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Delayed response in the Hawk Dove game

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Abstract. We consider a group of agents playing the Hawk-Dove game. These agents have a finite memory of past interactions which they use to optimize their play. By both analytical and numerical approaches, we show that an instability occurs at a critical memory length, and we provide its characterization. We show also that when the game is stable, having a long memory is beneficial but that instability, which may be produced by excessively long memory, hands the advantage to those with shorter memories.

1 Introduction

In the world of living creatures, some species behave aggressively to obtain resources; others adopt a more conservative attitude in order to minimize their potential incurred costs, such as injuries. This phenomenon was first characterized mathematically by J. Maynard Smith in 1973 [1] in the Hawk-Dove game model, where aggressive "hawks" compete with peaceful "doves" for survival. The model is related to the game "chicken" as well as the "Prisoner's Dilemma", and they have been the subject of serious research in game theory [2,3]. In general, the Hawk-Dove game can be regarded as a typical anti-coordination game, all of which have three Nash equilibria, two of which consist of pure strategies and a third of mixed strategies. The Hawk-Dove model has been successfully applied to explore animal behaviour and its evolution [4,5]. For example, recently it has been used to explain colour polymorphism in Gouldian finches which have red (aggressive) and black (passive) variants [6]. Many applications exist outside of biology. A "quantum" Hawk-Dove game [7], based on the recent financial crisis, leads to new evolutionary stable strategies which are non aggressive. In social science, a Hawk-Dove game approach may be used to manage customer expectation, and thus help firms achieve better financial performance [8]. The game has been used in political science to investigate international relations in a number of ways [9].

This paper explores memory effects in the classical Hawk-Dove game. The game is played by a group of agents who can recall a finite set of past interactions, which they use to make decisions about their future behaviour. In games in general, the methods used by agents to make decisions in competitive situations can have dramatic effects on the dynamics of strategies [10–13] including the

creation of limit cycles, chaotic behaviour and noise sustained cycling. In all cases, the nature of agents' memory of past experiences plays a role. The presence of memory naturally leads to delay differential equations [14] for game dynamics, and the effects of delay on stability have been studied for a number of evolutionary games [15–17]. The relative weight given to memories from different times in the past can also be important [11], and the nature of information recall in humans and animals is still being explored [18]. The effect on game dynamics of memory based strategies [19] is also the subject of experimental research in the ecological context.

We investigate the effect of memory length on game dynamics and the individual effectiveness of agents when their memory is used as an imperfect statistical sample of the population. Our agents use their recollection of a finite number of past interactions to adjust their probabilistic strategy after each new meeting toward the choice which, according to their memory, appears optimal. We have previously discovered in the case of a resource sharing game [20] and "Rock Scissors Paper" [21], that under this continuous updating method, known as "online" learning (as opposed to "batch" learning [10] where a new sample is gathered before each new adjustment) there exists a Hopf bifurcation marking a transition from stable equilibrium behaviour to regular population–wide strategy oscillations. In each case memory size, m, may be seen as the bifurcation parameter [20,21] and the analytical relationship between the critical value m_c of memory and the rate, ϵ, of strategy adjustment takes the form $m_c \propto \epsilon^\alpha$ where $\alpha = -1$ and $\alpha = -1/2$ for the sharing and rock scissor paper games respectively. Here we show that in the Hawk-Dove game, $\alpha = -2/3$. For a given ϵ, typically $\ll 1$, a larger α leads to smaller m_c, and a potentially less stable scenario. The appearance of strategy oscillations has also recently been discovered in delayed replicator dynamics for two strategy games [22]. For sufficiently

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large populations our learning rule also results in a deterministic delay equation for the average strategy of the group. However the form of this equation is quite different to replicator dynamics, because the statistical properties of the contents of agents' memories are explicitly encoded within it.

Our paper is structured as follows. We first define our Hawk-Dove game with memory and present our simulation results. We show that the game possesses a stable fixed point when memory size is sufficiently short, but that this stability gives way to a limit cycle at critical memory length. We demonstrate that the appearance of this cycle destroys the competitive statistical advantage of a long memory. Finally we derive a deterministic delay equation for the average strategy of the population and extract from this an analytical expression for the bifurcation point.

2 Model

The Hawk-Dove game is defined as follows [23]: two agents have at their disposal some resource such as food, territory or access to mating opportunities, having numerical value V. Each agent can choose to offer cooperation and share the resource, or attempt to aggressively claim it. A meeting of two aggressors (Hawks) leads to one winner who claims the prize, and one injured party who experiences a cost C; then symmetry dictates that the average gain to each agent is $(V - C)/2$. Two cooperators (Doves) simply share the resource equally, but a Hawk will leave a Dove with nothing. We therefore have the following payoff matrix, where the element in row X, column Y gives the payoff to strategy X against strategy Y where $X, Y \in \{H, D\}$

$$
\begin{array}{c}
\quad\quad\quad H \quad\quad D \\
\begin{array}{c} H \\ D \end{array}
\begin{pmatrix} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{pmatrix}.
\end{array} \tag{1}
$$

A group of $L \geq 2$ agents play this Hawk-Dove game in continuous time. Interactions take place via random pairing between agents at a rate of $L/2$ per unit time, so that each agent has one interaction per unit time on average. At interaction, each agent adopts either Hawk or Dove, according to its specific probabilistic strategy, $\phi_i(t)$, defined as the probability of agent i adopting Hawk at time t. Each interaction results in a payoff which is memorized by the pair. Each agent has a memory of the last m of its interactions, which it uses to adapt its strategy toward the choice that currently *appears* to be optimal. If H_i is the number of Hawks encountered in agent i's memory of m interactions, then $\tilde{\phi}_i := H_i/m$ is its estimate of the average strategy of other agents. If $\tilde{\phi}_i$ were an accurate measure of the current average probabilistic strategy of the whole population

$$
\phi(t) := \frac{1}{L} \sum_{i=1}^{L} \phi_i(t) \tag{2}
$$

then given the payoff matrix (1), it is readily verified that its optimal strategy would be to adopt Hawk if $\tilde{\phi}_i < \frac{V}{C}$ and Dove if $\tilde{\phi}_i > \frac{V}{C}$. Instead of acting in this way, in our model the memory $\tilde{\phi}_i$ is used to slowly adjust the probabilistic strategy $\phi_i(t)$. After each new interaction an agent will evolve either toward pure Hawk ($\phi_i = 1$) or toward pure Dove ($\phi_i = 0$). The probability of evolving toward Hawk is

$$
\tilde{p}(\tilde{\phi}_i) := \mathcal{H}\left(\frac{V}{C} - \tilde{\phi}_i\right) \tag{3}
$$

where $\mathcal{H}$ is the Heaviside function, defined using the half-maximum convention ($\mathcal{H}(0) = \frac{1}{2}$). That is, after an interaction, if $\tilde{\phi}_i < \frac{V}{C}$ then the agent will *definitely* evolve towards the pure Hawk strategy; if $\tilde{\phi}_i > \frac{V}{C}$ then the agent will *definitely* evolve towards pure Dove, otherwise if $\tilde{\phi}_i = \frac{V}{C}$ the agent will choose randomly (with equal probability) which strategy to evolve toward. In a small time interval δt, an agent has a probability $\delta t + o(\delta t)$ of interacting, so (neglecting terms which are $o(\delta t)$)

$$
\phi_i(t + \delta t) = \begin{cases} \phi_i(t) & \text{w.p.} \quad (1 - \delta t) \\ \phi_i(t) + \epsilon[1 - \phi_i(t)] & \text{w.p.} \quad \tilde{p}\delta t \\ \phi_i(t) - \epsilon\phi_i(t) & \text{w.p.} \quad (1 - \tilde{p})\delta t \end{cases} \tag{4}
$$

where $\epsilon \in [0, 1]$ is the "update" or "learning" rate which regulates how fast agents adapt to change. Note that the abbreviation w.p. means "with probability".

Using equation (4) we can compute the expected change in ϕ_i during the interval $(t, t + \delta t]$. We condition on the state of the agent's memory at time t so, neglecting any change in $\tilde{\phi}_i$ during the interval

$$
\mathbb{E}\left[\frac{\delta\phi_i}{\delta t}\middle|\tilde{\phi}_i(t)\right] = \epsilon\left[\tilde{p}(\tilde{\phi}_i) - \phi_i\right]. \tag{5}
$$

An interaction during the interval $(t, t + \delta t]$ can change $\tilde{\phi}_i$ by $\pm 2m^{-1}$ so this evolution equation is exact provided $|V/C - \tilde{\phi}_i(t)| > 2m^{-1}$, and therefore exact for all $\tilde{\phi}_i(t) \neq V/C$ in the limit $m \to \infty$.

3 Simulation results

We numerically simulate a population of identical agents. Figure 1 shows the evolution of probability weight $\phi(t)$. The values of update rate and memory are $\epsilon = 5 \times 10^{-3}$ and $m = 101$ respectively. For short time $t \approx m$, we observe transient behaviours which decay over a few times m, they do not affect the rest of the simulation and are of no interest here (the same applies to the next figure). For longer time, we find a stable state with small fluctuations, which may be further suppressed by larger population sizes. In Figure 2, the memory length is changed to $m = 151$, and the behaviour of the population has changed to stable strategy oscillations. These oscillations make their appearance at a critical memory length which depends on the value of ϵ. A lower update rate (ϵ) leads
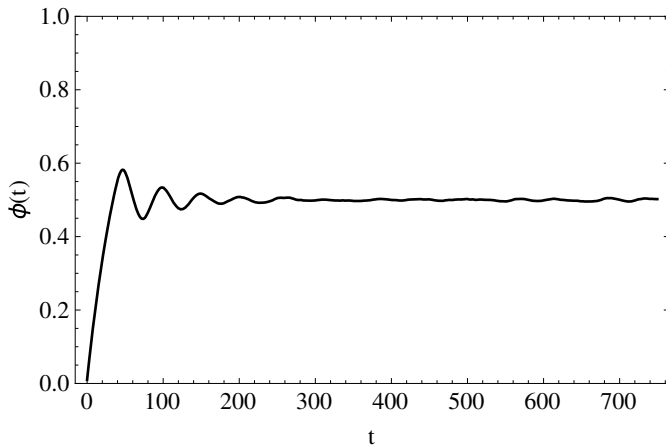
Fig. 1. Average simulated probability weight of Hawk strategy in a group of size $L = 1000$ with $V = 1, C = 2$. All agents have memory $m = 101$ and update rate $\epsilon = 5 \times 10^{-3}$.
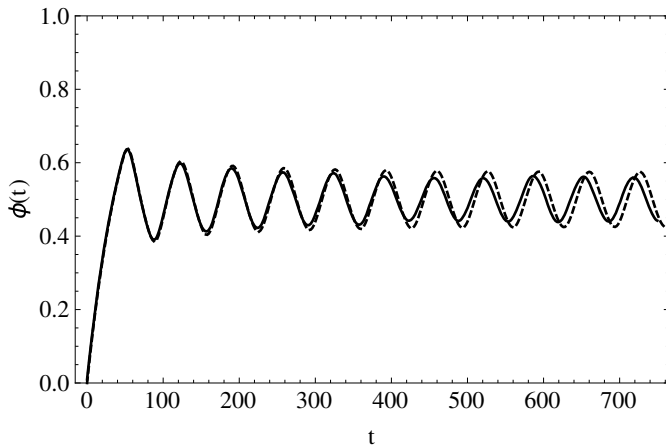

Fig. 2. Continuous black line shows simulated average probability weight of Hawk strategy in a group of size $L = 1000$ with $V = 1, C = 2$. All agents have memory $m = 151$ and update rate $\epsilon = 5 \times 10^{-3}$. Dashed line shows solution to approximate deterministic delay equation (14).

Fig. 3. Dependence of average payoff per interaction on update rate ϵ in a mixed population of $L = 1000$ where 100 agents have memory $m = 10$ (solid line) and 900 agents have memory $m = 200$ (dashed line). The game parameters are $V = 1$ and $C = 4$. Open circles show amplitude of $\phi(t)$; as ϵ passes through the bifurcation point, oscillations form and increase in amplitude as ϵ increases. These oscillations destroy the statistical advantage of the longer memory agents. Dashed vertical line shows theoretical bifurcation point ($\epsilon_c \approx 0.0027$) for a system consisting exclusively of agents with $m = 200$. Solid vertical line shows approximate bifurcation point ($\epsilon_c \approx 0.0038$) for this mixed system.

to a higher critical memory length. This "Hopf Bifurcation" can also be achieved by varying the update rate with memory length fixed, so either ϵ or m may be considered as the bifurcation parameter. We will demonstrate the relationship between the critical values of update rate for stable oscillations and agents' memory lengths analytically later.

We now consider the relative competitive advantages of different memory lengths in a population consisting of a mixture of long memory and short memory agents. When the strategies of the population are in a stable state, a long memory is beneficial because it gives a more accurate picture of your opponents' strategies. This allows agents to accurately tune their strategy toward the best possible response. However, when an excessive number of agents with memory lengths above the critical threshold are present or, equivalently, an excessively large update rate is used, then oscillations form. Under these circumstances, older
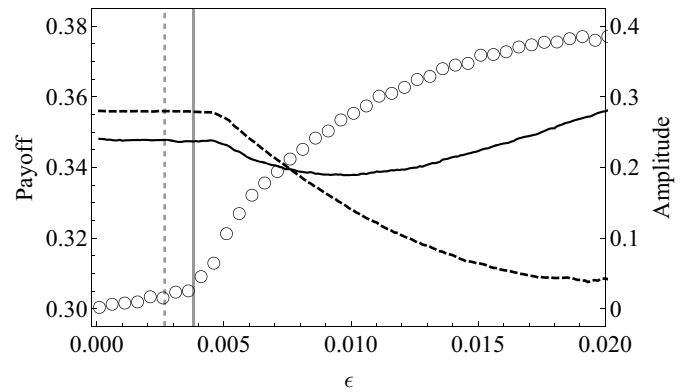
data loses its value as a predictive tool: it may be better to sacrifice long memory in order to utilize up-to-date information. We illustrate this effect in Figure 3 where we have estimated the average payoff per interaction for a mixed population of memory lengths, as a function of the update rate. This was achieved in a single simulation by recording a moving average payoff while increasing update rate, ϵ, sufficiently slowly so that it was approximately constant over the time scale of the moving average. As with a population of identical agents, a bifurcation occurs at critical ϵ, leading to the formation of strategy oscillations in the population as a whole. The amplitude of these oscillations is also shown in the figure and we may estimate the critical value of ϵ as the point at which the amplitude begins to increase rapidly. From Figure 3 we see that the average payoff for the long memory agents (dashed line) suffers a clear drop soon after the bifurcation point, and as ϵ increases, the payoff becomes significantly worse than that achieved by short memory agents. The effectiveness of the short memory agents is explained by the fact that they can adapt quickly to exploit strategy oscillations generated by the long memory agents. If the system is stable then the advantage of a long memory is greater for smaller ratios of V/C because in this case a short memory is more likely to incorrectly identify the current optimal strategy. We also emphasise that although oscillations are clearly present on a population level, estimates of the Hawk fraction obtained from individual agents' memories will be subject to fluctuations of order $m^{-1/2}$ which, in the example of Figure 3, would make the limit cycle difficult to perceive for an individual.

The critical value of ϵ in a mixed memory population will be a function of both the proportions of agents

with each memory, and the lengths of their memories. While we do not attempt to compute this function, we see in Figure 3 that the bifurcation point, as determined by the appearance of oscillations, is above that predicted for a pure population of agents with $m = 200$. This is consistent with a small stabilizing effect provided by the short memory agents. There are, in principle, other ways in which a bifurcation can be induced in a mixed population: for example at fixed ϵ, oscillations could be created by the addition of sufficient numbers of agents with sufficiently long memory (long enough to destabilize themselves in isolation). We may conclude that it is useful to use a lot of historical data to predict the current best strategy, provided not too many other agents are also doing this, or that the rate at which strategies are updated is not too fast.

4 Theory

We now return to the case of a population of identical agents, all with memory m, and derive a deterministic equation for system dynamics in the limit of large m and L. This then allows us to characterise system stability.

4.1 Delay equation

To obtain an evolution equation for the average strategy of the population (2), we average the evolution equation (5) over all agents

$$\frac{1}{L} \sum_{i=1}^{L} \mathbb{E} \left[\frac{\delta \phi_i}{\delta t} \bigg| \tilde{\phi}_i(t) \right] = \mathbb{E} \left[\frac{\delta \phi(t)}{\delta t} \bigg| \left\{ \tilde{\phi}_i(t) \right\}_{i=1}^{L} \right] \quad (6)$$

$$= \epsilon \left[\frac{1}{L} \sum_{i=1}^{L} \tilde{p}(\tilde{\phi}_i) - \phi(t) \right]. \quad (7)$$

As $L \to \infty$, fluctuations in the stochastic process $\phi(t)$ disappear and it becomes a deterministic function of time obeying, in the limit $\delta t \to 0$

$$\frac{d\phi}{dt} = \epsilon \left[\lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^{L} \tilde{p}(\tilde{\phi}_i) - \phi(t) \right]. \quad (8)$$

We now seek an expression for the average of $\tilde{p}(\tilde{\phi}_i)$ over the population, in the limit $L \to \infty$. We therefore require an expression for the probability distribution of $\tilde{\phi}_i$ for a randomly selected agent, conditional on the history of the average strategy of the population $\phi(t)$. We first define the average memory of the population

$$\bar{\phi}_m := \lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^{L} \tilde{\phi}_i. \quad (9)$$

Note that we have suppressed t dependence here for notational compactness. For large m, this approaches the time average of $\phi(t)$ over agent memory length so

$$\bar{\phi}_m \sim \frac{1}{m} \int_0^m \phi(t - s) ds \text{ as } m \to \infty, \quad (10)$$

where $\sim$ indicates asymptotic equality. This relation is true asymptotically because at time t, the distribution of the set of interaction times in the agents' memories approaches a uniform distribution on $[t - m, t]$ as $m \to \infty$. Equation (10) is the time average of $\phi(t)$ over this distribution. See reference [21] for a detailed discussion of this point.

The number H of Hawks observed amongst an agent's memory of m interactions will have a probability mass function $\mathbb{P}(H = h)$ which is approximately binomial

$$\mathbb{P}(H = h) \approx \frac{m! \, \bar{\phi}_m^h (1 - \bar{\phi}_m)^{m-h}}{h! \, (m - h)!} =: f(h, \bar{\phi}_m). \quad (11)$$

This relationship is an approximation because the probability of observing a Hawk at each interaction changes over the course of each agent's memory. We approximate this probability as constant and equal to the average $\bar{\phi}_m$. The binomial approximation approaches the true distribution as $m \to \infty$ provided $\phi(t)$ is constant.

Armed with the approximate probability distribution of the number of Hawks in an agent's memory we can perform the average of $\tilde{p}(\tilde{\phi}_i)$ over the population

$$\lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^{L} \tilde{p}(\tilde{\phi}_i) \approx \sum_{h=0}^{m} \tilde{p}(h/m) f(h, \bar{\phi}_m) \quad (12)$$

$$=: p(\bar{\phi}_m). \quad (13)$$

This defines $p(\bar{\phi}_m)$, the probability that a randomly selected agent will evolve towards a pure Hawk strategy, whilst in interaction with a population having time averaged strategy $\bar{\phi}_m$. The strategy evolution equation (8) is therefore

$$\frac{d\phi}{dt} \approx \epsilon \left[p(\bar{\phi}_m) - \phi \right] \quad (14)$$

which is a delay equation where $\bar{\phi}_m$ contains the memory of the previous time interval $[t - m, t]$. The equation may be solved numerically, and an example solution is plotted in Figure 2, where we see that it closely matches the simulation. We note that we expect to see small discrepancies between the delay equation solution and simulation, due to the finite number of agents (introducing stochasticity into $\phi(t)$), with finite memory. As noted above, our binomial approximation for the distribution of agent memory becomes exact in the limit of large m and small fluctuations. However Figure 2 demonstrates that the approximation remains effective when these conditions are not met.

4.2 Linear stability analysis

In order to determine the critical condition for stability, we now seek to linearise the delay equation around its fixed point. For simplicity, we set $V = 1, C = 2$, and therefore the fixed point is $\phi^* = \frac{V}{C} = \frac{1}{2}$. We introduce a function describing fluctuations about this point

$$\psi(t) := \phi(t) - \phi^* = \phi(t) - \frac{1}{2} \quad (15)$$

and its time average

$$\bar{\psi}_m(t) := \frac{1}{m} \int_{t-m}^{t} \psi(\tau) d\tau. \tag{16}$$

To evaluate p, as given in equation (13), we replace the summation with a continuous integral, and we approximate the binomial distribution with a Gaussian distribution [24] of the same mean and variance. Assuming small fluctuations, $\psi(t)$, this leads to the following linear approximation

$$p\left(\frac{1}{2} + \bar{\psi}_m\right) \approx \frac{1}{2} - \frac{\sqrt{2m}}{\sqrt{\pi}} \bar{\psi}_m. \tag{17}$$

We have verified this approximation by comparing the coefficient of $\bar{\psi}(t)$ to the exact values of the derivatives of $p(\phi)$ evaluated at $\phi = \frac{1}{2}$. The delay equation (14) is finally linearised to

$$\frac{1}{\epsilon} \frac{d\psi(t)}{dt} = -\frac{\sqrt{2m}}{\sqrt{\pi}} \bar{\psi}_m(t) - \psi(t). \tag{18}$$

We substitute $\psi(t) = e^{\lambda t}$ as a trial solution to obtain the characteristic equation

$$\lambda^2 + \epsilon\lambda + \frac{\epsilon\sqrt{2}}{\sqrt{\pi m}}(1 - e^{-\lambda m}) = 0. \tag{19}$$

Writing $\lambda = x + iy$, we separate the real and imaginary parts of the characteristic equation as follows

$$x^2 - y^2 + \epsilon x + \frac{\epsilon\sqrt{2}}{\sqrt{\pi m}}(1 - e^{-mx}\cos(my)) = 0 \tag{20}$$

$$2xy + \epsilon y + \frac{\epsilon\sqrt{2}}{\sqrt{\pi m}}e^{-mx}\sin(my) = 0. \tag{21}$$

Numerical solution of these equations shows that with fixed memory length m the real part, x, of the solution λ is negative when the value of ϵ is sufficiently small. Under these conditions the fixed point is stable in the sense that the system will return to the fixed point after perturbation. As ϵ increases past a critical value instability appears when the real part, x, becomes positive. In this case oscillations about the fixed point grow in amplitude. Numerical solution of the full non-linear differential equation reveals that the amplitude of these oscillations is bounded giving rise eventually to oscillations of fixed amplitude: a "limit cycle". The transition from stable fixed point to limit cycle is known as a "Hopf Bifurcation". To calculate the critical value of the update rate ϵ_c at which the bifurcation occurs we set $x = 0$ in equation (21) giving

$$\frac{\sin(my)}{my} = -\frac{\sqrt{\pi}}{\sqrt{2m}}. \tag{22}$$

The number of solutions to this equation increases with increasing memory length m, with higher magnitude solutions corresponding to higher frequency oscillations. Simulations of the system for large m reveal that these higher
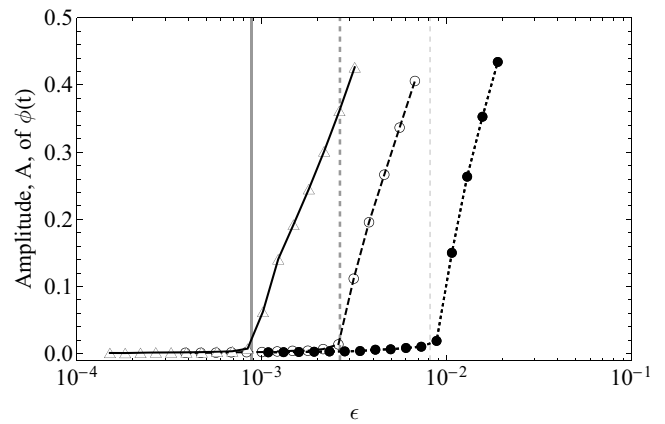

Fig. 4. Dependence of the steady state amplitude of $\phi(t)$ on ϵ for $m = 101$ (black circles), $m = 201$ (open circles) and $m = 401$ (triangles). $L = 10^4$ in all cases. Theoretical critical values $\epsilon_c(m)$, given by equation (24), are shown as vertical lines for the three cases: $m = 101$ (thin dashed), $m = 201$ (thick dashed) and $m = 401$ (thick solid).

frequency components are transient, so that only the lowest frequency oscillation persists at large times. Choosing the root near $y = \pi/m$, corresponding to the lowest frequency oscillation, we replace the left hand side of (22) with its series to second order about this root, and solve for y

$$y \sim \frac{\pi}{m} + \sqrt{\frac{\pi^3}{2m^3}} + \frac{\pi^2}{2m^2} \text{ as } m \to \infty. \tag{23}$$

Substitution of this expression into (20), and again setting $x = 0$, yields

$$\epsilon_c \sim \frac{\pi^{5/2}}{2\sqrt{2}m^{3/2}} + \frac{\pi^3}{2m^2} + \frac{\pi^{7/2}(12 + \pi^2)}{16\sqrt{2}m^{5/2}} \text{ as } m \to \infty. \tag{24}$$

We may view either the update rate ϵ or the memory length m as the bifurcation parameter by varying one while the other is fixed. The critical memory length m_c may be obtained by inverting expression (24), for which we see that to leading order $m_c \propto \epsilon^{-2/3}$. For an infinitely large system, equation (24) is a true asymptotic equality as $m \to \infty$, because our stability analysis involved a perturbation about the stable state, and the approximations which lead to the linearised delay equation (18) all become exact in the limit $m \to \infty$ in the absence of oscillations.

4.3 Numerical tests of stability

Our analytical result may be verified by considering the amplitude, A, of the simulated population average Hawk probability once early transient behaviour has dissipated. Below ϵ_c we expect A to be zero because the fixed point is stable and $\phi(t)$ does not oscillate, but to rise once the critical point is passed. We verify that this is the case for a series of memory values, as shown in Figure 4. It is clear that our analytical prediction of ϵ_c accurately captures the onset of oscillations.

5 Conclusion

We have explored the Hawk-Dove game played by L agents who have a simple memory for past interactions. They use this memory as a statistical sample, constantly updating their probabilistic strategy by weighting it toward the behaviour which currently appears optimal. When their memory length (sample size) is sufficiently small, the system possesses a stable fixed point, but excessive memory, or update rate, destabilises the fixed point, creating a limit cycle via a Hopf bifurcation. Either memory length or update rate may be viewed as the Bifurcation parameter, and we have analytically characterized the bifurcation point.

In a mixed population of short and long memory agents, when the system has a stable fixed point, agents with longer memories are better at selecting an optimal strategy because they are able to be more accurate in determining the average Hawk probability in the population. Although a long memory endows agents with better judgements in the stable game, it is no longer an advantage when strategies oscillate. Such oscillations can be created by excessively rapid strategy update rate, ϵ, or by an excess of agents with long memories. It is therefore possible for a "weaker" agent to prosper due to an excess of "stronger" competitors.

Author contribution statement

The contributions of the three authors are equal.

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