

# The Tsai-Wu Failure Criterion Rationalised in the Context of UD Composites

Shuguang Li<sup>1†</sup>, Elena Sitnikova<sup>1</sup>, Yuning Liang<sup>1</sup> and Abdul-Salam Kaddour<sup>2</sup>

<sup>1</sup>Faculty of Engineering, University of Nottingham, Nottingham NG7 2RD, UK

<sup>2</sup>QinetiQ, Farnborough, GU14 0LX, UK

<sup>†</sup>Corresponding author's email: shuguang.li@nottingham.ac.uk

## Abstract

This paper is to rationalise the empirical aspect of the Tsai-Wu failure criterion in the context of UD composites associated with the determination of the interactive strength property  $F_{12}$  based on the analytic geometry. It reveals that the condition of closed failure envelope cannot be satisfied by all UD composites and hence the restriction should be abandoned. Depending on the way the failure envelope opens, UD composites can be classified into two categories. (a)  $F_{12}$  can be determined uniquely using the conventional strength properties with an additional assumption that the material exhibits very high or infinite strength under triaxial compression at a specific stress ratio; or (b) The Tsai-Wu criterion leads to one of the two scenarios: either allowing infinite strength for an in-plane stress state or allowing infinite strength under triaxial stresses involving tension along fibres.

**Keywords:** A. Tsai-Wu criterion; B. Failure envelope; C. Strength; D. Triaxial compression

## 1 Introduction

Failure criteria have been one of the central subjects in the study of composites and even more so in their applications. It is fair to describe that the failure criteria available so far are mostly applicable practically to unidirectionally fibre reinforced composites (UD), whether as a ply in a laminate or a tow in a textile composite, which can be classified in terms of material anisotropy as a transversely isotropic material. The state-of-the-art has been well-reflected through the series of World Wide Failure Exercises (WWFEs) spanning the past two decades [1-3]. The Tsai-Wu failure criterion is one of the earliest failure criteria proposed originally for materials of the most general anisotropy in a non-phenomenological manner employing a tensorial expression as the failure function [4] whilst most of its practical applications have been made to UD composites. Over the past decades, the criterion has enjoyed remarkable success as it has been employed by researchers and designers all over the world. It has been included in most textbooks on the subject of composites, e.g. [5]. It has also been incorporated in commercial FE codes, such as Abaqus [6]. On the other hand, it has been subjected to criticisms for being non-phenomenological without due consideration

of the failure mechanisms by employing a single quadratic function to account for all possible failure modes observed in experiments [7], although the quadratic function stemmed from a more generic and comprehensive expression [8] as a simplification. More recent attempts of appraising it can be found in [9,10]. The debate on its suitability for engineering applications is likely to continue. Leaving the controversy aside, the performances of this criterion as published in the WWFE-I and II have been appropriately appraised. It is therefore not the purpose of this paper to reproduce such an account in any form. The authors have noticed that Tsai and his co-workers have continued to work to enable more convenient applications, e.g. [11,12].

Nevertheless, there has been one issue of the criterion which has never been thoroughly investigated, and that is the determination of the interactive coefficients  $F_{12}$ . Tsai and his co-workers tended modestly to consider the Tsai-Wu criterion as empirical, e.g. [13-15]. However, the authors would argue that the Tsai-Wu criterion rested on a reasonably rational footing in the form as it was proposed [4], if one defines rationalism as formulation obtained based on logical deductions from a set of predefined assumptions and conditions. On the other hand, if one defines empiricism as opposed to rationalism, it is associated with measures taken based mostly on experience or intuition with limited justifications. It should be pointed out that employment of experimental data does not compromise rationality whilst extrapolating experimental data or logical consequences does. In this sense, the Tsai-Wu criterion was rational conceptually with the failure function introduced as an assumption, if all coefficients could be determined experimentally. In reality, though, the determination of all coefficients has never been achieved, in particular, those associated with interactive stress terms, and it does not seem likely that it will be achieved in any foreseeable future. Users of the criterion have therefore been left with a room for empiricism. Putting in the context of UD composites as a type of transversely isotropic materials, which accounts for most applications of the criterion and indeed most applications of composites as a whole, the empiricism in the Tsai-Wu criterion was associated only with the determination of  $F_{12}$ , if its special application to this specific type of materials is considered as a predefined condition.

Given the nature of the Tsai-Wu failure criterion, determination of  $F_{12}$  through experimental means proves to be difficult as applying biaxial loads up to the failure of the material at a representative stress ratio has always been a challenge. The way  $F_{12}$  was obtained in the Tsai-Wu criterion is empirical supported by limited justifications. Various attempts of determining it have been summarised and presented in Table 1 [16-25]. Although the Tsai-Wu criterion has been a well-known theory in the composites community, favoured by some while objected by others, it is the authors' view that users' judgements had been made largely without the full facts. The objective of this paper is to offer one missing facet associated with the determination of the interactive coefficient,  $F_{12}$ , on a rational basis to eliminate the empiricism, as far as its applications to UD composites are

concerned. For this reason, the aim of this paper is neither to promote the use of the criterion in engineering nor to undermine it, but to develop the criterion as a theory to its full maturity for UD composites such that those who favour the theory will have a better reason to continue their practices with likely improved level confidence while those objecting it will have the full facts in front of them to support their views. Equally true, having the full facts established, some of the users might change their views either way. It is not for this paper to dictate the readers' views and users still have to come to their own assessments objectively and intellectually. For this reason, no efforts will be made to compare with experimental results to justify or disapprove the criterion. Ultimately, experimental data will serve as the proof but such data are unavailable. There is therefore a desperate need for them to be made available in order to address the failure of composites.

## 2 The Original Tsai-Wu criterion

With conventional assumptions, such as homogeneity and linear elasticity up to failure, the Tsai-Wu criterion [4] was originally proposed in the context generally anisotropic materials by using a quadratic polynomial expression of stresses with tensorial coefficients as a simplified form from a more comprehensive but less practical form [8]. The tensorial expressions employed enables its general applicability in terms of coordinate systems to be adopted to describe the problem. However, generally anisotropic materials are not often encountered in practice, and the world is in fact not quite ready for them, if one is honest, given the coverage of existing industrial standards [26,27]. If any, they are likely to be orthotropic materials but put in a coordinate system off their materials principal axes. As a result, the most familiar form of the Tsai-Wu criterion employs the following failure function for orthotropic materials in their materials' principal axes

$$F = F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{33}\sigma_3^2 + 2F_{23}\sigma_2\sigma_3 + 2F_{13}\sigma_1\sigma_3 + 2F_{12}\sigma_1\sigma_2 + F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3 + F_{44}\tau_{23}^2 + F_{55}\tau_{13}^2 + F_{66}\tau_{12}^2 \quad (1)$$

which has reflected the material symmetries as present in orthotropic materials. In this function,  $F_{ij}$  and  $F_i$  ( $i,j=1,2,\dots,6$ ) are contracted forms of 4<sup>th</sup> and 2<sup>nd</sup> ranked tensors, whilst the components not appearing in the function, such as  $F_{16}$  and  $F_4$ , are deemed to vanish due to the material symmetries. To deliver a failure criterion, it is claimed that the material is safe if

$$F < 1 \quad (2)$$

while the critical condition for failure is predicted when

$$F = 1. \quad (3)$$

In fact, there have not been many genuine generally orthotropic materials. It is generally unsatisfactory to consider laminated composites, e.g. cross-ply, or 3D textile composites as a material of orthotropy as far as their strength predictions are concerned, even they might exhibit orthotropic

elastic behaviour macroscopically. Unlike elastic properties which are dominated by the global behaviour at a macroscale, strengths are dictated by localised features at a meso/micro level. Even in so-called non-local theories, e.g. [28], it is still local enough with the focus placed in a small neighbourhood of a point of singularity, such as a crack tip. Fortunately, most composites are made of UD building blocks, whether as laminae of a laminate or tows of a textile composite. There is always sufficient interest in understanding the behaviour of UD composites. The present paper should limit the discussion of the Tsai-Wu criterion within this category of composites in terms of its applicability.

Given the random distribution of fibres in the cross-section of most UD composites, transverse isotropy is usually a sufficiently satisfactory description of behaviour of UD composites, for which one has

$$F_{33} = F_{22}, \quad F_{13} = F_{12}, \quad F_3 = F_2, \quad F_{55} = F_{66},$$

and 
$$F_{23} = F_{22} - \frac{1}{2}F_{44}. \quad (4)$$

The Tsai-Wu failure function can then be reduced to [4]

$$F = F_{11}\sigma_1^2 + F_{22}(\sigma_2^2 + \sigma_3^2) + (2F_{22} - F_{44})\sigma_2\sigma_3 + 2F_{12}\sigma_1(\sigma_3 + \sigma_2) + F_1\sigma_1 + F_2\sigma_2 + F_2\sigma_3 + F_{44}\tau_{23}^2 + F_{66}\tau_{13}^2 + F_{66}\tau_{12}^2 \quad (5)$$

where most of the coefficients involved can be determined as follows from the conventional strengths of UD composites as

$$\begin{aligned} F_{11} &= \frac{1}{\sigma_{1t}^* \sigma_{1c}^*} & F_{22} &= \frac{1}{\sigma_{2t}^* \sigma_{2c}^*}, \\ F_1 &= \frac{1}{\sigma_{1t}^*} - \frac{1}{\sigma_{1c}^*}, & F_2 &= \frac{1}{\sigma_{2t}^*} - \frac{1}{\sigma_{2c}^*}, \\ F_{44} &= \frac{1}{(\tau_{23}^*)^2} & \text{and} & \quad F_{66} = \frac{1}{(\tau_{12}^*)^2} \end{aligned} \quad (6)$$

with  $\sigma_{1t}^*$  and  $\sigma_{1c}^*$  being the tensile and compressive strengths of the material along fibres,  $\sigma_{2t}^*$  and  $\sigma_{2c}^*$  those in the direction transverse to the fibres, and  $\tau_{12}^*$  and  $\tau_{23}^*$  the shear strengths along and transverse to fibres. These conventional strength properties of typical UD composites should be obtained when they are loaded under uniaxial stress states or pure shear stress states in their material's principal axis. There are standards available for the experimental measurement of them, e.g. [26,27].

However, there is still one coefficient  $F_{12}$  which has not yet been specified above and it should ideally be determined through biaxial stress tests. Given the difficulties in conducting this type of tests and the lack of standard testing procedure, no standard experimental method is available to determine it.

In [4], Tsai and Wu insisted that the failure envelope must be an ellipsoid and hence remains closed. According to analytic geometry [29], this condition offers some constraints on  $F_{12}$  but the constraints are given as ranges rather than any fixed value for  $F_{12}$ .

For most applications under in-plane stresses, (1) or (5) can be rewritten in its 2D form as

$$F = F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2. \quad (7)$$

As  $F_{12}$  involves only direct stresses  $\sigma_1$  and  $\sigma_2$ , some considerations can be made below when the material is subject to biaxial direct stresses. The critical condition (3) can be simplified in this case to

$$F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_1\sigma_1 + F_2\sigma_2 = 1. \quad (8)$$

This defines a typical conic section in the  $\sigma_1$ - $\sigma_2$  plane. The condition for the failure locus in the  $\sigma_1$ - $\sigma_2$  plane to be an ellipse is given largely as

$$F_{12}^2 < F_{11}F_{22}. \quad (9)$$

However, this only defines a range for  $F_{12}$ , which appears to be rather wide in most cases. The complete determination of  $F_{12}$  remains as an issue to be resolved. It has been left as an empirical parameter [14,15]. One form of it has been suggested as [13,30]

$$F_{12} = -\frac{1}{2}\sqrt{F_{11}F_{22}} \quad (10)$$

which was expressed in terms of conventional strength properties. The justifications for the particular form (10) are

- (a) It falls in the range as defined by (9), and
- (b) It allows itself to be degenerated to that of von Mises if the material is specialised to isotropic having equal tensile and compressive strengths.

While some may argue that it fits well with some of the experiments, others can always produce a counterargument in other cases. Expression (10) satisfies both consideration (a) and (b) as stated above, but it cannot be deduced from considerations (a) and (b). A different choice from (10) for  $F_{12}$  could be  $-F_{11}/2$ , which falls within the range (9) for and also degenerates to the von Mises criterion. This, in fact, reproduces the Hoffman criterion [31] which came to light in the same era as [4] but slightly earlier. The focus of the present paper is on this coefficient where hitherto empiricism has been the norm [32], given the presence of attempts of bringing rationality into the determination of this strength coefficient, e.g. [20,21].

For the sake of clarity, the following terminologies will be adopted in this paper. The function of stresses, such as those given in (1), (5) and (7) will be called failure functions, based on which the failure criterion (3) is presented. The spatial surface as given by (3) will be called the failure envelope in the six dimensional stress space, in general. Without any confusion, that in a three dimensional

subspace associated with the direct stresses only is also called the failure envelope. The intersection of the failure envelope with any plane, e.g. coordinate plane  $\sigma_1$ - $\sigma_2$ , is called a failure locus in this plane.

Plotting (8) as a quadratic failure locus in the  $\sigma_1$ - $\sigma_2$  plane, the four conventional strength properties of tensile and compressive strengths along and transverse to fibres offer four anchoring points at the intersections with the coordinate axes, as illustrated in Fig. 1. They are apparently insufficient to determine the ellipse completely as a conic section involves five independent constants, in general, as illustrated in Fig. 1 where both ellipses pass those four anchoring points but they are apparently different. The interactive term  $F_{12}$  plays exactly the role of providing another anchoring point in the  $\sigma_1$ - $\sigma_2$  plane so that the ellipse can be uniquely determined. Different values of  $F_{12}$  tend to tilt the ellipse as illustrate in Fig. 1.

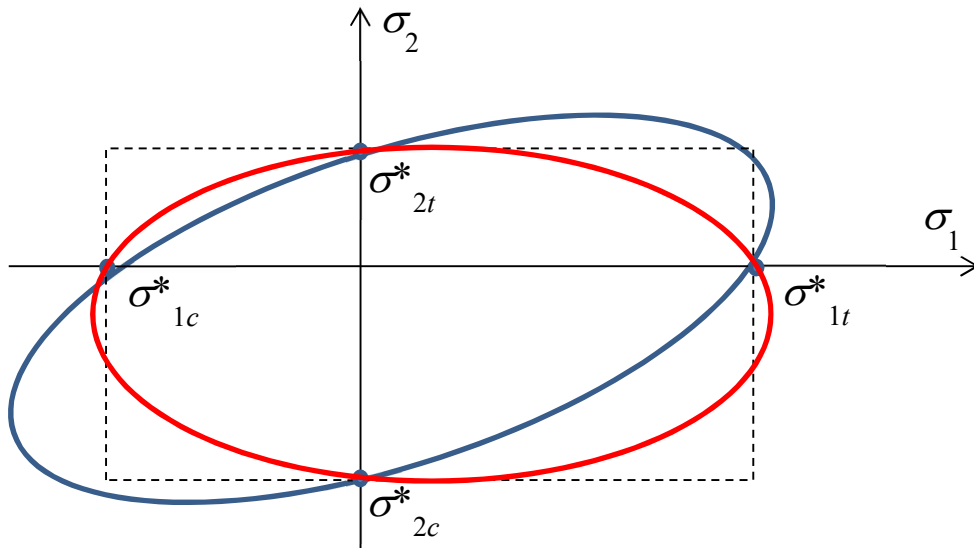


Fig.1 Elliptic failure loci determined to four anchor points on the axes

Failure criteria based on failure functions, such as (1), (5) and (7), for special material systems or stress states, are special forms of its general presentation for completely anisotropic materials under 3D stress states as proposed in [4], as had been provided as special cases also in [4], and they have all been referred to as the Tsai-Wu criterion in the literature in the special context concerned. When they were presented in [4], the coefficient(s) to the interactive terms were meant to be determined through biaxial tests. The lack of success in achieving their experimental determination must have triggered empirical alternatives, e.g. expression (10) as was found in [30] which seemed to have dominated the practical applications of the Tsai-Wu criterion since then. In the context of the present paper, the Tsai-Wu criterion employing expressions like (10) to determine the coefficients to the interactive terms in the failure function will be referred to as *the original Tsai-Wu criterion*, whilst the objective of this paper is to tackle the empirical aspect of the determination of such a coefficient.

### 3 Abandonment of the restriction of closed failure envelopes

Tsai and Wu only considered the possibility of closed failure envelopes in their original paper [4]. This will be disputed in this section as a starting point in the context of its applications to transversely isotropic materials for which the failure function is given by (5). It will be revealing to examine the failure locus in the  $\sigma_2$ - $\sigma_3$  plane where  $\sigma_1=0$ . In this case, the critical condition (3) becomes

$$F_{22}\sigma_2^2 + F_{22}\sigma_3^2 + (2F_{22} - F_{44})\sigma_2\sigma_3 + F_2\sigma_2 + F_2\sigma_3 = 1. \quad (11)$$

According to the established rules in analytic geometry [29], the above equation defines an ellipse in the  $\sigma_2 - \sigma_3$  plane if the second invariant of the above conic section as a discriminator satisfies

$$D = F_{22}^2 - \frac{1}{4}(2F_{22} - F_{44})^2 = \frac{1}{4}F_{44}(4F_{22} - F_{44}) = \frac{1}{4}F_{44}^2\delta > 0 \quad (12)$$

$$\text{where } \delta = 4 - \frac{F_{44}}{F_{22}} = 4 - \frac{\sigma_{2t}^* \sigma_{2c}^*}{(\tau_{23}^*)^2}. \quad (13)$$

It will be shown later that  $\delta$  is a characteristic parameter for UD composites. For the conic section to give a real ellipse, as opposed to an imaginary one, there is in fact another condition to be placed on the first and third invariants. However, it seems to be implied by (12) for most practical materials. For conic sections, the concept of being a closed locus is equivalent to being an ellipse as otherwise the locus would be either a parabola or a hyperbola, which is deemed to be open. Condition (12) is equivalent to

$$\delta > 0. \quad (14)$$

This places a restrictive condition on some of the strength properties for the failure locus (11) to be elliptic in the  $\sigma_2$ - $\sigma_3$  plane.

Unfortunately, this condition cannot be satisfied by every material. To facilitate the discussions to follow, nine different UD composites as employed in the WWFE-I, II & III [1-3] (composites of the same effective strength properties are considered as the same material for the present discussion) are quoted with their relevant strength properties listed in Table 2. Among the nine different materials, two of them do not satisfy condition (14). They produce negative values for  $\delta$  as shown in the shaded columns in Table 2, i.e.

$$\delta < 0 \quad (15)$$

and fall outside the category of (14). As a result, the failure locus is hyperbolic in the  $\sigma_2$ - $\sigma_3$  plane, which is open.

According to the analytic geometry [29], if

$$\delta = 0 \quad (16)$$

the conic section in the  $\sigma_2$ - $\sigma_3$  plane as given by (11) will be a parabola which is also open. A parabolic

failure locus in the  $\sigma_2$ - $\sigma_3$  plane as defined by (16) gives a critical position between ellipse and hyperbola but will unlikely be satisfied precisely by a material with strengths  $\sigma_{2t}^*$ ,  $\sigma_{2c}^*$  and  $\tau_{23}^*$  physically measured from experiments.

A locus in the  $\sigma_2$ - $\sigma_3$  plane is the intersection of the failure envelope in the six dimensional stress space to the  $\sigma_2$ - $\sigma_3$  plane. If the locus as an intersection is open in the  $\sigma_2$ - $\sigma_3$  plane, the failure envelope in the stress space cannot be closed. A clear conclusion obtained from the above elaboration is that, for the failure criterion as defined by failure function (5), an open failure envelope should be allowed for it to be applicable to all materials consistently. The restriction of closed failure envelopes contradicts with the rule of analytic geometry for some practical materials. This should therefore be abandoned as a rational step forward.

In general terms, having an open failure envelope is not really entirely unacceptable. The failure envelope given by the von Mises criterion is open as hydrostatic stress makes no contribution to the failure and therefore the strength against this stress condition is infinite, although the failure mechanisms in materials suitable for the von Mises criterion, typically metals, and those in composites are characteristically different. The Tsai-Wu criterion was supposed to degenerate to that of von Mises for isotropic materials of equal tensile and compressive strengths [13,30], which would not be possible without allowing an open failure envelope in general. The Hashin criteria [33] also assumed infinite strength for equal biaxial transverse compression, which seemed having been well-accepted without implying authors' approval. This was duplicated in [20,21] for the same purpose of determining  $F_{12}$  which will be seen as an inferior assumption as there is a better one as will be adopted in the present paper. The openness of the failure envelope does not necessarily exclude the closeness of failure loci as the intersections between the open failure envelope and some of the representative planes in the stress space, as will be elaborated later in this paper.

#### **4 A rational way of determining $F_{12}$**

As another attempt of determining  $F_{12}$  for the Tsai-Wu criterion, the closed failure envelope restriction will be will be abandoned. The underlying consideration is that, instead of prohibiting the openness of the failure envelope, it would be more productive to manage it. As a result, the Tsai-Wu criterion can be fully rationalised in the context of transversely isotropic composites. A logical outcome can be deduced from the following three basic assumptions for the present endeavour.

- i) The failure is determined using a single quadratic function of stresses;
- ii) The material is transversely isotropic; and
- iii) The composites exhibit much higher strength, which can be taken as infinite for mathematical convenience, under triaxial compression at a specific stress ratio, than any



of the strengths obtained under a uniaxial stress or pure shear in principal axes.

The third assumption above, whilst widens the range for  $F_{12}$  greatly, has been set as tightly as possible in order to narrow the openness as much as possible.

Since the framework based on the above assumptions represents a substantial deviation from the original Tsai-Wu criterion, with (ii) as a specialisation whilst (iii) as a fundamental extension, the outcome will therefore be called *the rationalised Tsai-Wu criterion* to distinguish it from *the original Tsai-Wu criterion* as previously defined. A similar assumption to (iii) was made in the Hashin criterion [33] for equal biaxial compression in the plane transverse to the fibres. Its counterpart in the von Mises criterion is infinite strength under hydrostatic pressure. However, it will be shown that the reproduction of the von Mises criterion from the rationalised Tsai-Wu criterion for isotropic materials of equal tensile and compressive strength is not a requirement a priori. It can be obtained as a natural consequence. These two stress conditions, viz. equal biaxial compression in the plane transverse to the fibres and hydrostatic compression, were taken in [20,21] and the expressions for  $F_{12}$  were derived from each of them. In the context of the paper, a search will be conducted for the stress ratio which maximises the strength. If a more favourable stress ratio can be identified to exhibit higher strength than these two specific stress ratios, it will be more representative for an ‘infinite strength’. The approach taken here is considered to be rational because it will be based on the above three assumptions only and the rest are purely logical and mathematical deduction.

The coefficient  $F_{12}$  characterises only the interaction between direct stresses  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  (under triaxial stresses with  $F_{13}=F_{12}$  under the assumption of transverse isotropy) and therefore it is sufficient to consider stress states involving these direct stresses. Given the transverse isotropy, under any stress state with  $\sigma_2 \neq \sigma_3$ , distortion arises in the  $\sigma_2$ - $\sigma_3$  plane which undermines the strength. In order to achieve the highest strength, it is therefore sufficient to consider only stress states having  $\sigma_2 = \sigma_3$  in addition to an independent  $\sigma_1$  by assuming a triaxial compressive stress state

$$\sigma_1 = -r\sigma^* \quad \text{and} \quad \sigma_2 = \sigma_3 = -\sigma^* \quad (\text{with} \quad \tau_{23} = \tau_{13} = \tau_{12} = 0) \quad (17)$$

where  $\sigma^*$  is a positive value indicating the level of triaxial compression when failure takes place and  $r$  the ratio between the longitudinal stress and the transverse stresses. For the stress state to be triaxial compression, one must have  $r > 0$ . Substituting these stresses into the failure function (5), one obtains

$$F = (r^2 F_{11} + 4F_{22} - F_{44} + 4rF_{12})(\sigma^*)^2 - (rF_1 + 2F_2)\sigma^*. \quad (18)$$

Failure is characterised by the critical condition

$$f = (r^2 F_{11} + 4F_{22} - F_{44} + 4rF_{12})(\sigma^*)^2 - (rF_1 + 2F_2)\sigma^* - 1 = 0. \quad (19)$$

This defines  $\sigma^*$  as an implicit function of  $r$

$$\sigma^* = \sigma^*(r). \quad (20)$$

It is conceivable that  $\sigma^*$  varies with  $r$  and  $\sigma^*$  can take its highest value at an appropriate value of the stress ratio  $r$ . Finding the favourable stress ratio to achieve the highest strength can be presented as an extreme value problem of an implicit function which is solved mathematically below. In order to find the extreme value of  $\sigma^*$ , the derivative of  $\sigma^*$  with respect to  $r$  is required. Following the rule of derivatives of implicit functions [29], one has

$$\frac{d\sigma^*}{dr} = - \frac{\partial f}{\partial r} / \frac{\partial f}{\partial \sigma^*} \quad (21)$$

where  $f$  as given in (19) is considered as a function of  $\sigma^*$  and  $r$ . Its partial derivatives can be found as follows

$$\begin{aligned} \frac{\partial f}{\partial \sigma^*} &= 2(r^2 F_{11} + 4F_{22} - F_{44} + 4rF_{12})\sigma^* - rF_1 - 2F_2 \\ \frac{\partial f}{\partial r} &= 2(rF_{11} + 2F_{12})(\sigma^*)^2 - F_1\sigma^*. \end{aligned} \quad (22)$$

For the implicit derivative (21) to exist, one should have

$$\frac{\partial f}{\partial \sigma^*} = 2(r^2 F_{11} + 4F_{22} - F_{44} + 4rF_{12})\sigma^* - rF_1 - 2F_2 \neq 0 \quad (23)$$

which can be later verified if one wishes. Thus

$$\frac{d\sigma^*}{dr} = - \frac{2(rF_{11} + 2F_{12})(\sigma^*)^2 - F_1\sigma^*}{2(r^2 F_{11} + 4F_{22} - F_{44} + 4rF_{12})\sigma^* - rF_1 - 2F_2}. \quad (24)$$

It vanishes to give the necessary condition for an extreme value of  $\sigma^*$  with respect to  $r$ , i.e.

$$2(rF_{11} + 2F_{12})(\sigma^*)^2 - F_1\sigma^* = 0. \quad (25)$$

This can be re-written into

$$r = - \frac{4F_{12}\sigma^* - F_1}{2F_{11}\sigma^*} = - \frac{2F_{12}}{F_{11}} + \frac{F_1}{2F_{11}\sigma^*}. \quad (26)$$

It is anticipated that, when the stress ratio  $r$  satisfies the above, the material would be able to sustain a stress level significantly higher than its strengths under uniaxial stresses, i.e.  $\sigma^*$  can be taken to a value significantly greater than any of  $\sigma_{1t}^*$ ,  $\sigma_{1c}^*$ ,  $\sigma_{2t}^*$ ,  $\sigma_{2c}^*$  and  $\tau_{23}^*$ . In other words, the failure envelope in the  $\sigma_1$ - $\sigma_2$ - $\sigma_3$  space is significantly elongated or open in the triaxial compressive octant. If so, the term with  $\sigma^*$  in the denominator can be neglected and  $r$  can therefore be approximated as

$$r = - \frac{2F_{12}}{F_{11}}. \quad (27)$$

The same ratio as above can be equally obtained if  $F_1$  is negligibly small, i.e. the tensile and compressive strengths in fibre direction are sufficient close to each other. On the other hand, rearranging (19), one has

$$F_{12} = \frac{1}{4r(\sigma^*)^2} + \frac{rF_1 + 2F_2}{4r\sigma^*} - \frac{1}{4r}(r^2F_{11} + 4F_{22} - F_{44}) \quad (28)$$

where the first two terms on the right hand side of the above equation have  $\sigma^*$  in the denominator. With the same argument as above, these two terms can be neglected to give

$$F_{12} = -\frac{rF_{11}}{4} - \frac{1}{4r}(4F_{22} - F_{44}). \quad (29)$$

It should be pointed out that the smallness of  $F_1$  and  $F_2$  alone would not be sufficient to deliver (29) from (28) and one has to resort to the largeness of  $\sigma^*$  in this case.

Substituting (27) into (29) to eliminate  $r$ , one obtains a simple quadratic expression for  $F_{12}$  as follows

$$F_{12}^2 = \frac{F_{11}}{4}(4F_{22} - F_{44}). \quad (30)$$

It can then be presented as

$$F_{12} = \pm \frac{1}{2}k\sqrt{F_{11}F_{22}} \quad (31)$$

where  $k = \sqrt{\delta} \geq 0$  with  $\delta$  being defined in (13). (32)

The dimensionless parameter  $\delta$  is an important material property defined completely by the strength properties of the material. It must be non-negative for (32) to produce a real value for  $k$ . It has significant implications on the nature of the failure envelope if it becomes negative as will be explored later in this paper.

In order to determine the sense of  $F_{12}$  as obtained in (31), substituting (31) into (27)

$$r = \mp \frac{k\sqrt{F_{11}F_{22}}}{F_{11}} = \mp k\sqrt{\frac{F_{22}}{F_{11}}} \quad (33)$$

where the plus or minus signs are kept consistent with those in (31). For the stress state as defined in (17) to be triaxial compression,  $r$  must take the positive sign, i.e.

$$r = k\sqrt{\frac{F_{22}}{F_{11}}} = \sqrt{4 - \frac{\sigma_{2t}^*\sigma_{2c}^*}{(\tau_{23}^*)^2}} \sqrt{\frac{\sigma_{1t}^*\sigma_{1c}^*}{\sigma_{2t}^*\sigma_{2c}^*}}. \quad (34)$$

Accordingly,  $F_{12}$  in (31) should take the negative sign to be consistent with the sense of  $r$ , i.e.

$$F_{12} = -\frac{1}{2}k\sqrt{F_{11}F_{22}} = -\frac{1}{2}\sqrt{4 - \frac{\sigma_{2t}^*\sigma_{2c}^*}{(\tau_{23}^*)^2}} \sqrt{\frac{1}{\sigma_{1t}^*\sigma_{1c}^*\sigma_{2t}^*\sigma_{2c}^*}}. \quad (35)$$

For the sake of mathematical rigor in validating the above manipulations, one can verify that the denominator of (24) does not vanish.

Expression (35) can be viewed as a corrected form of the empirical expression of  $F_{12}$  in the original Tsai-Wu criterion as given in (10) with a correction factor of  $k$ . The correction is obtained

here in a completely rational manner based on a set of three clearly stated assumptions, in particular the third one, i.e. UD composites exhibit much higher strengths under triaxial compression at an appropriate stress ratio than any of the strengths under a uniaxial stress or pure shear stress state in the materials' principal axes, after abandoning the closed failure envelope restriction. It is uniquely determined without any empiricism. The identical result can also be obtained alternatively by finding the principal axes of the conic section instead of the extreme value approach as adopted above.

It can be observed that the  $F_{12}$  obtained here also enables (5) or (7) to degenerate to the failure function of the von Mises criterion for isotropic materials of equal tensile and compressive strengths,

where  $\sigma_{2t}^* = \sigma_{2c}^* = \sigma^*$  and  $\tau_t^* = \tau^* = \sigma^*/\sqrt{3}$ , and, as a result,  $\delta = 4 - \frac{\sigma_{2t}^* \sigma_{2c}^*}{(\tau_{23}^*)^2} = 1$  and  $k=1$ . It was

mentioned previously that the Hoffman criterion also degenerates to the von Mises criterion. It has been shown here once again that (10) is indeed not the only expression to reproduce the von Mises criterion, neither a rational one.

It is obvious to observe from (35) that with  $\delta \geq 0$ ,

$$0 \leq k \leq 2 \quad \text{thus} \quad -\sqrt{F_{11} F_{22}} \leq F_{12} \leq 0. \quad (36)$$

The failure locus in the  $\sigma_1$ - $\sigma_2$  plane is therefore always closed, i.e. an ellipse, like the one as shown in Fig. 1, as well as in the  $\sigma_1$ - $\sigma_3$  and  $\sigma_2$ - $\sigma_3$  planes. However, in the  $\sigma_1$ - $\sigma_2$ - $\sigma_3$  space, the failure envelope is open, in fact a paraboloid. The openness of the failure envelope can be easily shown in the  $\sigma_2=\sigma_3$  plane where the analytic geometry dictates that the failure locus as a conic section is a parabola, as dictated by condition (30) as the condition for a parabola. The  $\sigma_2=\sigma_3$  plane should not be confused with the  $\sigma_2$ - $\sigma_3$  plane. Putting in a form of an equation for the plane analytically in consistence with the  $\sigma_2=\sigma_3$  plane, the  $\sigma_2$ - $\sigma_3$  plane should be given as  $\sigma_1=0$ . In the  $\sigma_2$ - $\sigma_3$  plane, the failure locus is an ellipse as previously stated, in contrast with that in the  $\sigma_2=\sigma_3$  plane. The central axis of the paraboloid in the  $\sigma_1$ - $\sigma_2$ - $\sigma_3$  space is  $\sigma_1 = r\sigma_2 = r\sigma_3$ . It opens towards the triaxial compression octant and intersects with coordinate planes finitely, giving elliptic failure loci as intersections with these planes.

It can be proven that if  $r$  is assigned any value other than that given in (34), finite strength will be obtained. Infinite strength is obtainable only at a unique stress ratio which is neither hydrostatic compression ( $r=1$ ) nor equal biaxial compression transverse to fibres ( $r=0$ ), in general. It is completely determined by conventional strength properties and hence varies from material to material as shown in Table 2.

Another observation is the involvement of the transverse shear strength  $\tau_t^*$  in the expression of  $F_{12}$  and it is necessary even for the assessment of failure under in-plane stress conditions. However, this is not uncommon and the similar observation can be made on the Puck criterion [34] although the explicit inclusion of it was avoided by the introduction of a hypothetical relationship between the

transverse and in-plane shear strengths along with some assumed slopes to the failure envelope. It does affect the practicality to an extent, especially when only in-plane stress states are of interest. In Table 2, the case for T300/BSL914C could not be addressed with the present formulation because of the lack of this particular strength properties. However, it is an established experimental observation that the failure could take a mode out of the plane whilst under in-plane stresses as illustrated in [34]. Although the Tsai-Wu criterion is independent of failure mode, the present rationalisation must have hit some right cords if it captures a genuine feature of physics purely from mathematical and logical deduction. For potential users of the Tsai-Wu criterion in its present rationalised form, one would have to get used to a culture that the complete set of strength properties should include  $\tau_{23}^*$  even for plane stresses. If one cares to look back in history of the early days of solid mechanics, Young's modulus had been perceived as the only elastic property required for some time before Poisson's ratio was eventually introduced. The ability of adapting itself to embrace advances in development and improvements in understanding is the spirit of science.

## 5 Examples and discussions

For the six materials satisfying  $\delta > 0$  as listed in Table 2, their values of  $\delta$ ,  $k$  and  $r$  have been calculated and listed in Table 2. The corrected values of  $F_{12}$  are comparable to the values as suggested in the original Tsai-Wu criterion but the differences are sometimes significant enough, e.g. for one of the most common types of composites, IM7 carbon/epoxy.

Using the obtained corrections for 2D in-plane stress conditions, the failure loci have been plotted in Fig. 2 for the materials shown in Table 2 for those satisfying  $\delta > 0$  and they are compared with their original form as directly obtained from the Tsai-Wu criterion. The differences that the rationalism has made are marginal in most parts of these failure loci, except for E-glass/LY556 and IM7 carbon/8551-7. The most pronounced discrepancies are found in the compression-compression quadrant. This is a natural consequence of the due consideration given to the triaxial compressive strength in the present formulation. It is admirable that Tsai and Wu's empiricism hit the target with such accuracy intuitively. However, it should be pointed out that differences would be significantly more pronounced when 3D stresses are involved.

The corrected form of  $F_{12}$  as obtained in (35) is subject to condition  $\delta > 0$ . Obviously, there are real materials falling outside of this category and one cannot turn a blind eye to these materials. This aspect will be pursued in the next section.

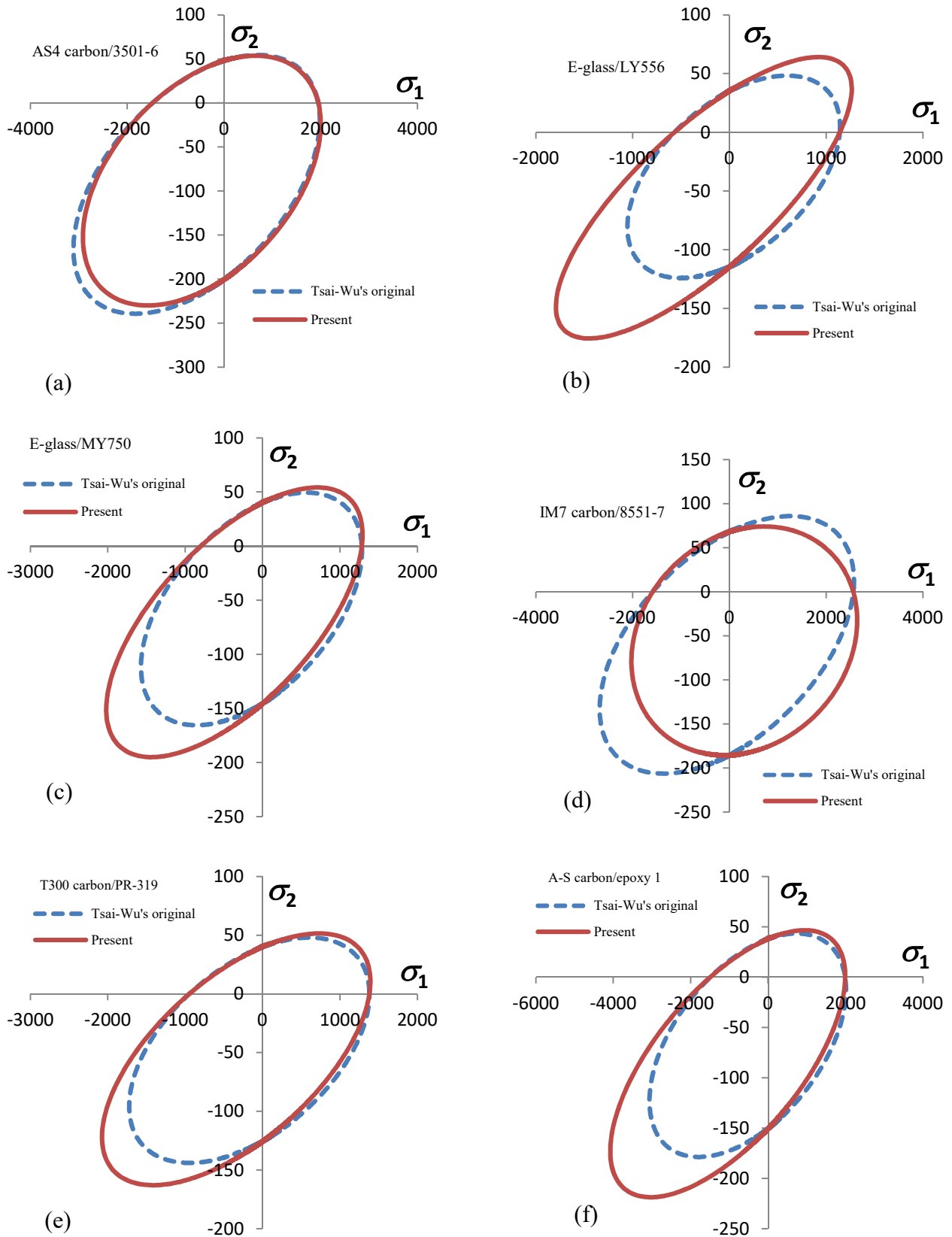


Fig. 2 In-plane failure loci for UD composites as indicated: comparison between the present theory and the original Tsai-Wu criterion (a) AS4 carbon/3501-6, (b) E-glass/LY556, (c) E-glass/MY750, (d) IM7 carbon/8551-7, (e) T300 carbon/PR-319 and (f) A-S carbon/epoxy 1

It can also be observed from Table 2 that the stress ratios corresponding to the infinite strength is quite different from  $r=1$ , i.e. hydrostatic compression. In fact, the strength under hydrostatic compression is bounded, typically not very high if one evaluates it. In the light of this observation, the discussions on the infinite strength or the lack of it under hydrostatic compression [2] would not seem relevant. The expression of  $F_{12}$  obtained under this assumption in [20-21] would therefore not be the most appropriate. Apparently, hydrostatic pressure is not the favourable stress ratio for UD composites to exhibit highest strength. Rather on the contrary, much higher longitudinal stresses than the transverse stresses should be applied in general in order to achieve the highest strength due to the anisotropy of the strength characteristics of UD composites. Existence of infinite strength as obtained here is a logical consequence of the fact that failure function is described as a quadratic function and that the material is transversely isotropic. It is neither a proven physical fact nor mathematical necessity if one introduced different failure functions. It should not be dismissed light-heartedly, in the authors' opinion, simply based on physical or ideal experiments under hydrostatic pressure as it is not necessarily to be the strongest aspect of the material given its anisotropy.

The Hashin criterion [33] was also based on a quadratic failure function. In its derivation of for matrix failure under compression, it assumed infinite strength under equal biaxial compression transverse to the fibres, which corresponded to  $r=0$ . The present study also challenges the validity of the assumption Hashin made there.

## 6 Cases where $\delta < 0$

The conclusion from the previous sections is only applicable if  $\delta \geq 0$  and it is deduced from the three assumptions introduced at the beginning of Section 4. In the case of  $\delta < 0$ , as is the subject of this section, the failure envelope in the stress space will have to be open as argued in Section 3 and hence the following discussion will be made with two basic assumptions.

- i) The failure is determined using a single quadratic function of stresses; and
- ii) The material is transversely isotropic.

One will then find that it is no longer possible to identify any meaningful condition to obtain a real value for  $F_{12}$ . To facilitate the elaboration in this section, the discussion can be made on two mutually exclusive but collective comprehensive scenarios, i.e.  $F_{12}^2 \geq F_{11}F_{22}$  or  $F_{12}^2 < F_{11}F_{22}$ . Given the transverse isotropy,  $\sigma_1$ - $\sigma_2$  and  $\sigma_1$ - $\sigma_3$  planes are equivalent and the discussion can be made to only one of them, say,  $\sigma_1$ - $\sigma_2$ .

- a)  $F_{12}^2 \geq F_{11}F_{22}$ : This allows the failure locus in the  $\sigma_1$ - $\sigma_2$  plane to be open.
- b)  $F_{12}^2 < F_{11}F_{22}$ : This means that the failure locus in the  $\sigma_1$ - $\sigma_2$  plane must be closed.

Consider scenario (a) first. A conic section has three possibilities as sketched in Fig. 3, i.e. an ellipse, a parabola or a pair of hyperbolae. Relatively, the  $\sigma_1$ - $\sigma_2$  plane is by large the best known aspect of the UD composites, although data sets are still wanting under many stress ratios to offer a complete assessment. From what are available, there does not seem to be any evidence or any sensible justification remotely suggesting the likelihood of infinite strength under any stress state in the  $\sigma_1$ - $\sigma_2$  plane. To expect  $F_{12}$  to fall within scenario (a) above, i.e. to allow infinite strength at a certain stress ratio, one would have to justify such a possibility as his/her first formidable task. As it is most unlikely to be true, it will not be explored in this paper. Readers are reminded that the objective of this paper is not to draw conclusion on applicability of the Tsai-Wu criterion. Rather it is to reveal all logical implications such that users could make their choice in an informed and objective manner.

If one chooses to reject the possibility of any open failure locus in the  $\sigma_1$ - $\sigma_2$  plane, i.e.  $F_{12}$  falls within the range defined by scenario (b), then he/she will observe the logical consequence according to the analytic geometry. Within this range, the following exercise can be carried out.

The failure loci in a special plane,  $\sigma_2=\sigma_3$ , is examined, after introducing the intersection of this plane with the  $\sigma_1=0$  plane as a new axis  $\sigma_t=\sigma_2/\sqrt{2}=\sigma_3/\sqrt{2}$  so that it is in the same scale as  $\sigma_2$  and  $\sigma_3$ . The  $\sigma_2=\sigma_3$  plane can also be called the  $\sigma_t$ - $\sigma_1$  plane, as the shaded plane in Fig. 4. The failure locus on this plane is analytically given as

$$F_{11}\sigma_1^2 + 2(4F_{22} - F_{44})\sigma_t^2 + 4\sqrt{2}F_{12}\sigma_1\sigma_t + F_1\sigma_1 + 2\sqrt{2}F_2\sigma_t = 1. \quad (37)$$

It can be easily seen that the second invariant as a discriminator of the above conic section

$$D = 2F_{11}(4F_{22} - F_{44}) - 8F_{12}^2 < 0 \quad (38)$$

given  $\delta = 4 - \frac{F_{44}}{F_{22}} < 0$  whilst  $F_{11}$ ,  $F_{22}$  and  $F_{12}^2$  are all positive. As a result, the conic section as defined

by (37) gives a pair of hyperbolae for any given value of  $F_{12}$  within the range of  $F_{12}^2 < F_{11}F_{22}$ . If one plots these hyperbolae in the  $\sigma_t$ - $\sigma_1$  plane for all permissible values of  $F_{12}$ , i.e.  $-\sqrt{F_{11}F_{22}} < F_{12} < \sqrt{F_{11}F_{22}}$ , a family of loci can be obtained, as shown in Figs 5(a) and (b) for S-2 glass/Epoxy 2 and G40-800/5026, respectively, the two materials in Table 2 falling into the category of  $\delta < 0$ . The loci are bounded by the limits of the range, i.e.  $F_{12} = \pm\sqrt{F_{11}F_{22}}$ , shown as red and blue line in Figs 5(a) and (b), respectively. The shaded areas on the inner side of the bounds correspond to condition  $F < 1$ , defining the safe zone of the stress states for the material, although the disconnected shaded part on the right in Fig. 5(b) is not accessible as loading process must start from the origin and any accessible state should be connected to the origin. Despite the disparity in appearance between Figs. 5(a) and (b), some common observations can be made: (1) There is an infinite number of stress ratios between  $\sigma_1$  and  $\sigma_t$ , at which infinite strengths can be obtained and (2) Some of such stress ratios



involve tensile stress in the fibre direction.

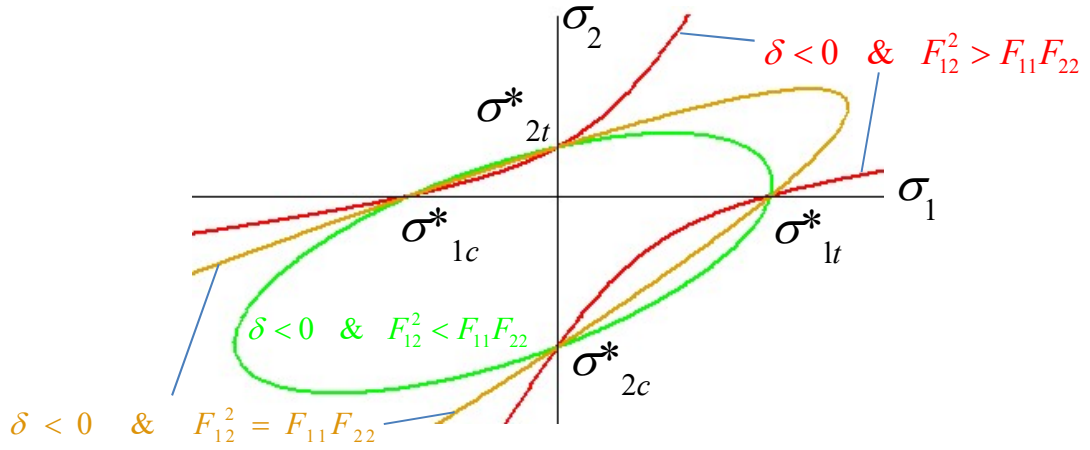


Fig. 3 Failure loci in the  $\sigma_1$ - $\sigma_2$  plane for different ranges of  $F_{12}$

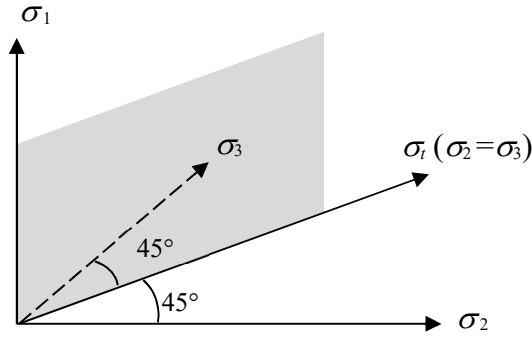


Fig. 4 The  $\sigma_t$ - $\sigma_1$  plane

As a more conservative measure of the safe zone, one can take the darkly shaded subzone bounded by the dashed lines which are parallel to the asymptotes to the loci boundaries next them. These two dashed lines can be expressed as

$$\sigma_1 = \pm \sqrt{\frac{2\sigma_{1t}^* \sigma_{1c}^*}{\sigma_{2t}^* \sigma_{2c}^*}} \left( \sqrt{1 - \frac{\delta}{4}} - 1 \right) \sigma_t \quad \text{bearing in mind } \sqrt{1 - \frac{\delta}{4}} > 1. \quad (39)$$

In other words, with equal biaxial transverse compression, i.e.  $\sigma_t \leq 0$ , within the range from

$\sigma_t : \sigma_1 = 1 : \sqrt{\frac{2\sigma_{1t}^* \sigma_{1c}^*}{\sigma_{2t}^* \sigma_{2c}^*}} \left( \sqrt{1 - \frac{\delta}{4}} - 1 \right)$  to  $\sigma_t : \sigma_1 = 1 : -\sqrt{\frac{2\sigma_{1t}^* \sigma_{1c}^*}{\sigma_{2t}^* \sigma_{2c}^*}} \left( \sqrt{1 - \frac{\delta}{4}} - 1 \right)$ , an infinite strength can be expected at

any stress ratio. Between stress ratios  $\sigma_t : \sigma_1 = 1 : 0$  and  $\sigma_t : \sigma_1 = 1 : -\sqrt{\frac{2\sigma_{1t}^* \sigma_{1c}^*}{\sigma_{2t}^* \sigma_{2c}^*}} \left( \sqrt{1 - \frac{\delta}{4}} - 1 \right)$ , the stress in the

fibre direction  $\sigma_1$  is tensile. UD composites are strong against tension in fibre direction but having an infinite strength in tension is hard to imagine.

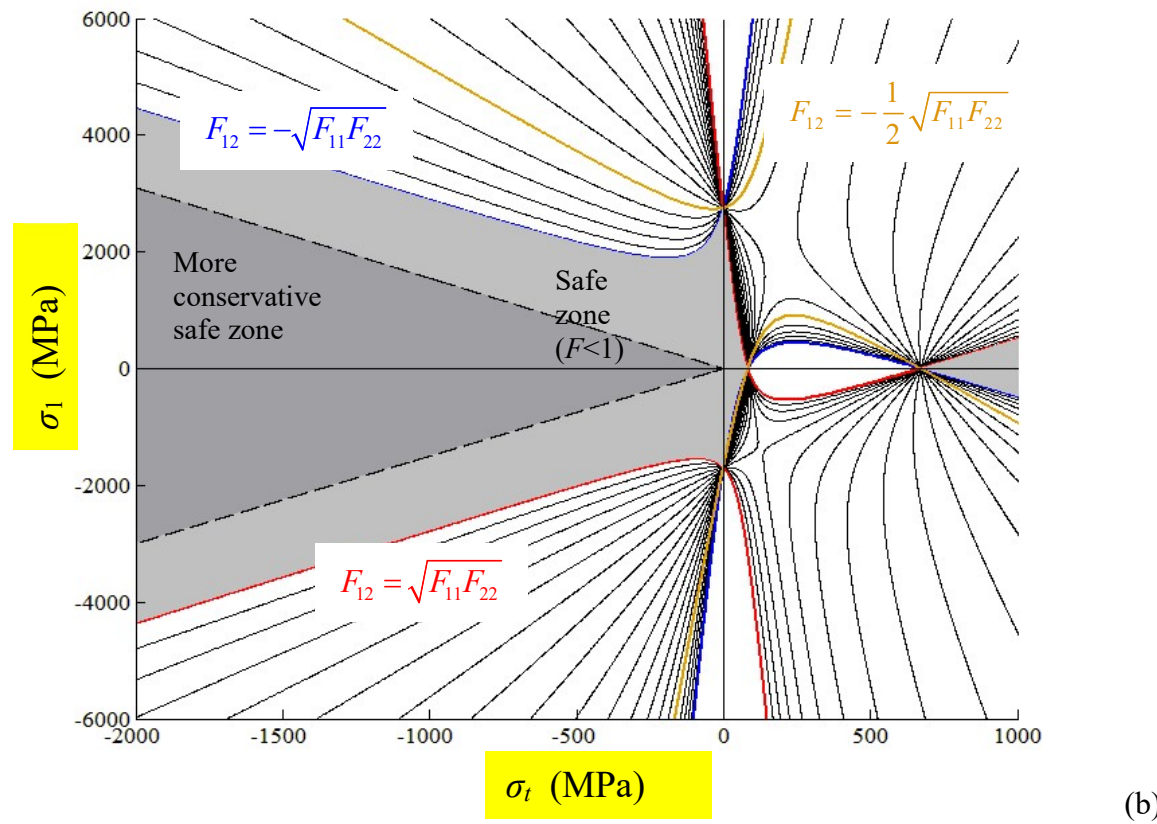
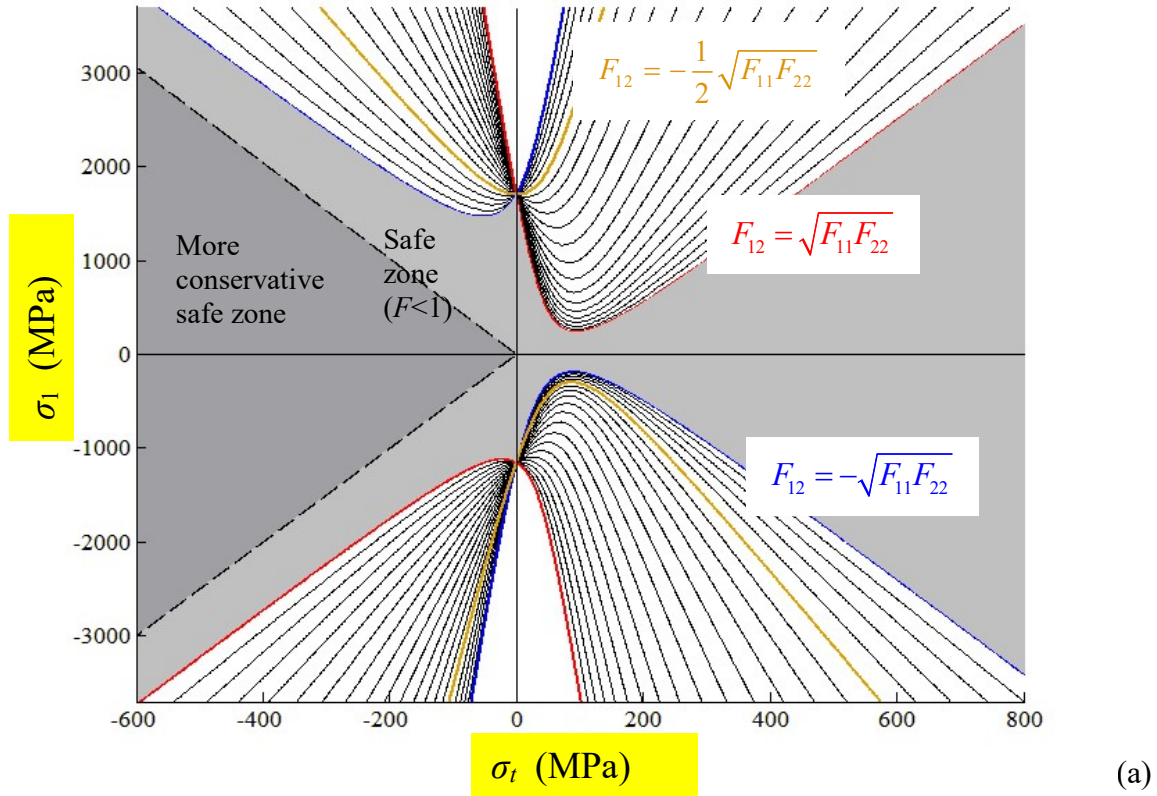


Fig. 5 The envelope of failure loci in the  $\sigma_t$ - $\sigma_1$  plane for all permissible values of  $F_{12}$  if the failure loci in the  $\sigma_1$ - $\sigma_2$  and  $\sigma_1$ - $\sigma_3$  planes remain closed (a) S-2 glass/Epoxy 2 and (b) G40-800/5026 (Blue and red correspond to the extremes of the range  $F_{12} = \pm\sqrt{F_{11}F_{22}}$  while yellow to the case of original Tsai-Wu criterion)

To summarise the observation over the case of  $\delta < 0$ , it is clear that any choice of the value of  $F_{12}$  will lead to a scenario of infinite strength under stress states other than triaxial compression, which are hard to defend. If the value of  $F_{12}$  is chosen so that  $F_{12}^2 \geq F_{11}F_{22}$ , the failure loci in the  $\sigma_1$ - $\sigma_2$  plane will be open, either as a parabola or hyperbola. Alternatively, if  $F_{12}^2 < F_{11}F_{22}$ , to keep the failure locus closed in the  $\sigma_1$ - $\sigma_2$  plane, i.e. the ellipse as shown in Fig.3, infinite strengths are inevitable under triaxial stresses at infinite number of stress ratios, e.g. at  $\sigma_1 : \sigma_2 : \sigma_3 = \sqrt{\frac{\sigma_{1t}^* \sigma_{1c}^*}{\sigma_{2t}^* \sigma_{2c}^*}} \left( \sqrt{1 - \frac{\delta}{4}} - 1 \right) : -1 : -1$ , where the stress along fibres is tensile while the transverse stresses are compressive, as shown in Fig. 5(a) and (b). The dilemma arrived in the case of  $\delta < 0$  is a logical consequence of the use of a single quadratic failure function and the transverse isotropy of the material. The position cannot be altered in presence of these two assumptions, which challenges directly the applicability of the Tsai-Wu criterion to this category of materials.

## 7 Concluding remarks

In this paper, the quadratic failure function as proposed by Tsai and Wu [4] has been subjected to a systematic re-examination from the mathematical perspective guided by the principles of analytic geometry in the context of transversely isotropic materials such as UD composites. It has been first argued that the failure envelope in the stress space cannot be kept closed for all materials and it is therefore more appropriate to manage the opening than prohibiting it. A non-dimensional parameter  $\delta = 4 - \frac{\sigma_{2t}^* \sigma_{2c}^*}{(\tau_{23}^*)^2}$  has been introduced which is completely determined by materials' conventional strength properties. The sense of this parameter divides all UD composites into two categories, i.e.  $\delta \geq 0$  and  $\delta < 0$ . For the former category, allowing infinite strength under a certain triaxial compressive stress ratio provides a condition for the unique determination of the interactive coefficient  $F_{12}$ , as well as the associated stress ratio, in a rational manner. This helps to eliminate the empiricism associated with this coefficient as a longstanding issue of the Tsai-Wu criterion. The obtained rational expression of  $F_{12}$  can be considered as a corrected form from that in the original Tsai-Wu criterion. It has also been shown that such rationalised Tsai-Wu criterion is also capable of reproducing the von Mises criterion for isotropic materials of equal tensile and compressive strengths. As the rationalisation is based on the assumption of the existence of much higher strength under a specific triaxial compressive stress state than its strengths under uniaxial loading conditions or pure shear in the material's principal axes, and hence approximated as infinite, its validation could only come from the experimental evidences observed under triaxial compression at an appropriate stress ratios, usually, with much lower transverse stresses than the longitudinal one. Hydrostatic stress is not the

right one for this purpose, in general. Experimental data from triaxial compression tests over a reasonable range of different stress ratios are scarce in the literature [2]. No solid understanding will be truly established without such experimental data. Readers are reminded of the role of experiments by Bridgman [35] at high hydrostatic pressures on metals which laid the foundation for the systematic theory of plasticity [36]. It is hoped that some experimentalists would be incentivised to fill this gap to pave the way to the establishment of the next generation of composites failure criteria of higher level of fidelity.

For the category of  $\delta < 0$ , if one chose to apply the Tsai-Wu criterion, it has been established in this paper that any value of  $F_{12}$  would imply features hard to defend, either to allow the failure locus in the  $\sigma_1$ - $\sigma_2$  plane to be open, i.e. allowing infinite strength under an in-plane stress state, or to allow a range of triaxial stress states to exhibit infinite strength, including some involving tension in the fibre direction.

Although the discussions in the present paper are made in the context of transversely isotropic materials as a special case of general orthotropic or completely anisotropic materials, logic dictates that the similar behaviours, if not more serious, can be expected for general orthotropic or completely anisotropic materials, at least for some of them. In this sense, whilst restricted to the UD composites, as a type of for transversely isotropic materials, a degree of generality of the considerations put forward in this paper should preserve for general orthotropic or completely anisotropic materials.

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Table 1 Previous studies on the determination of the interactive term  $F_{12}$  in the Tsai-Wu failure criterion

Author (year)	Method	Lay-up and loading	Remarks
Tsai and Wu (1971)	They noted that $F_{12}$ could be determined using infinite number of combined stresses, mentioned five cases: (a) Biaxial tension (b) Tension of off-axis 45° lamina (c) Compression of off-axis 45° lamina (d) Positive shear (e) Negative shear	(a) $\sigma_1=\sigma_2=P$ (P: tensile stress) (b) $\sigma_1=\sigma_2=\sigma_6=T/2$ (T: tensile strength of the 45° lamina) (c) Off-axis compression of 45° lamina: $\sigma_1=\sigma_2=\sigma_6=-T'/2$ (T': compressive strength of -45° lamina) (d) $\sigma_1=-\sigma_2=V$ (V: positive shear strength of 45° lamina) (e) $\sigma_1=-\sigma_2=-V'$ (V': negative shear strength of 45° lamina)	See also the work of Narayanaswami and Adelman (1977)
Wu (1972)	Biaxial loading of a lamina	0° lamina under various $\sigma_1$ versus $\sigma_2$ loadings	Large scatter, $F_{12}$ close to 0.
Clouston and Lam (2001)	Probabilistic approach	$\pm 15^\circ$ laminates under compression	Large scatter in results (c.a. 15%)
DeTeresa and Larsen (2001)	Assuming infinite strength under hydrostatic compression	Hydrostatic pressure $\sigma_1=\sigma_2=\sigma_3=-P_{hyd}$	$F_{12} = -F_{11}/4$
DeTeresa and Larsen (2001)	Assuming infinite strength under equal biaxial transverse compression	Equal biaxial compression $\sigma_2=\sigma_3 = -P_{bia}$	$F_{23} = -F_{22}$
Hansen (1992)	Experimental	10° Off-Axis Strength	
Hansen (1992) and Hansen et al (1993)	Experimental	Modified uniaxial strain test	For $\sigma_1 > \sigma_2$ and $\sigma_2 > \sigma_1$
Evans and Zhang (1987)	Combined experimental and analytical	Uniaxial strain tests on unidirectional lamina for both $\varepsilon_2=0$ and $\varepsilon_1=0$	Focussed on in-plane formulation. $F_{12}$ given in terms of lamina failure strains and moduli of the lamina
Pipe and Cole (1973)	Combined testing and analysis	Tension and compression of off-axis coupons with various ply angles: 15°, 30°, 45° and 60°	Boron/epoxy materials. Large variation of $F_{12}$ for tension tests and acceptable variation for compression tests.
Kallas and Hahn (1990)	Testing and analysis	Diametral compression	Valid for $F_{23}$ $F_{23} = -F_{22}/2$

<p>Narayanaswami and Adelman (1977)</p>	<p>Experimental</p>	<p>Off-axis tension : <math>\sigma_1=\sigma_2=\sigma_6=T/2</math>  Off-axis compression: <math>\sigma_1=\sigma_2=\sigma_6=-C/2</math>  Positive shear: <math>\sigma_1=-\sigma_2=V, \sigma_6=0</math>  Negative shear: <math>\sigma_1=-\sigma_2=V', \sigma_6=0</math>  Biaxial tension: <math>\sigma_1=\sigma_2=P, \sigma_6=0</math>  Biaxial compression: <math>\sigma_1=-\sigma_2=P', \sigma_6=0</math></p>	<p>They suggested setting <math>F_{12}=0</math>.</p>
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Table 2 UD composite properties and the results from the present formulation

Data from WWFE	I				II				III
Material	AS4 3501-6	T300 BSL914C	E-glass LY556	E-Glass MY750	IM7 8551-7	T300 PR-319	A-S Epoxy 1	S-2 glass Epoxy 2	G40-800 5026
$\sigma_{1t}^*$ (MPa)	1950	1500	1140	1280	2560	1378	1990	1700	2750
$\sigma_{1c}^*$ (MPa)	1480	900	570	800	1590	950	1500	1150	1700
$\sigma_{2t}^*$ (MPa)	48	27	35	40	68 <sup>1</sup>	40	38	56.5 <sup>2</sup>	70 <sup>3</sup>
$\sigma_{2c}^*$ (MPa)	200	200	114	145	185	125	150	180	210
$\tau_{23}^*$ (MPa)	55 <sup>4</sup>	N/Available	50 <sup>5</sup>	50 <sup>6</sup>	57	45	50	40	57
$\delta$	0.8264	N/Available	2.404	1.680	0.1280	1.531	1.720	-2.356	-0.5245
$k$	0.9091	N/Available	1.550	1.296	0.3578	1.237	1.311	N/Applicable	N/Applicable
$r$	15.76	N/Available	19.79	17.22	6.437	20.02	30.01	N/Applicable	N/Applicable
Present $F_{12}$ ( $\times 10^{-6}$ ) (1/MPa <sup>2</sup> )	-2.731	N/Available	-15.23	-8.409	-0.7906	-7.647	-5.027	N/Applicable	N/Applicable
Tsai-Wu's $F_{12}$ ( $\times 10^{-6}$ ) (1/MPa <sup>2</sup> )	-3.004	-5.856	-9.820	-6.488	-2.210	-6.180	-3.833	-3.546	-1.907

<sup>1</sup> Average taken from  $\sigma_{2t}^* = 73$  and  $\sigma_{3t}^* = 63$  MPa in direction 2 and 3, respectively

<sup>2</sup> Average taken from  $\sigma_{2t}^* = 63$  and  $\sigma_{3t}^* = 50$  MPa in direction 2 and 3, respectively

<sup>3</sup> Average taken from  $\sigma_{2t}^* = 75$  and  $\sigma_{3t}^* = 65$  MPa in direction 2 and 3, respectively

<sup>4</sup> From WWFE-III [3]

<sup>5</sup> From WWFE-III [3]

<sup>6</sup> From WWFE-II [2]