Exact solution of a boundary time-crystal phase transition: time-translation symmetry breaking and non-Markovian dynamics of correlations

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The breaking of the continuous time-translation symmetry manifests, in Markovian open quantum systems, through the emergence of non-stationary dynamical phases. Systems that display nonequilibrium transitions into these phases are referred to as time-crystals, and they can be realized, for example, in many-body systems governed by collective dissipation and long-ranged interactions. Here, we provide a complete analytical characterization of a boundary time-crystal phase transition. This involves exact expressions for the order parameter and for the dynamics of quantum fluctuations, which, in the time-crystalline phase, remains asymptotically non-Markovian as a consequence of the time-translation symmetry breaking. We demonstrate that boundary time-crystals are intrinsically critical phases, where fluctuations exhibit a power-law divergence with time. Our results show that a dissipative time-crystal phase is far more than merely a classical non-linear and non-stationary (limit cycle) dynamics of a macroscopic order parameter. It is rather a genuine many-body phase where the properties of correlations distinctly differs from that of stationary ones.

Introduction.— Dissipation and irreversible effects are typically associated with the convergence of a quantum system towards an asymptotic stationary state. Effectively time-translation invariant states can also occur in closed quantum systems when considering the dynamics of local observables [1–3]. Recently, this paradigm has been challenged by the observation that non-stationary asymptotic behavior can emerge in open quantum systems [4], not only in the presence of decoherence-free subspaces [5–8], but also as a consequence of nonequilibrium transitions toward many-body dynamical phases [9, 10]. Much of the interest in these asymptotic non-stationary states is due to the discovery of time-crystals [11, 12]. In simple terms, such time-crystal constitutes a non-stationary nonequilibrium phase of matter, in which the long-time dynamics does not reflect a time-translation symmetry of its generator, see e.g. Refs. [4, 9–33].

Specifically, in the context of Markovian open quantum systems the quantum state obeys the master equation $\dot{\rho}(t) = \mathcal{L}[\rho(t)]$ [34–38], with time-independent dynamical generator $\mathcal{L}$. The formal solution, $\rho(t) = e^{t\mathcal{L}}[\rho(0)]$, introduces the *time-translation operator* $e^{t\mathcal{L}}$ which propagates the system for a time t. Since $\mathcal{L}$ is time independent, one has $[e^{t\mathcal{L}}, \mathcal{L}] = 0$, showing that time-translation is a continuous "symmetry" of the generator. In these settings, the state $\rho(t)$ is expected to approach a time-independent stationary state $\rho_{\rm SS}$ [see sketch in Fig. 1(a)]. Such a state reflects the symmetry of the generator, since $e^{t'\mathcal{L}}[\rho_{\rm SS}] = \rho_{\rm SS}$, and may be regarded as a symmetric "ground state" of $\mathcal{L}$. However, Markovian open quantum systems can also feature non-stationary asymptotic behavior [4]. For large times, the state may approach a limit cycle, i.e. an asymptotic regime characterized by sustained (usually periodic) oscillations, $\rho(t) \to \rho_{\rm LC}(t)$ [see Fig. 1(b)]. The "absolute position" within the limit cycle is thus relevant, and further time-translations generically modify the quantum state, $e^{t'\mathcal{L}}[\rho_{\rm LC}(t)] = \rho_{\rm LC}(t'+t)$. The continuous time-translation symmetry of the generator is thus broken and the system forms a crystalline structure in time.

Paradigmatic models featuring a *dissipative* continuous time-crystal phase transition are the so-called *boundary time-crystals* [9, 10, 31–33]. For these models, it has been numerically demonstrated that average (mean-field) operators —acting as order parameter— show asymptotic limit-cycle dynamics [cf. Fig. 1(b)] [9, 10]. However, an analytic understanding of their behavior, as well as of the behavior of fluctuations and correlations, is still missing, leaving the characterization of these phases incomplete. Here, we provide an exact solution of a boundary time-crystal phase transition. We derive analytical expressions for the order parameter as well as for the dynamics of quantum fluctuations. The latter becomes asymptotically Markovian in the stationary phase, while it remains asymptotically non-Markovian in the time-crystalline phase, as witnessed by the persistent time-dependence of the dynamical generator [39]. Our results show that continuous time-crystals are critical many-body phases, displaying an algebraic growth of fluctuations with time [cf. Fig. 1(b)], typically observed at critical points of second-order phase transitions [40–43].

The model.— We consider the original boundary time-crystal model introduced in Ref. [9], which consists of an ensemble of N quantum spin-1/2 systems. As a basis for the single-particle algebra, we consider the set $\{v_\alpha\}_{\alpha=0}^3$ of (rescaled) Pauli matrices, $v_\alpha = \sigma_\alpha/\sqrt{2}$, with σ_0 proportional to the identity. With the notation $v_\alpha^{(k)}$ we denote

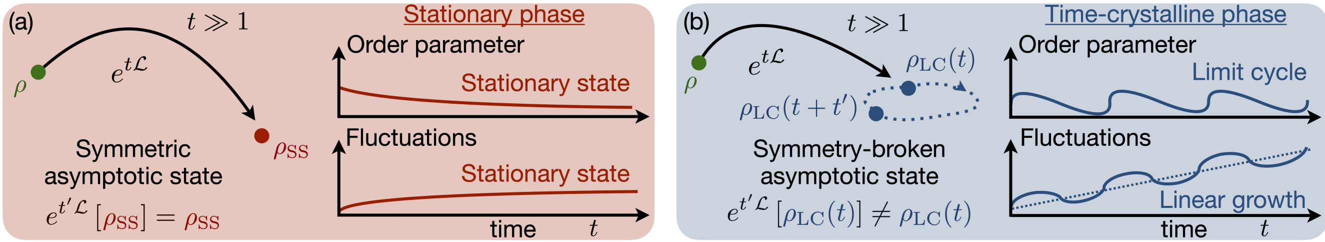

FIG. 1. **Stationary vs time-crystalline phase.** (a) A Markovian open quantum dynamics —implemented by a time-independent generator $\mathcal{L}$— typically brings the system towards an asymptotic time-invariant state, $\rho_{\rm SS}$. Such a stationary state reflects the symmetry of $\mathcal{L}$, since time-translations do not change its properties, $e^{t'\mathcal{L}}[\rho_{\rm SS}] = \rho_{\rm SS}$. In this regime, system observables, as well as fluctuations, converge to a stationary value, at least far from critical points. (b) In a (continuous) time-crystalline phase, Markovian open quantum systems approach a limit cycle with time-dependent state $\rho_{\rm LC}(t)$. Here, the symmetry of the generator is broken given that continuous time-translations modify the position of the quantum state in the limit cycle, i.e. $e^{t'\mathcal{L}}[\rho_{\rm LC}(t)] = \rho_{\rm LC}(t+t') \neq \rho_{\rm LC}(t)$. In this case, an appropriate observable —order parameter— can witness the persistent oscillations of the asymptotic quantum state. In this paper, we show that in the presence of long-range (collective) dissipative effects, continuous time-crystalline phases feature a critical growth of fluctuations of the order parameter.

the operator v_α acting on the kth particle. The model is defined in terms of collective operators $V_\alpha = \sum_{k=1}^N v_\alpha^{(k)}$ and its dynamics is governed a Lindblad dynamical generator [34] of the form [9]

$$\mathcal{L}^*[o] = i[H, o] + \sum_{\alpha,\beta=1}^{3} \frac{C_{\alpha\beta}}{N}\left(V_\alpha o V_\beta - \frac{1}{2}\{o, V_\alpha V_\beta\}\right), \quad (1)$$

which yields the evolution of an observable o, through the Heisenberg equation $\dot{o}(t) = \mathcal{L}^*[o(t)]$. We note that the map $\mathcal{L}^*$ is the dual of $\mathcal{L}$, introduced above. The Hamiltonian H solely consists of single-particle terms and is written as $H = (\omega/\sqrt{2})V_1$, with coherent rate (Rabi frequency) ω a real number. The second term in Eq. (1) accounts for dissipative contributions, which are parametrized by the matrix C. This matrix must be positive semi-definite and can be decomposed into a symmetric part, $A = A^T$, and an anti-symmetric one, $B = -B^T$, as $C = A + iB$. For the model considered, we have that $A_{11} = A_{22} = \gamma$ and $B_{21} = -B_{12} = \gamma$, with all other elements being zero. This yields long-range dissipation which can be formulated in terms of collective jump operators of the form $J = \sqrt{\gamma}(V_1 - iV_2)$. The scaling $1/N$ appearing in Eq. (1) ensures the existence of a well-defined thermodynamic ($N \to \infty$) limit [44].

Before proceeding, we note that other boundary time-crystal models have been recently proposed [10, 33]. These systems belong to a class of models subject to a dissipative collective dynamics generalizing Eq. (1) to higher dimensional single-particle algebras. This class of open quantum systems has been thoroughly investigated in Ref. [44], to which we refer for a mathematical discussion of the methodology employed here.

Order parameter.— To detect the emergence of a time-crystalline phase, we need to identify an appropriate order parameter, see Fig. 1(b). The usual choice falls on "mean-field" operators $m_\alpha^N = V_\alpha/N$ [9, 10], accounting for collective macroscopic properties of the system. For initial clustering states, i.e. states with short-range correlations, the time-evolved operators $m_\alpha^N(t)$ converge, in the large N limit, to multiples of the identity $m_\alpha^N(t) \to m_\alpha(t) = \lim_{N\to\infty}\langle m_\alpha^N(t)\rangle$ [44, 45], where $\langle\cdot\rangle = \mathrm{Tr}(\rho\,\cdot)$ is the quantum expectation. As such, mean-field operators provide a collective dynamical description of the many-body system in terms of classical variables.

Under the dynamics generated by the map $\mathcal{L}^*$ in Eq. (1), the evolution of mean-field operators is implemented, in the thermodynamic limit, by a system of non-linear differential equations [44, 46]. These equations are

$$\begin{aligned} \dot{m}_1(t) &= \gamma\sqrt{2} m_1(t) m_3(t)\,, \\ \dot{m}_2(t) &= \gamma\sqrt{2} m_2(t) m_3(t) - \omega m_3(t)\,, \\ \dot{m}_3(t) &= \omega m_2(t) - \gamma\sqrt{2}\left[m_1^2(t) + m_2^2(t)\right]\,, \end{aligned} \quad (2)$$

and need to be solved with initial conditions $m_\alpha(0) = \bar{m}_\alpha$. From now on, we set $\gamma = 1$ which amounts to measuring time in units of $1/\gamma$ and the energy scale ω in units of γ. The norm of the vector $m(t) = [m_1(t), m_2(t), m_3(t)]$ is a conserved quantity. The above equations also feature another conserved quantity [9] and we consider here initial states with $\bar{m}_1 = 0$, which implies $m_1(t) = 0\ \forall t$, and $\|m(t)\|^2 = 1/2$.

To solve the system in Eq. (2), we make the ansatz

$$\begin{aligned} m_2(t) &= \cos[f(t)]\bar{m}_2 + \sin[f(t)]\bar{m}_3\,, \\ m_3(t) &= \cos[f(t)]\bar{m}_3 - \sin[f(t)]\bar{m}_2\,. \end{aligned} \quad (3)$$

and substitute this in Eq. (2) to obtain a self-consistent differential equation for $f(t)$. This function completely determines the long-time behavior of the model (see Supplemental Material [47] for details).

For $\omega < 1$, the function $f(t)$ converges to a fixed value $f(\infty) := \lim_{t\to\infty} f(t)$. As such, we have that the limit $m_\alpha(\infty) := \lim_{t\to\infty} m_\alpha(t)$ exists for all α, ensuring that the system has approached a stationary state [cf. Fig. 2(a)]. In particular, the stationary value of the order parameter components is given by $m_2(\infty) = \omega/\sqrt{2}$ and $m_3(\infty) = -|\Delta|/\sqrt{2}$, with $\Delta = \sqrt{\omega^2 - 1}$, as reported in Fig. 2(a). At the critical coherent rate (Rabi frequency) $\omega = 1$, we observe the algebraic behavior (for $t \gg 1$)

$$m_2(t) - m_2(\infty) \sim -\frac{\sqrt{2}}{t^2} \quad \text{and} \quad m_3(t) \sim -\frac{\sqrt{2}}{t}. \quad (4)$$

On the other hand, for $\omega > 1$, the function $f(t)$ does not converge and $m_{2/3}(t)$ feature persistent oscillations witnessing the emergence of a time-crystalline phase [cf. Fig. 1(b)]. In order to retain a phenomenology similar to phase transitions among stationary phases we consider the time-averaged order parameter $\mu_\alpha(t) = t^{-1}\int_0^t du\, m_\alpha(u)$. In Fig. 2(a), we show that the asymptotic behavior of μ_2 and μ_3 suggests, that this boundary time-crystal phase transition effectively is a continuous second-order transition.

Quantum fluctuations and correlations.— The boundary time-crystal phase transition manifests through persistent oscillations of the order parameter. Such a phenomenology is well-known from simple non-linear dynamical systems, such as classical anharmonic oscillators [48]. However, the boundary time-crystal is a genuine many-body phenomenon with surprisingly non-trivial properties. This becomes evident when focusing on quantum fluctuations and on many-body correlations. Their dynamical behavior is not simply inherited from the non-linear (mean-field) system of equations (2).

To study them we introduce the fluctuation operators [44, 49–54]

$$F_\alpha^N = N^{-1/2}\left(V_\alpha - \langle V_\alpha\rangle\right), \text{ for } \alpha = 1, 2, 3, \quad (5)$$

which, by definition, account for fluctuations of the operators V_α around their average. Contrary to mean-field operators, they provide a collective description of the many-body system which retains a quantum character, i.e., the limiting operators $F_\alpha := \lim_{N\to\infty} F_\alpha^N$ are indeed bosonic operators and not classical variables. Their commutation relations are $[F_\alpha, F_\beta] = i\Omega_{\alpha\beta}$ with the matrix $\Omega_{\alpha\beta} = \sum_\gamma \sqrt{2}\epsilon_{\alpha\beta\delta} m_\delta$ and $\epsilon_{\alpha\beta\delta}$ being the fully antisymmetric tensor. For a rigorous discussion about the convergence of quantum fluctuations to bosonic operators, see e.g. Refs. [44, 52, 54].

The rationale for considering fluctuation operators is two-fold: first, they account for two-body correlations, and, second, they quantify fluctuations of the order parameter. In practice, fluctuations provide the susceptibility parameter for the m_α, as becomes evident, for example, from the variance of F_α:

$$\langle F_\alpha^2\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{k,h=1}^{N} \left(\langle v_\alpha^{(k)} v_\alpha^{(h)}\rangle - \langle v_\alpha^{(k)}\rangle\langle v_\alpha^{(h)}\rangle \right). \quad (6)$$

The time-evolution of quantum fluctuations under the generator in Eq. (1) has been rigorously derived in Ref. [44]. Fluctuations undergo a Gaussian dissipative dynamics [55] and their full information is contained in the covariance matrix $\Sigma_{\alpha\beta} = \langle\{F_\alpha, F_\beta\}\rangle/2$ [44]. The precise structure of the dynamics is complicated by the fact that commutation relations between fluctuation operators, as specified by the matrix Ω, are in principle time dependent. This gives rise to an emergent hybrid quantum-classical dynamical system, formed by quantum fluctuations and (classical) mean-field operators [44]. As we discuss here, this problem can be simplified by looking at the quantum fluctuations $\tilde{F}_\alpha$ defined in the frame rotating with the mean-field operators (see e.g. Refs. [56–58] for a similar approach in closed systems).

For completeness, we now briefly sketch the main technical steps for the derivation of the dynamical generator for quantum fluctuations [cf. Eqs. (9)-(10)]. As transparent from Eq. (3), the time-evolution of mean-field operators can be written through a matrix $R(t)$ as $m(t) = R(t)m(0)$ (see also the general discussion in Ref. [44]). For the model considered, we have

$$R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[f(t)] & \sin[f(t)] \\ 0 & -\sin[f(t)] & \cos[f(t)] \end{pmatrix}. \quad (7)$$

The time-evolved covariance matrix $\tilde{\Sigma}(t)$ in the frame rotating with the mean-field operators, can be obtained by subtracting from $\Sigma(t)$ the evolution implemented by the unitary matrix $R(t)$, as $\tilde{\Sigma}(t) = R^T(t)\Sigma(t)R(t)$. This covariance matrix obeys the differential equation [47]

$$\dot{\tilde{\Sigma}}(t) = Q(t)\tilde{\Sigma}(t) + \tilde{\Sigma}(t)Q^T(t) + \Omega A(t)\Omega^T \quad (8)$$

with $Q(t) = \Omega B(t)$, where we have used the relation $K(t) = R^T(t)KR(t)$, for $K = A, B$. Note, that the matrix Ω is time independent and fixed to its initial value since, in this frame, mean-field operators do not evolve. Remarkably, for the time-evolution of fluctuations in Eq. (8), we can derive a proper bosonic dynamical generator. We find that the dynamics of any operator O is implemented by a propagator Λ_t, $O(t) = \Lambda_t[O]$, obeying the equation $\dot{\Lambda}_t = \Lambda_t \circ \mathcal{W}_t^*$, with $\mathcal{W}_t^*$ being the time-dependent Lindblad generator

$$\mathcal{W}_t^*[O] = \sum_{\alpha,\beta=1}^{3} C_{\alpha\beta}(t) \left[\tilde{F}_\alpha O \tilde{F}_\beta - \frac{1}{2}\left\{ \tilde{F}_\alpha\tilde{F}_\beta, O \right\} \right], \quad (9)$$

where $C(t) = A(t) + iB(t)$. This result is rather general and can be adapted to the collective spin models in [44].

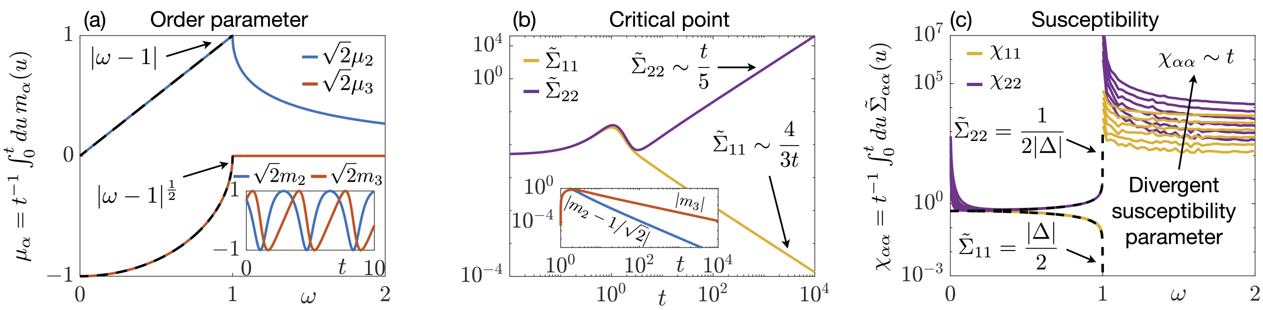

FIG. 2. **Order parameter and susceptibility.** (a) Time-averaged order parameter μ_α, for $t \to \infty$, from an initial state with $m_2 = m_3 = 1/2$. When the coherent rate $\omega < 1$, μ_α converges to the stationary value $m_\alpha(\infty)$ (dashed lines). Close to criticality, the stationary order parameter components show power-law behavior in $|\omega - 1|$ with different exponents. For $\omega > 1$, we observe limit-cycle oscillations (shown in the inset for $\omega = 2$). The oscillations of m_3 average to zero, as shown by μ_3, while μ_2 assumes an ω-dependent value. (b) At the critical point, $\omega = 1$, the order parameter components show different power-law decays [see Eq. (4) and inset in log-log scale]. The fluctuation $\tilde\Sigma_{11}$ algebraically tends to zero (large spin-squeezing) while the fluctuation $\tilde\Sigma_{22}$ diverges with a linear behavior in time, demonstrating a critical building up of (classical) correlations. (c) Linear-log plot of the susceptibility parameters $\chi_{\alpha\alpha}$ as a function of ω, for different times $t = 2^i \cdot 500$, with $i = 0, 1, 2, \ldots 5$. For $\omega < 1$, $\chi_{\alpha\alpha}$ converges to $\tilde\Sigma_{\alpha\alpha}(\infty)$ (dashed lines). In the time-crystalline phase, the susceptibility increases linearly with time.

For the subsequent analysis we use as initial state the one with all spins aligned with the positive direction of σ_3. Given that we look at the system from the frame rotating with the mean-field variables, we have that $[\tilde F_1, \tilde F_2] = i$ for all times, so that we can proceed with the identification $\tilde F_1 = x$ and $\tilde F_2 = p$, where x and p behave as position and momentum operators, respectively. We also find that $\tilde F_3 = 0$ since $\langle \tilde F_3 \rangle = 0$ and $\langle \tilde F_3^2 \rangle = 0$. Exploiting the result reported in Eq. (8), we find that the dynamics of quantum fluctuations is governed by the time-dependent generator

$$\mathcal{W}_t^*[O] = J_t^\dagger O J_t - \frac{1}{2}\left\{ J_t^\dagger J_t, O \right\}, \tag{10}$$

with the "jump operator" $J_t = x - i\cos[f(t)]p$. This generator clearly shows that fate of quantum fluctuations is also strongly linked with the asymptotic behavior of the time-dependent function $f(t)$.

For $\omega < 1$ the function $f(t)$ rapidly converges to a stationary value so that the generator $\mathcal{W}_t^*$ becomes asymptotically Markovian and we can easily compute the asymptotic state of quantum fluctuations, which is a squeezed vacuum state [47]. This implies that this phase features spin squeezing [59–61] with a squeezing parameter $\xi = |\Delta| = |\sqrt{\omega^2 - 1}|$ (see also Ref. [32]).

At criticality, $\omega = 1$, we find the following behavior for fluctuations (elements of the covariance matrix in the rotated frame)

$$\tilde\Sigma_{11}(t) \sim \frac{4}{3t}, \qquad \tilde\Sigma_{22}(t) \sim \frac{t}{5}. \tag{11}$$

The element $\tilde\Sigma_{11}$ tends to zero [see also Fig. 2(b)] and thus shows that, at criticality, the spin-squeezing parameter ξ tends to zero. This is witnessing the presence of strong quantum correlations in the spin ensemble. On the other hand, the element $\tilde\Sigma_{22}$ diverges linearly with time, as shown in Fig. 2(b). This growth of fluctuations is associated with the divergence of correlations in the ensemble, which usually occurs in second-order phase transitions.

For $\omega > 1$, the generator in Eq. (10) remains time-dependent, signalling that the dynamics of fluctuations is effectively non-Markovian [39]. In this regime, fluctuations show interesting critical behavior. For a stationary nonequilibrium phase transition, one would expect fluctuations to remain bounded away from the critical point, i.e. when $\omega \neq 1$. However, we find that the whole time-crystalline regime is characterized by divergent fluctuations. This is associated with the fact that the sustained oscillations of the mean-field operators determine, through the (dissipative) driving term in $\mathcal{W}_t^*$, an effective "diffusion" of fluctuations. This behavior is markedly different from an exponential "heating" of fluctuations which would instead be related to an instability of the mean-field behavior [53]. Specifically, we find

$$\tilde\Sigma_{11}(t) \sim \frac{\left(\Delta^4 + \Delta^2\right) t}{2\left(\Delta^2 + 2\cos^2\left[\frac{\Delta(t+k)}{2}\right] - \Delta\sin\left[\Delta(t+k)\right]\right)^2},$$
$$\tilde\Sigma_{22}(t) \sim \frac{\left(2\Delta^4 + 5\Delta^2 + 3\right) t}{2\left(\Delta^2 + 2\cos^2\left[\frac{\Delta(t+k)}{2}\right] - \Delta\sin\left[\Delta(t+k)\right]\right)^2}, \tag{12}$$

where $k = 2\tan^{-1}(1/\Delta)/\Delta$ [47]. The above quantities, which are well-defined for $\Delta \neq 0$, show an overall linear growth of fluctuations with time. To see this, we define the time-averaged susceptibility parameters $\chi_{\alpha\alpha}(t) = t^{-1}\int_0^t du\, \hat\Sigma_{\alpha\alpha}(u)$, which is plotted in Fig. 2(c). In the stationary regime, this converges to the stationary value $\tilde\Sigma_{\alpha\alpha}(\infty)$. For $\omega > 1$, however, these susceptibility parameters diverge with time, indicating that the boundary time-crystal phase is characterized by a critical (unbounded) build-up of correlations which usually solely occurs at a phase transition point.

Discussion.— We have provided an exact solution for the paradigmatic boundary time-crystal model introduced in Ref. [9]. In the thermodynamic limit, the dynamical behavior of the order parameter is captured by a set of nonlinear differential equations [see Eq. (2)], which, in certain regimes, can feature persistent oscillations. Our analytical solution shows that a boundary time-crystal is indeed an intricate many-body phase whose physics is much richer than that of a (classical) non-linear system. Remarkably, we have shown that the breaking of the time-translation symmetry becomes manifest in the quantum fluctuations, which evolve under a non-Markovian dynamics (in the sense of Ref. [39]) in the time-crystalline phase. In contrast, in the stationary phase their dynamics is asymptotically Markovian. Moreover, the analysis of the long-time behavior of quantum fluctuations reveals that the boundary time-crystalline phase is characterized by a diverging power-law growth (with dynamical exponent 1) of correlations. It thus appears that the entire time-crystal phase is in fact critical.

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