Improving the accuracy of asset price bubble start and end date estimators*

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Abstract

Recent research has proposed using recursive right-tailed unit root tests to date the start and end of asset price bubbles. In this paper an alternative approach is proposed that utilises model-based minimum sum of squared residuals estimators combined with Bayesian Information Criterion model selection. Conditional on the presence of a bubble, the dating procedures suggested are shown to offer consistent estimation of the start and end dates of a fixed magnitude bubble, and can also be used to distinguish between different types of bubble process, i.e. a bubble that does or does not end in collapse, or a bubble that is ongoing at the end of the sample. Monte Carlo simulations show that the proposed dating approach out-performs the recursive unit root test methods for dating periods of explosive autoregressive behaviour in finite samples, particularly in terms of accurate identification of a bubble's end point. An empirical application involving Nasdaq stock prices is discussed.

Keywords: Rational bubble; Explosive autoregression; Regime change; Break date estimation.

JEL Classification: C22; C13; G14.

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1 Introduction

There is now a substantial body of research suggesting that asset price bubbles and their ensuing collapses can have a significant impact on a country's macroeconomic performance.[1] Hence, detecting the presence of an asset price bubble and the timing of its termination is of crucial importance to central banks and financial regulators, as well as investors. Of particular interest to researchers in this area are "rational bubbles" – where the real price of an asset is assumed to be equal to the present value of relevant fundamentals and a bubble component that grows in expectation at the real interest rate, and investors are assumed to have rational expectations. Under these assumptions investing in the asset can be a rational choice for investors even though its current observed price is higher than the price level that is justified by relevant fundamentals. To define a rational bubble algebraically consider the simple case of a single stock, where P_t denotes the observed real stock price, D_t denotes the observed real dividend for the stock and r denotes the real interest rate used for discounting expected future cash flows. Define the observed price as consisting of a fundamentals component and a bubble component

$$P_t = P_t^f + B_t$$

where the fundamentals component P_t^f is given by

$$P_t^f = \sum_{i=1}^{\infty} (1+r)^{-i} E_t(D_{t+i}).$$

If the bubble component satisfies the stochastic difference equation

$$B_{t+1} = (1+r)B_t + u_t$$

where $E_{t-i}(u_t) = 0$ for all $i \geq 0$, then a rational bubble is said to exist (cf. Diba and Grossman, 1988).

It is clear from the algebraic representation given above that in the presence of a rational bubble, since the bubble grows at an explosive rate, the observed price will be a statistically explosive process (even if the fundamentals component of the

[1] See for example Bernanke (1995, 2013) and Greenspan (2007), Ch. 8, and the references therein, where examples of causal links between speculative bubbles, crises in banking systems and subsequent falls in aggregate demand leading to major macroeconomic recessions are discussed (including for the US stock price bubble in September/October 1929, the real estate and stock price bubbles in Japan in the late 1980s, and the US house price bubble in 2006.)

price is not statistically explosive). Recognition of this feature of rational bubbles led Diba and Grossman (1988) to propose statistical testing for the presence of a rational stock price bubble by attempting to detect explosive autoregressive behaviour in the stock price series that is not driven by similar explosive behaviour in the dividend series, using orthodox unit root tests such as the Dickey-Fuller (DF) test applied to the price and dividend series in levels and first differences. Since differencing an explosive autoregressive process does not lead to a stationary process, a rejection from the DF test for the first difference of the price and dividend series, with no rejection for the series in levels, suggests that no rational bubble exists.

The research on testing for rational bubbles by Diba and Grossman (1988) focuses on the specific case of an explosive rational bubble that does not collapse. As noted by Evans (1991) however, this type of bubble is empirically unrealistic because it implies that the asset price will perpetually grow at an explosive rate. Evans (1991) proposes a more realistic rational bubble model where the explosive bubble periodically collapses to a lower level, and the frequency of the collapses is controlled by a Bernoulli process. Using simulations, it is shown that even when the probability of collapse at each observation is extremely small so that there are just one or two collapses over the sample period considered, the use of orthodox unit root tests to detect bubbles as suggested by Diba and Grossman (1988) will tend to lead to the erroneous conclusion that a bubble is not present. This is due to the large adjustment in the price series caused by the bubble process collapsing. In effect, this reversion an appearance of mean reversion, causing the series to appear to be a stationary process with no explosive behaviour.

Recognizing this weakness of orthodox DF tests, researchers have focused on developing methods for detecting asset price bubbles that are more robust to the presence of collapses in the bubble process. Initial developments in this area employed unit root tests derived from regime switching models, such as Markov-switching models (e.g. Van Norden and Vigfusson, 1998; Hall *et al.*, 1999) and smooth transition autoregressive models (e.g. McMillan, 2006). Markov-switching models combined with Bayesian estimation techniques have also been found to be informative about the presence of bubbles that periodically collapse (e.g. Balke and Wohar, 2008). Whilst these methods have considerable advantages in the presence of periodically collapsing bubbles relative to using orthodox unit root tests, they can be computationally expensive, and the asymptotic distributions of unit root test statistics computed using these types of regime switching models are in some cases impossible to derive analytically.

Many of the more recently developed techniques for testing for bubbles retain use of DF-type tests; however, rather than applying traditional left-tailed DF tests to the price and fundamentals data in levels and differences, this research has recommended the use of *right-tailed* DF tests of the unit root null hypothesis against the alternative hypothesis of explosive autoregression applied to the relevant series in levels only. For example, see the papers by Phillips *et al.* (2011) (PWY), Homm and Breitung (2012), and Phillips *et al.* (2014) (PSY). PWY suggest constructing right-tailed DF tests recursively, and taking the supremum of this sequence of test statistics to test the unit root null hypothesis against the explosive alternative. Homm and Breitung (2012) consider a modified version of the PWY methodology, based on taking the supremum of backward recursive DF statistics. PSY recommend a statistic based on the supremum of both forward and backward recursively computed DF statistics. Simulations show that the proposed tests have very good finite sample power to detect a rational bubble, even if the bubble periodically collapses as in Evans (1991). Note also that as pointed out by PSY, a further attractive feature of the test statistics proposed in this line of research is that as well as being able to detect rational bubbles, the test statistics will have non-trivial finite sample power to detect other types of explosive bubble processes, including for example intrinsic bubbles (Froot and Obstfeld, 1991), herd behavior (Avery and Zemsky, 1998; Abreu and Brunnermeier, 2003), and bubbles generated by time varying discount factor fundamentals (Phillips and Yu, 2011).

At least as important as being able to detect the *presence* of a bubble is the issue of being able to accurately determine the *start and end dates* of a bubble regime that is deemed to exist. Such information can be crucial ex post for reconciling the origination and termination of a bubble with other economic and financial events. Both PWY and PSY address this important issue, proposing estimators for the timing of explosive behaviour that are based on the sequences of DF recursive statistics exceeding threshold values. For example, PWY apply their approach to data on the Nasdaq composite index 1972:3-2005:6, and find evidence of explosiveness that started in 1995, predating comments made by the Federal Reserve Board Chairman, Alan Greenspan, in December 1996 on "irrational exuberance" affecting the US stock market (Greenspan, 1996).

In this paper we suggest alternative estimators of the origination and termination of a bubble period. Specifically, rather than using sequences of recursive DF statistics to date the bubble regime, we propose estimating regime change-points on the basis of model-based minimum sum of squared residuals estimators (in the spirit of, *inter alia*, Bai and Perron, 1998, and Kejriwal *et al.*, 2013) combined with Bayesian Information Criterion (BIC) model selection. The proposed dating algorithms also allow identification of the particular form of bubble among a set of candidate bubble processes, which allow variously for a bubble that ends within the sample period, possibly with some form of collapse, or a bubble that is ongoing at the end of the sample. For a fixed magnitude bubble, we find that our BIC-based approach delivers consistent estimation of the exact bubble start and (where appropriate) end dates. Moreover, finite sample simulations suggest that, conditional on having detected the presence of a bubble using the PSY test, the new procedure offers considerably improved dating accuracy relative to the recursive DF statistic-based approaches of PSY, particularly with respect to the bubble's end date.

The next section of the paper sets out the basic framework and outlines the set of bubble data generating processes (DGPs) considered. Section 3 presents our proposed start and end date estimators for each model and their respective asymptotic properties. In section 4 we present the BIC-based algorithms for selecting between the alternative models and bubble date estimators outlined in section 3, and show that this approach results in correct model selection in the limit. Section 5 outlines a number of details concerning the practical implementation of the procedure, before section 6 presents a finite sample evaluation of the new date estimators relative to those proposed by PSY, using Monte Carlo simulations. In section 7 we revisit the Nasdaq composite index series considered by PWY to examine whether the new estimators can shed any further light on the timing of the asset price bubble in this data. Finally, some conclusions are offered in section 8.

Throughout the rest of the paper, y_t can be thought of as denoting the relevant asset price. The following notation is also used: '$\lfloor \cdot \rfloor$' denotes the integer part, '$\xrightarrow{p}$' denotes convergence in probability and '$1(\cdot)$' denotes the indicator function. We use the order notation $O_p^+(\cdot)$ to imply that the term concerned is positive. Finally we use '$x_T >_p 0$' to imply x_T becomes positive with probability one as $T \to \infty$.

2 The bubble DGPs

We consider the following generic DGP for y_t, $t = 1, ..., T$,

$$y_t = \mu + u_t \tag{1}$$

$$u_t = \begin{cases} u_{t-1} + v_t, & t = 2, ..., \lfloor \tau_{1,0} T \rfloor, \\ (1+\delta_1) u_{t-1} + v_t, & t = \lfloor \tau_{1,0} T \rfloor + 1, ..., \lfloor \tau_{2,0} T \rfloor, \\ (1-\delta_2) u_{t-1} + v_t, & t = \lfloor \tau_{2,0} T \rfloor + 1, ..., \lfloor \tau_{3,0} T \rfloor, \\ u_{t-1} + v_t, & t = \lfloor \tau_{3,0} T \rfloor + 1, ..., T \end{cases}$$

where $\delta_1 \geq 0$ and $\delta_2 \geq 0$. We assume that the initial condition u_1 satisfies $u_1 = O_p(1)$, while the innovation process $\{v_t\}$ satisfies the following linear process assumption:

Assumption 1. *The stochastic process $\{v_t\}$ is such that*

$$v_t = C(L)\eta_t,\ C(L) := \sum_{j=0}^{\infty} C_j L^j$$

with $C(1)^2 > 0$ and $\sum_{i=0}^{\infty} i|C_i| < \infty$, and where $\{\eta_t\}$ is an IID sequence with mean zero, unit variance and finite fourth moment. The short run variance of v_t is defined as $\sigma_v^2 = \sum_{j=0}^{\infty} C_j^2$.

The DGP imposes a unit root on y_t up to time $\lfloor \tau_{1,0} T \rfloor$, after which y_t is explosive (when $\delta_1 > 0$) until time $\lfloor \tau_{2,0} T \rfloor$. If $\tau_{2,0} = 1$, the explosive regime continues to the end of the sample period. However, if $\tau_{2,0} < 1$, the explosive regime terminates at some in-sample date, at which point a number of possibilities exist for the post-explosive period. If $\tau_{2,0} = \tau_{3,0}$, the series reverts to unit root behaviour for the remainder of the sample. Alternatively, if $\tau_{2,0} < \tau_{3,0}$ (with $\delta_2 > 0$), the explosive period is followed by a stationary regime, which either runs to the end of the sample if $\tau_{3,0} = 1$, or terminates within sample if $\tau_{3,0} < 1$; in this last case, the series reverts to unit root behaviour for the final regime.

In terms of modelling potential asset price bubble behaviour, our DGP allows for a number of specifications for y_t when applied to asset price series (assuming unit root dividends). When $\delta_1 > 0$, the series initially starts as a unit root process for $\lfloor \tau_{1,0} T \rfloor$ observations before a bubble regime begins. Given the presence of such a bubble, four possibilities are admitted by the DGP: (i) the bubble runs to the end of the sample, (ii) the bubble terminates and unit root behaviour is restored, (iii) the bubble terminates with some form of collapse modelled by the stationary regime, which then continues to the end of sample, or (iv) the bubble terminates with a collapse regime that also finishes in-sample, after which unit root behaviour resumes. The magnitude of δ_2 and the duration of the collapse regime ($\lfloor \tau_{3,0} T \rfloor - \lfloor \tau_{2,0} T \rfloor$) control the rapidity and extent to which a collapse occurs. Our approach offers a flexible way of modelling potential collapse behaviour that might be expected when an asset price bubbles terminates, from relatively slow gradual adjustments in the price level to more rapid corrections, the rate being a reflection of how long agents need to adjust their behaviour. Modelling the collapse in this manner as opposed to an instantaneous collapse is credible from an empirical viewpoint, as it is hard to imagine all agents can react immediately and in unison upon a bubble's termination.

When a collapse regime is present, the mean reversion implicit in the stationary process generates a model of a collapsing bubble, as the underlying autoregressive process for u_t creates an exponential decay from the final bubble observation towards zero, thereby "offsetting" the explosive period to some extent. Of course, if left completely unrestricted, this decay process will eventually flatten out and merely resemble a zero mean stationary process, which is unappealing as a model for the price level. As a result, we wish to introduce a constraint on the stationary regime to ensure that exponential decay remains the dominant feature of the process when the collapse regime terminates. The condition we impose is that

$$(1+\delta_1)^{(\tau_{2,0}-\tau_{1,0})}(1-\delta_2)^{(\tau_{3,0}-\tau_{2,0})} \geq 1 \tag{2}$$

and, drawing on the proof of Theorem 1, it then follows that

$$\begin{aligned} y_{\lfloor\tau_{3,0}T\rfloor} \approx\ & (1-\delta_2)^{\lfloor\tau_{3,0}T\rfloor-\lfloor\tau_{2,0}T\rfloor}(1+\delta_1)^{\lfloor\tau_{2,0}T\rfloor-\lfloor\tau_{1,0}T\rfloor} y_{\lfloor\tau_{1,0}T\rfloor} \\ & + \textstyle\sum_{j=0}^{\lfloor\tau_{3,0}T\rfloor-\lfloor\tau_{2,0}T\rfloor-1}(1-\delta_2)^j v_{\lfloor\tau_{3,0}T\rfloor-j} \end{aligned} \tag{3}$$

where the first term of (3) dominates the second. Hence at the point where the collapse period terminates, the effect of the decay from the explosive period is still dominant.

To summarise, given $\delta_1 > 0$ and $\delta_2 > 0$, the following possibilities are considered for the behaviour of y_t:

DGP 1: $0 < \tau_{1,0} < 1$, $\tau_{2,0} = 1$
(unit root, then bubble to sample end)

DGP 2: $0 < \tau_{1,0} < \tau_{2,0} < 1$, $\tau_{2,0} = \tau_{3,0}$
(unit root, then bubble, then unit root to sample end)

DGP 3: $0 < \tau_{1,0} < \tau_{2,0} < 1$, $\tau_{3,0} = 1$
(unit root, then bubble, then collapse to sample end)

DGP 4: $0 < \tau_{1,0} < \tau_{2,0} < \tau_{3,0} < 1$
(unit root, then bubble, then collapse, then unit root to sample end)

Our focus in this paper is on dating an asset price bubble that is assumed to be present (or has been inferred to be present on the basis of a test for a bubble), and we concentrate on estimating the bubble start and finish timings ($\lfloor\tau_{1,0}T\rfloor+1$ and $\lfloor\tau_{2,0}T\rfloor$) that arise under DGPs 1-4.

3 Estimating the regime change points

We first consider the case where we assume knowledge as to which of DGP 1, DGP 2, DGP 3 or DGP 4 is the true generating process. For DGP j, we estimate the regime change point(s) on the basis of minimising the residual sum of squares across all candidate dates, using the fitted OLS regressions corresponding to Model j below:

$$\begin{aligned}
&\text{Model 1:} \quad \Delta y_t = \hat{\mu}_1 D_t(\tau_1, 1) + \hat{\delta}_1 D_t(\tau_1, 1) y_{t-1} + \hat{v}_{1t} \\
&\text{Model 2:} \quad \Delta y_t = \hat{\mu}_1 D_t(\tau_1, \tau_2) + \hat{\delta}_1 D_t(\tau_1, \tau_2) y_{t-1} + \hat{v}_{2t} \\
&\text{Model 3:} \quad \Delta y_t = \hat{\mu}_1 D_t(\tau_1, \tau_2) + \hat{\mu}_2 D_t(\tau_2, 1) + \hat{\delta}_1 D_t(\tau_1, \tau_2) y_{t-1} + \hat{\delta}_2 D_t(\tau_2, 1) y_{t-1} + \hat{v}_{3t} \\
&\text{Model 4:} \quad \Delta y_t = \hat{\mu}_1 D_t(\tau_1, \tau_2) + \hat{\mu}_2 D_t(\tau_2, \tau_3) + \hat{\delta}_1 D_t(\tau_1, \tau_2) y_{t-1} + \hat{\delta}_2 D_t(\tau_2, \tau_3) y_{t-1} + \hat{v}_{4t}.
\end{aligned}$$

Here, the dummy variables are defined by $D_t(a, b) = 1(\lfloor aT \rfloor < t \leq \lfloor bT \rfloor)$. The constant dummy variables associated with $\hat{\mu}_1$ and $\hat{\mu}_2$ are included to ensure invariance of the residuals $\hat{v}_{jt}$, $j = 1, ..., 4$, to the series mean μ. Given these models, the change point estimators obtained from each are as follows:

$$\begin{aligned}
&\text{Model 1:} \quad \hat{\tau}_1 = \arg\min\nolimits_{0<\tau_1<1, y_T > y_{\lfloor \tau_1 T \rfloor}} SSR_1(\tau_1) \\
&\text{Model 2:} \quad (\hat{\tau}_1, \hat{\tau}_2) = \arg\min\nolimits_{0<\tau_1<\tau_2<1, y_{\lfloor \tau_2 T \rfloor} > y_{\lfloor \tau_1 T \rfloor}} SSR_2(\tau_1, \tau_2) \\
&\text{Model 3:} \quad (\hat{\tau}_1, \hat{\tau}_2) = \arg\min\nolimits_{0<\tau_1<\tau_2<1, y_{\lfloor \tau_2 T \rfloor} > y_{\lfloor \tau_1 T \rfloor}, y_{\lfloor \tau_2 T \rfloor} > y_T} SSR_3(\tau_1, \tau_2) \\
&\text{Model 4:} \quad (\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3) = \arg\min\nolimits_{0<\tau_1<\tau_2<\tau_3<1, y_{\lfloor \tau_2 T \rfloor} > y_{\lfloor \tau_1 T \rfloor}, y_{\lfloor \tau_2 T \rfloor} > y_{\lfloor \tau_3 T \rfloor}} SSR_4(\tau_1, \tau_2, \tau_3)
\end{aligned}$$

where $SSR_j(.) = \sum_{t=2}^{T} \hat{v}_{jt}^2$, $j = 1, ..., 4$, and the constraints $y_{\lfloor \tau_2 T \rfloor} > y_{\lfloor \tau_1 T \rfloor}$ and $y_{\lfloor \tau_2 T \rfloor} > y_{\lfloor \tau_3 T \rfloor}$ are incorporated to ensure that the period between the sample fractions τ_1 and τ_2 is associated with a (putative) upward explosive regime, and τ_2 to τ_3 associates with a downward stationary collapse regime.[2]

We now consider the asymptotic behaviour of the regime change point estimators for Models 1-4, assuming a correct pairing between the DGP and the corresponding Model. The results are given in the following theorem.

Theorem 1

(I) For DGP 1 and Model 1, $\lfloor \hat{\tau}_1 T \rfloor - \lfloor \tau_{1,0} T \rfloor \xrightarrow{p} 0$.

[2] Homm and Breitung (2012) propose a similar estimator of a bubble start date in the context of a model where the bubble originates from unit root behaviour within-sample and continues to the sample endpoint (i.e. DGP 1), using the argmax of a sequence of Chow tests for structural change in the autoregressive parameter of an AR(1) regression. Breitung and Kruse (2013) suggest using a similar SSR-based estimator for the bubble end date in a model where the bubble originates at the sample start-point and terminates in-sample, reverting to unit root behaviour (i.e. DGP 2, but without the initial unit root regime).

(II) For DGP 2 and Model 2 $\lfloor\hat{\tau}_i T\rfloor - \lfloor\tau_{i,0} T\rfloor \xrightarrow{p} 0$*,* $i = 1, 2$*.*
(III) For DGP 3 and Model 3, $\lfloor\hat{\tau}_i T\rfloor - \lfloor\tau_{i,0} T\rfloor \xrightarrow{p} 0$*,* $i = 1, 2$*.*
(IV) For DGP 4 and Model 4, under the condition (2), $\lfloor\hat{\tau}_i T\rfloor - \lfloor\tau_{i,0} T\rfloor \xrightarrow{p} 0$*,* $i = 1, 2, 3$*.*

Theorem 1 shows that, for each correct DGP/Model pairing, the actual date of the start of the bubble period is consistently estimated by $\lfloor\hat{\tau}_1 T\rfloor + 1$. For DGPs/Models 2-4, which have a bubble ending within the sample, the bubble end date is consistently estimated by $\lfloor\hat{\tau}_2 T\rfloor$. Finally, for DGP/Model 4, the end date of the collapse period is consistently estimated by $\lfloor\hat{\tau}_3 T\rfloor$. Should condition (2) not hold, we would still obtain $\lfloor\hat{\tau}_i T\rfloor - \lfloor\tau_{i,0} T\rfloor \xrightarrow{p} 0$, $i = 1, 2$, but for the end date of the collapse regime, consistency would only apply to the break fraction (not the exact date), i.e. $\hat{\tau}_3 - \tau_{3,0} \xrightarrow{p} 0$. In the next section we consider selection between these alternative Models and their estimated break points in the practical case where it is not known which of DGPs 1-4 corresponds to the true generating process.

4 Selecting between the models

To obtain efficient break date estimation it is important to select the correct model, i.e. the model that corresponds to the true DGP. We propose selecting between the models on the basis of the BIC as follows. In the leading case of interest, where we assume that a bubble is present in the series, we choose Model j_{opt}, and the change point estimates associated with Model j_{opt}, according to

$$j_{opt} = \arg \min_{j \in \{1,2,3,4\}} BIC_j$$

where

$$\begin{aligned}
BIC_1 &= T \ln\{T^{-1} SSR_1(\hat{\tau}_1, 1)\} + (2+1)\ln(T), \\
BIC_2 &= T \ln\{T^{-1} SSR_2(\hat{\tau}_1, \hat{\tau}_2)\} + (2+2)\ln(T), \\
BIC_3 &= T \ln\{T^{-1} SSR_3(\hat{\tau}_1, \hat{\tau}_2, 1)\} + (4+2)\ln(T), \\
BIC_4 &= T \ln\{T^{-1} SSR_4(\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3)\} + (4+3)\ln(T).
\end{aligned}$$

The scalar multiplying the penalty $\ln(T)$ represents the number of columns in the associated regressor matrix, i.e. the number of coefficients being estimated, plus the number of estimated regime change points. In what follows we refer to this model/break date selection algorithm as BIC_{opt}.

In order to establish the asymptotic behaviour of BIC$_{opt}$, we first analyze the large sample behaviour of the (scaled) minimised sum of squared residuals for each Model across the various DGPs considered, i.e. $T^{-1}SSR_1(\hat{\tau}_1, 1)$, $T^{-1}SSR_2(\hat{\tau}_1, \hat{\tau}_2)$, $T^{-1}SSR_3(\hat{\tau}_1, \hat{\tau}_2, 1)$, $T^{-1}SSR_4(\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3)$ under DGPs 1-4. Given the results in Theorem 1 for the break date estimators, the following results are straightforward to establish:

	$T^{-1}SSR_1(.)$	$T^{-1}SSR_2(.)$	$T^{-1}SSR_3(.)$	$T^{-1}SSR_4(.)$
DGP 1:	$\sigma_v^2 + O_p(T^{-1/2})$	$O_p^+(T^\kappa)$	$\sigma_v^2 + O_p(T^{-1/2})$	$O_p^+(T^\kappa)$
DGP 2:	$O_p^+(T^\kappa)$	$\sigma_v^2 + O_p(T^{-1/2})$	$\sigma_v^2 + O_p(T^{-1/2})$	$\sigma_v^2 + O_p(T^{-1/2})$
DGP 3:	$O_p^+(T^\kappa)$	$O_p^+(T^\kappa)$	$\sigma_v^2 + O_p(T^{-1/2})$	$\lambda^2 + O_p(T^{-1/2})$
DGP 4:	$O_p^+(T^\kappa)$	$O_p^+(T^\kappa)$	$O_p^+(T^\kappa)$	$\sigma_v^2 + O_p(T^{-1/2})$

Here, $\kappa > 0$ is used generically and λ^2 is a constant satisfying $\lambda^2 > \sigma_v^2$. From the above results, we see that a true DGP/Model combination always yields $T^{-1}SSR_j(.) = \sigma_v^2 + O_p(T^{-1/2})$. Employing a minimum BIC rule means that any competing model for which $T^{-1}SSR_j(.) = O_p^+(T^\kappa)$ will, in the limit, never be selected since

$$\begin{aligned} T\ln\{O_p^+(T^\kappa)\} - T\ln\{\sigma_v^2 + O_p(T^{-1/2})\} &= \kappa T\ln\{TO_p^+(1)\} - T\ln\{\sigma_v^2 + O_p(T^{-1/2})\} \\ &= \kappa T\ln(T) + \kappa T\ln\{O_p^+(1)\} \\ &\quad -T\ln\{\sigma_v^2 + O_p(T^{-1/2})\} \end{aligned}$$

which diverges to $+\infty$ at a rate $T\ln(T)$, thereby dominating any order $\ln(T)$ penalty term involved. Also, for DGP 3, when comparing Model 3 and Model 4 we have

$$T\ln\{\lambda^2 + O_p(T^{-1/2})\} - T\ln\{\sigma_v^2 + O_p(T^{-1/2})\} = T\ln\left\{\frac{\lambda^2}{\sigma_v^2} + O_p(T^{-1/2})\right\}$$

which diverges to $+\infty$ at a rate T since $\lambda^2/\sigma_v^2 > 1$, again dominating any order $\ln(T)$ penalty term involved, so that minimising the BIC results in a preference for the true Model 3 over Model 4 in the limit. Elsewhere, the comparison is between a true DGP/Model combination and other models that are overspecified, such that $T^{-1}SSR_j(.) = \sigma_v^2 + O_p(T^{-1/2})$ in each case. However, a minimum BIC approach always selects the true model in the limit because, denoting the true model by j and its overspecified counterpart by k, we find that

$$T\ln\{T^{-1}SSR_k\} - T\ln(T^{-1}SSR_j) = -O_p^+(1)$$

and hence the penalty term associated with the overspecified model in BIC_k ensures that Model k is not selected in the limit.

The above results establish that model selection on the basis of minimising the BIC will, in the limit, lead to selection of the true model. Moreover, the constants multiplying the $\ln(T)$ penalties are not unique in this regard. As a consequence, the BIC_{opt} algorithm results in correct asymptotic selection between DGPs 1-4, i.e. between the alternative bubble models.

Finally, while the large sample properties of the $\lfloor \hat{\tau}_i T \rfloor$ of Theorem 1, and the BIC_{opt} algorithm, have been established under an assumption of homoskedastic innovations v_t according to Assumption 1, we conjecture that the same results will continue hold under most forms of heteroskedasticity, including nonstationary volatility. This conjecture stems from the fact that the asymptotic analysis involves only establishing stochastic orders of magnitude for the relevant quantities, and never any limiting distributions. The orders of magnitude are unlikely to be affected by departures from homoskedasticity.

5 Practical implementation of the algorithm

The large sample results in sections 3 and 4 above rely on setting $0 < \tau_{1,0} < \tau_{2,0} < \tau_{3,0} < 1$, that is, any particular regime present has a duration of $O(T)$ time periods. When implementing the BIC_{opt} algorithm in what follows, we estimate Models 1-4 imposing $\tau_1 \geq s$ for the initial unit root regime. In general, we impose $\tau_2 - \tau_1 \geq s$ for any potential explosive regime present, so that any bubble has duration of at least $\lfloor sT \rfloor$ observations, and impose $\tau_3 - \tau_2 \geq s/2$ for any potential collapse regime present. This allows the possibility of the bubble collapsing over a shorter time period than that over which it emerges, as might be expected empirically. In doing this, we are restricting the bubble and collapse regimes to each being of $O(T)$ duration. Such an approach precludes modelling a collapse as occurring instantaneously (such as in equation (14) of PWY), but instantaneous adjustment is arguably less realistic empirically as agents do not typically react in unison.

In practical applications, when computing the BIC_{opt} algorithm, enforcing an $O(T)$ duration on the final regimes of Models 1-4 prevents a segue from one model to another (e.g. moving from Model 2 to Model 1 would involve a discontinuity from a model with an $O(T)$ duration final unit root regime to one with no final unit root regime). To allow for a smooth movement from one model to another, we relax the requirement of $O(T)$ durations when applied to the *final* regime of any given model. That is, for Model 1 we allow τ_1 to run up to the point $\lfloor \tau_1 T \rfloor = T - 1$, for Models 2 and 3 we let τ_2 run to $\lfloor \tau_2 T \rfloor = T - 1$, and for Model 4 τ_3 is permitted to run to $\lfloor \tau_3 T \rfloor = T - 1$. Note that in the case of Model 1 when $\lfloor \tau_1 T \rfloor = T-1$ and Model 3 when $\lfloor \tau_2 T \rfloor = T-1$ (where the final regime lasts only a single observation), the corresponding dummy variable regressors ($D_t(\tau_1, 1)$ and $D_t(\tau_1, 1)y_{t-1}$ for Model 1, $D_t(\tau_2, 1)$ and $D_t(\tau_2, 1)y_{t-1}$ for Model 3) become collinear; in these cases, we therefore replace these regressors with a single (one-time) dummy regressor; we also reduce the corresponding BIC penalty by $\ln(T)$ in these cases to reflect the reduced number of estimated parameters.

6 Finite sample performance

In this section we examine via Monte Carlo simulation the finite sample properties of the BIC_{opt} model selection algorithm in terms of its ability to correctly identify $\lfloor \tau_{1,0} T \rfloor$ and $\lfloor \tau_{2,0} T \rfloor$ when a bubble is present in the DGP.[3] Given that the algorithm is applicable only when a bubble is deemed to be present, we assess the dating performance of the procedure *conditional* on detecting a bubble in a given series from prior application of the PSY test (this being more powerful than the PWY test). We also examine the algorithm's performance in identifying which of Models 1-4 corresponds to the true DGP. Of course, correct model identification is not necessarily critical for accurate estimation of the bubble start and end dates, since, for example, it is possible that Model 4 may still prove informative about $\lfloor \tau_{1,0} T \rfloor$ and $\lfloor \tau_{2,0} T \rfloor$ when DGP 2 holds.

Figures 1-3 report measures of the accuracy of the change point estimators obtained by BIC_{opt} for series generated according to DGPs 2 and 4, using a variety of bubble and collapse timings and magnitudes. All simulations are conducted with a sample size of $T = 200$, and we set $s = 0.1$ in the computation of BIC_{opt}. As a measure of the accuracy of the change point estimators, we compute the simulated frequency with which the break date estimates $\lfloor \hat{\tau}_1 T \rfloor$ and $\lfloor \hat{\tau}_2 T \rfloor$ are within $\pm k$ observations of $\lfloor \tau_{1,0} T \rfloor$ and $\lfloor \tau_{2,0} T \rfloor$, respectively, computing this frequency across the subset of replications for which evidence of a bubble is found at the 0.05-level by the PSY test.[4] We report results for $k = \{0, 1, 5\}$; clearly, $k = 0$ corresponds to a correct dating of the precise observation associated with the regime change point. The duration of the bubble regime is set to $\lfloor 0.2T \rfloor$ in each case (i.e. $\tau_{2,0} - \tau_{1,0} = 0.2$), and the accuracy measures are plotted against a range of bubble magnitudes, with $\delta_1 = \{0.0400, 0.0425, ..., 0.1000\}$. The minimum value of δ_1 is chosen so that the PSY test has a decent level of power

[3] Identification of $\lfloor \tau_{3,0} T \rfloor$ is arguably of less importance and is not considered here.

[4] When implementing the PSY test, we follow PSY's recommendation to use a small fixed lag length in the ADF regressions, setting the lag length to one. We adopt PSY's recommended minimum window width of $\lfloor (0.01 + 1.8/\sqrt{T})T \rfloor$, and simulate finite sample null critical values for the test using 10,000 replications.

for all experiments conducted, ensuring that the accuracy frequencies are computed over a sufficient number of replications; for $\delta_1 = 0.04$, test power exceeds 0.45 across all DGPs considered, rising to powers in excess of 0.95 across all DGPs when $\delta_1 = 0.1$. For DGP 4, where a stationary collapse regime is present, we set the duration to $\lfloor 0.1T \rfloor$ (i.e. $\tau_{3,0} - \tau_{2,0} = 0.1$) in recognition of the empirical observation that the duration of a collapse phase is typically shorter than the duration of the corresponding bubble phase. We consider two cases for the magnitude of the stationary parameter in these regimes, setting $\delta_2 = \delta_1$ or $\delta_2 = \delta_1/2$, the latter allowing for more partial collapses relative to the former. For a given DGP, we simulate y_t according to (1), with $\mu = 0$ and $v_t \sim IIDN(0,1)$; in cases where such a simulated DGP resulted in a downward explosive regime (i.e. if $y_{\lfloor \tau_{2,0}T \rfloor} < y_{\lfloor \tau_{1,0}T \rfloor}$), typically due to the explosive period originating with negative values, we multiplied the simulated series by -1, so as to ensure that all generated series had upward explosive regimes. All simulations were conducted using 2,000 Monte Carlo replications, and were programmed in GAUSS 9.0.

By way of comparison, we also report the same conditional accuracy measures for the dating scheme proposed by PSY. Specifically, we adopt their backward supremum ADF-based start and end date estimators (denoted by $\lfloor \hat{r}_e T \rfloor$ and $\lfloor \hat{r}_f T \rfloor$ in PSY).[5] We restrict identification of a valid bubble regime to cases where the sequence of backward supremum statistics exceeds the corresponding threshold critical values for at least $\lfloor \ln T \rfloor$ contiguous observations, ensuring that only bubbles with a minimum duration of $\lfloor \ln T \rfloor$ are used for dating purposes. In the case that more than one bubble is identified in a series, the bubble with the longest contiguous sequence of rejections is used for dating purposes.

We begin by considering the case of DGP 2, where a bubble begins and ends within the sample period, but no collapse occurs at the end of the bubble regime. Figure 1 reports results for a bubble beginning at the sample mid-point ($\tau_{1,0} = 0.5, \tau_{2,0} = 0.7$).[6] Focusing first on the estimated start dates, for a given value of k, the accuracy of both dating procedures increases monotonically with δ_1 as would be expected. For both bubble timings and for any k, it is clear that the BIC$_{opt}$ algorithm delivers the more precise estimator, with the accuracy levels for the PSY approach quite markedly below those for the proposed BIC-based approach. This reflects the fact that the estimator based on BIC$_{opt}$ involves direct modelling of the start of the bubble period in the DGP,

[5] We again use a fixed lag length of one in the ADF regressions, use the minimum window width $\lfloor (0.01 + 1.8/\sqrt{T})T \rfloor$, and simulate finite sample null 0.05-level critical values for the sequence of backward supremum statistics using 10,000 replications.

[6] Qualitatively similar results were obtained for an earlier bubble timing ($\tau_{1,0} = 0.2, \tau_{2,0} = 0.4$); these results are available from the authors on request.

as opposed to relying on the more indirect approach of recursive unit root statistics exceeding threshold values.

Turning attention now to the end date estimators, for BIC$_{opt}$ we again see that for a given k, accuracy improves with δ_1, but here the accuracy levels are much higher than were seen for the corresponding start date. Indeed, the *exact* true end date $\lfloor \tau_{2,0}T \rfloor$ is much more readily identified (conditional on a bubble being detected), with a frequency close to one for the larger values of δ_1. In contrast, the PSY end date estimators display substantially lower accuracy levels, particularly when $k = 0$ or $k = 1$, where the frequencies are at, or are very close to, zero for much of the δ_1 range. This property arises since the PSY estimators of the end point rely on the sequence of recursive unit root statistics returning to magnitudes below the corresponding thresholds, and the delay inherent in this methodology (in finite samples) has the tendency to place the end date later than the true bubble's actual end point.

We next consider DGPs where a stationary collapse regime occurs following the termination of the bubble regime. In Figures 2-3, accuracy measures for DGP 4 are reported for the bubble/collapse timings $\tau_{1,0} = 0.5, \tau_{2,0} = 0.7, \tau_{3,0} = 0.8$, and for two settings for the stationary parameter, with $\delta_2 = \delta_1/2$ in Figure 2 and $\delta_2 = \delta_1$ in Figure 3.[7] Comparing Figures 2 and 3 with Figure 1 (where the same bubble component was present but without collapse), we again observe BIC$_{opt}$ outperforming PSY. The accuracy measures for the BIC$_{opt}$ and PSY start dates are almost identical to their respective counterparts in Figure 1, while the end dates are more accurately identified by both methodologies now that a collapse occurs, particularly for PSY. The greatest accuracy differences between BIC$_{opt}$ and PSY are again seen for the end dates and for $k = \{0, 1\}$, where the accuracy gains of BIC$_{opt}$ over PSY are substantial. It is only for $k = 5$ for the faster collapse DGP (Figure 3) that PSY achieves similar accuracy levels to BIC$_{opt}$.[8]

In addition to evaluating the dating performance of the BIC$_{opt}$ algorithm, it is also of interest to establish, for a given DGP j, the frequency with which the algorithm correctly selects Model j. Using the same simulations as outlined above for the dating

[7] Broadly similar results, available from the authors on request, were again obtained for an earlier bubble timing ($\tau_{1,0} = 0.2, \tau_{2,0} = 0.4, \tau_{3,0} = 0.5$).

[8] We also conducted a number of additional simulations, the results of which are unreported but are available from the authors on request. We simulated DGP 3, where the stationary collapse regime runs to the end of the sample, with the settings $\tau_{1,0} = 0.7, \tau_{2,0} = 0.9, \tau_{3,0} = 1$, and found the pattern of results to be similar to those for DGP 4. We also simulated DGP 1, where a bubble is present without collapse, but with the bubble running to the sample end ($\tau_{1,0} = 0.8, \tau_{2,0} = 1$); in that case, results for the bubble start date were qualitatively similar to those in Figure 1 for DGP 2 (as the bubble is ongoing at the end of the sample, there is no meaningful concept of a bubble end date for DGP 1).

evaluation, we also computed the correct model selection frequencies for BIC_{opt}, again conditional on detection of a bubble, measured across replications for which the PSY test indicated rejection of the null. Figure 4 presents the results for a representative example of each of DGPs 1-4. It is clear that as the magnitude of the bubble component (δ_1) increases (and as the magnitude of the collapse parameter δ_2 for DGPs 3-4 increases), the correct model is chosen with increasing probability. For a given δ_1, it is clear that the algorithm is best at correctly identifying DGP 1, and at its least effective for correctly identifying DGP 4, with correct identification of DGP 2 and DGP 3 lying between these two extremes. Overall, though, the tendency towards correctly identifying the underlying DGP as the true regime changes become more substantial is an attractive feature of the BIC_{opt} algorithm, and allows inference to be made as to the form of bubble/collapse, in addition to providing accurate estimates of the bubble start and end dates.

Overall, on the basis of our simulation experiments, it is clear that the BIC_{opt} algorithm offers a good approach for dating the start and end dates of a bubble regime, outperforming the comparator dating methodology of PSY, particularly where end-point detection is concerned. We envisage that these new methods of dating, when used in conjunction with the PSY test for the presence of a bubble, should be very useful for practitioners wanting to date the timing of a bubble episode, offering worthwhile improvements in the estimation accuracy of the regime change-points.

7 Empirical application

To demonstrate the usefulness of the new procedure we consider an empirical application, dating an explosive rational bubble in the Nasdaq composite stock price index using monthly data on the index over the period 1973:2-2005:6 (a repeat of the empirical application in PWY). The US consumer price index is used to convert the data from nominal to real values and following PWY the natural logarithm of the real data is used. As in PWY, the Nasdaq composite data is collected from the Datastream database and the CPI data from the Federal Reserve Bank of St. Louis FRED database.

As we have briefly mentioned in the introduction, the study of this data by PWY reveals statistically significant evidence of explosive autoregressive behaviour in the price index; given that no evidence of explosive behaviour was found in the dividend index, this result is consistent with the presence of an explosive rational bubble. A particularly interesting feature of PWY's results concerns the dates obtained for the start and end of the bubble. More specifically, PWY find that the bubble starts in mid-1995 and ends in mid-2001. Therefore the PWY dating procedure reveals that the bubble began some time before the famous comments on the outlook for asset prices made by the Federal Reserve Board Chairman, Alan Greenspan, on December 5th, 1996, in his speech to the American Enterprise Institute:

"Clearly, sustained low inflation implies less uncertainty about the future, and lower risk premiums imply higher prices of stocks and other earning assets. We can see that in the inverse relationship exhibited by price/earnings ratios and the rate of inflation in the past. But how do we know when irrational exuberance has unduly escalated asset values, which then become subject to unexpected and prolonged contractions as they have in Japan over the past decade?" (Greenspan, 1996)

It is generally believed that these comments had a significant impact on global financial markets, leading to falls the next day in several stock markets of as much as 4% (e.g. both Frankfurt and London stock markets fell by 4%), although the Nasdaq composite index went on to rise to unprecedented levels. As PWY note at the start of their paper: "...it is of interest to determine whether the Greenspan perception of exuberance was supported by empirical evidence in the data or if Greenspan actually foresaw the outbreak of exuberance and its dangers when he made the remark" (PWY, p.202). The results obtained by PWY indicate that the rational bubble was well under way when the Greenspan comments were made. As such, it appears that his comments were grounded in empirical realities but could not be considered anticipatory.

In our analysis of this data we apply the PSY test for the presence of a bubble, and then, having found a rejection of the null of no bubble, compute the estimated bubble start and end dates using the PSY procedure plus our recommended BIC_{opt} dating algorithm. Given that the sample size of $T = 389$ observations is roughly twice that used in our finite sample simulations, we set $s = 0.05$ in the computation of BIC_{opt}. For the PSY procedure, as in the simulations we adopt a minimum duration for identifying a valid bubble episode to be $\lfloor \ln T \rfloor$ contiguous observations, use PSY's recommended minimum window width of $\lfloor (0.01 + 1.8/\sqrt{T})T \rfloor$, include one lagged difference term in the regressions, and simulate 0.05-level critical values for testing and dating using 10,000 Monte Carlo replications.

The results are reported in Panel A of Table 1. Strong evidence of explosive behaviour (which can be interpreted as a rational bubble given the lack of evidence for explosive behaviour in the dividends) is detected by the PSY test, with the null of no bubble rejected at the 0.01-level, in line with the results of PWY. Focusing on the longest contiguous segment of bubble evidence, the PSY dating scheme estimates the bubble to be present from 1998:11 to 2000:12, although evidence for explosive behaviour is present in 2001:2, and the PSY dating statistic is also very close to detecting explosivity in 2001:1; moreover, evidence for explosive behaviour is found in a number of periods between 1995:8 and 1998:8, although only one contiguous explosive period that exceeds $\lfloor \ln T \rfloor$ observations is identified over these dates (1997:6-1997:12). In addition to this primary bubble period in the data, the PSY approach also identifies a short-lived period of explosive behaviour between 1987:2 and 1987:10.

Turning to application of the BIC$_{opt}$ algorithm, we find that the selected model is Model 3 (unit root, then bubble, then collapse to sample end), with a bubble present from 1998:11 to 2000:9. The start date therefore matches that of the PSY approach, but the BIC$_{opt}$ procedure identifies an earlier end date for the period of explosive behaviour. Given that the PSY approach identifies a potential second earlier bubble (1987:2-1987:10), we also checked the robustness of the BIC$_{opt}$ results by splitting the sample to exclude this earlier potential explosive regime. Applying the BIC$_{opt}$ procedure to the sample period 1987:11-2005:6, and setting $s = 0.1$ for this smaller sample (as in the simulations), we again find that Model 3 is selected with exactly the same period identified for the presence of bubble behaviour (1998:11-2000:9).

As discussed at the beginning of this section, an interesting and important finding by PWY is that their dating technique suggests the bubble began in mid-1995, predating comments made in December 1996 by the then Chairman of the Federal Reserve Alan Greenspan regarding the presence of irrational exuberance affecting US stock prices. Compared to the PWY results, the estimated beginning of sustained explosive behaviour that the PSY and BIC$_{opt}$ methods identify (1998:11) is much later than when the Greenspan comments were made. Thus these procedures raise the interesting possibility that rather than simply responding to current empirical explosivity, Alan Greenspan's comments might actually have anticipated irrational exuberance.

When interpreting and comparing the results from these types of econometric procedures for dating stock market bubbles it is also important to consider the results within the wider context of global macroeconomic events and relevant monetary policy at the time. In this particular case, it is interesting that a start date for the bubble of 1998:11 is consistent with the fact that the Federal Reserve opted to cut interest rates in late-1998 as a response to the East Asian financial crisis in the summer of 1997, the Russian default on their huge dollar debt in August 1998, and the collapse of the Long Term Capital Management (LTCM) hedge fund that followed. Specifically, between 1998:8 and 1998:12, the Federal Open Market Committee (FOMC) lowered interest rates (specifically their target federal funds rate) on three separate occasions in the hope of avoiding a financial crisis (September 29, 1998; October 15, 1998; November

17, 1998) by a total of 75 basis points, and this easing of monetary policy by the Federal Reserve was repeated by several other European and Asian central banks under their G7 commitment. This significant shift in monetary policy has been interpreted as having been successful, in the sense that it prevented a significant financial and macroeconomic downturn, but it also is thought to have contributed to the huge inflation of equity prices that occurred in 1999 (e.g. see Klein, 2015). Indeed, between late-1998 and late-1999, the Nasdaq index nearly doubled. In his own account of this period, Alan Greenspan notes:

"I suppose we might have guessed that the last year of the millennium would be the wildest, giddiest boom year of all. Euphoria swept the U.S. markets in 1999, partly because the East Asian crises hadn't done us in. If we'd made it through those, the thinking went, then the future was bright for as far as the eye could see." (Greenspan, 2007, p.294)

Thus, it appears there are coherent economic arguments to support the conclusion that the explosive Nasdaq bubble began in earnest in late-1998.

In terms of the estimated end dates obtained using the different dating procedures, Figure 5 provides a plot of the time series with the different end (and start) dates superimposed. The end date identified by the BIC_{opt} algorithm (2000:9) is only six months after the Nasdaq composite's numerical peak (2000:3), and appears to be the more plausible of the two estimated end dates, with the PSY end date placed a few months later and during the collapse phase identified by BIC_{opt}. Our finding that the estimated PSY end date occurs later than the BIC_{opt} end date is what might be expected given the discussion in section 6, i.e. that a potential exists in finite samples for the PSY scheme to estimate a delayed end date relative to the true termination of the bubble phase. The end date identified by PWY was either 2001:2 or 2001:7, so both PSY and BIC_{opt} detect an earlier end of bubble; interestingly, the BIC_{opt} end date of 2000:9 is also consistent with the date suggested by the rolling regression robustness check reported by PWY (p.215).

Although primarily designed for dating historical episodes of explosive (and collapse) behaviour, the BIC_{opt} procedure can also be implemented in a real-time manner to detect the end of a bubble regime. As an illustration of this, suppose first that we had been conducting an analysis of the Nasdaq composite index in 2000:9 (the last date where a bubble is deemed to be present according to our BIC_{opt} results). The results reported in Panel B of Table 1 show that in such a scenario, we would have concluded that a bubble exists (due to the strong rejections obtained by the PSY test), and both the PSY and BIC_{opt} dating methods would have identified the bubble as running up to the end of the sample period (note that here BIC_{opt} selects Model 1, consistent with a bubble running to the sample end). Moving forwards in a pseudo real-time fashion, for the next two months (2000:10 and 2000:11), Panel B of Table 1 shows that the same analysis would again have concluded that a bubble was still continuing, regardless of which dating method was used. However, if the analysis had been done in 2000:12, while the PSY approach would have indicated that the bubble was still ongoing, the BIC_{opt} algorithm would now have switched into Model 3 (where the bubble terminates prior to the sample end and begins to collapse) with the end date identified as 2000:9. In 2001:1, the BIC_{opt} algorithm would again have indicated that the bubble had terminated in 2000:9; at this point, the PSY dating approach would also no longer have found evidence for a bubble at the 0.05-level, also signifying that the bubble regime had come to an end.[9] However, the evidence for bubble behaviour at that point would have been only just below the 0.05-level threshold, and if the analysis had been done in 2001:2, the PSY approach would again have detected bubble behaviour at the final observation, introducing a lack of clarity as to whether or not the bubble regime had terminated. It would not have been until 2001:3 that the PSY approach would have delivered clear evidence that the bubble had terminated. Potentially then, in situations where rapid determination of a bubble's end date is considered important, the BIC_{opt} procedure could be of use.

8 Conclusion

In this paper we have proposed estimates of the start and end dates of a single bubble episode in an asset price series. Our method utilises a minimum sum of squared residuals type approach for a variety of potential bubble specifications, which are then distinguished using BIC-based model selection. The proposed procedure differs from existing methods of bubble dating which rely on sequences of recursive DF statistics exceeding certain threshold values. Conditional on detecting a bubble using PSY's test, our simulation results demonstrate that the new BIC_{opt} approach can offer improved levels of accuracy in dating the bubble's origination and, especially, termination point. A by-product of the dating scheme is that the form of bubble process is correctly identified asymptotically; in particular, this may inform us as to whether the bubble is ongoing or has terminated in some form of collapse.

A potential criticism of our current analysis is that it only dates a single bubble

[9] Note that similar results are obtained when applying BIC_{opt} to the sub-samples beginning in 1987:11 (using $s = 0.1$).

episode. We have envisaged how our proposed techniques might be extended to dating multiple bubbles. The PSY test will reject in the presence of multiple bubbles, and the PSY dating methodology provides an indication of the number of distinct bubbles. Conditional on this information on the number of bubbles, two natural approaches can in principle be taken for dating along the lines of the BIC_{opt} procedure. One method is to use an extended version of the algorithm that admits multiple bubble and collapse regimes. However, we would not recommend using such a procedure to fit multiple bubble episodes jointly, due to the number of model possibilities involved when more than one bubble is present. For example, if PSY identifies two bubble episodes, an extended BIC_{opt} algorithm would require allowing for 12 possible model combinations to select between, while in the case of three bubble episodes, there are 28 possible models. This, we feel, is unlikely to prove successful in samples of typical size as the number of competing models is simply too large. A second approach is to split the data and apply our one-bubble BIC_{opt} procedure *separately* on data subsets. For example, if the PSY approach identified and dated two bubble episodes, this approach would bifurcate the data at some point inbetween the two episodes, and then apply the BIC_{opt} algorithm to the two sub-samples individually. In effect, we would then be using the PSY dates as initial estimates, then using the BIC_{opt} procedure to subsequently improve the accuracy of the start and end date estimates for each bubble episode. In addition, for historical bubble episodes there is usually consensus amongst economists as to approximately when they occurred, hence such *a priori* information could also be brought to bear in determining the appropriate points to split the sample. Finally, sample splitting and applying the one-bubble BIC_{opt} procedure to each sub-sample is unlikely to have a significant cost in terms of efficiency, relative to using a full-sample BIC-based procedure to fit multiple bubble episodes jointly, because it is difficult to envisage that there is much information in one bubble episode that is of direct relevance to another. Our preferred approach would therefore be for the sample splitting method, although it would be interesting to examine the two possible procedures more formally in future research.

In summary, we feel that our results indicate that there is a worthwhile role for the new BIC_{opt} algorithm for dating the timeline of a bubble episode, complementing the existing testing procedures of PSY for detecting the presence of a bubble regime.

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Appendix: Proof of Theorem 1

We will show the result of part (IV) of Theorem 1; the other parts of Theorem 1 follow in a similar fashion. In what follows we assume that $\mu = 0$ in (1) and simplify to the case where there is no sub-sample demeaning i.e. Model 4 reduces to

$$\Delta y_t = \hat{\delta}_1 D_t(\tau_1, \tau_2) y_{t-1} + \hat{\delta}_2 D_t(\tau_2, \tau_3) y_{t-1} + \hat{v}_{4t}. \tag{A.1}$$

The same results obtain for the sub-sample demeaned case, however, since our approach only relies on orders of magnitude. In what follows we make repeated use of the order results collected in the following lemma:

Lemma 1 *Let* $S_T = \lfloor \tau_{1,0}T \rfloor (1+\delta_1)^{2(\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor)}$ *and* $R_T = (1-\delta_2)^{2(\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor)}$.
Then:
(i) $y^2_{\lfloor \tau_{2,0}T \rfloor} = O_p(S_T)$,
(ii) $\sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y^2_{t-1} = O_p(S_T)$,
(iii) $\sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} v_t y_{t-1} = O_p(S_T^{1/2})$,
(iv) $y^2_{\lfloor \tau_{3,0}T \rfloor} = O_p(R_T S_T)$.

Proof of Lemma 1

(i) We can write

$$y_{\lfloor \tau_{2,0}T \rfloor} = (1+\delta_1)^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor} y_{\lfloor \tau_{1,0}T \rfloor} + \sum_{j=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor - 1} (1+\delta_1)^j v_{\lfloor \tau_{2,0}T \rfloor - j}$$

so that

$$\begin{aligned} S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor} &= S_T^{-1/2} (1+\delta_1)^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor} y_{\lfloor \tau_{1,0}T \rfloor} + S_T^{-1/2} \sum_{j=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor - 1} (1+\delta_1)^j v_{\lfloor \tau_{2,0}T \rfloor - j} \\ &= O_p(1) + O_p(T^{-1/2}). \end{aligned}$$

Hence $y_{\lfloor \tau_{2,0}T \rfloor} = O_p(S_T^{1/2})$ and $y^2_{\lfloor \tau_{2,0}T \rfloor} = O_p(S_T)$.

(ii) Now write

$$\begin{aligned} y_{\lfloor \tau_{2,0}T \rfloor} &= (1+\delta_1) y_{\lfloor \tau_{2,0}T-1 \rfloor} + v_{\lfloor \tau_{2,0}T \rfloor} \\ S_T^{-1} y^2_{\lfloor \tau_{2,0}T \rfloor} &= (1+\delta_1)^2 S_T^{-1} y^2_{\lfloor \tau_{2,0}T \rfloor - 1} + o_p(1). \end{aligned}$$

Rearranging we obtain

$$S_T^{-1} y^2_{\lfloor \tau_{2,0}T-1 \rfloor} = (1+\delta_1)^{-2} S_T^{-1} y^2_{\lfloor \tau_{2,0}T \rfloor} + o_p(1)$$

which leads to the following recursion for $0 \le t \le \lfloor \tau_{2,0}T - \tau_{1,0}T - 1 \rfloor$

$$S_T^{-1} y^2_{\lfloor \tau_{2,0}T-t \rfloor} = (1+\delta_1)^{-2t} S_T^{-1} y^2_{\lfloor \tau_{2,0}T \rfloor} + o_p(1).$$

Then

$$\begin{aligned}
S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 &= S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor}^{\lfloor \tau_{2,0}T \rfloor} y_t^2 - S_T^{-1} y_{\lfloor \tau_{2,0}T \rfloor}^2 \\
&= S_T^{-1} \sum_{t=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor -1} (1+\delta_1)^{-2t} y_{\lfloor \tau_{2,0}T \rfloor}^2 + S_T^{-1} y_{\lfloor \tau_{1,0}T \rfloor}^2 - S_T^{-1} y_{\lfloor \tau_{2,0}T \rfloor}^2 + o_p(1) \\
&= S_T^{-1} y_{\lfloor \tau_{2,0}T \rfloor}^2 \sum_{t=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor -1} (1+\delta_1)^{-2t} - S_T^{-1} y_{\lfloor \tau_{2,0}T \rfloor}^2 + o_p(1) \\
&= \frac{(1+\delta_1)^2}{(1+\delta_1)^2 - 1} S_T^{-1} y_{\lfloor \tau_{2,0}T \rfloor}^2 - S_T^{-1} y_{\lfloor \tau_{2,0}T \rfloor}^2 + o_p(1) \\
&= \frac{1}{\delta_1(\delta_1+2)} S_T^{-1} y_{\lfloor \tau_{2,0}T \rfloor}^2 + o_p(1)
\end{aligned}$$

and so $\sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 = O_p(S_T)$ in view of Lemma 1(i).

(iii) Write

$$S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor} = (1+\delta_1) S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor -1} + o_p(1).$$

Rearranging we obtain

$$S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor -1} = (1+\delta_1)^{-1} S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor} + o_p(1)$$

which leads to the following recursion for $0 \leq t \leq \lfloor \tau_{2,0}T - \tau_{1,0}T - 1 \rfloor$

$$S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor -t} = (1+\delta_1)^{-t} S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor} + o_p(1).$$

Then, since $S_T^{-1/2} v_{\lfloor \tau_{2,0}T \rfloor+1} y_{\lfloor \tau_{2,0}T \rfloor}$ and $\sum_{t=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor -1} (1+\delta_1)^{-t} v_{\lfloor \tau_{2,0}T \rfloor -t+1}$ are $O_p(1)$,

$$\begin{aligned}
S_T^{-1/2} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} v_t y_{t-1} &= S_T^{-1/2} \sum_{t=\lfloor \tau_{1,0}T \rfloor}^{\lfloor \tau_{2,0}T \rfloor} v_{t+1} y_t - S_T^{-1/2} v_{\lfloor \tau_{2,0}T \rfloor+1} y_{\lfloor \tau_{2,0}T \rfloor} \\
&= \sum_{t=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor -1} v_{\lfloor \tau_{2,0}T - t+1 \rfloor} S_T^{-1/2} y_{\lfloor \tau_{2,0}T - t \rfloor} + S_T^{-1/2} v_{\lfloor \tau_{1,0}T \rfloor+1} y_{\lfloor \tau_{1,0}T \rfloor} \\
&\quad - S_T^{-1/2} v_{\lfloor \tau_{2,0}T \rfloor+1} y_{\lfloor \tau_{2,0}T \rfloor} \\
&= S_T^{-1/2} y_{\lfloor \tau_{2,0}T \rfloor} \sum_{t=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor -1} (1+\delta_1)^{-t} v_{\lfloor \tau_{2,0}T \rfloor -t+1} \\
&\quad - S_T^{-1/2} v_{\lfloor \tau_{2,0}T \rfloor+1} y_{\lfloor \tau_{2,0}T \rfloor} + o_p(1) \\
&= O_p(1).
\end{aligned}$$

(iv) Write

$$\begin{aligned}
y_{\lfloor \tau_{3,0}T \rfloor} &= (1-\delta_2)^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor} y_{\lfloor \tau_{2,0}T \rfloor} + \sum_{j=0}^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor -1} (1-\delta_2)^j v_{\lfloor \tau_{3,0}T \rfloor -j} \\
&= (1-\delta_2)^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor} \{(1+\delta_1)^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor} y_{\lfloor \tau_{1,0}T \rfloor} \\
&\quad + \sum_{j=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor -1} (1+\delta_1)^j v_{\lfloor \tau_{2,0}T \rfloor -j}\} + \sum_{j=0}^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor -1} (1-\delta_2)^j v_{\lfloor \tau_{3,0}T \rfloor -j}
\end{aligned}$$

using the expansion for $y_{\lfloor \tau_{2,0}T \rfloor}$ from Lemma 1(i). Now (2) implies

$$\begin{aligned} (1+\delta_1)^{-(\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor)} (1-\delta_2)^{-(\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor)} &= o(1) \\ R_T^{-1/2} S_T^{-1/2} &= o(T^{-1/2}) \end{aligned}$$

and, since

$$\begin{aligned} S_T^{-1/2} \textstyle\sum_{j=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor - 1} (1+\delta_1)^j v_{\lfloor \tau_{2,0}T \rfloor - j} &= O_p(T^{-1/2}), \\ \textstyle\sum_{j=0}^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor - 1} (1-\delta_2)^j v_{\lfloor \tau_{3,0}T \rfloor - j} &= O_p(1) \end{aligned}$$

we can write

$$\begin{aligned} R_T^{-1/2} S_T^{-1/2} y_{\lfloor \tau_{3,0}T \rfloor} &= R_T^{-1/2} (1-\delta_2)^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor} S_T^{-1/2} (1+\delta_1)^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor} y_{\lfloor \tau_{1,0}T \rfloor} \\ &\quad + R_T^{-1/2} (1-\delta_2)^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor} S_T^{-1/2} \textstyle\sum_{j=0}^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor - 1} (1+\delta_1)^j v_{\lfloor \tau_{2,0}T \rfloor - j} \\ &\quad + R_T^{-1/2} S_T^{-1/2} \textstyle\sum_{j=0}^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor - 1} (1-\delta_2)^j v_{\lfloor \tau_{3,0}T \rfloor - j} \\ &= R_T^{-1/2} (1-\delta_2)^{\lfloor \tau_{3,0}T \rfloor - \lfloor \tau_{2,0}T \rfloor} S_T^{-1/2} (1+\delta_1)^{\lfloor \tau_{2,0}T \rfloor - \lfloor \tau_{1,0}T \rfloor} y_{\lfloor \tau_{1,0}T \rfloor} + o_p(1) \\ &= O_p(1) \end{aligned}$$

and hence $y^2_{\lfloor \tau_{3,0}T \rfloor} = O_p(R_T S_T)$.

Proof of main result

For (A.1), we may write

$$SSR_4(\tau_1, \tau_2, \tau_3) - \textstyle\sum_{t=2}^{T} \Delta y_t^2 = -\dfrac{(\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_1 T \rfloor+1}^{\lfloor \tau_2 T \rfloor} y_{t-1}^2} - \dfrac{(\sum_{t=\lfloor \tau_2 T \rfloor+1}^{\lfloor \tau_3 T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_2 T \rfloor+1}^{\lfloor \tau_3 T \rfloor} y_{t-1}^2}$$

and it follows that

$$\begin{aligned} SSR_4(\hat\tau_1, \hat\tau_2, \hat\tau_3) - SSR_4(\tau_{1,0}, \tau_{2,0}, \tau_{3,0}) &= \frac{(\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor \hat\tau_1 T \rfloor+1}^{\lfloor \hat\tau_2 T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \hat\tau_1 T \rfloor+1}^{\lfloor \hat\tau_2 T \rfloor} y_{t-1}^2} \\ &\quad + \frac{(\sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{3,0} T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{3,0} T \rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor \hat\tau_2 T \rfloor+1}^{\lfloor \hat\tau_3 T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \hat\tau_2 T \rfloor+1}^{\lfloor \hat\tau_3 T \rfloor} y_{t-1}^2} \le 0. \end{aligned}$$

Now suppose that $\hat{\tau}_1$, $\hat{\tau}_2$ and $\hat{\tau}_3$ are such that $\lfloor \hat{\tau}_i T \rfloor = \lfloor \tau_{i,0} T \rfloor + k_i$, $i = 1, 2, 3$, where k_1, k_2 and k_3 are $O(1)$ integers. Let

$$\begin{aligned} F(k_1, k_2, k_3) \;=\;& \frac{(\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor \tau_{1,0} T \rfloor+1+k_1}^{\lfloor \tau_{2,0} T \rfloor+k_2} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0} T \rfloor+1+k_1}^{\lfloor \tau_{2,0} T \rfloor+k_2} y_{t-1}^2} \\ & + \frac{(\sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{3,0} T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{3,0} T \rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor \tau_{2,0} T \rfloor+1+k_2}^{\lfloor \tau_{3,0} T \rfloor+k_3} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{2,0} T \rfloor+1+k_2}^{\lfloor \tau_{3,0} T \rfloor+k_3} y_{t-1}^2}. \end{aligned}$$

We next consider the behaviour of $F(k_1, k_2, k_3)$ where one of k_1, k_2 or k_3 is non-zero, with the other two quantities set to zero. The next three sub-sections deal with these three cases, beginning with the more involved case of $k_2 \neq 0$.

Case 1: $k_2 \neq 0$, $k_1 = k_3 = 0$

Here,

$$\begin{aligned} F(0, k_2, 0) \;=\;& \frac{(\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} y_{t-1}^2} \\ & + \frac{(\sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{3,0} T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{3,0} T \rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor \tau_{2,0} T \rfloor+1+k_2}^{\lfloor \tau_{3,0} T \rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{2,0} T \rfloor+1+k_2}^{\lfloor \tau_{3,0} T \rfloor} y_{t-1}^2}. \end{aligned}$$

We consider the two cases $k_2 > 0$ and $k_2 < 0$ separately.

When $k_2 > 0$, we can write

$$\begin{aligned} S_T^{-1} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} \Delta y_t y_{t-1} \;&=\; S_T^{-1} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1} + S_T^{-1} \sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} \Delta y_t y_{t-1} \\ &=\; \delta_1 S_T^{-1} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2 + S_T^{-1} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} v_t y_{t-1} \\ &\quad + S_T^{-1} \sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} \Delta y_t y_{t-1} \\ &=\; \delta_1 S_T^{-1} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2 - \delta_2 S_T^{-1} \sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} y_{t-1}^2 + o_p(1) \end{aligned}$$

which is $O_p(1)$ using the scalings implied by Lemma 1(i)-1(iii) and the fact that k_2 is finite. Also,

$$S_T^{-1} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} y_{t-1}^2 = S_T^{-1} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2 + S_T^{-1} \sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor+k_2} y_{t-1}^2.$$

Then

$$\frac{(S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}\Delta y_t y_{t-1})^2}{S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2}-\frac{(S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}\Delta y_t y_{t-1})^2}{S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}y_{t-1}^2}$$

$$=\frac{\{\delta_1 S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2+o_p(1)\}^2}{S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2}-\frac{\{\delta_1 S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2-\delta_2 S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}y_{t-1}^2+o_p(1)\}^2}{S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2+S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}y_{t-1}^2}$$

which is $>_p 0$ since the second term involves a numerator (denominator) that is less than (greater than) the corresponding numerator (denominator) in the first term. Next,

$$\begin{aligned}
&\frac{(S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor}\Delta y_t y_{t-1})^2}{S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2}-\frac{(S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1+k_2}^{\lfloor\tau_{3,0}T\rfloor}\Delta y_t y_{t-1})^2}{S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1+k_2}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2}\\
&= \frac{(-\delta_2 S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2)^2}{S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2}-\frac{(-\delta_2 S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1+k_2}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2)^2}{S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1+k_2}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2}+o_p(1)\\
&= \delta_2^2(S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2-S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1+k_2}^{\lfloor\tau_{3,0}T\rfloor}y_{t-1}^2)+o_p(1)
\end{aligned}$$

which is $>_p 0$. It then follows that $S_T^{-1}F(0,k_2,0)>_p 0$.

When $k_2<0$, write

$$\begin{aligned}
S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}\Delta y_t y_{t-1} &= \delta_1 S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}y_{t-1}^2+S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}v_t y_{t-1}\\
&= \delta_1 S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor+k_2}y_{t-1}^2+o_p(1)\\
&= \delta_1(1+\delta_1)^{2k_2}S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2+o_p(1)
\end{aligned}$$

where the last line follows since, for $t=\lfloor\tau_{1,0}T\rfloor+1-k_2,...,\lfloor\tau_{2,0}T\rfloor$,

$$y_{t-1}=(1+\delta_1)^{-k_2}y_{t-1+k_2}+\sum_{j=0}^{-k_2-1}(1+\delta_1)^j v_{t-j-1}$$

from which we find

$$\begin{aligned}
(1+\delta_1)^{2k_2}S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2 &= (1+\delta_1)^{2k_2}S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1-k_2}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2\\
&\quad+(1+\delta_1)^{2k_2}S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor-k_2}y_{t-1}^2\\
&= (1+\delta_1)^{2k_2}S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1-k_2}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1}^2+o_p(1)\\
&= S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1-k_2}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1+k_2}^2+o_p(1)\\
&= S_T^{-1}\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor}y_{t-1+k_2}^2+o_p(1)
\end{aligned}$$

$$
\begin{aligned}
&= S_T^{-1} \textstyle\sum_{t=\lfloor \tau_{1,0}T \rfloor+1+k_2}^{\lfloor \tau_{2,0}T \rfloor+k_2} y_{t-1}^2 + o_p(1) \\
&= S_T^{-1} \textstyle\sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor+k_2} y_{t-1}^2 + o_p(1).
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{(S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} \Delta y_t y_{t-1})^2}{S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2} - \frac{(S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor+k_2} \Delta y_t y_{t-1})^2}{S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor+k_2} y_{t-1}^2} \\
&= \frac{\{\delta_1 S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 + o_p(1)\}^2}{S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2} \\
&\quad - \frac{\{\delta_1 (1+\delta_1)^{2k_2} S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 + o_p(1)\}^2}{(1+\delta_1)^{2k_2} S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 + o_p(1)} \\
&= \delta_1^2 \{1 - (1+\delta_1)^{2k_2}\} S_T^{-1} \sum_{t=\lfloor \tau_{1,0}T \rfloor+1}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 + o_p(1)
\end{aligned}
$$

which is $>_p 0$ since $1 - (1+\delta_1)^{2k_2} > 0$. Next,

$$
\begin{aligned}
&\frac{(S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} \Delta y_t y_{t-1})^2}{S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2} - \frac{(S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1+k_2}^{\lfloor \tau_{3,0}T \rfloor} \Delta y_t y_{t-1})^2}{S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1+k_2}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2} \\
&= \frac{(-\delta_2 S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2)^2}{S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2} - \frac{(-\delta_2 S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2 + \delta_1 S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1+k_2}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2)^2}{S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2 + S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1+k_2}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2} + o_p(1)
\end{aligned}
$$

is $>_p 0$ since

$$
S_T^{-1} \textstyle\sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2 + S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1+k_2}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 > S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2
$$

and

$$
(-\delta_2 S_T^{-1} \textstyle\sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2)^2 - (-\delta_2 S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2 + \delta_1 S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1+k_2}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2)^2 >_p 0.
$$

The second result is true when, in the limit,

$$
\delta_1 S_T^{-1} \textstyle\sum_{t=\lfloor \tau_{2,0}T \rfloor+1+k_2}^{\lfloor \tau_{2,0}T \rfloor} y_{t-1}^2 < 2\delta_2 S_T^{-1} \sum_{t=\lfloor \tau_{2,0}T \rfloor+1}^{\lfloor \tau_{3,0}T \rfloor} y_{t-1}^2.
$$

Expanding these terms separately, it can be easily shown that

$$\delta_1 S_T^{-1} \sum_{t=\lfloor \tau_{2,0} T \rfloor+1+k_2}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2 = \delta_1 (1+\delta_1)^{-2} \frac{1-(1+\delta_1)^{-2|k_2|}}{1-(1+\delta_1)^{-2}} y_{\lfloor \tau_{2,0} T \rfloor}^2 + o_p(1)$$

and

$$\delta_1 (1+\delta_1)^{-2} \frac{1-(1+\delta_1)^{-2|k_2|}}{1-(1+\delta_1)^{-2}} y_{\lfloor \tau_{2,0} T \rfloor}^2 = \frac{1-(1+\delta_1)^{-2|k_2|}}{2+\delta_1} y_{\lfloor \tau_{2,0} T \rfloor}^2 < \frac{1}{2+\delta_1} y_{\lfloor \tau_{2,0} T \rfloor}^2.$$

Also,

$$2\delta_2 S_T^{-1} \sum_{t=\lfloor \tau_{2,0} T \rfloor+1}^{\lfloor \tau_{3,0} T \rfloor} y_{t-1}^2 = \frac{2\delta_2}{1-(1-\delta_2)^2} y_{\lfloor \tau_{2,0} T \rfloor}^2 + o_p(1)$$

and

$$\frac{2\delta_2}{1-(1-\delta_2)^2} y_{\lfloor \tau_{2,0} T \rfloor}^2 = \frac{2}{2-\delta_2} y_{\lfloor \tau_{2,0} T \rfloor}^2.$$

Then, since $\delta_1 > 0$ and $0 < \delta_2 < 2$,

$$\frac{1}{2+\delta_1} < 0.5, \qquad \frac{2}{2-\delta_2} > 1, \qquad \frac{1}{2+\delta_1} < \frac{2}{2-\delta_2}$$

giving the required result. It then follows that $S_T^{-1} F(0, k_2, 0) >_p 0$.

In combination, the results for $k_2 > 0$ and $k_2 < 0$ imply that when $k_2 \neq 0$, $S_T^{-1} F(0, k_2, 0) >_p 0$.

Case 2: $k_1 \neq 0$, $k_2 = k_3 = 0$

Here,

$$F(k_1, 0, 0) = \frac{\left(\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1}\right)^2}{\sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2} - \frac{\left(\sum_{t=\lfloor \tau_{1,0} T \rfloor+1+k_1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1}\right)^2}{\sum_{t=\lfloor \tau_{1,0} T \rfloor+1+k_1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2}.$$

As in Case 1, we consider the cases $k_1 > 0$ and $k_1 < 0$ in turn.

For $k_1 > 0$, we have

$$\begin{aligned} \sum_{t=\lfloor \tau_{1,0} T \rfloor+1+k_1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1} &= \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1} - \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{1,0} T \rfloor+k_1} \Delta y_t y_{t-1} \\ &= \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} \Delta y_t y_{t-1} - \delta_1 \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{1,0} T \rfloor+k_1} y_{t-1}^2 - \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{1,0} T \rfloor+k_1} v_t y_{t-1} \end{aligned}$$

and

$$\sum_{t=\lfloor \tau_{1,0} T \rfloor+1+k_1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2 = \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{2,0} T \rfloor} y_{t-1}^2 - \sum_{t=\lfloor \tau_{1,0} T \rfloor+1}^{\lfloor \tau_{1,0} T \rfloor+k_1} y_{t-1}^2.$$

Thus

$$
\begin{aligned}
&F(k_1,0,0)\\
&= \frac{(\delta_1 \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2}\\
&\quad - \frac{(\delta_1 \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 - \delta_1 \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} y_{t-1}^2 + \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} v_t y_{t-1} - \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} y_{t-1}^2}\\
&= \delta_1^2 \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} y_{t-1}^2 + 2\delta_1 \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} v_t y_{t-1}\\
&\quad - \frac{(\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} v_t y_{t-1})^2 - 2\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} v_t y_{t-1} \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} v_t y_{t-1}}{\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} y_{t-1}^2}\\
&\quad - \frac{\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} y_{t-1}^2 (\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 (\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} y_{t-1}^2)}
\end{aligned}
$$

after some simplification. Hence, recalling that k_1 is finite,

$$T^{-1}F(k_1,0,0) = \delta_1^2 T^{-1} \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{1,0}T\rfloor+k_1} y_{t-1}^2 + o_p(1) >_p 0.$$

For $k_1 < 0$, write

$$
\begin{aligned}
\sum_{t=\lfloor\tau_{1,0}T\rfloor+1+k_1}^{\lfloor\tau_{2,0}T\rfloor} \Delta y_t y_{t-1} &= \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} \Delta y_t y_{t-1} + \sum_{t=\lfloor\tau_{1,0}T\rfloor+1+k_1}^{\lfloor\tau_{1,0}T\rfloor} \Delta y_t y_{t-1}\\
&= \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} \Delta y_t y_{t-1} + \sum_{t=\lfloor\tau_{1,0}T\rfloor+1+k_1}^{\lfloor\tau_{1,0}T\rfloor} v_t y_{t-1}
\end{aligned}
$$

and

$$\sum_{t=\lfloor\tau_{1,0}T\rfloor+1+k_1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 = \sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{1,0}T\rfloor+1+k_1}^{\lfloor\tau_{1,0}T\rfloor} y_{t-1}^2.$$

Then

$$
\begin{aligned}
&F(k_1,0,0)\\
&= \frac{(\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} \Delta y_t y_{t-1} + \sum_{t=\lfloor\tau_{1,0}T\rfloor+1+k_1}^{\lfloor\tau_{1,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{1,0}T\rfloor+1}^{\lfloor\tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{1,0}T\rfloor+1+k_1}^{\lfloor\tau_{1,0}T\rfloor} y_{t-1}^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\delta_1 \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2} \\
&\quad - \frac{(\delta_1 \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} v_t y_{t-1} + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2} \\
&= \delta_1^2 \frac{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2 \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2}{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2} \\
&\quad -2\delta_1 \frac{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} v_t y_{t-1} - \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2 \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} v_t y_{t-1}}{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2} \\
&\quad - \frac{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} v_t y_{t-1} (\sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} v_t y_{t-1} + 2 \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} v_t y_{t-1})}{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2} \\
&\quad + \frac{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2 (\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 (\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2)}
\end{aligned}
$$

after some simplification. Finally, we obtain

$$
\begin{aligned}
T^{-1}F(k_1,0,0) &= \delta_1^2 T^{-1} \frac{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2 \sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2}{\sum_{t=\lfloor \tau_{1,0}T\rfloor+1}^{\lfloor \tau_{2,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2} + o_p(1) \\
&= \delta_1^2 T^{-1} \sum_{t=\lfloor \tau_{1,0}T\rfloor+1+k_1}^{\lfloor \tau_{1,0}T\rfloor} y_{t-1}^2 + o_p(1)
\end{aligned}
$$

which is $>_p 0$.

These two results show that when $k_1 \neq 0$, $T^{-1}F(k_1,0,0) >_p 0$.

Case 3: $k_3 \neq 0$**,** $k_1 = k_2 = 0$

Now in this case,

$$
F(0,0,k_3) = \frac{(\sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor+k_3} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor+k_3} y_{t-1}^2}
$$

and once again we consider $k_3 > 0$ and $k_3 < 0$ separately.

When $k_3 > 0$,

$$
\begin{aligned}
\sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor+k_3} \Delta y_t y_{t-1} &= \sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor} \Delta y_t y_{t-1} + \sum_{t=\lfloor \tau_{3,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor+k_3} \Delta y_t y_{t-1} \\
&= -\delta_2 \sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor \tau_{2,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor} v_t y_{t-1} + \sum_{t=\lfloor \tau_{3,0}T\rfloor+1}^{\lfloor \tau_{3,0}T\rfloor+k_3} v_t y_{t-1}.
\end{aligned}
$$

Given that

$$\begin{aligned}
S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 &= S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor}^{\lfloor\tau_{3,0}T\rfloor-1} y_t^2 \\
&= S_T^{-1}\sum_{t=\lfloor\tau_{2,0}T\rfloor}^{\lfloor\tau_{3,0}T\rfloor-1}(1-\delta_2)^{2(t-\lfloor\tau_{2,0}T\rfloor)}y_{\lfloor\tau_{2,0}T\rfloor}^2 + o_p(1) \\
&= S_T^{-1}\sum_{t=0}^{\lfloor\tau_{3,0}T\rfloor-\lfloor\tau_{2,0}T\rfloor-1}(1-\delta_2)^{2t}y_{\lfloor\tau_{2,0}T\rfloor}^2 + o_p(1) \\
&= \frac{1}{1-(1-\delta_2)^2}S_T^{-1}y_{\lfloor\tau_{2,0}T\rfloor}^2 + o_p(1) \\
&= \frac{1}{\delta_2(2-\delta_2)}S_T^{-1}y_{\lfloor\tau_{2,0}T\rfloor}^2 + o_p(1)
\end{aligned}$$

is $O_p(1)$, we can write

$$\begin{aligned}
&F(0,0,k_3) \\
&= \frac{(\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor}\Delta y_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor}\Delta y_t y_{t-1} + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2} \\
&= \frac{(-\delta_2\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2} \\
&\quad - \frac{(-\delta_2\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1} + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2} \\
&= \delta_2^2\frac{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2} \\
&\quad +2\delta_2\frac{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} v_t y_{t-1} - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2 \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1}}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2} \\
&\quad - \frac{\sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} v_t y_{t-1}(\sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} v_t y_{t-1} + 2\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1})}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2} \\
&\quad + \frac{\sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2(\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2(\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2)}
\end{aligned}$$

and, using the scaling in Lemma 1(iv) and the fact that k_3 is finite,

$$\begin{aligned}
R_T^{-1}S_T^{-1}F(0,0,k_3) &= \delta_2^2 R_T^{-1}S_T^{-1}\frac{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2} + o_p(1) \\
&= \delta_2^2 R_T^{-1}S_T^{-1}\sum_{t=\lfloor\tau_{3,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2 + o_p(1)
\end{aligned}$$

which is $>_p 0$.

When $k_3 < 0$,

$$\begin{aligned}
\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} \Delta y_t y_{t-1} &= \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} \Delta y_t y_{t-1} - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} \Delta y_t y_{t-1} \\
&= -\delta_2 \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1} \\
&\quad +\delta_2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1}.
\end{aligned}$$

Then

$$\begin{aligned}
&F(0,0,k_3) \\
=\ & \frac{(\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2} - \frac{(\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} \Delta y_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor+k_3} y_{t-1}^2} \\
=\ & \frac{(-\delta_2 \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2} - \\
& \frac{(-\delta_2 \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + \sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1} + \delta_2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1})^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2} \\
=\ & \delta_2^2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 - 2\delta_2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1} \\
& + \frac{\sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1} (2\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1} - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1})}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2} \\
& - \frac{(\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} v_t y_{t-1})^2 \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2}{\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 (\sum_{t=\lfloor\tau_{2,0}T\rfloor+1}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 - \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2)}
\end{aligned}$$

and so

$$R_T^{-1} S_T^{-1} F(0,0,k_3) = \delta_2^2 R_T^{-1} S_T^{-1} \sum_{t=\lfloor\tau_{3,0}T\rfloor+1+k_3}^{\lfloor\tau_{3,0}T\rfloor} y_{t-1}^2 + o_p(1)$$

which is $>_p 0$.

These two Case 3 results therefore show that when $k_3 \neq 0$, $R_T^{-1} S_T^{-1} F(0,0,k_3) >_p 0$.

Taken together, the results of Cases 1-3 imply that when at least one of k_1, k_2 and k_3 is non-zero,

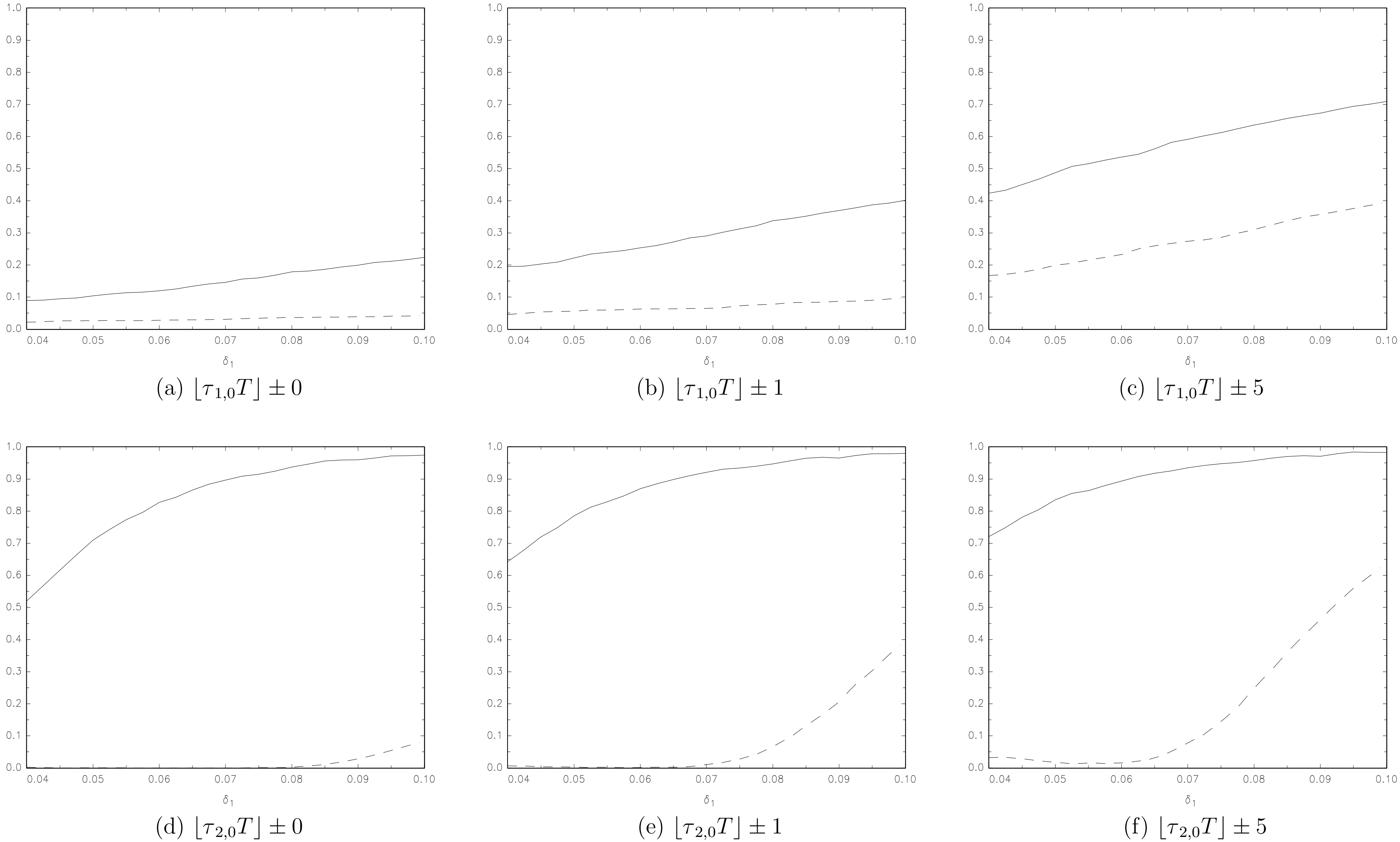
$$F(k_1,k_2,k_3) >_p 0.$$

In order that $F(k_1,k_2,k_3)$ is not positive in the limit, we require that $k_1 = k_2 = k_3 = 0$. Hence, it must hold that $\lfloor\hat{\tau}_i T\rfloor - \lfloor\tau_{i,0}T\rfloor \xrightarrow{p} 0$, $i = 1,2,3$.

Table 1. Application to Nasdaq composite real price index

	PSY			BIC$_{opt}$		
	Test	Start	End	Model	Start	End
Panel A. Full sample results						
1973:2–2005:6	3.07***	1998:11	2000:12	3	1998:11	2000:9
Panel B. Pseudo real-time results						
1973:2–2000:9	3.07***	1998:11	2000:9	1	2000:1	2000:9
1973:2–2000:10	3.07***	1998:11	2000:10	1	2000:1	2000:10
1973:2–2000:11	3.07***	1998:11	2000:11	1	1999:12	2000:11
1973:2–2000:12	3.07***	1998:11	2000:12	3	1998:11	2000:9
1973:2–2001:1	3.07***	1998:11	2000:12	3	1998:11	2000:9

Note: *** denotes rejection at the 0.01-level.



(a) $\lfloor \tau_{1,0} T \rfloor \pm 0$

(b) $\lfloor \tau_{1,0} T \rfloor \pm 1$

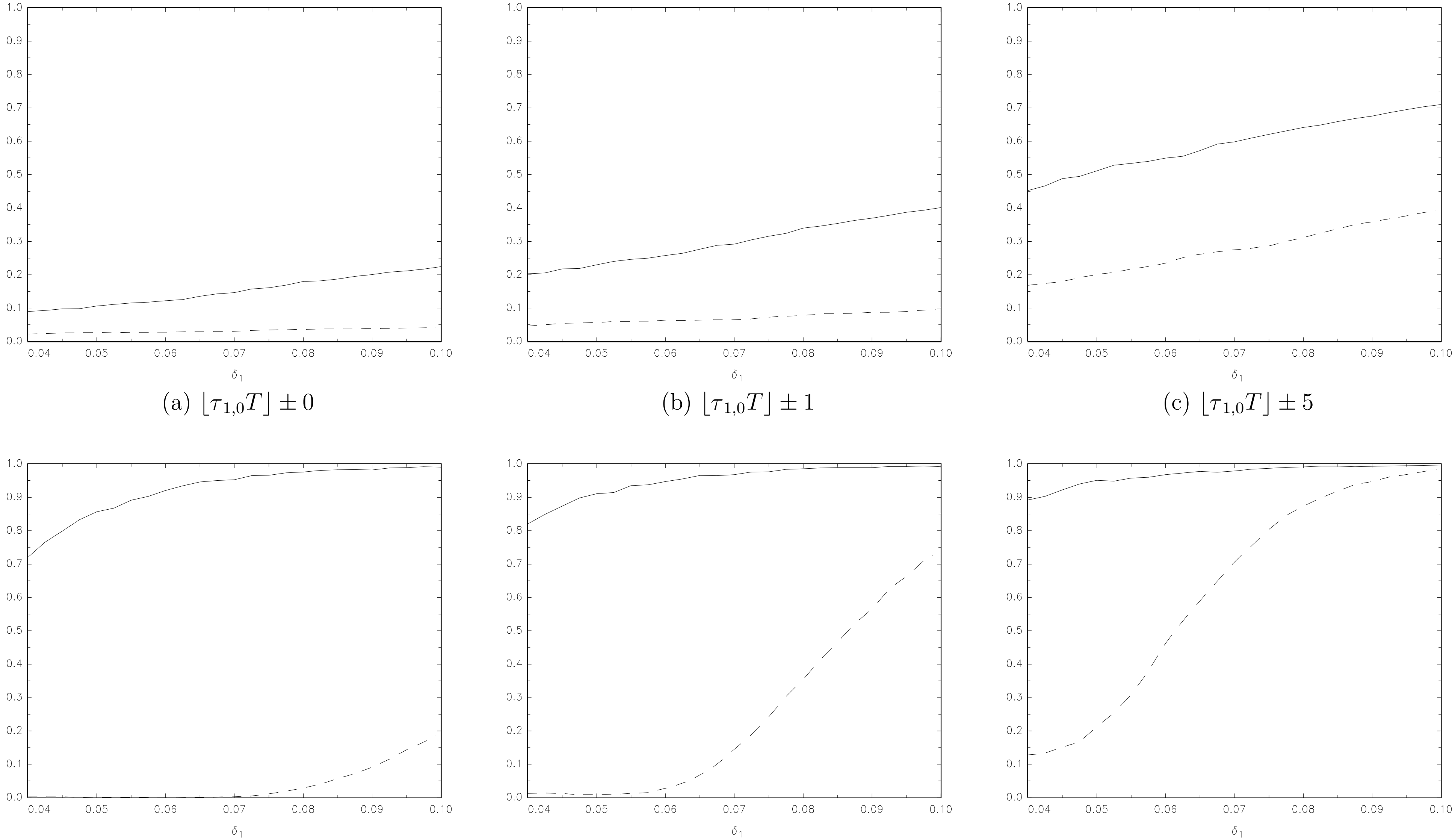
(c) $\lfloor \tau_{1,0} T \rfloor \pm 5$

(d) $\lfloor \tau_{2,0} T \rfloor \pm 0$

(e) $\lfloor \tau_{2,0} T \rfloor \pm 1$

(f) $\lfloor \tau_{2,0} T \rfloor \pm 5$

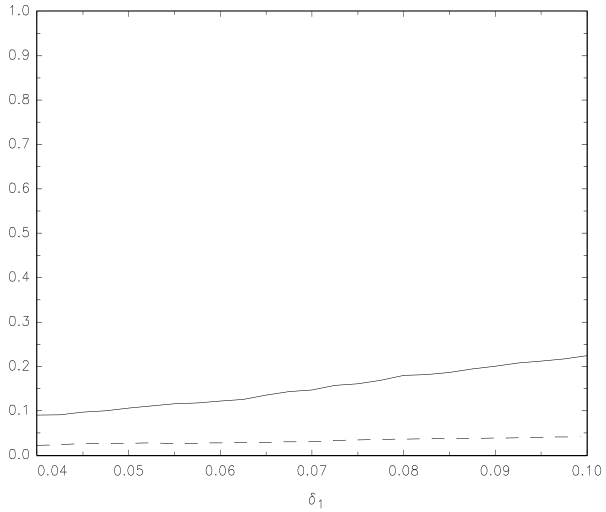
Figure 1. Conditional accuracy of bubble start and end date estimators: DGP 2, $\tau_{1,0} = 0.5$, $\tau_{2,0} = 0.7$;
BIC$_{opt}$: —— , PSY: – –



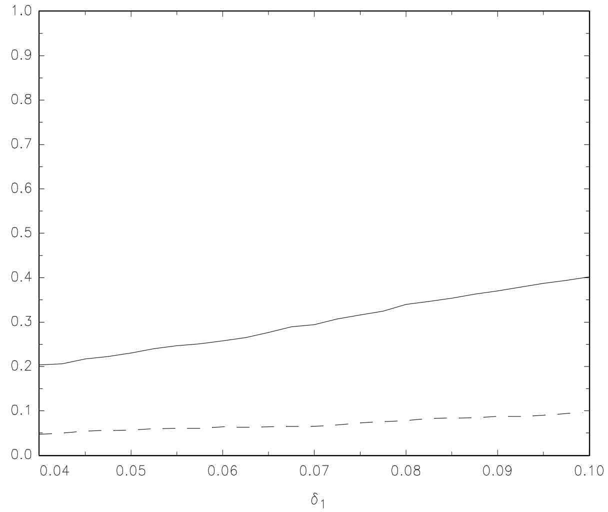
(a) $\lfloor \tau_{1,0} T \rfloor \pm 0$ (b) $\lfloor \tau_{1,0} T \rfloor \pm 1$ (c) $\lfloor \tau_{1,0} T \rfloor \pm 5$

(d) $\lfloor \tau_{2,0} T \rfloor \pm 0$ (e) $\lfloor \tau_{2,0} T \rfloor \pm 1$ (f) $\lfloor \tau_{2,0} T \rfloor \pm 5$

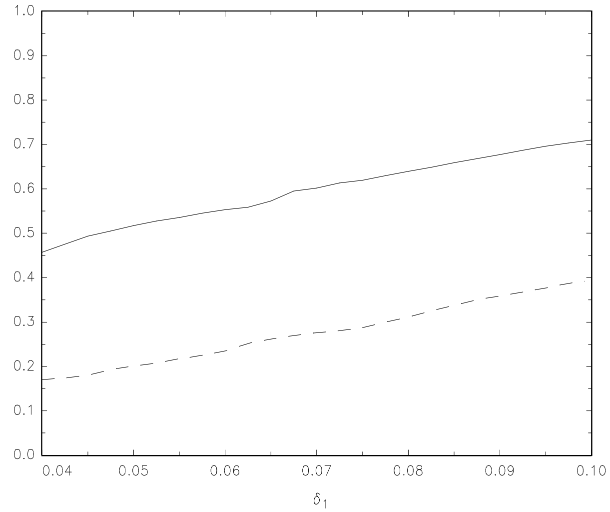
Figure 2. Conditional accuracy of bubble start and end date estimators: DGP 4, $\tau_{1,0} = 0.5$, $\tau_{2,0} = 0.7$, $\tau_{3,0} = 0.8$, $\delta_2 = \delta_1/2$; BIC$_{opt}$: —— , PSY: – –



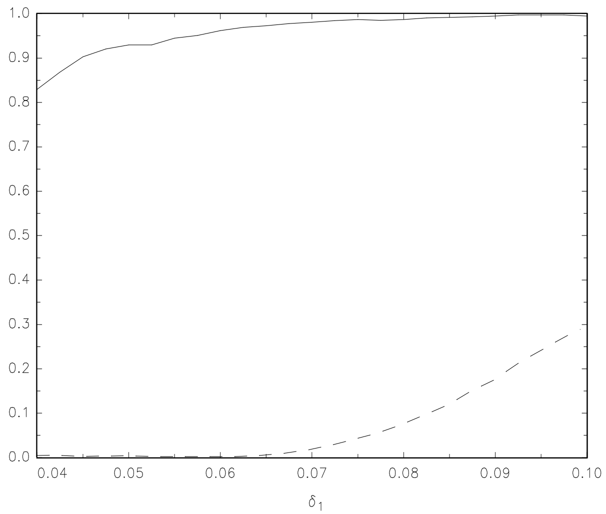
(a) $\lfloor \tau_{1,0} T \rfloor \pm 0$



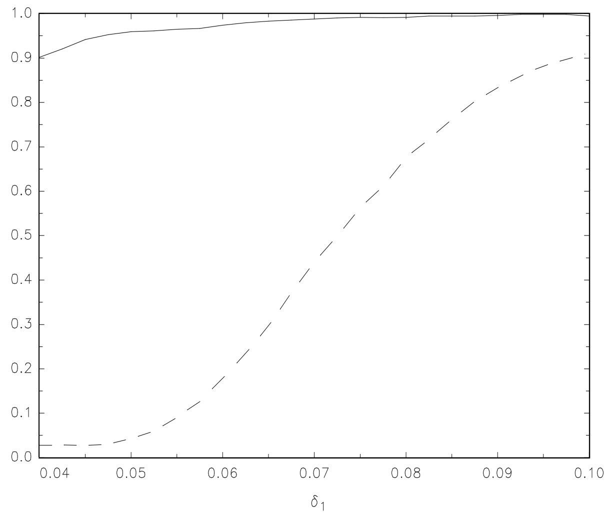
(b) $\lfloor \tau_{1,0} T \rfloor \pm 1$



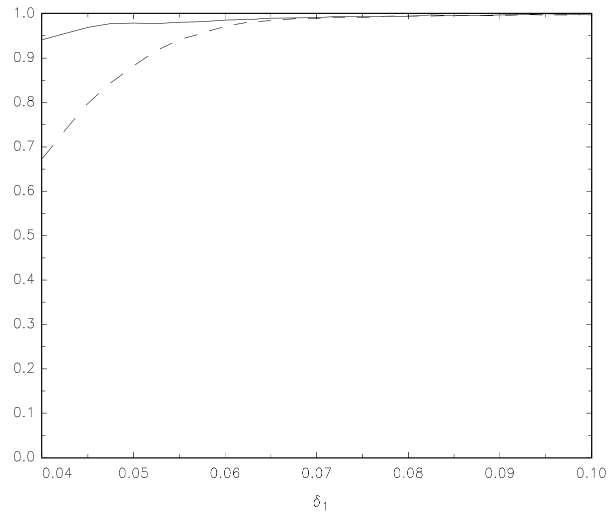
(c) $\lfloor \tau_{1,0} T \rfloor \pm 5$



(d) $\lfloor \tau_{2,0} T \rfloor \pm 0$

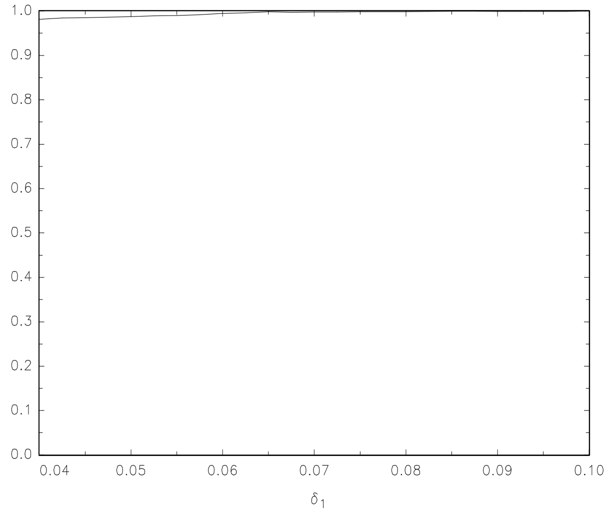


(e) $\lfloor \tau_{2,0} T \rfloor \pm 1$

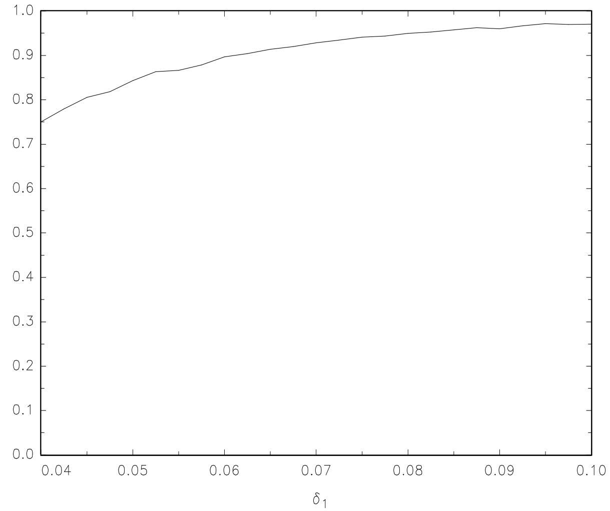


(f) $\lfloor \tau_{2,0} T \rfloor \pm 5$

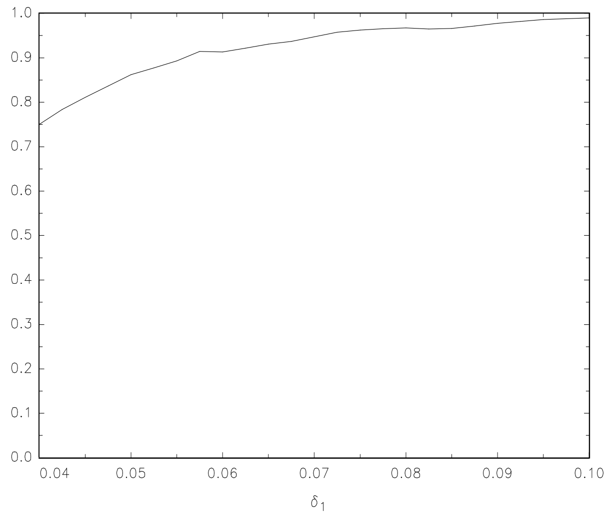
Figure 3. Conditional accuracy of bubble start and end date estimators: DGP 4, $\tau_{1,0} = 0.5$, $\tau_{2,0} = 0.7$, $\tau_{3,0} = 0.8$, $\delta_2 = \delta_1$; BIC$_{opt}$: —— , PSY: – –



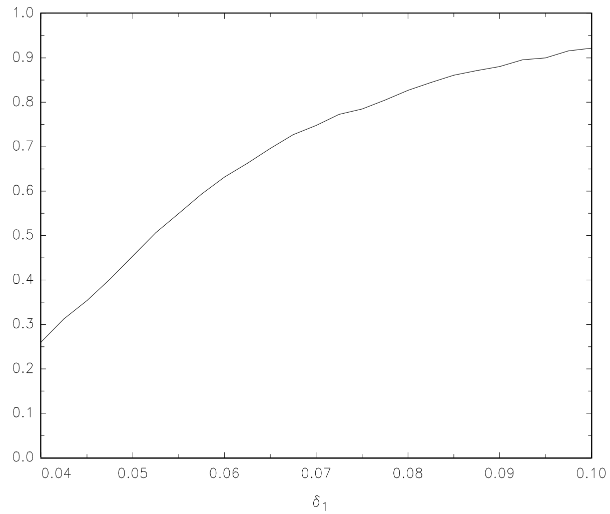
(a) DGP 1, $\tau_{1,0} = 0.8$, $\tau_{2,0} = 1$



(b) DGP 2, $\tau_{1,0} = 0.5$, $\tau_{2,0} = 0.7$



(c) DGP 3, $\tau_{1,0} = 0.7$, $\tau_{2,0} = 0.9$, $\tau_{3,0} = 1$, $\delta_2 = \delta_1/2$



(d) DGP 4, $\tau_{1,0} = 0.5$, $\tau_{2,0} = 0.7$, $\tau_{3,0} = 0.8$, $\delta_2 = \delta_1/2$

Figure 4. Correct model selection conditional frequencies: BIC_{opt}: —

F.5

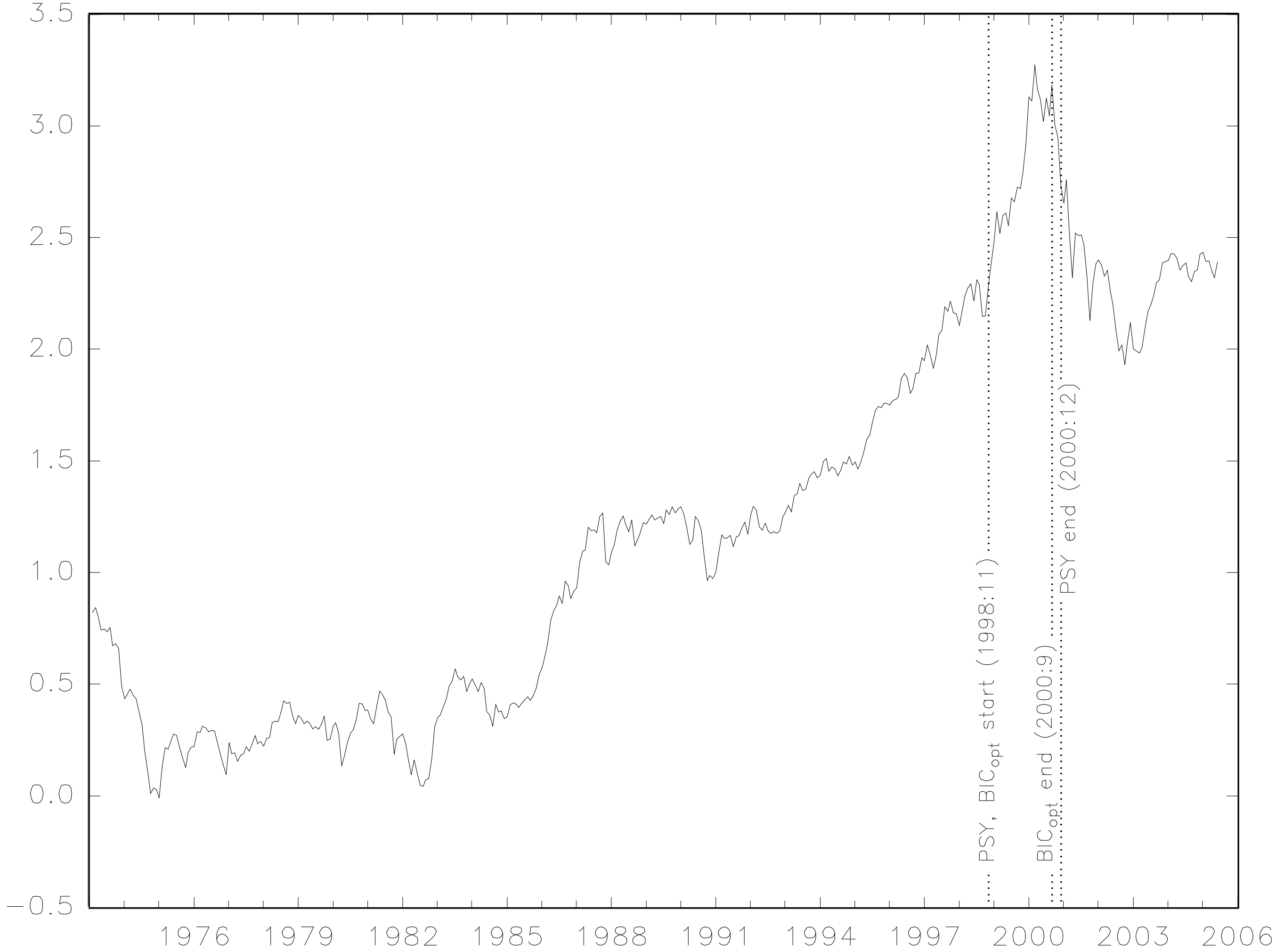


Figure 5. Logarithms of Nasdaq composite real price index, 1973:2-2005:6, and estimated bubble start and end dates