n}(p)]^{(1-\rho)/\gamma}\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\int_{-t}^{\infty} v^{p-2}(q_{\tau_n}(p)+v+t)^{(\rho-1)/\gamma}dv(1+\mathrm{o}(1)) \\
= \;& \mathrm{O}\left([q_{\tau_n}(p)]^{p-1}\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{-t}^{\infty} g(v)\mathbb{P}(q_{\tau_n}(p)+v+t<X<q_{\tau_n}(p)+v)dv \;=\; & \frac{-t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1)) \\
& + \;\mathrm{o}\left([q_{\tau_n}(p)]^{p-1}\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\right).
\end{aligned}$$

When $t>0$ and $\boldsymbol{\mathcal{H}_1(\gamma)}$ holds, we get in a similar fashion

$$\int_{-t}^{0} g(v)\mathbb{P}(q_{\tau_n}(p)<X<q_{\tau_n}(p)+v+t)dv = \frac{t^p}{p}\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1))$$

and

$$\int_{0}^{\infty} g(v)\mathbb{P}(q_{\tau_n}(p)+v<X<q_{\tau_n}(p)+v+t)dv = \frac{t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1)).$$

If $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$ holds, we have

$$\int_{-t}^{0} g(v)\mathbb{P}(q_{\tau_n}(p)<X<q_{\tau_n}(p)+v+t)dv = \frac{t^p}{p}\frac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1))+\mathrm{o}\left(\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)t^{p-1}\right)$$

and

$$\begin{aligned}
\int_{0}^{\infty} g(v)\mathbb{P}(q_{\tau_n}(p)+v<X<q_{\tau_n}(p)+v+t)dv \;=\; & \frac{t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1)) \\
+ \;& \mathrm{o}\left([q_{\tau_n}(p)]^{p-1}\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\right).
\end{aligned}$$

All in all, under $\boldsymbol{\mathcal{H}_1(\gamma)}$, using (B.15) entails:

$$\begin{aligned}
& T_{2,2,n}(t) \\
&= \begin{cases}
-\dfrac{t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1))+\dfrac{(-t)^p}{p}(p-1)\dfrac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)) & \text{if } t<0 \\
-\dfrac{t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1))-\dfrac{t^p}{p}\dfrac{\overline{F}(q_{\tau_n}(p))}{\gamma q_{\tau_n}(p)}(1+\mathrm{o}(1)) & \text{if } t>0
\end{cases} \\
&= -\frac{t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1)).
\end{aligned}$$

Working under $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$ gives instead:

$$T_{2,2,n}(t) = -\frac{t}{\gamma}(p-1)[q_{\tau_n}(p)]^{p-2}\overline{F}(q_{\tau_n}(p))B(p-1,2+\gamma^{-1}-p)(1+\mathrm{o}(1))+\mathrm{o}\left([q_{\tau_n}(p)]^{p-1}\overline{F}(q_{\tau_n}(p))A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\right)$$

because of (B.15) again. By Proposition 1 and the identity

$$\forall x,y>0,\; \frac{B(x,y+1)}{B(x+1,y)} = \frac{\Gamma(x)}{\Gamma(x+1)}\frac{\Gamma(y+1)}{\Gamma(y)} = \frac{y}{x}$$

this reads

$$\tau_n T_{2,2,n}(t) = -t(\gamma^{-1} - (p-1))[q_{\tau_n}(p)]^{p-2}(1-\tau_n)(1+\mathrm{o}(1)) \tag{B.18}$$

under $\boldsymbol{\mathcal{H}_1(\gamma)}$, and

$$\tau_n T_{2,2,n}(t) = -t(\gamma^{-1} - (p-1))[q_{\tau_n}(p)]^{p-2}(1-\tau_n)(1+\mathrm{o}(1)) + \mathrm{o}\left([q_{\tau_n}(p)]^{p-1} n^{-1/2}\sqrt{1-\tau_n}\right) \tag{B.19}$$

under $\boldsymbol{\mathcal{C}_2(\gamma, \rho, A)}$.

Third term $T_{2,3,n}(t)$: Write

$$T_{2,3,n}(t) \quad = \quad \mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}} dv\right).$$

Split then the above integral as

$$\begin{aligned}\mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{X \leqslant q_{\tau_n}(p)\}} dv\right) &= \mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{X \leqslant q_{\tau_n}(p)/2\}} dv\right) \\ &+ \mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{q_{\tau_n}(p)/2 < X \leqslant q_{\tau_n}(p)\}} dv\right).\end{aligned}$$

The first term in the rhs above is

$$(p-1)\mathbb{E}\left(|X - q_{\tau_n}(p)|^{p-2}[X - q_{\tau_n}(p)]\int_1^{1-t/(X-q_{\tau_n}(p))} |w|^{p-2} dw \mathbb{1}_{\{X - q_{\tau_n}(p) \leqslant -q_{\tau_n}(p)/2\}}\right).$$

Because $\sup_{|t|\in I_n(u)} |t|/q_{\tau_n}(p) \to 0$ and $|X - q_{\tau_n}(p)| \geqslant q_{\tau_n}(p)/2$ in the integrand, this term is equivalent to

$$\begin{aligned}-t(p-1)\mathbb{E}\left(|X - q_{\tau_n}(p)|^{p-2}\mathbb{1}_{\{X - q_{\tau_n}(p) \leqslant -q_{\tau_n}(p)/2\}}\right) &= -t(p-1)[q_{\tau_n}(p)]^{p-2}\mathbb{E}\left(\left|\frac{X}{q_{\tau_n}(p)} - 1\right|^{p-2}\mathbb{1}_{\{X \leqslant q_{\tau_n}(p)/2\}}\right) \\ &= -t(p-1)[q_{\tau_n}(p)]^{p-2}(1+\mathrm{o}(1))\end{aligned}$$

by the dominated convergence theorem. The second term, meanwhile, is equal to

$$\begin{aligned}&\mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{q_{\tau_n}(p)/2 < X \leqslant q_{\tau_n}(p)\}} dv\right) \\ &= \begin{cases}\mathbb{E}\left(\int_{\mathbb{R}} g(v)\mathbb{1}_{\{X - q_{\tau_n}(p) < v < X - q_{\tau_n}(p) - t,\, q_{\tau_n}(p)/2 < X \leqslant q_{\tau_n}(p)\}} dv\right) & \text{if } t < 0 \\ -\mathbb{E}\left(\int_{\mathbb{R}} g(v)\mathbb{1}_{\{X - q_{\tau_n}(p) - t < v < X - q_{\tau_n}(p),\, q_{\tau_n}(p)/2 < X \leqslant q_{\tau_n}(p)\}} dv\right) & \text{if } t > 0\end{cases} \\ &= \begin{cases}\int_{\mathbb{R}} g(v)\mathbb{P}(q_{\tau_n}(p) + \max(-q_{\tau_n}(p)/2, v+t) < X < q_{\tau_n}(p) + \min(0, v))dv & \text{if } t < 0 \\ -\int_{\mathbb{R}} g(v)\mathbb{P}(q_{\tau_n}(p) + \max(-q_{\tau_n}(p)/2, v) < X < q_{\tau_n}(p) + \min(0, v+t))dv & \text{if } t > 0.\end{cases}\end{aligned}$$

When $\mathcal{H}_1(\gamma)$ holds, we then have by Lemma 5(i):

$$\begin{aligned}
&\left|\mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{q_{\tau_n}(p)/2<X\leqslant q_{\tau_n}(p)\}}dv\right)\right| \\
=&\begin{cases} \int_{-q_{\tau_n}(p)/2}^{-t} g(v)\mathbb{P}(q_{\tau_n}(p)+\max(-q_{\tau_n}(p)/2, v+t) < X < q_{\tau_n}(p)+\min(0,v))dv & \text{if } t<0 \\ \int_{-t-q_{\tau_n}(p)/2}^{0} g(v)\mathbb{P}(q_{\tau_n}(p)+\max(-q_{\tau_n}(p)/2, v) < X < q_{\tau_n}(p)+\min(0,v+t))dv & \text{if } t>0 \end{cases} \\
\leqslant&\begin{cases} \int_{-q_{\tau_n}(p)/2}^{-t} g(v)\mathbb{P}(q_{\tau_n}(p)+v+t < X < q_{\tau_n}(p)+v)dv & \text{if } t<0 \\ \int_{-t-q_{\tau_n}(p)/2}^{0} g(v)\mathbb{P}(q_{\tau_n}(p)+v < X < q_{\tau_n}(p)+v+t)dv & \text{if } t>0 \end{cases} \\
=&\begin{cases} \int_{-q_{\tau_n}(p)/2}^{-t} g(v)\overline{F}(q_{\tau_n}(p)+v+t)\dfrac{-t}{\gamma(q_{\tau_n}(p)+v+t)}dv(1+\mathrm{o}(1)) & \text{if } t<0 \\ \int_{-t-q_{\tau_n}(p)/2}^{0} g(v)\overline{F}(q_{\tau_n}(p)+v)\dfrac{t}{\gamma(q_{\tau_n}(p)+v)}dv(1+\mathrm{o}(1)) & \text{if } t>0 \end{cases} \\
\leqslant&\ \overline{F}((q_{\tau_n}(p)/2)-|t|)\frac{|t|}{\gamma((q_{\tau_n}(p)/2)-|t|)}\begin{cases} \int_{-q_{\tau_n}(p)/2}^{-t} g(v)dv(1+\mathrm{o}(1)) & \text{if } t<0 \\ \int_{-t-q_{\tau_n}(p)/2}^{0} g(v)dv(1+\mathrm{o}(1)) & \text{if } t>0 \end{cases} \\
=&\ \overline{F}(q_{\tau_n}(p)/2)\frac{|t|}{\gamma}\left(\frac{q_{\tau_n}(p)}{2}\right)^{p-2}(1+\mathrm{o}(1)) = \mathrm{o}\left(|t|[q_{\tau_n}(p)]^{p-2}\right).
\end{aligned}$$

Similarly, when $\mathcal{C}_2(\gamma, \rho, A)$ holds, we have by Lemma 5(ii):

$$\begin{aligned}
&\left|\mathbb{E}\left(\int_{X-q_{\tau_n}(p)}^{X-q_{\tau_n}(p)-t} g(v)\mathbb{1}_{\{q_{\tau_n}(p)/2<X\leqslant q_{\tau_n}(p)\}}dv\right)\right| \\
\leqslant&\begin{cases} \int_{-q_{\tau_n}(p)/2}^{-t} g(v)\overline{F}(q_{\tau_n}(p)+v+t)\left[\dfrac{-t}{\gamma(q_{\tau_n}(p)+v+t)}(1+\mathrm{o}(1))+\mathrm{o}\left(A\left(\dfrac{1}{\overline{F}(q_{\tau_n}(p)+v+t)}\right)\right)\right]dv & \text{if } t<0 \\ \int_{-t-q_{\tau_n}(p)/2}^{0} g(v)\overline{F}(q_{\tau_n}(p)+v)\left[\dfrac{t}{\gamma(q_{\tau_n}(p)+v)}(1+\mathrm{o}(1))+\mathrm{o}\left(A\left(\dfrac{1}{\overline{F}(q_{\tau_n}(p)+v)}\right)\right)\right]dv & \text{if } t>0 \end{cases} \\
=&\ \mathrm{O}\left(|t|\overline{F}(q_{\tau_n}(p))[q_{\tau_n}(p)]^{p-2}\right)+\mathrm{o}\left(\overline{F}(q_{\tau_n}(p))[q_{\tau_n}(p)]^{p-1}A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))}\right)\right) \\
=&\ \mathrm{o}\left(|t|[q_{\tau_n}(p)]^{p-2}\right)+\mathrm{o}\left([q_{\tau_n}(p)]^{p-1}n^{-1/2}\sqrt{1-\tau_n}\right).
\end{aligned}$$

As a conclusion, if $\mathcal{H}_1(\gamma)$ holds:

$$(1-\tau_n)T_{2,3,n}(t) = -t(p-1)[q_{\tau_n}(p)]^{p-2}(1-\tau_n)(1+\mathrm{o}(1)) \tag{B.20}$$

and if $\mathcal{C}_2(\gamma, \rho, A)$ holds then:

$$(1-\tau_n)T_{2,3,n}(t) = (1-\tau_n)\left[-t(p-1)[q_{\tau_n}(p)]^{p-2}(1+\mathrm{o}(1))+\mathrm{o}\left([q_{\tau_n}(p)]^{p-1}n^{-1/2}\sqrt{1-\tau_n}\right)\right]. \tag{B.21}$$

Combining (B.16), (B.18) and (B.20), we get

$$(1-2\tau_n)T_{2,1,n}(t)+\tau_n T_{2,2,n}(t)+(1-\tau_n)T_{2,3,n}(t) = -\frac{t}{\gamma}[q_{\tau_n}(p)]^{p-2}(1-\tau_n)(1+\mathrm{o}(1))$$

and thus

$$\begin{aligned}
T_{2,n}(u) &= -\frac{n}{[q_{\tau_n}(p)]^p}\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} [\mathbb{E}(\varphi_{\tau_n}(X-q_{\tau_n}(p)-t;p))-\mathbb{E}(\varphi_{\tau_n}(X-q_{\tau_n}(p);p))]dt \\
&= \frac{n(1-\tau_n)}{\gamma[q_{\tau_n}(p)]^2}\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} t\,dt(1+\mathrm{o}(1)) \\
&= \frac{u^2}{2\gamma}(1+\mathrm{o}(1))
\end{aligned}$$

if $\boldsymbol{\mathcal{H}_1(\gamma)}$ is assumed. If we work under $\boldsymbol{\mathcal{C}_2(\gamma,\rho,A)}$, we have by combining (B.17), (B.19) and (B.21) that:

$$\begin{aligned}
(1-2\tau_n)T_{2,1,n}(t)+\tau_n T_{2,2,n}(t)+(1-\tau_n)T_{2,3,n}(t) &= -\frac{t}{\gamma}[q_{\tau_n}(p)]^{p-2}(1-\tau_n)(1+\mathrm{o}(1)) \\
&+ \mathrm{o}\left([q_{\tau_n}(p)]^{p-1}n^{-1/2}\sqrt{1-\tau_n}\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
T_{2,n}(u) &= -\frac{n}{[q_{\tau_n}(p)]^p}\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} [\mathbb{E}(\varphi_{\tau_n}(X-q_{\tau_n}(p)-t;p))-\mathbb{E}(\varphi_{\tau_n}(X-q_{\tau_n}(p);p))]dt \\
&= \frac{n}{[q_{\tau_n}(p)]^p}\left[\frac{1}{\gamma}[q_{\tau_n}(p)]^{p-2}(1-\tau_n)\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} t\,dt(1+\mathrm{o}(1))+\mathrm{o}\left(\frac{[q_{\tau_n}(p)]^p}{n}\right)\right] \\
&= \frac{u^2}{2\gamma}(1+\mathrm{o}(1)).
\end{aligned}$$

The proof is complete. □

Lemma 10. *Work under the conditions of Theorem 1. Let* $\mathcal{S}_{n,i}(v) := \varphi_{\tau_n}(X_i - v;p) - \mathbb{E}(\varphi_{\tau_n}(X-v;p))$ *and*

$$T_{3,n}(u) = -\frac{1}{[q_{\tau_n}(p)]^p}\sum_{i=1}^n \int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} [\mathcal{S}_{n,i}(q_{\tau_n}(p)+t)-\mathcal{S}_{n,i}(q_{\tau_n}(p))]dt.$$

Then

$$T_{3,n}(u) \xrightarrow{\mathbb{P}} 0 \quad as \quad n\to\infty.$$

Proof of Lemma 10. As in the proof of Lemma 9, let $I_n(u) = [0, |u|q_{\tau_n}(p)/\sqrt{n(1-\tau_n)}]$. In the present proof, all $\mathrm{o}(\cdot)$ and $\mathrm{O}(\cdot)$ terms are to be understood as uniform in $|t| \in I_n(u)$.

Let $\mathcal{S}_n(v) := \varphi_{\tau_n}(X-v;p) - \mathbb{E}(\varphi_{\tau_n}(X-v;p))$. By Lemma 7,

$$\mathrm{Var}(T_{3,n}(u)) = \mathrm{O}\left(\frac{n}{[q_{\tau_n}(p)]^{2p}}\,\mathrm{Var}\left(\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}}[\mathcal{S}_n(q_{\tau_n}(p)+t)-\mathcal{S}_n(q_{\tau_n}(p))]dt\right)\right).$$

Because for any t, $\mathcal{S}_n(q_{\tau_n}(p)+t)$ is centered and $[\varphi_{\tau_n}(X-q_{\tau_n}(p)-t;p)]^2$ is integrable w.r.t. t on the interval $[0, uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}]$, we get

$$\begin{aligned}
&\mathrm{Var}(T_{3,n}(u)) \\
= \ &\mathrm{O}\left(\frac{n}{[q_{\tau_n}(p)]^{2p}}\int_{[0,\,uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}]^2}\mathbb{E}([\mathcal{S}_n(q_{\tau_n}(p)+s)-\mathcal{S}_n(q_{\tau_n}(p))][\mathcal{S}_n(q_{\tau_n}(p)+t)-\mathcal{S}_n(q_{\tau_n}(p))])ds\,dt\right).
\end{aligned}$$

The Cauchy-Schwarz inequality now yields

$$\mathrm{Var}(T_{3,n}(u)) \leqslant \frac{n}{[q_{\tau_n}(p)]^{2p}}\left(\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}}\sqrt{\mathbb{E}(|\mathcal{S}_n(q_{\tau_n}(p)+t)-\mathcal{S}_n(q_{\tau_n}(p))|^2)}\,dt\right)^2. \tag{B.22}$$

Applying Lemma 6, we get for any t

$$\begin{aligned}
& |\mathcal{S}_n(q_{\tau_n}(p)+t)-\mathcal{S}_n(q_{\tau_n}(p))| \\
\leqslant\ & |\varphi_{\tau_n}(X-q_{\tau_n}(p)-t;p)-\varphi_{\tau_n}(X-q_{\tau_n}(p);p)| + \mathbb{E}|\varphi_{\tau_n}(X-q_{\tau_n}(p)-t;p)-\varphi_{\tau_n}(X-q_{\tau_n}(p);p)| \\
\leqslant\ & |t|^{p-1}\left(\mathbb{1}_{\{|X-q_{\tau_n}(p)|\leqslant|t|\}}+\mathbb{P}(|X-q_{\tau_n}(p)|\leqslant|t|)\right) \\
+\ & (1-\tau_n+\mathbb{1}_{\{X>q_{\tau_n}(p)\}})\begin{cases} 2|t|^{p-1} & \text{if } 1<p<2 \\ (p-1)(2^{p-2}+1)(|t|^{p-1}+|X-q_{\tau_n}(p)|^{p-2}|t|) & \text{if } p\geqslant 2 \end{cases} \\
+\ & (1-\tau_n)\begin{cases} 2|t|^{p-1} & \text{if } 1<p<2 \\ (p-1)(2^{p-2}+1)(|t|^{p-1}+|t|\mathbb{E}|X-q_{\tau_n}(p)|^{p-2}) & \text{if } p\geqslant 2 \end{cases} \\
+\ & \begin{cases} 2|t|^{p-1}\mathbb{P}(X>q_{\tau_n}(p)) & \text{if } 1<p<2 \\ (p-1)(2^{p-2}+1)(|t|^{p-1}\mathbb{P}(X>q_{\tau_n}(p))+|t|\mathbb{E}(|X-q_{\tau_n}(p)|^{p-2}\mathbb{1}_{\{X>q_{\tau_n}(p)\}})) & \text{if } p\geqslant 2. \end{cases}
\end{aligned}$$

By Lemma 1 with $H(x)=(x-1)^{p-2}\mathbb{1}_{\{x\geqslant 1\}}$ and Proposition 1, we have when $p\geqslant 2$ that

$$\begin{aligned}
\mathbb{E}(|X-q_{\tau_n}(p)|^{p-2}\mathbb{1}_{\{X>q_{\tau_n}(p)\}}) &= [q_{\tau_n}(p)]^{p-2}\mathbb{E}\left(\left[\frac{X}{q_{\tau_n}(p)}-1\right]^{p-2}\mathbb{1}_{\{X>q_{\tau_n}(p)\}}\right) \\
&= \mathrm{O}\left((1-\tau_n)[q_{\tau_n}(p)]^{p-2}\right)
\end{aligned}$$

and it is moreover a consequence of the dominated convergence theorem that

$$\mathbb{E}(|X-q_{\tau_n}(p)|^{p-2}) = [q_{\tau_n}(p)]^{p-2}\mathbb{E}\left(\left|\frac{X}{q_{\tau_n}(p)}-1\right|^{p-2}\right) = [q_{\tau_n}(p)]^{p-2}(1+\mathrm{o}(1)).$$

Recalling convergence (B.15), *i.e.* $|t|/q_{\tau_n}(p)\to 0$ uniformly in $|t|\in I_n(u)$, and using Proposition 1 again, we get

$$\begin{aligned}
& |\mathcal{S}_n(q_{\tau_n}(p)+t)-\mathcal{S}_n(q_{\tau_n}(p))| \\
\leqslant\ & |t|^{p-1}\left(\mathbb{1}_{\{|X-q_{\tau_n}(p)|\leqslant|t|\}}+\mathbb{P}(|X-q_{\tau_n}(p)|\leqslant|t|)\right) \\
+\ & (1-\tau_n+\mathbb{1}_{\{X>q_{\tau_n}(p)\}})\begin{cases} 2|t|^{p-1} & \text{if } 1<p<2 \\ (p-1)(2^{p-2}+1)(|t|^{p-1}+|X-q_{\tau_n}(p)|^{p-2}|t|) & \text{if } p\geqslant 2 \end{cases} \\
+\ & \begin{cases} \mathrm{O}((1-\tau_n)|t|^{p-1}) & \text{if } 1<p<2 \\ \mathrm{O}\left((1-\tau_n)[q_{\tau_n}(p)]^{p-2}|t|\right) & \text{if } p\geqslant 2. \end{cases}
\end{aligned}$$

Squaring, integrating and using convergence (B.15) once again, we obtain that there is a constant C with

$$\begin{aligned}
\mathbb{E}|\mathcal{S}_n(q_{\tau_n}(p)+t)-\mathcal{S}_n(q_{\tau_n}(p))|^2 &\leqslant C|t|^{2(p-1)}\mathbb{P}(|X-q_{\tau_n}(p)|\leqslant|t|) \\
&+ \begin{cases} \mathrm{O}((1-\tau_n)|t|^{2(p-1)}) & \text{if } 1<p<2 \\ \mathrm{O}\left((1-\tau_n)[q_{\tau_n}(p)]^{2(p-2)}|t|^2\right) & \text{if } p\geqslant 2. \end{cases}
\end{aligned}$$

When $\boldsymbol{\mathcal{H}_1(\gamma)}$ holds, then by Lemma 5(i) and Proposition 1 again,

$$\mathbb{P}(|X - q_{\tau_n}(p)| \leqslant |t|) = \overline{F}(q_{\tau_n}(p)) \frac{2|t|}{\gamma q_{\tau_n}(p)}(1 + \mathrm{o}(1)) = \mathrm{o}(1 - \tau_n).$$

If we work under $\boldsymbol{\mathcal{C}_2(\gamma, \rho, A)}$, then by Lemma 5(ii) and Proposition 1,

$$\mathbb{P}(|X - q_{\tau_n}(p)| \leqslant |t|) = \overline{F}(q_{\tau_n}(p)) \left[\frac{2|t|}{\gamma q_{\tau_n}(p)}(1 + \mathrm{o}(1)) + \mathrm{o}\left(A\left(\frac{1}{\overline{F}(q_{\tau_n}(p))} \right) \right) \right] = \mathrm{o}(1 - \tau_n).$$

In any case,

$$\begin{aligned} \mathbb{E}|\mathcal{S}_n(q_{\tau_n}(p) + t) - \mathcal{S}_n(q_{\tau_n}(p))|^2 \quad &\leqslant \quad \begin{cases} \mathrm{O}((1 - \tau_n)|t|^{2(p-1)}) & \text{if } 1 < p < 2 \\ \mathrm{O}\left((1 - \tau_n)[q_{\tau_n}(p)]^{2(p-2)}|t|^2\right) + \mathrm{O}\left((1 - \tau_n)|t|^{2(p-1)}\right) & \text{if } p \geqslant 2 \end{cases} \\ &= \quad \begin{cases} \mathrm{O}((1 - \tau_n)|t|^{2(p-1)}) & \text{if } 1 < p < 2 \\ \mathrm{O}\left((1 - \tau_n)[q_{\tau_n}(p)]^{2(p-2)}|t|^2\right) & \text{if } p \geqslant 2 \end{cases} \end{aligned}$$

by using (B.15). Because

$$\begin{aligned} \frac{n}{[q_{\tau_n}(p)]^{2p}} \left(\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} \sqrt{(1 - \tau_n)|t|^{2(p-1)}}\, dt \right)^2 &= \frac{n(1 - \tau_n)}{[q_{\tau_n}(p)]^{2p}} \left(\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} |t|^{p-1}\, dt \right)^2 \\ &= \mathrm{O}([n(1 - \tau_n)]^{1-p}) = \mathrm{o}(1) \end{aligned}$$

and

$$\begin{aligned} \frac{n}{[q_{\tau_n}(p)]^{2p}} \left(\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} \sqrt{(1 - \tau_n)[q_{\tau_n}(p)]^{2(p-2)}|t|^2}\, dt \right)^2 &= \frac{n(1 - \tau_n)}{[q_{\tau_n}(p)]^4} \left(\int_0^{uq_{\tau_n}(p)/\sqrt{n(1-\tau_n)}} |t|\, dt \right)^2 \\ &= \mathrm{O}([n(1 - \tau_n)]^{-1}) = \mathrm{o}(1) \end{aligned}$$

we get $T_{3,n}(u) \xrightarrow{\mathbb{P}} 0$ and the proof is complete. ☐

C Additional simulations

C.1 Extreme expectile estimation

We concentrate here on extreme L^2-quantiles, or equivalently, expectiles. A comparison of the three estimators $\widehat{q}^W_{\alpha_n}(2)$ in (11), $\widetilde{q}^W_{\alpha_n}(2)$ in (12) and $\check{q}^p_{\alpha_n}(2)$ in (13) (see the main paper) of the extreme expectile $q_{\alpha_n}(2)$ is shown in Figures 1 and 2, where we present the evolution of their relative MSE (in log scale) in terms of the value k. We used the same considerations as in Section 6 for the choice of $\widehat{\gamma}_n$ and the intermediate and extreme expectile levels τ_n and $\tau'_n = \alpha_n$. The experiments employ the Fréchet, Pareto and Student distributions with tail-indices $\gamma \in \{0.1, 0.45\}$ and various values of $p \in (1, 2)$ in the formulation (13) of $\check{q}^p_{\alpha_n}(2)$.

In the case of Fréchet and Pareto distributions, we already know that $\widetilde{q}^W_{\alpha_n}(2)$ behaves better than $\widehat{q}^W_{\alpha_n}(2)$ in terms of relative MSE. In this case, it turns out that the accuracy of the estimator $\check{q}^p_{\alpha_n}(2)$ is also superior to $\widehat{q}^W_{\alpha_n}(2)$ and is similar to that of $\widetilde{q}^W_{\alpha_n}(2)$ for very 'small' values of p (close to 1), as may be seen from Figure 1. In this Figure we only present the estimates of the relative MSE in a log scale. We do not graph the bias estimates to save space: most of the error is due to variance, the squared bias being much smaller in all cases.

In contrast, in the case of the Student distribution, we know that $\widehat{q}^{W}_{\alpha_n}(2)$ behaves overall better than $\widetilde{q}^{W}_{\alpha_n}(2)$: we do not graph the curve related to the latter estimator in Figure 2 as it exhibits considerable volatility. Also, it appears in this case that $\check{q}^{p}_{\alpha_n}(2)$ outperforms $\widetilde{q}^{W}_{\alpha_n}(2)$ as well and has a similar behavior compared to to $\widehat{q}^{W}_{\alpha_n}(2)$ for very 'large' values of p (close to 2), as may be seen from Figure 2. There is also a significant improvement for large γ in this case when using the estimator $\check{q}^{p}_{\alpha_n}(2)$, probably because this estimator benefits from increasing robustness (see the final lines of Section 6.2).

This might suggest the following strategy with a real data set. If the data set is concerned with a non-negative loss distribution, it is most efficient to use $\widetilde{q}^{W}_{\alpha_n}(2)$ and $\check{q}^{p}_{\alpha_n}(2)$ with values of p very close to 1. At the opposite, if the data set is concerned with a real-valued profit-loss random variable, we favor the use of $\widehat{q}^{W}_{\alpha_n}(2)$ and $\check{q}^{p}_{\alpha_n}(2)$ with values of p very close to 2. The important question of how to pick out p in practice, in order to get the best estimates $\check{q}^{p}_{\alpha_n}(2)$ from historical data, can be addressed by adapting the practical guidelines provided in Section 7 for selecting p in the extreme L^p-quantile estimates $\widetilde{q}^{W}_{\alpha_n}(p)$ and $\widehat{q}^{W}_{\alpha_n}(p)$.

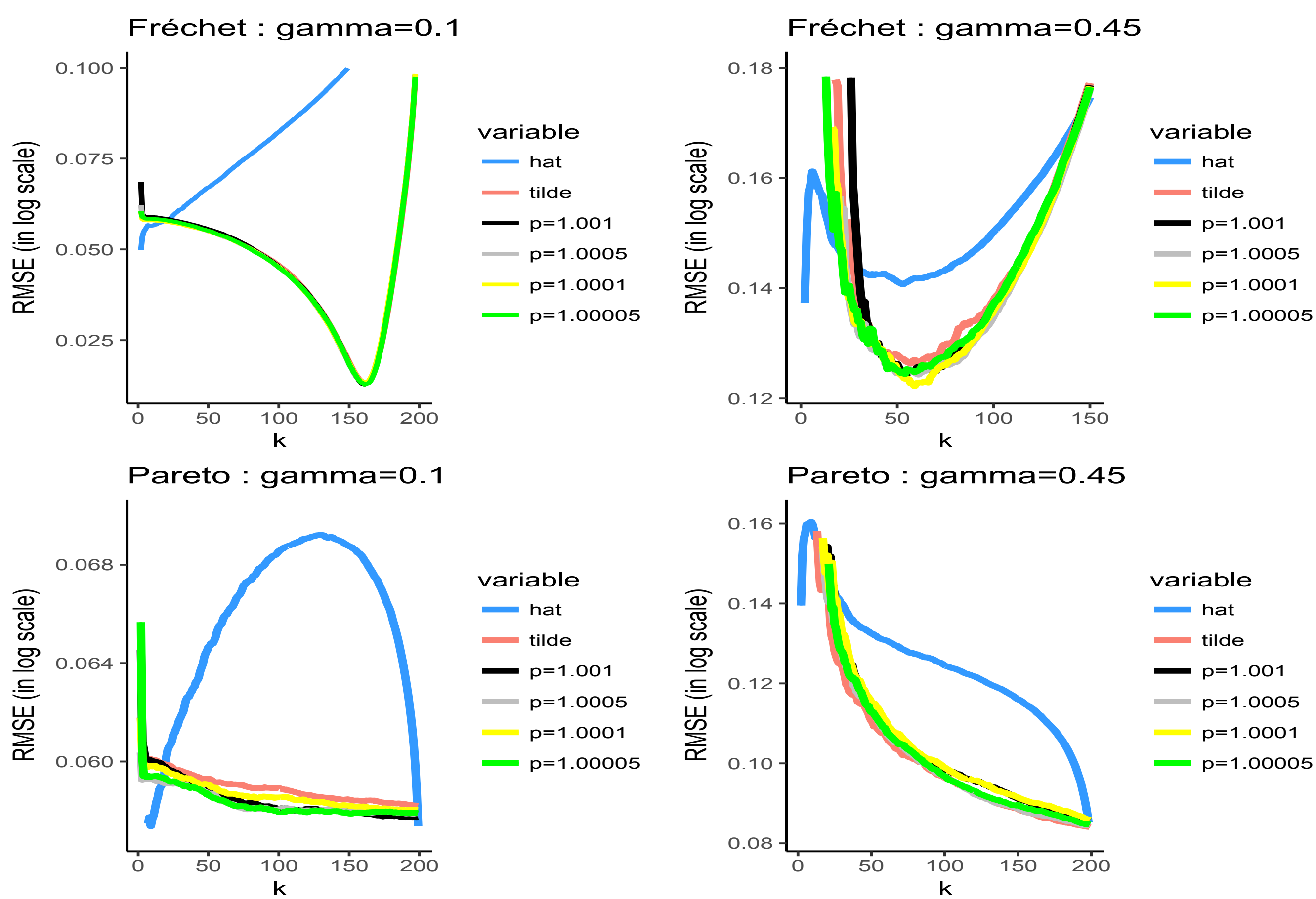


Figure 1: *Fréchet and Pareto distributions—RMSE (in log scale) of* $\widehat{q}^{W}_{\alpha_n}(2)$ *(blue),* $\widetilde{q}^{W}_{\alpha_n}(2)$ *(red), and* $\check{q}^{p}_{\alpha_n}(2)$ *with* $p = 1.001$ *(black),* $p = 1.0005$ *(grey),* $p = 1.0001$ *(yellow) and* $p = 1.00005$ *(green). From left to right,* $\gamma = 0.1, 0.45$. *From top to bottom, Fréchet and Pareto distributions.*

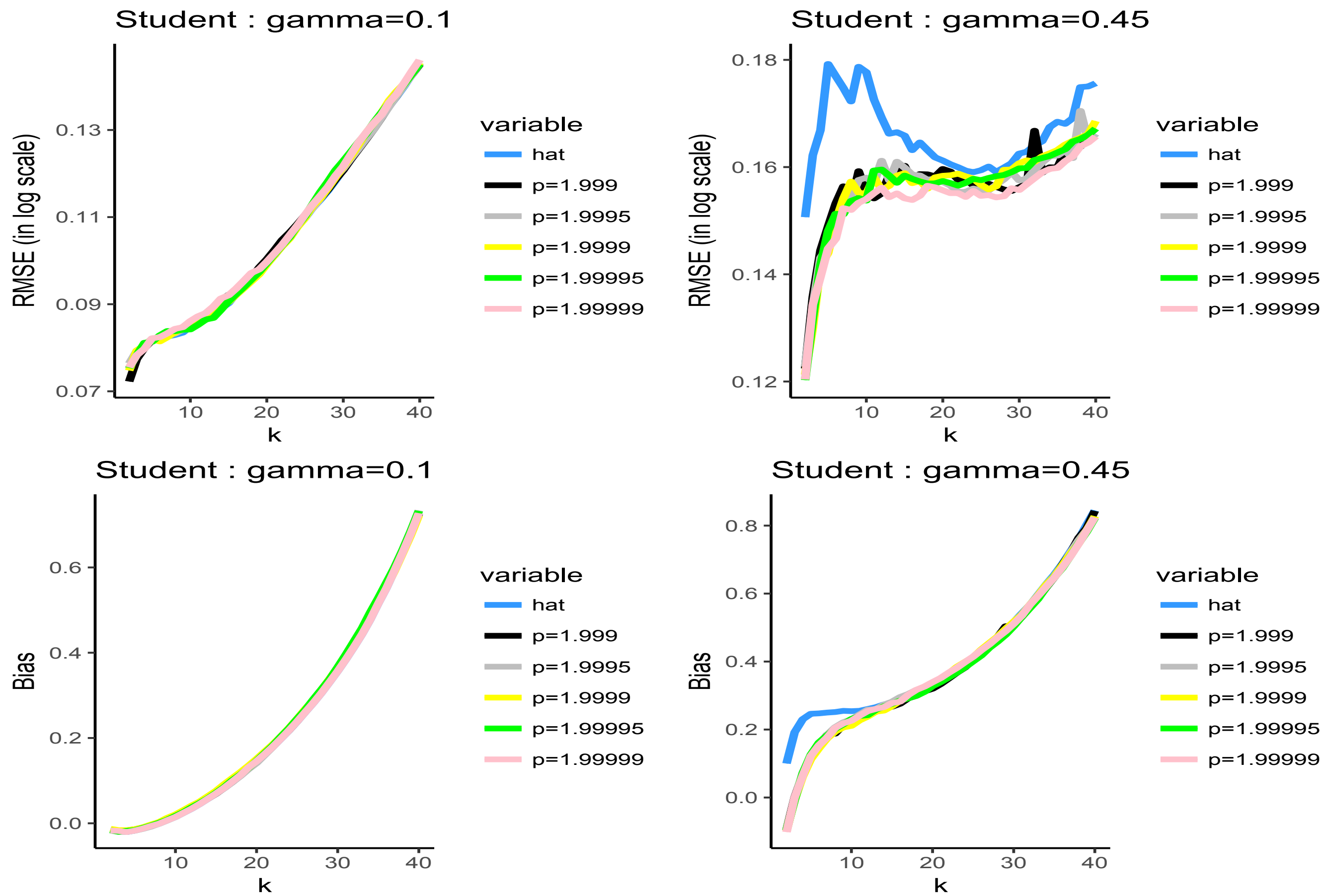


Figure 2: *Student distribution. Top—RMSE (in log scale) of* $\widehat{q}^W_{\alpha_n}(2)$ *(blue) and* $\check{q}^p_{\alpha_n}(2)$ *with* $p = 1.999$ *(black),* $p = 1.9995$ *(grey),* $p = 1.9999$ *(yellow),* $p = 1.99995$ *(green) and* $p = 1.99999$ *(pink). From left to right,* $\gamma = 0.1,\ 0.45$*. Bottom—Bias estimates.*

C.2 Extreme expectile composite estimation

Here, we focus on the composite L^p−estimator $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ in (16) of the extreme expectile $q_{\alpha_n}(2)$, where $\alpha_n = 1 - 1/n$. A comparison with the benchmark estimators $\widehat{q}^W_{\alpha_n}(2)$ in (11) and $\widetilde{q}^W_{\alpha_n}(2)$ in (12) is shown below in Figures 3 and 4. We used the same considerations as in Section 6 and Supplement C.1 for the choice of $\widehat{\gamma}_n$ and the intermediate and extreme expectile levels τ_n and $\tau'_n \equiv \alpha_n$. All the experiments employ the Fréchet, Pareto and Student distributions with tail-indices $\gamma \in \{0.1, 0.45\}$ and various values of the power $p \in (1, 2)$ in the formulation of $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$.

In Fréchet and Pareto models, where $\widetilde{q}^W_{\alpha_n}(2)$ is known to be superior to $\widehat{q}^W_{\alpha_n}(2)$ in terms of MSE, it may be seen from Figure 3 that $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ behaves similarly to $\widetilde{q}^W_{\alpha_n}(2)$ for very small values of p (close to 1). Our simulations also indicate that most of the error is due to variance, the squared bias being much smaller in all cases. We only display in Figure 3 the estimates of the relative MSE (in a log scale) to save space.

In the Student model, where $\widehat{q}^W_{\alpha_n}(2)$ is known to be superior to $\widetilde{q}^W_{\alpha_n}(2)$, it may be seen from Figure 4 that $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ performs at least like $\widehat{q}^W_{\alpha_n}(2)$, for large values of p (close to 2), in terms of both MSE (top panels) and Bias (bottom panels). Interestingly, like $\check{q}^p_{\alpha_n}(2)$ in Supplement C.1, the composite estimator $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ seems to provide a better accuracy relative to $\widehat{q}^W_{\alpha_n}(2)$ in the case of heavier tails ($\gamma = 0.45$).

As regards the second composite estimator $\widetilde{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ in (17), the obtained Monte Carlo estimates do not provide evidence of any added value with respect to the benchmark estimators $\widehat{q}^W_{\alpha_n}(2)$ and $\widetilde{q}^W_{\alpha_n}(2)$, hence the results are not reported here.

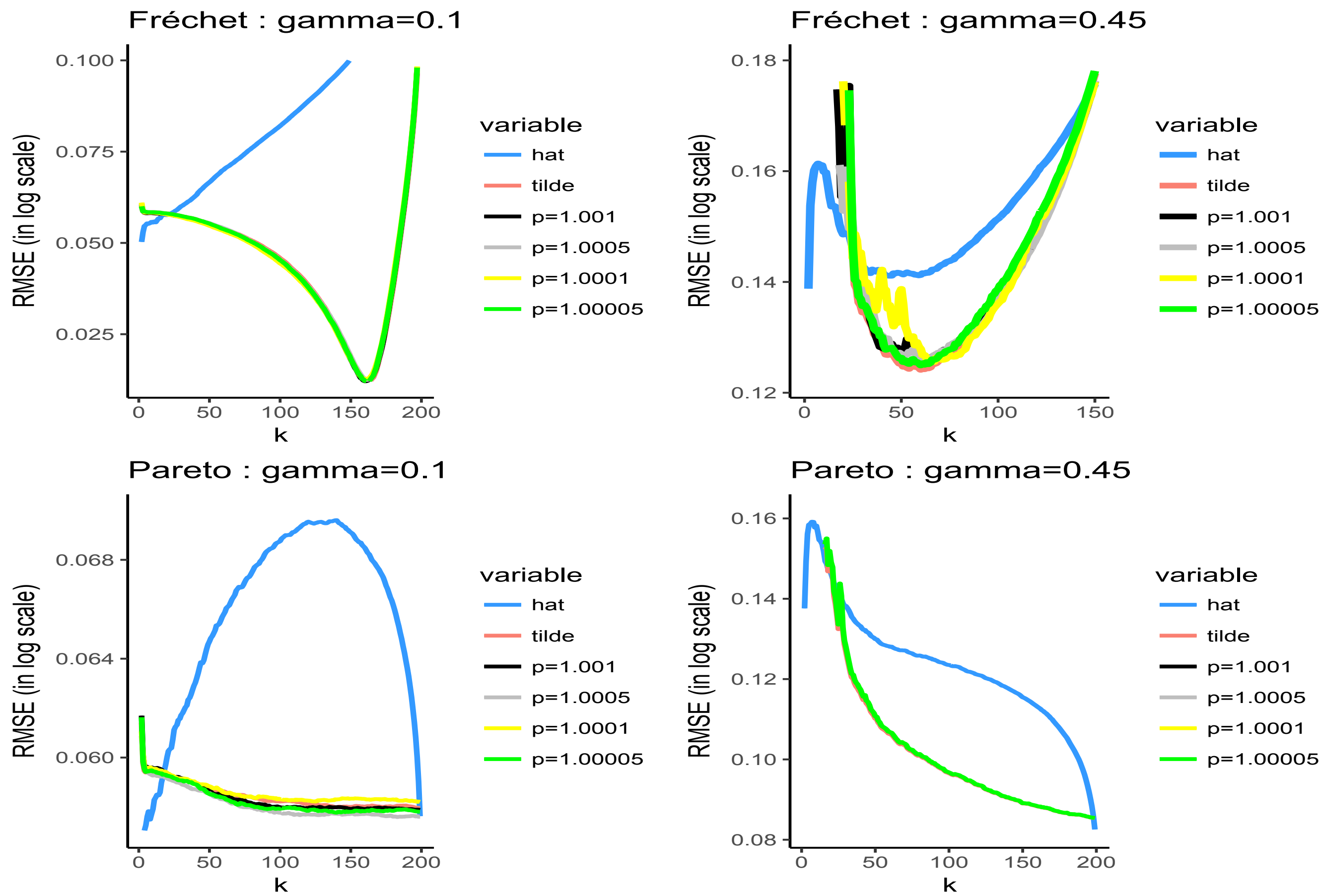


Figure 3: *Fréchet and Pareto distributions—RMSE (in log scale) of* $\widehat{q}^W_{\alpha_n}(2)$ *(blue),* $\widetilde{q}^W_{\alpha_n}(2)$ *(red), and* $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ *with* $p = 1.001$ *(black),* $p = 1.0005$ *(grey),* $p = 1.0001$ *(yellow) and* $p = 1.00005$ *(green). From left to right,* $\gamma = 0.1, 0.45$*. From top to bottom, Fréchet and Pareto distributions.*

C.3 Quality of asymptotic approximations

This section gives Monte Carlo evidence that our limit theorems provide adequate approximations for finite sample sizes. We first investigate the normality of the extrapolated least asymmetrically weighted L^p estimators $\widehat{q}^W_{\tau'_n}(p)$ in (6) and the plug-in Weissman estimators $\widetilde{q}^W_{\tau'_n}(p)$ in (7), for $\tau'_n = 1 - 1/n$ and $p \in \{1.2, 1.5, 1.8\}$. Hereafter we restrict our simulation study to the Student distribution with independent observations. The asymptotic normality of $\widehat{q}^W_{\tau'_n}(p)/q_{\tau'_n}(p)$ in Theorem 2 can be expressed as $r_n \log(\widehat{q}^W_{\tau'_n}(p)/q_{\tau'_n}(p)) \xrightarrow{d} \zeta$, with $r_n = \frac{\sqrt{k}}{\log[k/(n(1-\tau'_n))]}$. Likewise, the asymptotic normality of $\widetilde{q}^W_{\tau'_n}(p)/q_{\tau'_n}(p)$ in Theorem 3 can be expressed as $r_n \log(\widetilde{q}^W_{\tau'_n}(p)/q_{\tau'_n}(p)) \xrightarrow{d} \zeta$. Following Theorems 2.4.1 and 3.2.5 in de Haan and Ferreira (2006, p.50 and p.74), the limit distribution ζ of the Hill estimator under independence is $\mathcal{N}(\lambda/(1-\rho), \gamma^2)$, where $\lambda = \lim_{n\to\infty} \sqrt{k} A(\frac{n}{k})$. It can be shown that a Student t_ν distribution

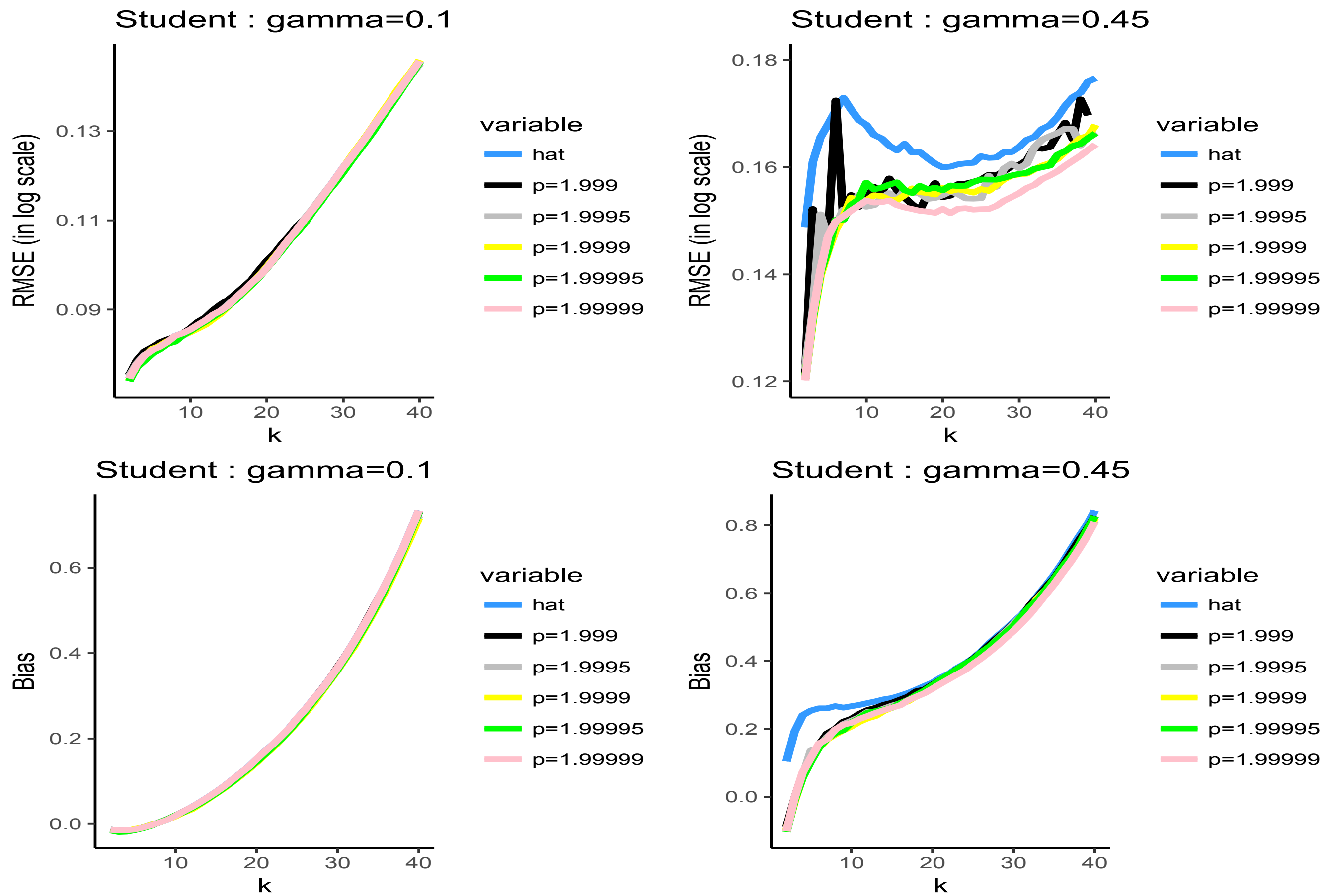


Figure 4: *Student distribution. Top—RMSE (in log scale) of* $\widehat{q}^{W}_{\tau'_n}(2)$ *(blue) and* $\widehat{q}^{W}_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ *with* $p = 1.999$ *(black),* $p = 1.9995$ *(grey),* $p = 1.9999$ *(yellow),* $p = 1.99995$ *(green) and* $p = 1.99999$ *(pink). From left to right,* $\gamma = 0.1$, 0.45. *Bottom—Bias estimates.*

satisfies the conditions of the two aforementioned theorems with $\gamma = 1/\nu$, $\rho = -2/\nu$ and

$$A(t) \sim \frac{\nu+1}{\nu+2}(c_\nu t)^{-2/\nu}, \quad c_\nu = \frac{2\Gamma((\nu+1)/2)\nu^{(\nu-1)/2}}{\sqrt{\nu\pi}\Gamma(\nu/2)}.$$

Hence, we can compare the distributions of

$$\widehat{W}_n := \left[r_n \log(\widehat{q}^{W}_{\tau'_n}(p)/q_{\tau'_n}(p)) - \lambda/(1-\rho)\right]/\gamma \quad \text{and} \quad \widetilde{W}_n := \left[r_n \log(\widetilde{q}^{W}_{\tau'_n}(p)/q_{\tau'_n}(p)) - \lambda/(1-\rho)\right]/\gamma$$

with the limit distribution $\mathcal{N}(0,1)$, with $\lambda \approx \sqrt{k}A(\frac{n}{k})$ for n large enough. The Q–Q-plots in Figures 5 and 6 present, respectively, the sample quantiles of $\widehat{W}_n$ and $\widetilde{W}_n$, based on 3,000 simulated samples of size $n = 1000$, versus the theoretical standard normal quantiles. For each estimator, we used the optimal k selected by the data-driven method described in Section 6.3 of the main article. It may be seen that the scatters for the Student $t_{1/\gamma}$ distributions, with $\gamma = 0.1$, 0.45 displayed respectively from left to right, are quite encouraging for all values of p.

Next, we investigate the normality of the estimators $\check{q}^{p}_{\alpha_n}(2)$ in (13) and $\widehat{q}^{W}_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)$ in (16) of the extreme expectile $q_{\alpha_n}(2)$, where $\alpha_n = 1 - 1/n$. For the Student distribution we used large values of p (close to 2) as suggested by our experiments in Sections C.1 and C.2, but also smaller values of p, namely $p \in \{1.2, 1.5, 1.8, 1.99, 1.999, 1.9999\}$. The asymptotic normality of $\check{q}^{p}_{\alpha_n}(2)/q_{\alpha_n}(2)$ in Theorem 7 can be expressed as $v_n \log(\check{q}^{p}_{\alpha_n}(2)/q_{\alpha_n}(2)) \xrightarrow{d} \zeta$, with

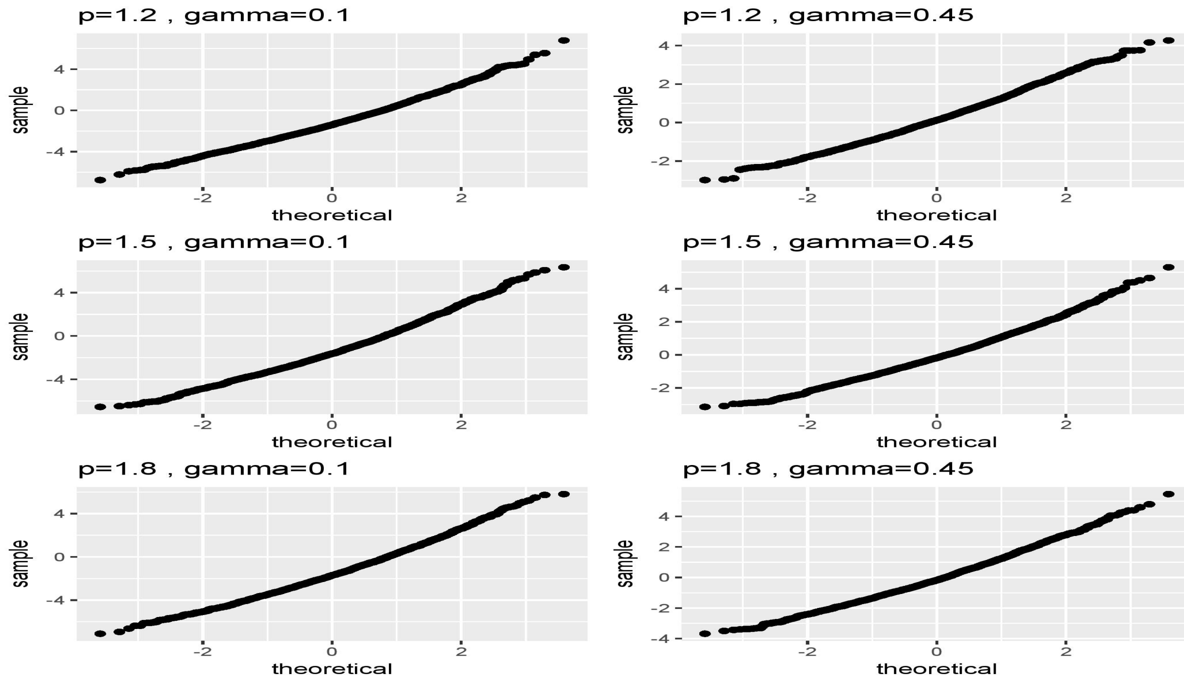


Figure 5: *Q–Q-plots on quality of asymptotic approximations. Each plot shows the sample quantiles of* $\widehat{W}_n$ *versus the theoretical standard normal quantiles, based on* $3,000$ *samples of size* $n = 1000$*. Data are simulated from the Student* $t_{1/\gamma}$ *with* $\gamma = 0.1$ *(left panels) and* $\gamma = 0.45$ *(right panels). From top to bottom,* $p = 1.2, 1.5, 1.8$*.*

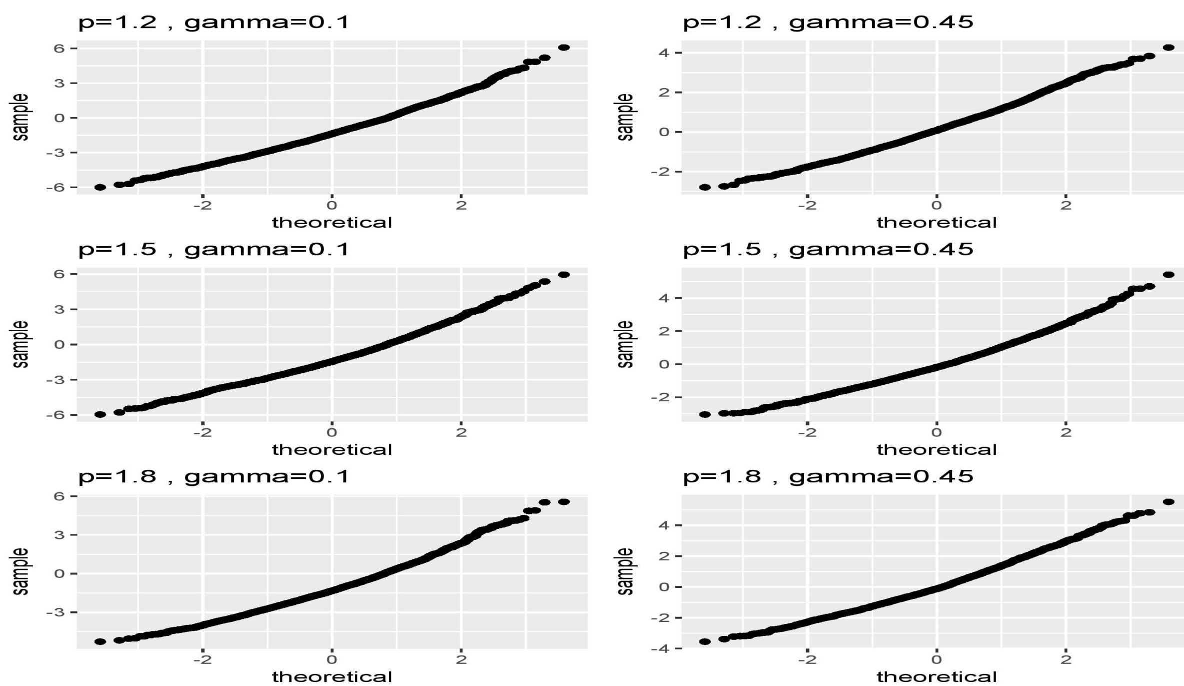


Figure 6: *As before with scatters for* $\widetilde{W}_n$*.*

$v_n = \frac{\sqrt{k}}{\log[k/(n(1-\alpha_n))]}$. Likewise, the asymptotic normality of $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)/q_{\alpha_n}(2)$ in Theorem 8 can be expressed as $v_n \log(\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)/q_{\alpha_n}(2)) \xrightarrow{d} \zeta$. Thus we can compare the distributions of

$$\breve{W}_n := \left[v_n \log(\breve{q}^p_{\alpha_n}(2)/q_{\alpha_n}(2)) - \lambda/(1-\rho)\right]/\gamma \quad \text{and} \quad \widehat{W}_{\widehat{\tau}'_n(p,\alpha_n;2)} := \left[v_n \log(\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;2)}(p)/q_{\alpha_n}(2)) - \lambda/(1-\rho)\right]/\gamma$$

with the limit distribution $\mathcal{N}(0,1)$. The Q–Q-plots in Figures 7 and 8 present, respectively, the sample quantiles of $\breve{W}_n$ and $\widehat{W}_{\widehat{\tau}'_n(p,\alpha_n;2)}$, based on $3,000$ simulated samples of size $n = 1000$ as above, versus the theoretical standard normal quantiles. For each estimator, we used the optimal k selected by the data-driven method. The scatters for the Student $t_{1/\gamma}$ distributions, with $\gamma = 0.1$, 0.45 respectively from left to right, indicate that the limit Theorems 7 and 8 also provide adequate approximations for finite sample sizes.

D Medical insurance data example

We consider here the Society of Actuaries' Group Medical Insurance Large Claims Database which records all the claim amounts exceeding 25,000 USD over the period 1991-92. Similarly to Beirlant *et al.* (2004), we focus on the 75,789 claims for 1991 that we treat as the outcomes of i.i.d. non-negative loss random variables $X_1, \ldots, X_n$. The scatterplot and histogram of the log-claim amounts in Figure 9 (a) give evidence of an important right-skewness. The model assumption of a heavy-tailed loss severity distribution has been already verified in Beirlant *et al.* (2004, p.123) with Hill's estimate $\widehat{\gamma}_n$ around 0.35. Insurance companies typically are interested in an estimate of the claim amount that will be exceeded (on average) only once in 100,000 cases. This translates into estimating the extreme quantile $q_{\alpha_n}(1)$ with the relative frequency $\alpha_n = 1 - \frac{1}{100{,}000} > 1 - \frac{1}{n}$, or equivalently, the generalized L^p-quantile $q_{\tau'_n}(p) \equiv q_{\alpha_n}(1)$ with the extreme level $\tau'_n := \tau'_n(p, \alpha_n; 1)$ described in (9). The Value at Risk $q_{\alpha_n}(1) \equiv q_{\tau'_n(p,\alpha_n;1)}(p)$ can be estimated either by the traditional Weissman quantile estimator $\widehat{q}^W_{\alpha_n}(1)$ defined in (8), or by the composite L^p-quantile estimator $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;1)}(p)$ studied in Theorem 5. To calculate the two estimates $\widehat{q}^W_{\alpha_n}(1)$ and $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;1)}(p)$ of the VaR as well as the estimate $\widehat{\tau}'_n(p, \alpha_n; 1)$ of $\tau'_n(p, \alpha_n; 1)$ defined in (10), we used the optimal sample fraction k selected by the data-driven method described in Section 6.3. The final composite estimates $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;1)}(p)$ are plotted in Figure 9 (b) against the power p in blue, along with the constant traditional estimate $\widehat{q}^W_{\alpha_n}(1)$ in green and the sample maximum in red. None of the two extrapolated VaR estimates $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;1)}(p)$ and $\widehat{q}^W_{\alpha_n}(1)$ exceed the sample maximum $X_{n,n} = 4,518,420$ USD. The classical L^1-quantile based estimator $\widehat{q}^W_{\alpha_n}(1)$ relies on a single order statistic $\widehat{q}_{\tau_n}(1)$, and hence may not respond properly to infrequent large claims. By contrast, the composite L^p-quantile estimator $\widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;1)}(p)$ relies directly on the least asymmetrically weighted L^p estimator $\widehat{q}_{\tau_n}(p)$ given in (3), and hence it bears much better the burden of representing a conservative measure of risk. It may also be seen from the path $p \mapsto \widehat{q}^W_{\widehat{\tau}'_n(p,\alpha_n;1)}(p)$ that this risk measure tends to be more alert to infrequent large claims as the power p increases.

The resulting estimates $\widehat{\tau}'_n(p, \alpha_n; 1)$ of the extreme level $\tau'_n := \tau'_n(p, \alpha_n; 1)$ such that $q_{\tau'_n}(p) \equiv q_{\alpha_n}(1)$ are plotted in Figure 9 (c) against p in blue, along with the constant tail probability α_n in red horizontal line. This plot is of course of capital importance when it comes to use a generalized L^p-quantile $q_{\tau'_n}(p)$, for a given $p \in (1, 2]$, as an alternative risk measure to the quantile-VaR $q_{\alpha_n}(1)$, as it allows to select the value τ'_n such that $q_{\tau'_n}(p) \equiv q_{\alpha_n}(1)$. For instance, if the practitioner wishes to employ the expectile $q_{\tau'_n}(2)$ but still keep the probabilistic interpretation

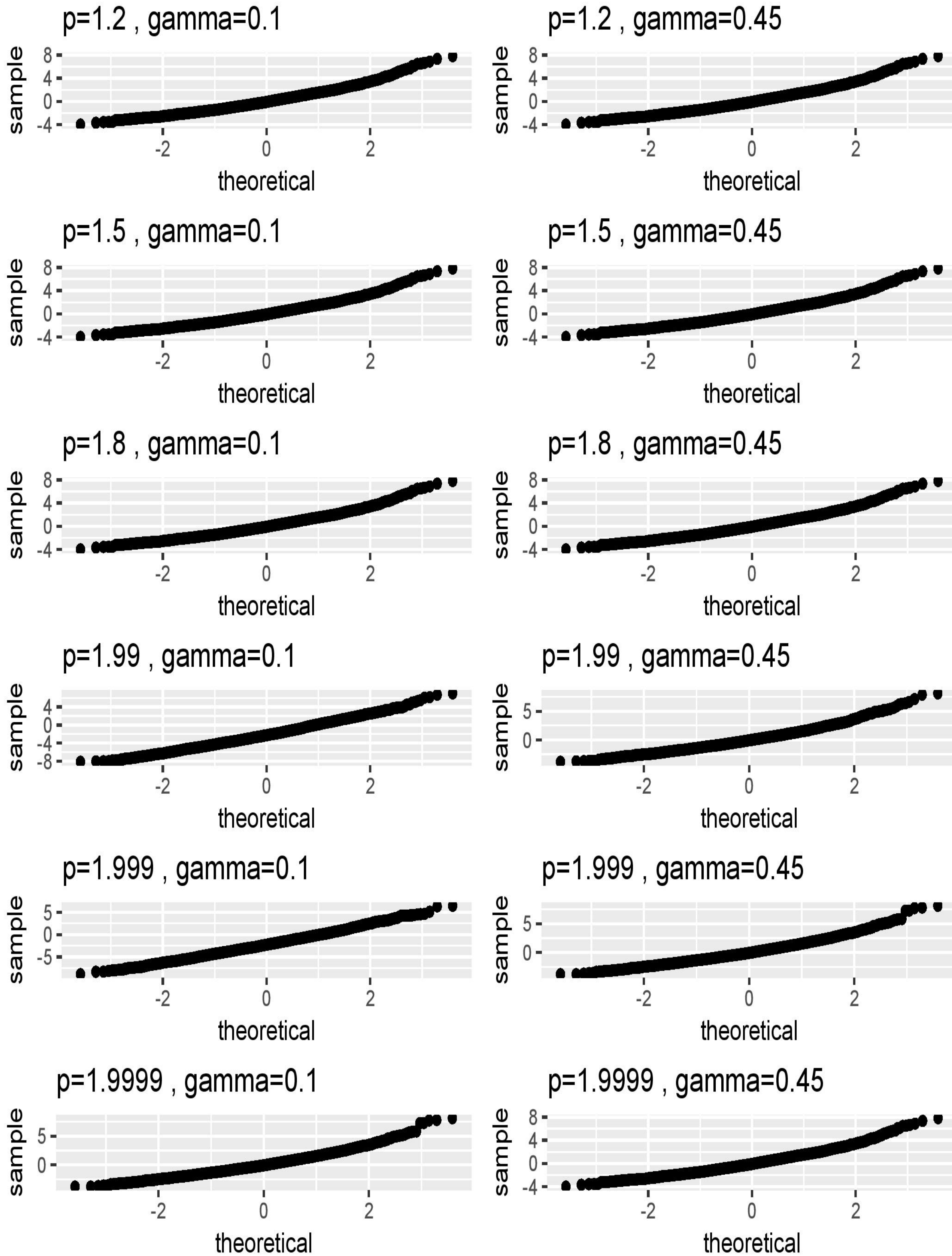


Figure 7: *Q–Q-plots on quality of asymptotic approximations. Each plot shows the sample quantiles of* $\breve{W}_n$ *versus the theoretical standard normal quantiles, based on* $3,000$ *samples of size* $n = 1000$*. Data are simulated from the Student* $t_{1/\gamma}$ *with* $\gamma = 0.1$ *(left panels) and* $\gamma = 0.45$ *(right panels). From top to bottom,* $p = 1.2, 1.5, 1.8, 1.99, 1.999, 1.9999$.

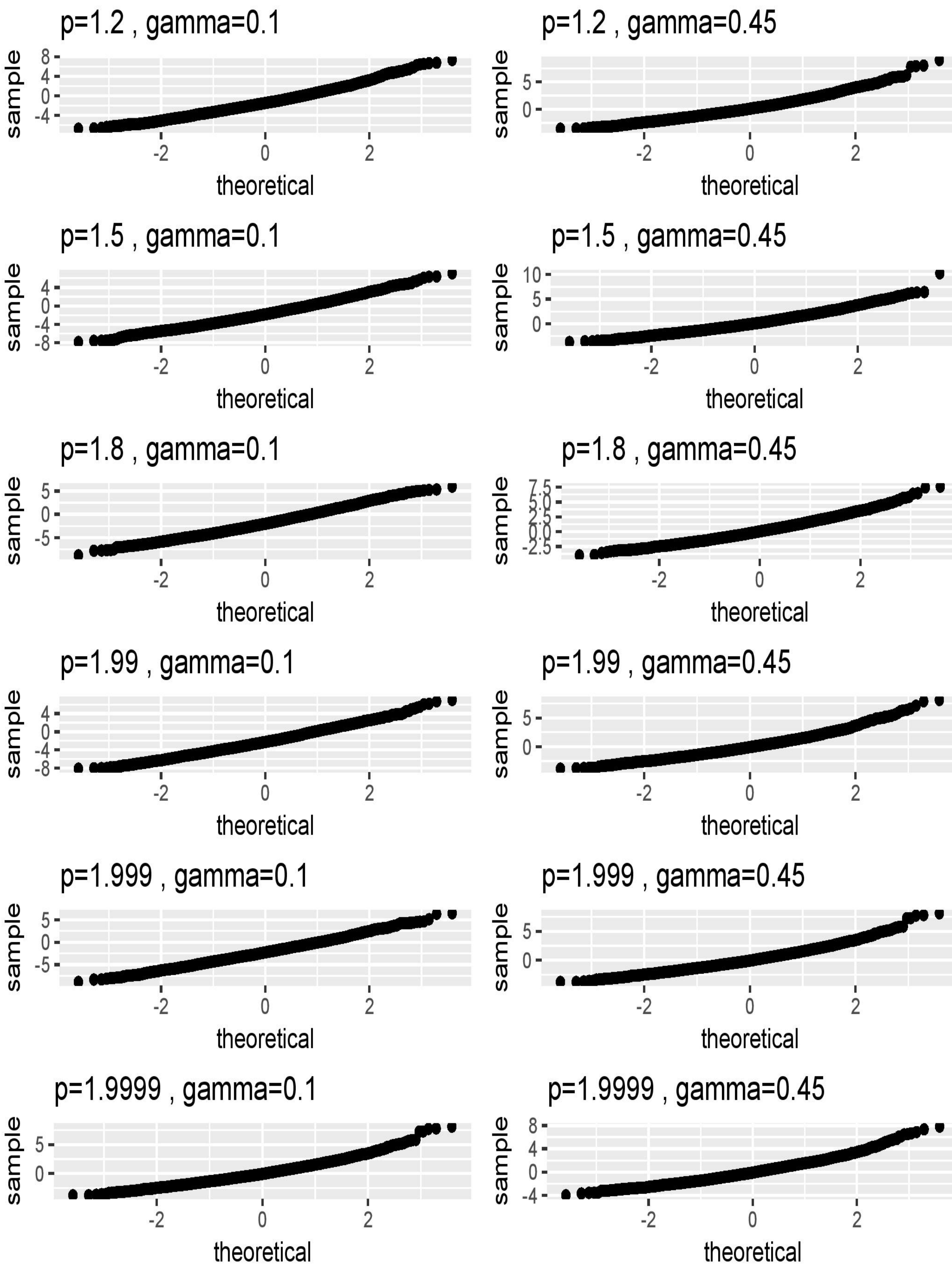


Figure 8: *As before with scatters for* $\widehat{W}_{\hat{\tau}'_n(p,\alpha_n;2)}$.

of $q_{\alpha_n}(1)$, Figure 9 (b) shows that the corresponding expectile level $\tau_n' := \tau_n'(2, \alpha_n; 1)$ may be approximated in the present setup by its estimate $\widehat{\tau}_n'(2, \alpha_n; 1) = 0.9999942$.

If the interest now is in estimating the expectile $q_{0.9999942}(2)$, one may wish to check how the L^p-quantile estimators $\check{q}^{p}_{0.9999942}(2)$ in (13) and $\widehat{q}^{W}_{\widehat{\tau}_n'(p,0.9999942;2)}(p)$ in (16) differ from the benchmark estimators $\widehat{q}^{W}_{0.9999942}(2)$ in (11) and $\widetilde{q}^{W}_{0.9999942}(2)$ in (12) when the power p varies between 1 and 2. In Figure 10 (a) we plot the optimal estimates $p \mapsto \check{q}^{p}_{0.9999942}(2)$ in green, $p \mapsto \widehat{q}^{W}_{\widehat{\tau}_n'(p,0.9999942;2)}(p)$ in blue, $\widehat{q}^{W}_{0.9999942}(2)$ in black, $\widetilde{q}^{W}_{0.9999942}(2)$ in orange, and the sample maximum in red. As is to be expected, the asymmetric least squares estimate $\widehat{q}^{W}_{0.9999942}(2)$, in black line, is clearly more pessimistic than the plug-in estimate $\widetilde{q}^{W}_{0.9999942}(2)$, in orange line, that heavily depends on the optimistic Weissman quantile estimator $\widehat{q}^{W}_{0.9999942}(1)$ as can be seen from (12). The more sophisticated expectile estimate $p \mapsto \check{q}^{p}_{0.9999942}(2)$, as green curve, steers overall a middle course behavior since it approaches $\widehat{q}^{W}_{0.9999942}(2)$ as p tends to 2 and $\widetilde{q}^{W}_{0.9999942}(2)$ when p tends to 1. By contrast, the composite expectile estimate $p \mapsto \widehat{q}^{W}_{\widehat{\tau}_n'(p,0.9999942;2)}(p)$, as blue curve, appears to be the most conservative risk measure, especially as p decays to 1. The evolution of the extrapolated estimator $\widehat{\tau}_n'(p, 0.9999942; 2)$ in (15) of $\tau_n'(p, 0.9999942; 2)$ in (14) is plotted in Figure 10 (b) against p in blue, along with the expectile level 0.9999942 in red line.

References

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. (2004). *Statistics of extremes: Theory and applications*, Wiley.

Bingham, N. H., Goldie, C. M., Teugels, J. L. (1987). *Regular Variation*, Cambridge University Press, Cambridge.

Geyer, C.J. (1996). On the asymptotics of convex stochastic optimization, unpublished manuscript.

de Haan, L., Ferreira, A. (2006). *Extreme Value Theory: An Introduction*, Springer-Verlag, New York.

Ibragimov, I.A. (1962). Some limit theorems for stationary processes, *Theory of Probability and its Applications* **7**(4): 349–382.

Lehmann, E.L. (1966). Some concepts of dependence, *Annals of Mathematical Statistics* **37**(5): 1137–1153.

Utev, S.A. (1990). On the central limit theorem for $\varphi-$mixing arrays of random variables, *Theory of Probability and its Applications* **35**(1): 131–139.

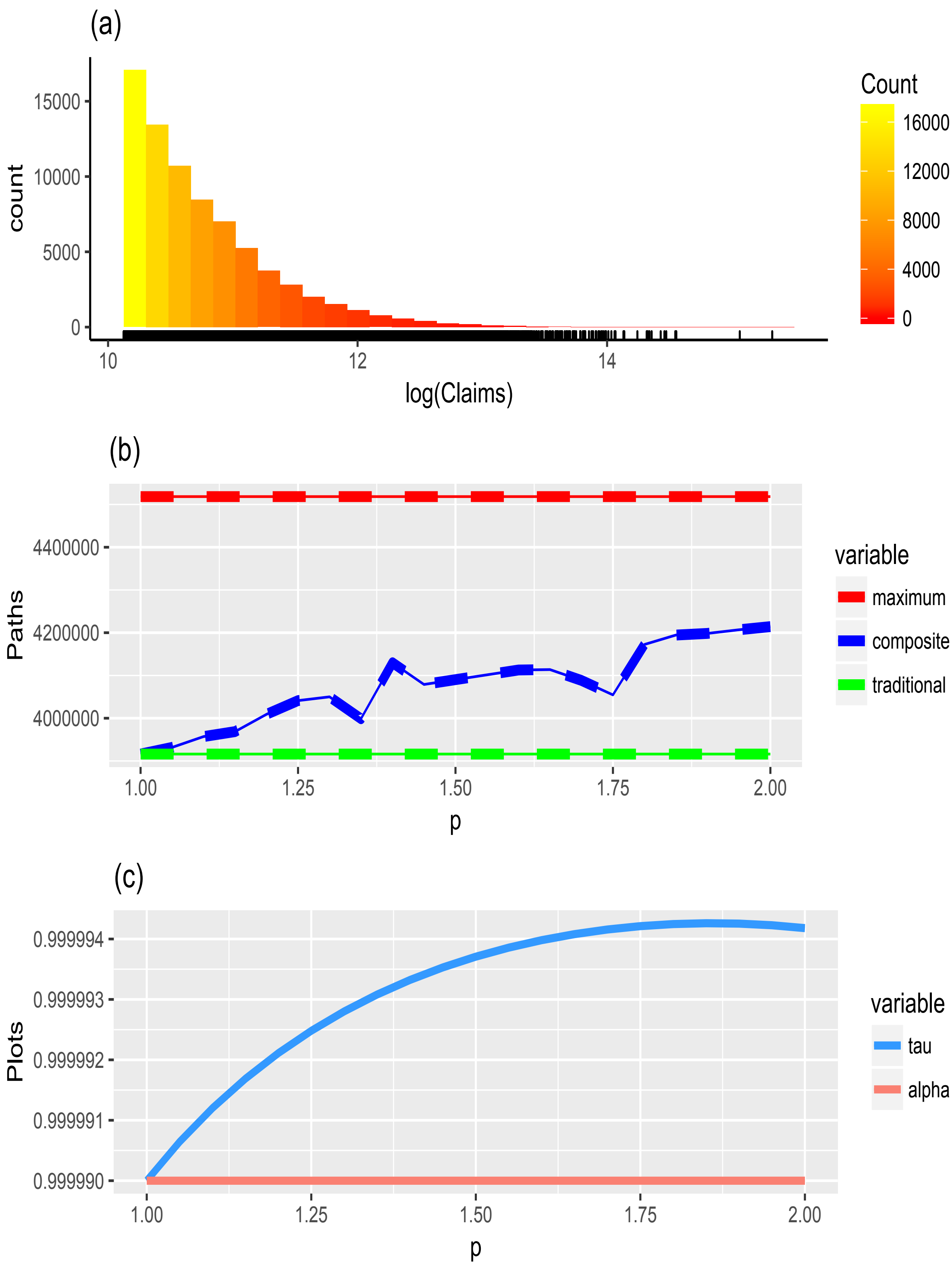
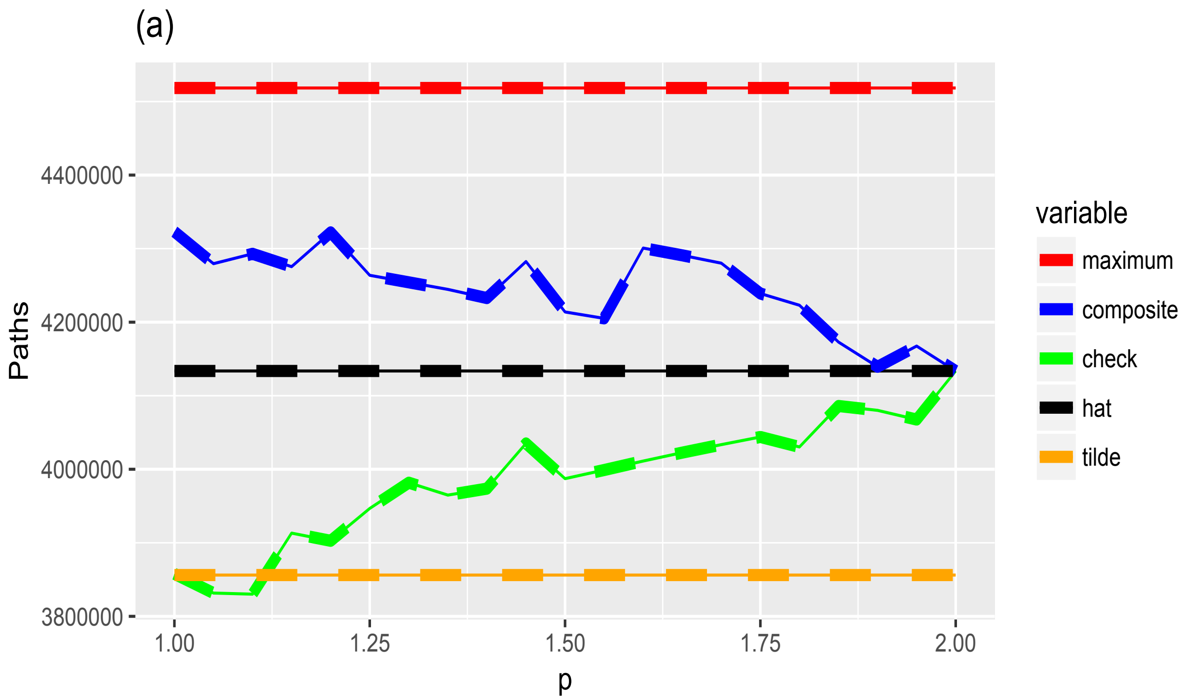


Figure 9: *Medical insurance data. (a)—Scatterplot and histogram. (b)—Composite estimates* $p \mapsto \widehat{q}^{W}_{\widehat{\tau}'_n(p,\alpha_n;1)}(p)$ *in blue, along with the traditional estimate* $\widehat{q}^{W}_{\alpha_n}(1)$ *in green and the sample maximum* $X_{n,n}$ *in red. (c)—Extrapolated estimates* $p \mapsto \widehat{\tau}'_n(p, \alpha_n; 1)$ *in blue, along with the tail probability* α_n *in red line.*



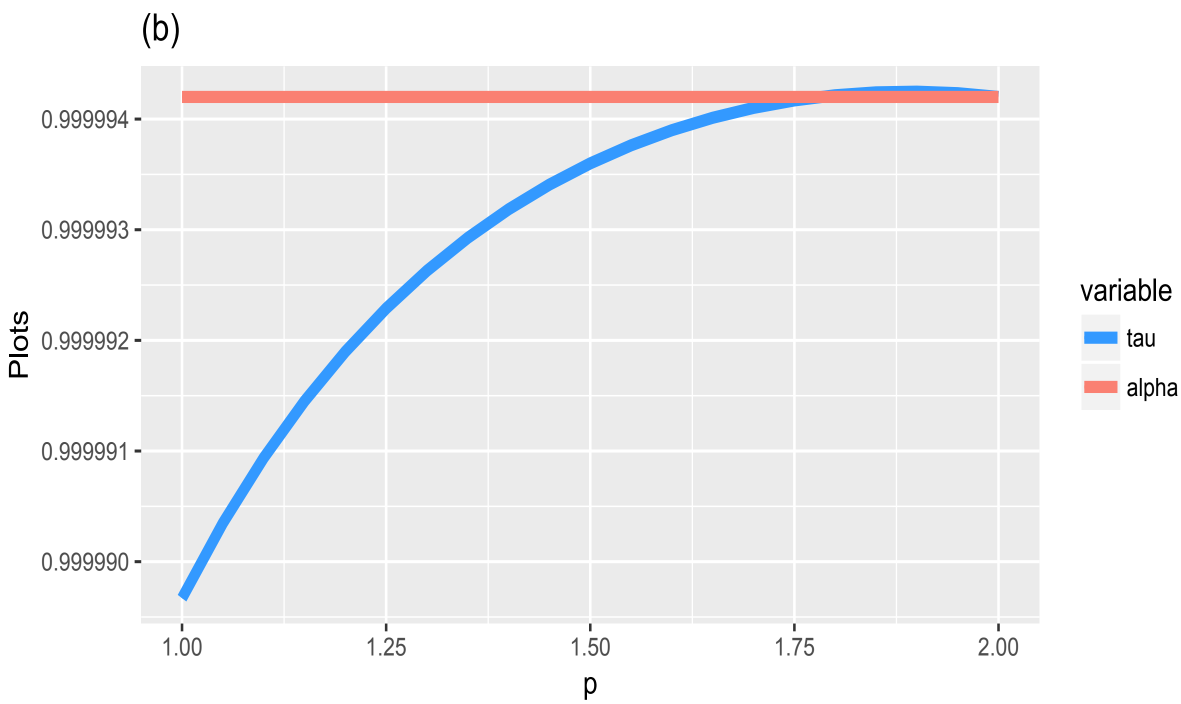


Figure 10: *(a)—The paths* $p \mapsto \check{q}^{p}_{0.9999942}(2)$ *in green and* $p \mapsto \widehat{q}^{W}_{\widehat{\tau}'_n(p,0.9999942;2)}(p)$ *in blue, along with* $\widehat{q}^{W}_{0.9999942}(2)$ *in black,* $\widetilde{q}^{W}_{0.9999942}(2)$ *in orange, and the sample maximum* $X_{n,n}$ *in red. (b)—Extrapolated estimates* $p \mapsto \widehat{\tau}'_n(p, 0.9999942; 2)$ *in blue, along with the expectile level* 0.9999942 *in red line.*