Evaluating decision making units under uncertainty using fuzzy multi-objective nonlinear programming

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Abstract

This paper proposes a new method to evaluate Decision Making Units (DMUs) under uncertainty using fuzzy Data Envelopment Analysis (DEA). In the proposed multi-objective nonlinear programming methodology both the objective functions and the constraints are considered fuzzy. This model is comprehensive in dealing with uncertainty, in the sense that coefficients of the decision variables in the objective functions and in the constraints, as well as the DMUs under assessment, are assumed to be fuzzy numbers with triangular membership functions. A comparison between the current fuzzy DEA models and the proposed method is illustrated by a numerical example.

Keywords: Fuzzy DEA; membership function; fuzzy multi-objective linear programming; possibility programming.

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1. Introduction

Data Envelopment Analysis (DEA) is a relatively recent approach in the assessment of performance of organizations and their functional units. DEA is able to evaluate the Decision Making Units (DMUs) based on multiple inputs and outputs. Since the first development of DEA (Banker, Charnes, & Cooper, 1984; Charnes, Cooper, & Rhodes, 1978), there have been many applications of DEA in a variety of different contexts (Emrouznejad & De Witte, 2010; Emrouznejad, Parker, & Tavares, 2008).

However in many real world applications, input or output variables are not always represented by crisp values. Hence, the traditional DEA models cannot be used for evaluating such DMUs. Several attempts have been made to develop fuzzy DEA models that are powerful tools for comparing the performance of a set of activities or organizations under uncertainty. For instance, Sengupta (1992) considered the objective function to be fuzzy when utilizing a standard DEA and used Zimmermann's method (Zimmermann, 1975, 1978) to obtain the results. León et al. (2003) transformed the fuzzy DEA into crisp DEA (Hougaard, 2005). Takeda and Satoh (2000) used both multicriteria decision analysis and DEA with incomplete data. Lertworasirikul et al., (2003a) and Lertworasirikul et al., (2003b) applied a possibilistic approach (Zarafat Angiz et al., 2006) to treat the constraints of the DEA as fuzzy events. Several other fuzzy models (Guo & Tanaka, 2001) have been proposed to evaluate DMUs with fuzzy data, using the concept of comparison of fuzzy numbers. Wen and Li (2009) proposed a hybrid method based on fuzzy simulation and genetic algorithms. Recently, Emrouznejad, Tavana, and Hatami-Marbini (2014) provided a taxonomy and review of fuzzy DEA (FDEA) methods which comprise a tolerance approach, the α-level based approach, the fuzzy ranking approach, the possibility approach, the fuzzy arithmetic, and the fuzzy random/type-2 fuzzy set.

The α-cut approach (Zerafat Angiz, Emrouznejad, & Mustafa, 2012) for fuzzy DEA is one of the most frequently used methods. It first solves a linear program to determine the upper bound of the weights, then a common set of weights are obtained by solving another linear programming problem. The shortcoming of this approach is that we lose some information about uncertainty. Further, since the nature of a fuzzy linear programming (FLP) model is nonlinear, to keep all information about uncertainty when solving the model, we need a nonlinear programming model. In other words, in order to use a mathematical programming problem to analyze the solution of an FLP problem, a multi-objective nonlinear programming has the most consistency with the nature of FLP problem.

Alternative methodologies based on multi-objective programming are seen in Zerafat Angiz, Emrouznejad, and Mustafa (2010) and Zerafat Angiz et al. (2012) who introduced a new concept called local α-level which approximates the optimal solution of an FLP problem by partitioning the interval of fuzzy numbers. The optimal solution in this approach is based on the closeness to defuzzified points. The benefit of this approach is that the multi-objective programming corresponding to FLP is linear. In fact, in this approach the authors impose α-cuts together, and solve a single linear programming problem. On the other hand, Zerafat Angiz et al. (2010) presented a model for ranking decision making units based on a non-radial approach. Saati et al. (2001) presented a non-radial model that assumed inputs and outputs are fuzzy. This paper deals with a primal form of an FLP problem. Because of the nature of the model, it is categorized as a pessimistic approach because the worst situation of the DMU under evaluation is compared with the best situation of other DMUs.

In this paper an optimistic approach will be presented. We propose a multi-objective programming model that can retain the uncertainty in many aspects including objective functions, coefficients of the decision matrix and the DMUs under assessment. The discrete approach (Zerafat Angiz et al., 2012) and the proposed approach follow two different views. In the discrete approach, the goal is achieving defuzzified points whereas the goal of fuzzy numbers in the proposed approach is the most possible values. One advantage of the proposed approach is that it retains information about uncertainty as much as possible, while the discrete approach approximates the solution, but it still loses some information about uncertainty. The benefit of applying the discrete approach is that a linear programming problem is used.

The rest of this paper is organized as follows. A brief description of standard DEA and fuzzy DEA is given in Section 2. A specific multi-objective model is discussed in Section 3 and we propose an alternative fuzzy DEA model under uncertainty. This is followed by a numerical illustration in Section 4. In Section 5 empirical data is analyzed to illustrate the proposed approach. Section 6 presents the discussion of the paper and conclusion is drawn in Section **7**.

2. DEA and Fuzzy DEA

DEA is a nonparametric technique for measuring the relative efficiency of a set of DMUs with multiple inputs and multiple outputs. Today, DEA has been adopted in many disciplines as a powerful tool for assessing efficiency and productivity. Hence, many other applications of DEA have been reported, for example hospital efficiency (Tiemann, Schreyögg, & Busse, 2012), banking (Paradi & Zhu, 2013), manufacturing efficiency (Jain, Triantis, & Liu, 2011), and productivity of Organization for Economic Co-operation and Development (OECD) countries (Emrouznejad, 2003; Lábaj, Luptáčik, & Nežinský, 2014; Prieto & Zofío, 2007). Many more applications can be found in the scientific literature (Emrouznejad et al., 2008; Liu, Lu, Lu, & Lin, 2013) which indicates that most of these studies have ignored the uncertainty in input and output values. This uncertainty could have an effect on the border defined by the standard DEA; hence the CCR-DEA (Charnes et al., 1978) model may not obtain the true efficiency of DMUs. Theoretically, the standard CCR-DEA model has its production frontier spanned by the linear combination of the observed DMUs.

The production frontier under uncertainty is different. The idea proposed in this research is to allow some flexibility in defining the frontiers with uncertain DMUs, using a fuzzy concept.

2.1 Preliminaries

Definition 1 (Lai & Hwang, 1992). The α-level set (α-cut) of a fuzzy set $\tilde{A}$ is a crisp subset of X and is denoted by

$$A_\alpha = \left\{ x \mid \mu_{\tilde{A}} \geq \alpha \,\&\, x \in X \right\}$$

Definition 2. A triangular fuzzy number $\tilde{A}$ is defined as follows

$$\mu_{\tilde{A}}(\bar{x}) = \begin{cases} \dfrac{\bar{x} - x^l}{x^m - x^l} & \text{for } x^l \leq \bar{x} \leq x^m \\ \dfrac{x^u - \bar{x}}{x^u - x^m} & \text{for } x^m \leq \bar{x} \leq x^u \end{cases} \tag{1}$$

x^m, x^l and x^u are the mean value, the lower bound and the upper bound of the interval of fuzzy number (Zimmermann, 1978). The interval of fuzzy number $[x^l, x^u]$ is the region where the value of $\bar{x}$ fluctuates. Symbolically, $\tilde{A}$ is denoted by (x^m, x^l, x^u). Notice that there are special concepts and terminology in the Fuzzy Sets Theory, when fuzzy numbers with possibilistic data are being used. In this case, x^m, x^l and x^u are called the most possible value, the most pessimistic and the most optimistic values of the imprecise parameter x represented by a triangular fuzzy number. For more details, see Torabi and Hassini (2008) and Pishvaee and Torabi (2010).

2.2 Fuzzy DEA

The DEA technique evaluates the relative efficiency of a set of homogenous DMUs by using a ratio of the weighted sum of outputs to the weighted sum of inputs. It generalizes the usual efficiency measurement from a single-input, single-output ratio to a multiple-input, multiple-output ratio.

Let inputs x_{ij} $(i = 1, 2, ..., m)$ and outputs y_{rj} $(r = 1, 2, ..., s)$ be given for DMU_j $(j = 1, 2, ..., n)$.

The linear programming statement for the CCR model is formulated as follows:

Model 1: CCR-DEA model

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rp} \\
\text{s.t.} \quad & \\
& \sum_{i=1}^{m} v_i x_{ip} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \qquad \forall j \\
& u_r ,\, v_i \geq 0 \qquad \forall r, i
\end{aligned}
$$

where v_i and u_r are the weight variables for i th and r th input and output, respectively.

At the turn of the present century, reducing complex real-world systems into precise mathematical models was the main trend in science and engineering. Unfortunately, real-world situations cannot usually be modelled with exact data. Thus precise mathematical models are not enough to tackle all practical problems. In practice there are many problems in which, all (or some) input–output levels are fuzzy numbers. It is difficult to evaluate DMUs in an accurate manner to measure the efficiency. Fuzzy DEA is a powerful tool for evaluating the performance of a set of organizations or activities under an uncertain environment.

Suppose that the inputs and outputs of DMUs are fuzzy, and they are denoted by $\tilde{x}_{ij}\,(i=1,2,...,m)$ and $\tilde{y}_{rj}\,(r=1,2,...,s)$ respectively. Then, the CCR model with fuzzy coefficients for assessing DMU_p is formulated as follows:

Model 2: Fuzzy CCR-DEA, multiplier model

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r \tilde{y}_{rp} \\
\text{s.t.} \quad & \\
& \sum_{i=1}^{m} v_i \tilde{x}_{ip} = 1 \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0 \qquad \forall j \\
& u_r, v_i \ge 0 \qquad \forall r, i
\end{aligned}
$$

Saati, Memariani, and Jahanshahloo (2002) proposed a fuzzy DEA by considering the α-cut of objective function and the α-cut of constraints; hence the following model is obtained.

Model 3: Fuzzy CCR-DEA, using α-cut approach

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r (\alpha y_{rp}^{m} + (1-\alpha) y_{rp}^{l}, \alpha y_{rp}^{m} + (1-\alpha) y_{rp}^{u}) \\
s.t. \quad & \sum_{i=1}^{m} v_i (\alpha x_{ip}^{m} + (1-\alpha) x_{ip}^{l}, \alpha x_{ip}^{m} + (1-\alpha) x_{ip}^{u}) = (\alpha + (1-\alpha) l^{l}, \alpha + (1-\alpha) l^{u}) \quad \forall_i \\
& \sum_{r=1}^{s} u_r (\alpha y_{rj}^{m} + (1-\alpha) y_{rj}^{l}, \alpha y_{rj}^{m} + (1-\alpha) y_{rj}^{u}) \\
& - \sum_{i=1}^{m} v_i (\alpha x_{ij}^{m} + (1-\alpha) x_{ij}^{l}, \alpha x_{ij}^{m} + (1-\alpha) x_{ij}^{u}) \le 0 \quad \forall j \\
& u_r, v_i \ge 0 \quad \forall r, i.
\end{aligned}
$$

If we substitute $\tilde{x}_{ij} = (x_{ij}^{m}, x_{ij}^{l}, x_{ij}^{u})$, $\tilde{y}_{ij} = (y_{ij}^{m}, y_{ij}^{l}, y_{ij}^{u})$ and $\tilde{1} = (1, 1^{l}, 1^{u})$, Model (3) is written as follows.

Model 4: Fuzzy CCR-DEA, using α-cut approach, interval programming

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r \hat{y}_{rp} \\
s.t. \quad & \sum_{i=1}^{m} v_i \hat{x}_{ip} = L \\
& \sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \le 0 \\
& \alpha y_{rj}^{m} + (1-\alpha) y_{rj}^{l} \le \hat{y}_{rj} \le \alpha y_{rj}^{m} + (1-\alpha) y_{rj}^{u} \\
& \alpha x_{ij}^{m} + (1-\alpha) x_{ij}^{l} \le \hat{x}_{ij} \le \alpha x_{ij}^{m} + (1-\alpha) x_{ij}^{u} \\
& \alpha + (1-\alpha) l^{l} \le L \le \alpha + (1-\alpha) l^{u} \\
& u_r, v_i \ge 0 \quad \forall r, i.
\end{aligned}
$$

As it is shown in Saati et al. (2002) we have $\alpha + (1-\alpha) l^{l} \le L \le 1$. One main drawback in Model 4 is that the optimum efficiency level occurs when the outputs of the evaluated DMU and the inputs of other DMUs are set to their upper bounds, while the inputs of the evaluated DMU and the outputs of other DMUs are set to their lower bounds. As a result the evaluated DMU will have the largest possible efficiency value; hence Model 4 may not obtain the true efficiency score.

In the next section we propose an alternative fuzzy DEA to tackle this problem. In the suggested method the evaluated DMU will have the efficiency value between the smallest and the largest possible values.

3. Multi-objective programming

Since we must solve a particular multi-objective model, a short discussion related to this kind of problem is presented.

Consider the following multi-objective problem

$$
\begin{aligned}
\max \quad & f_1(x), f_2(x), \ldots, f_n(x) \\
\text{s.t.} \quad & x \in X
\end{aligned}
$$

In the above model, functions $f_1(x), f_2(x), ..., f_n(x)$ are objective functions and X is considered as a feasible region. To solve the above mathematical problem, a two stage procedure is proposed.

1. Goal of function $f_i(x)$ $\mathrm{i} = 1, 2, ..., \mathrm{n}$ is obtained by the following mathematical programming:

$$\begin{aligned} & f_i^* = \max \ f_i(x) \\ & \text{s.t.} \quad \mathrm{x} \in \mathrm{X} \end{aligned}$$

2. In this stage scale β is introduced to move functions $\frac{f_i(x)}{f_i^*} \leq 1$ towards their optimality. For this purpose the following mathematical programming problem should be solved:

$$\begin{aligned} & \max \beta \\ & \text{s.t.} \quad \beta \leq \frac{f_i(x)}{f_i^*} \\ & \qquad x \in X \end{aligned}$$

3.1. A multi-objective fuzzy DEA model under uncertainty

This section proposes an alternative fuzzy DEA model. The main idea of the suggested method is based on the membership functions of the coefficients. We consider the coefficients as triangular fuzzy numbers (x^m, x^l, x^u). Hence, the membership functions of the coefficients can be defined as follows.

$$\mu_{\tilde{x}_{ij}}(\bar{x}_{ij}) = \begin{cases} \dfrac{\bar{x}_{ij} - x_{ij}^l}{x_{ij}^m - x_{ij}^l} & x_{ij}^l \leq \bar{x}_{ij} < x_{ij}^m \\ \dfrac{\bar{x}_{ij} - x_{ij}^u}{x_{ij}^m - x_{ij}^u} & x_{ij}^m \leq \bar{x}_{ij} \leq x_{ij}^u \end{cases} \quad \forall i, j \qquad (2)$$

$$\mu_{\tilde{y}_{rj}}(\bar{y}_{rj}) = \begin{cases} \dfrac{\bar{y}_{rj} - y^l_{rj}}{y^m_{rj} - y^l_{rj}} & y^l_{rj} \le \bar{y}_{rj} < y^m_{rj} \\ \dfrac{\bar{y}_{rj} - y^u_{rj}}{y^m_{rj} - y^u_{rj}} & y^m_{rj} \le \bar{y}_{rj} \le y^u_{rj} \end{cases} \quad \forall r, j \qquad (3)$$

Variables $\bar{x}_{ij}$ and $\bar{y}_{rj}$, in formulas (2) and (3), are representative of values in the corresponding intervals of fuzzy numbers.

We suggest the following multi-objective nonlinear program that maximizes both the objective function and the membership functions of the coefficients simultaneously.

<u>Model 5: A multi-objective nonlinear programming Fuzzy CCR-DEA</u>

$$\begin{aligned} \max \quad & \left\{ \mu_{\tilde{x}_{ij}}(\bar{x}_{ij}), \mu_{\tilde{y}_{rj}}(\bar{y}_{rj}) \right\} \forall j \\ \max \quad & \sum_{r=1}^{s} u_r \bar{y}_{rp} \\ s.t. \quad & \sum_{i=1}^{m} v_i \bar{x}_{ip} = 1 \\ & \sum_{r=1}^{s} u_r \bar{y}_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \le 0 \quad \forall j (j \ne p) \\ & x^l{}_{ip} \le \bar{x}_{ip} \le x^u{}_{ip} \quad \forall i \\ & y^l{}_{rp} \le \bar{y}_{rp} \le y^u{}_{rp} \quad \forall r \\ & x^l{}_{ij} \le \bar{x}_{ij} \le x^u{}_{ij} \quad \forall i, j \\ & y^l{}_{rj} \le \bar{y}_{rj} \le y^u{}_{rj} \quad \forall r, j \\ & u_r, v_i \ge 0 \quad \forall r, i \end{aligned}$$

Variables u_r, v_i indicate the coefficients of fuzzy outputs and inputs. Furthermore, variables $\bar{x}_{ij}$ and $\bar{y}_{rj}$ represent the intervals of fuzzy numbers $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$, respectively.

This is a multi-objective nonlinear fuzzy model that we suggest to solve in two stages as explained in the rest of this paper. Zimmermann's approach (Lai & Hwang, 1992) for solving FLP with fuzzy resources used a similar approach to solve the multi-objective linear programming model corresponding to FLP. Notice that the focus in this paper is to solve an FLP (Model 2) using a non-linear multi-objective programming model (Model 5), not a Fuzzy multi-objective programming model (FMOP). We refer readers interested in FMOP to Torabi and Hassini (2008).

Let us ignore the objective functions corresponding to membership functions in Model 5, that is, $\max\left\{\mu_{\tilde{x}_{ij}}(\bar{x}_{ij}), \mu_{\tilde{y}_{rj}}(\bar{y}_{rj})\right\}$. According to Zerafat Angiz et al. (2010), the optimal solution of the modified model will be as follows:

$$\begin{array}{lll} \bar{x}_{ij}^{*} = x_{ij}^{u} & j \neq p & \bar{x}_{ip}^{*} = x_{ip}^{l} \\ \bar{y}_{rj}^{*} = y_{rj}^{l} & j \neq p & \bar{y}_{rp}^{*} = y_{rp}^{u} \end{array}$$

This is because each DMU with inputs greater than and outputs less than inputs and outputs DMU_p respectively, will not be better than DMU_p. So the optimal value of Model (5) is equals to efficiency of DMU_p.

Ignoring the last objective function in Model (5), the optimal solution will be as follows:

$$\begin{array}{lll} \bar{x}_{ij}^{*} = x_{ij}^{m} & j \neq p & \bar{x}_{ip}^{*} = x_{ip}^{m} \\ \bar{y}_{rj}^{*} = y_{rj}^{m} & j \neq p & \bar{y}_{rp}^{*} = y_{rp}^{m} \end{array}$$

Interaction between two opposed objective functions specify the optimal solution.

Lemma1: Let's consider the optimistic point of view that is the best condition for DMU under evaluation and the worst condition for other DMUs.

a. The optimal solution for $\mu_{\tilde{x}_{ij}}(\bar{x}_{ij}), \mu_{\tilde{y}_{rp}}(\bar{y}_{rp})$ are obtained in the second condition of the membership functions (2) and (3), respectively.
b. The optimal solution for $\mu_{\tilde{x}_{ip}}(\bar{x}_{ip}), \mu_{\tilde{y}_{rj}}(\bar{y}_{rj})(j \neq p)$ are obtained in the first condition of the membership functions (2) and (3), respectively.

Proof: Suppose that objective function in Model (5) be only ($\max \sum_{r=1}^{s} u_r \bar{y}_{rp}$), as mentioned above, due the nature of the model the optimal solution will be:

$$\begin{array}{llll} \min \ \bar{x}_{ip} & \forall i & \max \ \bar{x}_{ij} & \forall i, j(j \neq p) \\ \max \ \bar{y}_{rp} & \forall r & \min \ \bar{y}_{rj} & \forall r, j(j \neq p) \end{array} \tag{4}$$

When considering the effect of the membership function, the values of $\bar{x}_{ij} \quad \forall i, j(j \neq p)$ and $\bar{y}_{rp} \quad \forall r$ will be decreased and the values of $\bar{x}_{ip} \quad \forall i$ and $\bar{y}_{rj} \quad \forall r, j(j \neq p)$ will be increased (membership numbers will be zero for the above mentioned values). So, to obtain the optimal solution of $\mu_{\tilde{x}_{ij}}(\bar{x}_{ij}), \mu_{\tilde{y}_{rp}}(\bar{y}_{rp})$ the second condition of the membership functions (2) and (3) are sufficient, respectively. Similarly to obtain the optimal value for $\mu_{\tilde{x}_{ip}}(\bar{x}_{ip}), \mu_{\tilde{y}_{rj}}(\bar{y}_{rj})(j \neq p)$ the first condition of the membership functions (2) and (3) are sufficient, respectively, i.e.

$$\mu_{\tilde{x}_{ip}}(\bar{x}_{ip}) = \frac{\bar{x}_{ip} - x_{ip}^{l}}{x_{ip}^{m} - x_{ip}^{l}} \quad \bar{x}_{ip} \in [x_{ip}^{l}, x_{ip}^{m}] \qquad \forall i \tag{5}$$

$$\mu_{\tilde{y}_{rp}}(\bar{y}_{rp}) = \frac{y^{u}{}_{rp} - \bar{y}_{rp}}{y_{rp}^{u} - y_{rp}^{m}} \quad \bar{y}_{rp} \in [y_{rp}^{m}, y_{rp}^{u}] \qquad \forall r \tag{6}$$

$$\mu_{\tilde{x}_{ij}}(\bar{x}_{ij}) = \frac{x_{ij}^{u} - \bar{x}_{ij}}{x_{ij}^{u} - x_{ij}^{m}} \quad \bar{x}_{ij} \in [x_{ij}^{m}, x_{ij}^{u}] \quad \forall i, j(j \neq p) \tag{7}$$

$$\mu_{\tilde{y}_{rj}}(\bar{y}_{rj}) = \frac{\bar{y}_{rj} - y^{l}{}_{rj}}{y_{rj}^{m} - y_{rj}^{l}} \quad \bar{y}_{rj} \in [y_{rj}^{l}, y_{rj}^{m}] \quad \forall r, j(j \neq p) \tag{8}$$

Let $\bar{x}_{ij}^{*}, \bar{y}_{rj}^{*}(j \neq p)$ and $\bar{x}_{ip}^{*}, \bar{y}_{rp}^{*}$ be the optimal solution for $\bar{x}_{ij}, \bar{y}_{rj}(j \neq p)$ and $\bar{x}_{ip}, \bar{y}_{rp}$. It is clear that there exist two values in the intervals $[x_{ij}^{l}, x_{ij}^{u}], [y_{rj}^{l}, y_{rj}^{u}](j \neq p)$ and $[x_{ip}^{l}, x_{ip}^{u}], [y_{rp}^{l}, y_{rp}^{u}]$ with the same membership function, say,

$$
\begin{aligned}
&\bar{x}^{*}_{ij1} \in [x^{l}_{ij}, x^{m}_{ij}], \bar{y}^{*}_{rj1} \in [y^{l}_{rj}, y^{m}_{rj}] \\
&\bar{x}^{*}_{ij2} \in [x^{m}_{ij}, x^{u}_{ij}], \bar{y}^{*}_{rj2} \in [y^{m}_{rj}, y^{u}_{rj}] \\
&\bar{x}^{*}_{ip1} \in [x^{l}_{ip}, x^{m}_{ip}], \bar{y}^{*}_{rp1} \in [y^{l}_{rp}, y^{m}_{rp}] \\
&\bar{x}^{*}_{ip2} \in [x^{m}_{ip}, x^{u}_{ip}], \bar{y}^{*}_{rp2} \in [y^{m}_{rp}, y^{u}_{rp}].
\end{aligned}
\qquad (9)
$$

In this view, the $\bar{x}_{ij}s$ are similar to the input values and the $\bar{y}_{rj}s$ are similar to the output values in the DEA models, so by considering constant values for $\bar{x}_{ij}s$ and $\bar{y}_{rj}s$, Model (5) will be converted to Model (4). According to Lemma 1, the best situation of the DMU under evaluation is compared with the worst situation of other DMUs, and this means that the evaluation is based on an optimistic approach. In Zerafat Angiz et al. (2010), it is proved that the worst situation of the DMU under evaluation is compared with the best situation of other DMUs, that is, a pessimistic view. A discrete approach is based on defuzzified points, and two other methodologies consider the mean value (most possible point) as their goals.

The discrete approach (Zerafat Angiz et al., 2012) and the proposed approach follow two different views. In the discrete approach, the goal is achieving defuzzified points whereas the goal of fuzzy numbers in the proposed approach is the most possible values. The discrete approach tries to keep information about uncertainty as much as possible as the new approach does. The discrete approach approximates the solution, but it still loses some information about uncertainty. The benefit of applying a discrete approach is that a linear programming model is used.

Assume that inputs and outputs of DMU_A and DMU_B are $(x'^{*}_{ip1}, x'^{*}_{ij1}, y'^{*}_{rp2}, y'^{*}_{rj2})(j \neq p)$ and $(x'^{*}_{ip2} x'^{*}_{ij2}, y'^{*}_{rp1}, y'^{*}_{rj1})(j \neq p)$, respectively. Obviously DMU_A is more efficient than DMU_B. In other words, DMU_B is dominated by DMU_A. This means only the second condition of the membership functions (2) and (3) are sufficient to obtain the optimal solution for $\mu_{\tilde{x}_{ip}}(x'_{ij}), \mu_{\tilde{y}_{rp}}(y'_{ip})$. Similarly the first condition of the membership function (2) and (3) are sufficient to obtain the optimum value for $\mu_{\tilde{x}_{ip}}(x'_{ip}), \mu_{\tilde{y}_{ij}}(y'_{ij})(j \neq p)$.

Hence, to solve Model (5), the methodology presented in section 3 is applied, and multi-objective programming problem (5) is converted to the following nonlinear programming problem:

<u>Model 6: A new Fuzzy CCR-DEA, non-linear programming</u>

$$
\begin{aligned}
\max \quad & Z = h \\
s.t. \quad & \\
& \sum_{i=1}^{m} v_i \bar{x}_{ip} = 1 \\
& h \le \left(\sum_{r=1}^{s} u_r \bar{y}_{rp}\right) / z_p^{*} \\
& \sum_{r=1}^{s} u_r \bar{y}_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \le 0 \quad \forall j (j \neq p) \\
& h \le \frac{x^u{}_{ij} - \bar{x}_{ij}}{x^u_{ij} - x^m_{ij}} \quad \forall i, j (j \neq p) \\
& h \le \frac{\bar{y}_{rj} - y^l_{rj}}{y^m_{rj} - y^l_{rj}} \quad \forall r, j (j \neq p) \\
& h \le \frac{\bar{x}_{ip} - x^l_{ip}}{x^m_{ip} - x^l_{ip}} \quad \forall i \\
& h \le \frac{y^u{}_{rp} - y_{rp}}{y^u_{rp} - y^m_{rp}} \quad \forall r \\
& x^m_{ij} \le \bar{x}_{ij} \le x^u_{ij} \quad \forall i, j (j \neq p) \qquad 6.1 \\
& y^l_{rj} \le \bar{y}_{rj} \le y^m_{rj} \quad \forall r, j (j \neq p) \qquad 6.2 \\
& x^l_{ip} \le \bar{x}_{ip} \le x^m_{ip} \quad \forall i \qquad 6.3 \\
& y^m_{rp} \le \bar{y}_{rp} \le y^u_{rp} \quad \forall r \qquad 6.4 \\
& u_r, v_i \ge 0 \quad \forall r, i
\end{aligned}
$$

In Model (6), z_p^* is obtained with the best situation (optimistic view point) of the DMUs as follows:

Model 7: A new Fuzzy CCR-DEA, estimation of Z^*_p

$$
\begin{aligned}
z_p = \max \quad & \sum_{r=1}^{s} u_r y_{rp}^{u} \\
s.t \quad & \sum_{i=1}^{m} v_i x_{ip}^{l} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj}^{l} - \sum_{i=1}^{m} v_i x_{ip}^{l} \le 0 \quad \forall j (j \ne p) \\
& u_r, v_i \ge 0 \quad \forall r, i
\end{aligned}
$$

Obviously, fluctuating between 0 and 1, the objective functions corresponding to membership functions do not need to follow the first stage of Section 3. Z*P indicates the best situation of the DMU under evaluation comparing to other DMUs. Notice that Model 7 finds the optimal solution ignoring the membership values. This is why we consider the largest value of outputs and smallest values of inputs corresponding to the DMU under evaluation, and the smallest outputs and largest inputs for the other DMUs. Therefore, in Model 6, $0 \le (\sum_{r=1}^{s} u_r \bar{y}_{rp}) / z_p^{*} \le 1$, and the goal will be maximum value that is 1.

The variable *h* in Model (6) is used to convert the multi-objective problem Model (5) to a nonlinear programming problem. This variable is within the interval $[0,1]$. Adding the concept of α-cut to Model (6), it is sufficient to replace the following constraints instead of 6-1, 6-2, 6-3 and 6-4.

$$
\begin{aligned}
x'^{m}_{ij} \le x'_{ij} \le \alpha x'^{m}_{ij} + (1-\alpha) x'^{u}_{ij} \quad & \forall i, j (j \ne p) \\
\alpha y'^{m}_{rj} + (1-\alpha) y'^{l}_{rj} \le y'_{rj} \le y'^{m}_{rj} \quad & \forall r, j (j \ne p) \\
\alpha x'^{m}_{ip} + (1-\alpha) x'^{l}_{ip} \le x'_{ip} \le x'^{m}_{ip} \quad & \forall i \\
y'^{m}_{rp} \le y'_{rp} \le \alpha y'^{m}_{rp} + (1-\alpha) y'^{u}_{rp} \quad & \forall r
\end{aligned}
$$

This is different from the standard α-cut used in the fuzzy DEA Model (4), because in each α-level the model still retains uncertainty information interior of the interval that was generated by α. Next section compares our results with the current fuzzy DEA model.

4. An illustration with a numerical example

In this section, a numerical example is presented to illustrate the difference between the results obtained using the proposed approach and the current fuzzy DEA models. Consider the data in Table 1 that is extracted from Guo and Tanaka (2001) and used by Lertworasirikul et al. (2003a) and Saati et al. (2002). There are 5 DMUs with two symmetrical triangular fuzzy inputs and 2 symmetrical triangular fuzzy outputs.

Table 1: Data for numerical example

	DMU				
Variable	D1	D2	D3	D4	D5
I1	(4.0, 3.5, 4.5)	(2.9, 2.9, 2.9)	(4.9, 4.4, 5.4)	(4.1, 3.4, 4.8)	(6.5, 5.9, 7.1)
I2	(2.1, 1.9, 2.3)	(1.5, 1.4, 1.6)	(2.6, 2.2, 3.0)	(2.3, 2.2, 2.4)	(4.1, 3.6, 4.6)
O1	(2.6, 2.4, 2.8)	(2.2, 2.2, 2.2)	(3.2, 2.7, 3.7)	(2.9, 2.5, 3..3)	(5.1, 4.4, 5.8)
O2	(4.1, 3.8, 4.4)	(3.5, 3.3, 3.7)	(5.1, 4.3, 5.9)	(5.7, 5.5, 5.9)	(7.4, 6.5, 8.3)

Using fuzzy CCR Model (4), the efficiency scores are summarized in the Table 2.

Table 2: The efficiencies using Model (4)

	DMU				
A	D1	D2	D3	D4	D5
0	1.107	1.506	1.276	1.525	1.296
.5	0.995	1.321	1.035	1.319	1.159
.75	0.906	1.237	0.936	1.230	1.086
1	0.852	1.000	0.863	1.000	1.000

Considering the above Lemma 1, the optimal solution given in Table 2 is equivalent to the optimal solution related to the optimistic part of Kao and Liu (2000) approach in its supper efficiency form. The methods based on the **α**-cut approach just extend number of membership values considered in the evaluation. Therefore the major part of the fuzzy concept is ignored. Differences between the proposed method and the **α**-cut based approach can be compared with differences between integration and numerical methods for integrals. The numerical methods do not cover the whole area under the curve in integration.

Results from the possibility approach of Lertworasirikul et al. (2003a) are shown in Table 3. As can be seen, the efficiency values in the above two models are very similar.

Table 3: The efficiencies using Lertworasirikul et al. (2003a) model

	DMU				
α	D1	D2	D3	D4	D5
0	1.107	1.238	1.276	1.520	1.3296
.5	0.963	1.112	1.035	1.258	1.159
.75	0.904	1.055	0.932	1.131	1.095
1	0.855	1.000	0.861	1.000	1.000

Using the proposed Model (6), the results are shown in Table 4.

Table 4: The efficiencies using the proposed model in this paper

	DMU				
α	D1	D2	D3	D4	D5
0	0.899	1.220	0.930	1.220	1.076
0.5	0.865	1.180	0.871	1.169	1.041
0.75	0.845	1.110	0.866	1.160	1.037
1	0.842	1.000	0.860	1.000	1.000

Due to the nature of the fuzzy CCR Model (4) the maximum efficiency occurs when the outputs of the evaluated DMU and the inputs of other DMUs are set to their upper bounds. It is obvious that the results in Table 2 are always greater than the results that we obtained in Table 4 since Model 4 always captures the efficiency under pessimistic circumstances. The results obtained using the proposed model in this paper have the efficiency values between the smallest and the largest possible values, hence they are more close to the true efficiency.

5. Empirical study

To illustrate the fuzzy DEA approach, we consider data given in Yeh and Chang (2009) which was presented for an aircraft selection problem. Five types of aircraft (B757-200, A-321, B767-200, MD-82, and A310-300) are to be evaluated. Four inputs and two outputs are introduced in Table 5 as follows:

Table 5: Inputs and outputs for aircrafts evaluation

Data	Description
Input1 (I1)	Maintenance requirements (Subjective assessment)
Input2 (I2)	Pilot adaptability (Subjective assessment)
Input3 (I3)	Maximum range (Kilometer)
Input4 (I4)	Purchasing price (US millions)
Output1 (O1)	Passenger preference (Subjective assessment)
Output2 (O2)	Operational productivity (Seat-kilometer per hour)

The first input is the aircraft maintenance capability (I1) which is concerned with the availability and the level of standardization of spare parts and post-sale services. The second input, pilot adaptability (I2) is related to the skills of available pilots and the specific features of the aircraft. Increasing pilot adaptability and maintenance capability will increase the outputs, so they are considered as inputs. To consider a datum (data) as an input we should look at the effect of the datum in producing outputs. The third input maximum range (I3) of an aircraft is determined by the maximum kilometers that the aircraft can travel at the maximum payload and the fourth input, purchasing price (I4) is the price to be paid for a new aircraft which correlates with reliability of the aircraft.

On the other hand for the outputs, passengers' preference (O1) reflects the social responsibility of the airline in order to establish a positive image in public and of the requirements imposed by various environment protection laws and regulations whilst operational productivity (O2) is determined by the number of seats available, the load rate, the travel frequency, and the aircraft travel speed.

In this research, the eight decision makers stated their opinion about 3 subjective inputs and outputs. They used a set of five linguistic terms {very low, low, medium, high, very high} which are associated with the corresponding numbers 1, 2, 3, 4 and 5, respectively, as in a 5-point Likert scale.

Table 6 shows the inputs and outputs of the five aircrafts. For example, B757-200 type of aircraft has two subjective inputs (I1 and I2) and one subjective output (O1), with triangular fuzzy numbers. For other two inputs and one output, the values are crisps.

Table 6: Data for numerical example

	DMU				
Variable	B757-200	A-321	B767-200	MD-82	A310-300
I1	(2.0, 3.064, 4)	(4, 4.229,5)	(3, 3.224, 4)	(1, 1.929, 3)	(3,3.464, 4)
I2	(2, 2.852, 3)	(2,2.000,2)	(2, 2.852, 3)	(4, 4.113, 5)	(2,2.000,2)
I3	5522	4350	5856	4032	7968
I4	56	54	69	33	80
O1	(4, 4.000, 4)	(2, 2.852, 3)	(4, 4.000, 4)	(3, 3.591, 4)	(3, 3.342, 4)
O2	116279	109063	129465	87662	130664

Using Model (6), the values of h*, the efficiency scores and rank of each aircraft are given in Table 7. The MD-82 aircraft type gives the highest efficiency score of

1.8520 and is ranked first, whilst B767-200 gives lowest score of 1.0949 and is ranked last.

Table 7: The rank of five types of aircrafts

DMU	h^*	Eff. scores	Rank
B757-200	0.6348	1.2696	2
A-321	0.9798	1.1720	3
B767-200	1.0000	1.0949	5
MD-82	0.9260	1.8520	1
A310-300	1.0000	1.1237	4

6. Discussion

According to Theorem 2, if the objective functions corresponding to membership functions in Model (5) are ignored, the optimal solution for inputs and outputs will beat the endpoints of the interval of fuzzy numbers. Furthermore, if the last objective function ($\max \sum_{r=1}^{s} u_r \overline{y}_{rp}$) in Model (5) is eliminated, Lemma 1 adopted the optimal solution will be in the mean value of fuzzy number. Figure 1 illustrates the above mentioned concept for evaluating DMU_P. This figure can also be seen in Zerafat Angiz et al. (2012). Since the discrete approach (Zerafat Angiz et al., 2012) assumes the defuzzified points as its goal, so the interpretation presented in Zerafat Angiz et al. (2012) is not appropriate for this specific application. The interior arrows represent the optimal solution when the last objective function ($\max \sum_{r=1}^{s} u_r \overline{y}_{rp}$) is absent in Model (5) and the arrows located under fuzzy numbers construct the optimal solution Model (5) when only the objective function ($\max \sum_{r=1}^{s} u_r \overline{y}_{rp}$) is present.

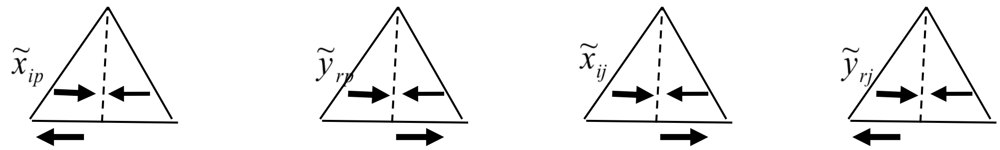

Figure 1: Concepts of evaluating DMUs

Interaction between the objective functions corresponding to objective functions and the last objective function ($\max \sum_{r=1}^{s} u_r \bar{y}_{rp}$) in Model (5), cause the fuzzy optimal solution.

7. Conclusion

In evaluating DMUs under uncertainty several fuzzy DEA models have been proposed in the literature. The **α**-cut approach is one of the most frequently used models. However, due to the nature of the **α**-cut approach the uncertainty in inputs and outputs is effectively ignored. This paper has proposed a multi-objective fuzzy DEA model to retain fuzziness of the model by maximizing the membership function of inputs and outputs. In the proposed method, both the objective functions and the constraints are considered fuzzy. A numerical example is used to show the difference between the proposed and the current fuzzy DEA models. For further studies, it is suggested that an exploration be done on: a) reducing the size of the converted (crisp equivalent) problem, b) possible linearization of the nonlinear model.

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